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GOING THE EXTRA MILE: EFFORT BY WORKERS AND JOB-SEEKERS

by Matthias S. Hertweck*, Vivien Lewis** and Stefania Villa***

Abstract

We introduce two types of effort into an otherwise standard labor search model to examine equilibrium determinacy. Indeterminacy occurs when wages rise sharply in response to a labor market tightening. Variable *labor* effort gives rise to short-run increasing returns to hours in production. This raises workers' marginal product and wages, expanding the region of indeterminacy. Variable *search* effort makes workers search more intensively in a tighter labor market, which limits the rise in wages and shrinks the region of indeterminacy. Indeterminacy disappears completely when vacancy posting costs are replaced with hiring costs.

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Contents

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1 Introduction[1]

We investigate how two types of effort, namely labor effort by the employed and search effort by the unemployed, affect existence and uniqueness of the equilibrium dynamics in an otherwise standard labor search-and-matching model à la Diamond-Mortensen-Pissarides, henceforth DMP (Pissarides, 2000). Why is introducing these two specific features in a search-and-matching model worthy of investigation? Variable search effort is well-documented and helps to replicate the sharp fall of the hiring rate during the Great Recession and its slow recovery thereafter (Leduc and Liu, 2020).[2] Variable labor effort is supported by evidence from time use surveys (Burda et al., 2019) and can explain the procyclicality of labor productivity observed across many countries and time periods.[3]

Krause and Lubik (2010) show that the standard search-and-matching model is indeterminate if the workers' share of the match surplus far exceeds their contribution to match success. In this paper, we elucidate the mechanism through which indeterminacy comes about in labor search models. It works as follows. A rise in labor market tightness has a positive effect on wages ('*wage channel*'), which reduces the returns to hiring. If this effect is very large relative to the rise in effective hiring costs ('*hiring cost channel*'), it can overturn the initial incentive to hire, leading to non-stationarity in vacancy posting and thereby to equilibrium indeterminacy. In response to a labor market tightening, the Nash wage rises strongly if the bargaining share of workers is high. This leads to indeterminacy if, at the same time, matches are very elastic to vacancies, such that effective hiring costs per worker become relatively unresponsive to changes in labor market tightness.

For a number of model variants, we examine equilibrium determinacy in the unit square spanned by the match elasticity to unemployment and the workers' bargaining share. The indeterminacy region is a triangle in the top left corner which is far away from the Hosios condition. First, we show that in the standard search model, indeterminacy regions are larger if labor markets are more fluid or if the value of leisure is close to that of working, as suggested in Hagedorn and Manovskii (2008). We also show how labor market policies affect equilibrium determinacy. Second, in an extended model where variable labor effort

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[2] See also the empirical evidence in Shimer (2004), Krueger and Mueller (2010), Faberman et al. (2017), Mukoyama et al. (2018), Pan (2019).

[3] See Lewis et al. (2019) for the euro area. Additional related works using variable labor effort include Bils and Cho (1994), Barnichon (2010), and Galí and van Rens (2014), among others.

leads to short-run increasing returns to hours, the wage becomes more volatile — relative to future hiring costs — expanding the region of indeterminacy, albeit to a small extent. Third, in an extended model with search effort, fluctuations in tightness are dampened as workers react to a tighter labor market by driving up their search intensity. The cost of searching reduces the worker's outside option and hence the Nash wage. This shrinks the region of indeterminacy. Our key insight is thus that introducing effort by workers and job-seekers has opposing effects on equilibrium uniqueness.

While the search-and-matching model of the labor market is undoubtedly the most popular model of labor market frictions, it does suffer from certain deficiencies (see Shimer, 2005). In a competing approach vacancy posting costs are replaced with hiring costs (see, for example, Gertler and Trigari, 2009; Pissarides, 2009; Hertweck, 2013; Galí and van Rens, 2014; Christiano et al., 2016). This leads to more sluggish vacancy dynamics in response to technology shocks. We show that indeterminacy completely disappears in this alternative model.

Our exercise is useful for the development and calibration of empirically sound business cycle models with labor search, and for understanding the role of beliefs in business cycles. Business cycles driven by fundamental shocks, i.e. to technology or preferences, are not generally inefficient and as such do not warrant any policy response. Instead, under equilibrium indeterminacy and multiple equilibria, self-fulfilling beliefs can lead to inefficient fluctuations and macroeconomic volatility. In that case, there is room for policy to stabilize the economy and raise economic welfare (Farmer and Guo, 1994).

Related literature and contribution. The paper speaks to two strands of the literature, labor search-and-matching models, and models with equilibrium indeterminacy. Regarding the former, it is now well established that labor markets are not perfectly flexible, but are instead characterized by considerable frictions as workers do not seamlessly move from one job to another. The search-and-matching framework presented by Diamond, Mortensen and Pissarides (Pissarides, 2000) has emerged as a consensus model to characterize the labor market, with Merz (1995) and Andolfatto (1996) integrating it into real business cycle theory.

The second strand of the literature is on indeterminacy in macroeconomics.[4] Farmer and Guo (1994) argue that sunspot shocks should be taken seriously as a potential source of business cycle fluctuations — rather than being a mere intellectual curiosity. As shown by Benhabib and Farmer (1994), increasing returns in the production function can be a source of indeterminacy, leading to multiple equilibria. Wen (1998) demonstrates that, in a real business cycle model with capacity utilization, indeterminacy can arise under an empirically

[4]For surveys of this literature, see Benhabib and Farmer (1999) and Farmer (2019).

plausible calibration in the case of (mildly) increasing returns. Our work is related in that variable labor effort generates increasing returns to hours in production, providing a potential source of indeterminacy.

Krause and Lubik (2010) and Lazaryan and Lubik (2019) bring the two strands of the literature together.[5] Our contribution is to elucidate and to extend their results. We characterize two channels through which vacancy posting affects the asset value of a worker, the wage channel and the hiring cost channel. If the first channel is comparatively strong, the equilibrium solution is indeterminate. First, we consider different calibration strategies of the baseline model without effort. Then, we analyze determinacy when the model is extended to include variable labor effort and variable search effort.

The remainder of the paper is structured as follows. Section 2 presents our model with two types of effort. In the linearized model written as a two-equation system, the transition matrix is derived and its roots determine whether a stable model solution exists and is unique. Then Section 3 derives the condition for indeterminacy in the standard labor search model and discusses how equilibrium uniqueness is related to search externalities. This is done analytically in the case of risk neutrality. For the more general case with risk aversion, it conducts a numerical exercise showing how determinacy depends on a set of parameter values. Section 4 shows first how variable labor effort affects the determinacy results. It then presents a similar analysis for variable search effort. Section 5 derives an alternative model with hiring costs, and discusses its determinacy properties. Finally, Section 6 concludes.

2 Model

In the following, we outline our search-and-matching model featuring two additional labor margins, hours and effort, as well as variable search intensity by job-seekers.[6]

Abstracting from a participation margin, we normalize the labor force to unity, such that

$$n_t + u_t = 1, \tag{1}$$

where n_t denotes employment and u_t is the unemployment rate. The law of motion for employment is

$$n_{t+1} = (1-\rho)(n_t + m_t), \tag{2}$$

[5] Zanetti (2006) studies a New Keynesian model with search-and-matching frictions where the monetary policy rule generates indeterminacy. Farmer (2019) argues that a meaningful theory of involuntary unemployment and large welfare losses must allow for *steady state* indeterminacy. The present paper focuses on *dynamic* equilibrium determinacy.

[6] We build on the analysis in Krause and Lubik (2010). Unlike Hashimzade and Ortigueira (2005), we abstract from physical capital.

with initial employment n_0 given. The parameter $\rho \in (0, 1)$ captures the job separation rate and m_t is the number of new job matches. A constant separation rate is justified by the observation that — in comparison with the job creation margin — the empirical counterpart of ρ is fairly stable over the US business cycle (for evidence on this, see Hall, 2005; Fujita and Ramey, 2009; Shimer, 2012). The matching technology is a function of unemployed workers, their search intensity s_t, and vacancies v_t,

$$m_t = \chi(s_t u_t)^{\xi} v_t^{1-\xi}, \tag{3}$$

where $\xi \in (0, 1)$ is the match elasticity to 'total search effort' (Merz, 1995) and $\chi > 0$ captures the efficiency of the matching process. Petrongolo and Pissarides (2001) argue that the Cobb-Douglas form for the matching function is a stylized fact compatible with a large number of empirical studies. By spending more time and resources searching for jobs, unemployed workers can raise the probability of match success. Search intensity entering the matching function multiplicatively with unemployment can be thought of as 'input-augmenting' (Pissarides, 2000), similar to technological progress in the production function for goods.

The representative household is composed of n_t workers whose wage income is $w_t h_t$ each, and u_t unemployed members who receive unemployment benefits b and spend resources $\mathcal{G}(s_t)$ on searching for a job. Households choose a path for consumption $\{C_t\}_{t=0}^{\infty}$ to maximize expected lifetime utility,

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma} - 1}{1 - \sigma} - n_t g(h_t, e_t) \right], \tag{4}$$

subject to the budget constraint $C_t + T_t = n_t w_t h_t + (1 - n_t)(b - \mathcal{G}(s_t))$, where T_t are lump-sum taxes, $\beta \in (0, 1)$ is the subjective discount rate, $\sigma \geq 0$ is the parameter of relative risk aversion and $g(h_t, e_t)$ measures individual disutility of providing hours of work h_t and effort per hour e_t. Unemployment benefits are financed through lump-sum taxes. For simplicity, we abstract from public debt and stipulate that the government budget constraint is balanced in each period, i.e. $T_t = u_t b$ for all t.

In period t, an employed worker receives the wage income $w_t h_t$. In the next period, he is either still employed with probability $(1 - \rho)$, in which case he has an expected value of $E_t\{\beta_{t,t+1} W_{t+1}\}$, or the employment relation is dissolved with probability ρ, then his expected value is $E_t\{\beta_{t,t+1} U_{t+1}\}$. The worker's asset value therefore is

$$W_t = w_t h_t + E_t\{\beta_{t,t+1}[(1 - \rho) W_{t+1} + \rho U_{t+1}]\}, \tag{5}$$

where $\beta_{t-1,t} = \beta\lambda_t/\lambda_{t-1}$ is the household's stochastic discount factor and $\lambda_t = C_t^{-\sigma}$ is the marginal utility of consumption. The value of being unemployed U_t is in turn given by

$$U_t = b - \mathcal{G}(s_t) + E_t\{\beta_{t,t+1}[p_t(1-\rho)W_{t+1} + (1 - p_t(1-\rho))U_{t+1}]\}. \tag{6}$$

The term $b - \mathcal{G}(s_t)$ can be thought of as the (net) value of leisure or non-market activity, measured in terms of consumption goods. In the next period, the unemployed person faces a probability p_t of finding a new job, which has an expected value of $E_t\{\beta_{t,t+1}W_{t+1}\}$, and a probability $1 - p_t$ of remaining unemployed, which has an expected value of $E_t\{\beta_{t,t+1}U_{t+1}\}$. The job finding rate is defined as the number of matches over unemployment, $p_t = m_t/u_t$. Defining the worker's surplus as $\mathcal{W}_t \equiv W_t - U_t$, we can subtract (6) from (5) to write the match surplus going to the household as

$$\mathcal{W}_t = w_t h_t - g(h_t, e_t)/\lambda_t - (b - \mathcal{G}(s_t)) + (1-\rho)E_t\left\{\beta_{t,t+1}(1-p_t)\mathcal{W}_{t+1}\right\}. \tag{7}$$

One-worker firms produce consumption goods y_t. Let $\mathcal{J}_t$ denote the firm's match surplus, i.e. the value to the firm of hiring a worker. It is the sum of current profits, i.e. output minus the wage bill $w_t h_t$, and the firm's continuation value. The latter is the expected future match surplus in case the employment relationship continues, which happens with probability $(1-\rho)$. The firm's value is zero in case the worker and the firm separate, which happens with probability ρ. Thus,

$$\mathcal{J}_t = y_t - w_t h_t + (1-\rho)E_t\{\beta_{t,t+1}\mathcal{J}_{t+1}\}. \tag{8}$$

The value of posting a vacancy is given by the negative of the vacancy posting cost c, plus the expected future value of the vacancy. The latter is a weighted average of the value of filling the vacancy, i.e. the firm's match value in the next period, which has probability $q_t(1-\rho)$, and the future value of the unfilled vacancy, V_{t+1}, which has probability $(1-q_t(1-\rho))$, where $q_t = m_t/v_t$ is the vacancy filling rate. Therefore,

$$V_t = -c + E_t\{\beta_{t,t+1}[q_t(1-\rho)\mathcal{J}_{t+1} + (1 - q_t(1-\rho))V_{t+1}]\}. \tag{9}$$

Free entry drives the value of a vacancy to zero at each point in time, such that $V_t = 0$ for all t and thus (9) becomes

$$c/q_t = (1-\rho)E_t\{\beta_{t,t+1}\mathcal{J}_{t+1}\}. \tag{10}$$

Combining the firm's asset value (8) and the free entry condition (10), we get the following expression for the firm's match surplus: $\mathcal{J}_t = y_t - w_t h_t + (1-\rho)c/q_t$. Finally, using this to substitute out $\mathcal{J}_{t+1}$ in the free entry condition (10), we obtain the vacancy posting condition,

$$c/q_t = (1-\rho)E_t\left\{\beta_{t,t+1}\left(y_{t+1} - w_{t+1}h_{t+1} + c/q_{t+1}\right)\right\}. \tag{11}$$

Hiring is an investment decision where the intertemporal dimension, more specifically the expected value of a marginal worker, is key. Equation (11) states that the current cost of posting a vacancy, c/q_t, must equal the expected benefit of posting a vacancy, which consists of three terms: (1) the output produced y_t, (2) wage payments $w_t h_t$, and (3) the savings on future vacancy posting costs due to a successful match. The transversality condition is

$$\lim_{T\to\infty} E_t\{\beta_{t,T}\mathcal{J}_T n_T\} = 0, \tag{12}$$

see also Mortensen (2009). Under Nash bargaining, the real wage maximizes the joint match surplus $\mathcal{W}_t^{\eta}\mathcal{J}_t^{1-\eta}$, where $\eta \in (0,1)$ is the worker's bargaining share. The surplus sharing rule is $(1-\eta)\mathcal{W}_t = \eta\mathcal{J}_t$, and the bargaining wage satisfies

$$w_t h_t = \eta(y_t + c\theta_t) + (1-\eta)[g(h_t, e_t)/\lambda_t + (b - \mathcal{G}(s_t))], \tag{13}$$

where $\theta_t \equiv v_t/u_t$ is labor market tightness. Equation (13) shows the direct effects of variable labor effort and search effort on the bargaining wage. The former increases output, y_t, and labor disutility, $g(h_t, e_t)$, while the latter increases search costs, $\mathcal{G}(s_t)$, which reduces the worker's outside option. The two types of effort have opposite effects on the bargaining wage.

Finally, goods market clearing requires that consumption equals net aggregate output, $C_t = Y_t$. In a symmetric equilibrium, the latter is total output produced by all firms, less the resources used up in vacancy posting and search activities, $Y_t = y_t n_t - cv_t - u_t\mathcal{G}(s_t)$.

Labor effort. The firm's production function is given by the product of hours of work and effort per hour, as follows:

$$y_t = e_t h_t. \tag{14}$$

Worker effort is modeled as in Bils and Cho (1994), who assume that labor disutility is given by

$$g(h_t, e_t) = \frac{\lambda_h h_t^{1+\sigma_h}}{1+\sigma_h} + h_t \frac{\lambda_e e_t^{1+\sigma_e}}{1+\sigma_e}. \tag{15}$$

The parameters $\sigma_h > 0$ and $\sigma_e > 0$ measure, respectively, the curvature of the labor disutility function in hours and effort, while $\lambda_h > 0$ and $\lambda_e > 0$ are the weights on hours and effort in labor disutility.

Every period, workers choose their supply of hours and effort in order to maximize expected lifetime utility (6), with labor disutility given by (15), subject to the firm's production technology. In other words, the firm requires a certain number of effective labor hours to meet its production target. However, the worker has discretion over the combination of hours and effort per hour that gets the job done while minimizing disutility. The firm pays the worker by the hour and does not monitor effort.

Equilibrium effort is an increasing and convex function of hours per worker,

$$e_t = e_0 h_t^{\sigma_h/(1+\sigma_e)}, \tag{16}$$

where $e_0 = (\frac{1+\sigma_e}{\sigma_e} \frac{\lambda_h}{\lambda_e})^{1/(1+\sigma_e)}$. Using the optimal effort choice, we can rewrite labor disutility as a function of hours only,

$$g(h_t) = \lambda_h \frac{1 + \sigma_h + \sigma_e}{(1 + \sigma_h)\,\sigma_e} h_t^{1+\sigma_h}, \tag{17}$$

and the production function becomes

$$y_t = e_0 h_t^{\phi}, \tag{18}$$

with $\phi = 1 + \sigma_h/(1 + \sigma_e)$ measuring the returns to hours in production. For a given elasticity of labor disutility to hours σ_h, a finite value for σ_e implies that there are increasing returns to hours in production ($\phi > 1$), i.e. a one percent increase in hours worked increases output by more than one percent. The constant-effort model is recovered as the limiting case where $\sigma_e \to \infty$; any incremental rise in effort would lead to an overwhelmingly large utility loss, such that in equilibrium effort does not change.

Hours worked are determined jointly by the firm and the worker to maximize the sum of the firm's and worker's surpluses, respectively $\mathcal{J}_t$ and $\mathcal{W}_t$. Hours per worker thus satisfy

$$\phi e_0 h_t^{\phi-1} = g'(h_t)/\lambda_t, \tag{19}$$

where $g'(h_t)$ denotes the worker's disutility from working an additional hour. By (19), the marginal product of hours must equal the marginal rate of substitution between hours and consumption.

Search effort. Suppose that the search cost function is given by

$$\mathcal{G}(s_t) = s_t^{1+\zeta}/(1 + \zeta). \tag{20}$$

The household chooses the optimal amount of search intensity up to the point where the marginal search costs and the benefits from searching just balance out. As explained in chapter 5 of Pissarides (2000), worker i chooses s_i, taking the aggregate job finding rate p_t and labor market tightness θ_t as given. His personal job finding rate does, however, depend upon his search intensity, $p_{it} = p_t(s_{it}; s_t, \theta_t)$. For each efficiency unit supplied in the search process, workers transition from unemployment to employment at rate $\frac{m_t}{s_t u_t}$. Therefore, the transition probability of worker i per period is given by $p_{it} = \frac{m_t}{s_t u_t} s_{it}$, and the derivative is $\frac{\partial p_{it}}{\partial s_{it}} = \frac{p_t}{s_t}$. In equilibrium, search intensity is positively related to labor market tightness,

$$s_t^{\zeta} = [\eta/(1-\eta)]c\theta_t/s_t. \tag{21}$$

The left hand side of (21) is the marginal cost of exerting search effort. The right hand side is the contribution of one efficiency unit of search to expected value of employment, $\frac{\partial p_{it}}{\partial s_{it}}(1-\rho)E_t\{\beta_{t,t+1}\mathcal{W}_{t+1}\}$, which we can combine with the surplus sharing rule and the free entry condition for vacancies (10), to obtain (21). Table 1 reports the equilibrium conditions. To summarize, equation (a) represents the definition of the unemployment rate; equation (b) the definition of the labor market tightness; equation (c) the definition of the job finding rate; equation (d) the definition of the probability of a vacancy being filled; equation (e) the law of motion for employment; equation (f) the matching function; equation (g) the optimality condition for hours; equation (h) the production function; equation (i) the aggregate resource constraint; equation (j) the vacancy posting condition; equation (k) the optimality condition for the bargaining wage; equation (l) the equilibrium search intensity; equation (m) the definition of labor disutility given the optimal effort choice; and equation (n) the definition of the search cost function.

Definition 1. *A decentralized equilibrium in the labor search model with two types of effort is a set of infinite sequences for quantities* $\{u_t, \theta_t, n_{t+1}, m_t, h_t, y_t, Y_t, v_t, s_t\}_{t=0}^{\infty}$, *matching rates* $\{q_t, p_t\}_{t=0}^{\infty}$ *and wages* $\{w_t\}_{t=0}^{\infty}$, *satisfying the transversality condition (12), such that:*

1. *given matching rates and wages, the quantities solve the household's problem,*
2. *given matching rates and wages, the quantities solve the firm's problem,*
3. *employment is determined by the law of motion (2),*
4. *matching rates are determined by the matching function (3),*
5. *wages solve the Nash bargaining problem,*
6. *goods markets clear.*

Table 1: Model equilibrium conditions

Unemployment, u_t	$(a)\, u_t = 1 - n_t$
Tightness, θ_t	$(b)\, \theta_t = v_t/u_t$
Job finding rate, p_t	$(c)\, p_t = m_t/u_t$
Vacancy filling rate, v_t	$(d)\, q_t = m_t/v_t$
Employment, n_t	$(e)\, n_{t+1} = (1-\rho)(n_t + m_t)$
Matches, m_t	$(f)\, m_t = \chi(s_t u_t)^{\xi} v_t^{1-\xi}$
Hours, h_t	$(g)\, e_0 \phi h_t^{\phi-1} = (1+\sigma_h+\sigma_e)\lambda_h h_t^{\sigma_h} Y_t^{\sigma}/\sigma_e$
Production, y_t	$(h)\, y_t = e_0 h_t^{\phi}$
GDP, Y_t	$(i)\, Y_t = y_t n_t - c v_t - \mathcal{G}(s_t) u_t$
Vacancies, q_t	$(j)\, c/q_t = (1-\rho)\beta E_t \left\{ (Y_t/Y_{t+1})^{\sigma} \left(y_{t+1} - w_{t+1} h_{t+1} + c/q_{t+1} \right) \right\}$
Wages, w_t	$(k)\, w_t h_t = \eta(y_t + c\theta_t) + (1-\eta)[g(h_t) Y_t^{\sigma} + (b - \mathcal{G}(s_t))]$
Search intensity, s_t	$(l)\, (1-\eta) s_t^{1+\zeta} = \eta c \theta_t$
Labor disutility, $g(h_t)$	$(m)\, \sigma_e(1+\sigma_h) g(h_t) = (1+\sigma_h+\sigma_e)\lambda_h h_t^{1+\sigma_h}$
Search cost, $\mathcal{G}(s_t)$	$(n)\, (1+\zeta)\mathcal{G}(s_t) = s_t^{1+\zeta}$

We linearize the equilibrium conditions around their non-stochastic steady state. Letting a hat above a variable denote that variable's linear approximation, the system can be condensed into two equilibrium conditions determining one control variable, $\hat{\theta}_t$, and one state variable, $\hat{n}_{t+1}$,

$$\alpha_1 E_t\{\hat{\theta}_{t+1}\} = \left[\xi + \sigma \frac{1}{\delta_1} \left(\frac{cv}{Y} + \frac{\mathcal{G}(s)u}{Y} \right) + \sigma\alpha_2 \left(1 - \xi + \frac{\xi}{1+\zeta} \right) \rho \right] \hat{\theta}_t - \sigma\alpha_2 \left(\frac{\rho}{u} - \frac{\delta_2 - 1}{\delta_2} \right) \hat{n}_t, \tag{22}$$

$$\hat{n}_{t+1} = \left(1 - \xi + \frac{\xi}{1+\zeta} \right) \rho \hat{\theta}_t + \frac{u-\rho}{u} \hat{n}_t. \tag{23}$$

In (22), α_1 and α_2 are defined as follows,

$$\alpha_1 = \beta(1-\rho)\frac{\zeta}{1+\zeta}\xi \left(1 - \frac{\eta}{\xi} p \right) + \sigma \frac{\delta_2}{\delta_1} \left(\frac{cv}{Y} + \frac{\mathcal{G}(s)u}{Y} \right),$$

$$\alpha_2 = \sigma \frac{\delta_2}{\delta_1} \left(1 + \frac{c\theta}{Y} + \frac{\mathcal{G}(s)}{Y} \right),$$

and we introduce the composite parameters $\delta_1 \geq 1$ and $\delta_2 \geq 1$,

$$\delta_1 = 1 + \sigma \left(1 + \frac{c\theta}{Y} + \frac{\mathcal{G}(s)}{Y} \right) \frac{\phi}{(1+\sigma_h) - \phi},$$

$$\delta_2 = 1 + \beta(1-\rho)(1-\eta)\frac{\phi}{1+\sigma_h}\frac{e_0 h^{\phi}}{c/q}.$$

We can write the two-equation system (22) and (23) in a more compact way:

$$\begin{bmatrix} E_t\{\hat{\theta}_{t+1}\} \\ \hat{n}_{t+1} \end{bmatrix} = \begin{bmatrix} \left\{\frac{\xi}{\alpha_1} + \frac{\sigma}{\alpha_1\delta_2}\left(\frac{cv}{Y} + \frac{\mathcal{G}(s)u}{Y}\right) + \sigma\frac{\alpha_2}{\alpha_1}\left(1-\xi+\frac{\xi}{1+\zeta}\right)\rho\right\} & -\sigma\frac{\alpha_2}{\alpha_1}\left(\frac{\rho}{u} - \frac{\delta_2 - 1}{\delta_2}\right) \\ \left(1-\xi+\frac{\xi}{1+\zeta}\right)\rho & 1-\frac{\rho}{u} \end{bmatrix} \begin{bmatrix} \hat{\theta}_t \\ \hat{n}_t \end{bmatrix}. \tag{24}$$

The model has a (locally) unique stable solution if and only if the transition matrix in (24) has one stable root (i.e. smaller than 1) and one unstable root (i.e. greater than 1). Then θ_t can be solved forward in terms of the state variable n_t and the model is characterized by saddle-path stability. If, instead, both roots are unstable, the model solution is non-existent. Finally, if both roots are stable, the solution is indeterminate and multiple equilibria exist. This means that any initial value of θ is consistent with the model's equilibrium condition in Table 1.[7]

3 Determinacy in the standard labor search model

Having derived the model and its linearized representation, we now analyze its determinacy properties. We proceed as follows. First, we characterize determinacy in the standard labor search model with constant (labor and search) effort. We do so analytically for the case of risk neutral households. Second, we discuss how search externalities may lead to indeterminacy in this framework. Third, to analyze the more general case with risk aversion, we resort to numerical methods.

3.1 Analytical results under risk neutrality

It is instructive to study the case of risk neutrality where $\sigma = 0$, also shown by Krause and Lubik (2010). In that case, $\alpha_2 = \delta_2 = 0$, $\delta_1 = 1$, and $\alpha_1 = \beta(1-\rho)\frac{\zeta}{1+\zeta}\xi(1-\frac{\eta}{\xi}p)$. The dynamic system simplifies to

$$\begin{bmatrix} E_t\{\hat{\theta}_{t+1}\} \\ \hat{n}_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\beta(1-\rho)(1-\frac{\eta}{\xi}p)} & 0 \\ (1-\xi+\frac{\xi}{\zeta})\rho & 1-\frac{\rho}{u} \end{bmatrix} \begin{bmatrix} \hat{\theta}_t \\ \hat{n}_t \end{bmatrix}. \tag{25}$$

[7] A more general condition for a determinate solution is that there are as many non-predetermined variables as non-explosive roots of Φ in the system $z_t = \Phi z_{t+1} + \Gamma e_{t+1}$, where z are the endogenous variables and e are the exogenous shocks (see Benhabib and Farmer, 1999).

Since the transition matrix in (25) is lower triangular, its roots are given by its diagonal elements. Thus,

1. the model solution is **unique** if and only

   (a) either $|\frac{1}{\beta(1-\rho)(1-\frac{\eta}{\xi}p)}| < 1$ and $|1 - \frac{\rho}{u} > 1|$, that is if $|\beta(1-\rho)(1-\frac{\eta}{\xi}p)| > 1$ and $2u < \rho$.

   (b) or $|\frac{1}{\beta(1-\rho)(1-\frac{\eta}{\xi}p)}| > 1$ and $|1 - \frac{\rho}{u}| < 1$, that is if $|\beta(1-\rho)(1-\frac{\eta}{\xi}p)| < 1$ and $\rho < 2u$.

2. the model solution is **indeterminate** if $|\beta(1-\rho)(1-\frac{\eta}{\xi}p)| > 1$ and $\rho < 2u$.

3. the model solution is **non-existent** if $|\beta(1-\rho)(1-\frac{\eta}{\xi}p)| < 1$ and $2u < \rho$.

Regardless of parameter choices, the model itself provides further restrictions that influence equilibrium existence and uniqueness.

Proposition 1. *Under risk neutrality, the labor search model has at least one stable solution for all admissable values of the steady state unemployment rate and the job separation rate, i.e. those that lie on the unit interval.*

Proof. Combining the law of motion for employment (2) and the constant labor force assumption (1) at the steady state, the job finding rate can be expressed as $p = \frac{\rho}{1-\rho}\frac{1-u}{u}$. For the job finding rate to be strictly lower than unity at the steady state, the separation rate must not exceed the steady state unemployment rate, $\rho < u$. Therefore, under our calibration strategy that fixes u, and given the above cross-parameter restriction, the second root of the transition matrix is stable. This rules out equilibrium non-existence, Case 3, as well as Case 1(a). □

Proposition 2. *Under risk neutrality, the labor search model is characterized by equilibrium indeterminacy if the worker's bargaining weight exceeds the match elasticity to unemployment, thereby violating the Hosios (1990) condition, to a sufficiently large degree.*[8]

Proof. Consider the root $1/[\beta(1-\rho)(1-\frac{\eta}{\xi}p)]$. Notice that, since β, $(1-\rho)$ and p all lie between 0 and 1, it is clear that under the so-called 'Hosios condition', $\eta = \xi$, we have that $|\beta(1-\rho)(1-p)| < 1$ and thus the first root of the transition matrix is unstable. Therefore, the Hosios condition ensures equilibrium uniqueness.[9] Indeterminacy arises if the first root is stable, which occurs if the Hosios condition is violated to a sufficiently large degree.

[8] The indeterminacy frontier derived below can also be found in Lazaryan and Lubik (2019) for the global solution to the simple search model.

[9] This result has been noted in Bhattacharya and Bunzel (2003), Krause and Lubik (2010), Lazaryan and Lubik (2019).

More specifically, we need that $|\beta(1-\rho)(1-\frac{\eta}{\xi}p)| > 1$, which requires that the workers' bargaining power, which measures the share of the match surplus going to workers, exceeds by a sufficiently large amount the workers' contribution to match success, which is captured by the match elasticity to unemployment, i.e. when $\eta >> \xi$. Rearranging the indeterminacy condition, we find that indeterminacy arises if and only if

$$\eta > \frac{1+\beta(1-\rho)}{\beta(1-\rho)p}\xi. \tag{26}$$

Condition (26) shows that, on the (ξ, η)-plane, the indeterminacy frontier is a straight line with slope greater than 1, where we again assume that $\rho < u$. All (ξ, η)-pairs above this line are associated with an indeterminate model solution. □

The slope of the indeterminacy frontier depends positively on the separation rate and negatively on the steady state job finding rate. This means that the indeterminacy region is larger, the lower is the separation rate and the higher is the steady state job finding rate.[10]

1. In labor markets characterized by frequent *outflows* from unemployment, indeterminacy is more prevalent than otherwise. To see this, consider that the slope of the indeterminacy frontier (26) is approximately $2/p$, since $\beta(1-\rho)$ is close to 1. Thus, a higher p reduces the slope, making it more likely for the inequality to be satisfied.

2. Labor markets characterized by infrequent *inflows* into unemployment are more prone to indeterminacy. Recall that in this framework, hiring is an investment activity with a stream of future benefits in the form of profits and vacancy posting costs saved. Discounting takes into account both impatience, captured by β, and separations, captured by ρ. Lower discounting — either due to a higher β or a lower ρ — makes the future benefits from hiring more elastic; this raises the probability of an indeterminate equilibrium.

Proposition 3. *Under risk neutrality, neither variable labor effort nor variable search effort has any bearing on equilibrium determinacy.*

Proof. The proof is straightforward from the fact that the two roots of the transition matrix are independent of ζ, as well as σ_h and σ_e. □

[10]Note that, in the steady state, unemployment outflows equal unemployment inflows. Hence, the two rates are not independent of each other, but related as follows: $p = [\rho/(1-\rho)][u/(1-u)]$.

3.2 Search externalities and indeterminacy

The search-and-matching model has inherent externalities. The probability of a vacancy being filled $q(\theta)$ and the job finding rate $p(\theta)$ both depend on labor market tightness. The vacancy filling rate decreases with the number of vacancies. As a firm posts more vacancies, it becomes more difficult for other firms to fill their open positions. This constitutes an externality, because a firm does not take into account that an additional vacancy increases the hiring costs to other firms. Similarly, the job finding rate decreases with the size of the unemployment pool. When an additional unemployed worker searches for a job, or when an unemployed worker exerts additional search effort, this reduces the chances of other job-hunters getting hired. This phenomenon of more agents searching on the same side of the market thus gives rise to a negative *congestion externality*. The probability of a vacancy being filled, instead, increases with the number of unemployed workers and the job finding rate increases with the number of vacancies. More agents searching on the other side of the market causes a positive *trading externality* (Yashiv, 2007) or *thick-market effect* (Petrongolo and Pissarides, 2001).

As explained by Petrongolo and Pissarides (2001), the elasticity of matches to the number of unemployed workers, parameter ξ in the matching function, governs the size of the search externalities. Given the Cobb-Douglas form of the matching function, a lower elasticity, i.e. a higher value of $1-\xi$, implies

1. less congestion of firms on each other. If $(1-\xi)$ is close to 1, the matching function is almost linear in vacancies. A variation in v thus leads to an almost one-to-one change in m. Consequently, the vacancy filling rate $q = m/v$ moves only little, which implies that also effective hiring costs are inelastic to variations in θ.

2. a greater positive trading externality (thick-market effect) caused by firms on searching workers. If $(1-\xi)$ is close to 1, the matching function responds very little to changes in u. Hence, the following negative feedback effect is very weak: Consider a rise in v, which induces a rise in m. This leads to a fall in u, which in turn dampens the rise in m. For this reason, fluctuations in the job finding rate $p = m/u$ are amplified.

In the following, we abstract from hours, labor effort, search effort, and risk aversion and thus consider the baseline DMP model with exogenous output, $y_t = y$. We can rewrite the vacancy posting condition (11) in terms of labor market tightness,

$$(c/\chi)\theta_t^{\xi} = (1-\rho)\beta E_t\{(1-\eta)(y-b) \underbrace{-\eta c\theta_{t+1}}_{\text{wage channel}} \underbrace{+(c/\chi)\theta_{t+1}^{\xi}}_{\text{hiring cost channel}} \}. \tag{27}$$

In (27), the current 'asset price' of a worker, represented by θ_t, depends on the expected future price. As explained in Blanchard (1979), uniqueness of the solution depends on whether the elasticity of the current price to next period's expected price is greater or less than 1 in absolute value.

When firms post more vacancies, this leads to a tightening of the labor market. In turn, this affects the returns to hiring — the right hand side of (27) — in two opposite ways. On the one hand, the congestion effect increases the returns to hiring through the *hiring cost channel*. By lowering the vacancy filling rate q, a labor market tightening increases vacancy duration $1/q$ and thus effective hiring costs c/q. This encourages vacancy posting today, since a successful hire today implies savings on future vacancy posting. On the other hand, a tighter labor market increases the workers' outside option of labor market search and thus the Nash wage. This reduces the returns to hiring through the *wage channel*.

The vacancy posting condition (27) balances out these two effects on profits under a standard calibration, leading to a unique equilibrium. However, if the strength of the wage channel far exceeds the strength of the hiring cost channel, such that the consolidated coefficient on $E_t\{\theta_{t+1}\}$ in (27) is smaller than -1, the model solution becomes indeterminate. This can happen when:

1. effective future hiring costs are inelastic with respect to θ, which is the case when ξ is small so that there is little congestion, and

2. the wage is very responsive to labor market conditions, which is the case when the workers' bargaining share η is high.

To sum up, indeterminacy in the standard DMP model with risk neutrality arises when firms exert a large thick-market externality on unemployed workers without being appropriately compensated; η is far in excess of ξ. Then the hiring cost channel of vacancy posting on the worker's asset value is weak and the wage channel is strong.

3.3 Numerical results under risk aversion

We calibrate and examine a two-dimensional continuum of models on the (ξ,η)-plane, where $\xi, \eta \in (0, 1)$. To maintain comparability, steady state unemployment should remain constant across all calibrated models. Hence, we adjust the leisure value b simultaneously whenever a change in the match elasticity ξ or the bargaining parameter η is examined.

Additionally, we present indeterminacy regions for two alternative calibration strategies. First, we pin down the replacement rate, i.e. the value of leisure relative to productivity, and let the vacancy posting cost be determined endogenously. Second, we normalize steady state labor market tightness and treat the leisure value as a residual.

Baseline calibration. We calibrate the model to a monthly frequency and set the discount factor to $\beta = 0.99^{1/3}$. A risk aversion parameter of $\sigma = 1$ yields logarithmic consumption utility and implies balanced growth. The steady state unemployment rate is calibrated to 6% in line with US post-war data. The cost of posting a vacancy is set to $c = 0.1$ as in Krause and Lubik (2010). This value is consistent with Hagedorn and Manovskii (2008), who propose a non-capital cost of posting a vacancy equal to 11% of labor productivity. We also note that our choice for c implies a share of vacancy posting costs over GDP close to 1%, which is in line with Andolfatto (1996) and Blanchard and Galí (2010) among others.

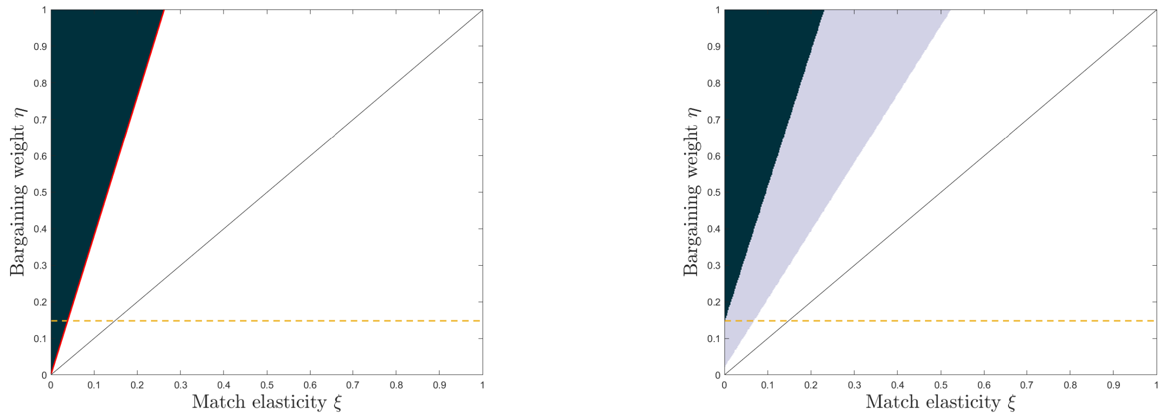
Our calibration for the separation rate is based on Shimer's (2012) estimate using US labor market micro data. His value $\rho = 0.034$ for the average monthly separation rate implies that jobs last for around two and a half years. Hobijn and Şahin (2009) present estimates of monthly separation rates, defined as the fraction of workers who leave their jobs, in different OECD countries ranging from 0.7% to 2%. Our calibration for the steady state vacancy filling rate, $q = 0.33$, follows den Haan et al. (2000). Christoffel et al. (2009) also propose this value based on European data.

The parameter governing the disutility of hours, σ_h, is calibrated to 2, which is in the middle of the range proposed by Keane and Rogerson (2012). The disutility of effort parameter, σ_e, is calibrated to target $\phi = 1.5$. This calibration of increasing returns in aggregate production follows Barnichon (2010) and is consistent with Bils and Cho (1994)'s work. In addition, Lewis et al. (2019) estimate the parameter ϕ in a New Keynesian model with variable capital and labor utilization and find a value greater than 1.5.

From this calibration, we derive several other steady state variables and parameters recursively. At the steady state, employment is $n = 1 - u$. The number of matches is derived from the law of motion for employment, $m = \frac{\rho}{1-\rho} n$. Given that we pin down the vacancy filling rate q, vacancies are given by $v = m/q$. Labor market tightness is $\theta = v/u$. Without loss of generality, we normalize search intensity to unity, $s = 1$. Matching efficiency is computed as $\chi = q(\theta/s)^{\xi}$. We set hours h to unity and find the value of λ_h which achieves this normalization. Similarly, we calibrate λ_e to obtain $e_0 = 1$, which yields $\lambda_e = \frac{1+\sigma_e}{\sigma_e}\lambda_h$. Firm output y is equal to $e_0 h^{\phi}$, see the production function. GDP is aggregate production minus vacancy posting costs and job search costs, $Y = yn - cv - \mathcal{G}(s)u$. Finally, we solve the steady state job creation condition for the value of leisure b.

Risk aversion. Under the assumption of risk averse households ($\sigma > 0$), we can no longer characterize the determinacy properties analytically and need to use numerical techniques instead. In our model, we set $e_t = s_t = 1$ and we assume constant hours as well, $h_t = 1$. Constant hours and effort can be achieved with a calibration that sets the elasticity of

Figure 1: Standard search model: risk neutrality ($\sigma = 0$) vs. risk aversion ($\sigma = 1$)



Note: Indeterminacy regions are shaded black, uniqueness regions with a negative stable root in vacancy posting condition are shaded gray, uniqueness regions with a positive stable root are white. The red line depicts the indeterminacy frontier (26). Below the dotted line, the implied leisure value b is negative.

disutility of hours and effort, as well as the elasticity of search costs, to a very high number.

Comparing the two panels in Figure 1, we see that indeterminacy is somewhat less likely under risk aversion. Why is this? Risk aversion implies greater intertemporal substitution, which leads to greater discounting, i.e. it lowers the effective discount factor. There is a negative relationship between the effective discount factor and the slope of the indeterminacy frontier in (26). Therefore, under risk aversion, the indeterminacy region lies further away from the Hosios condition.

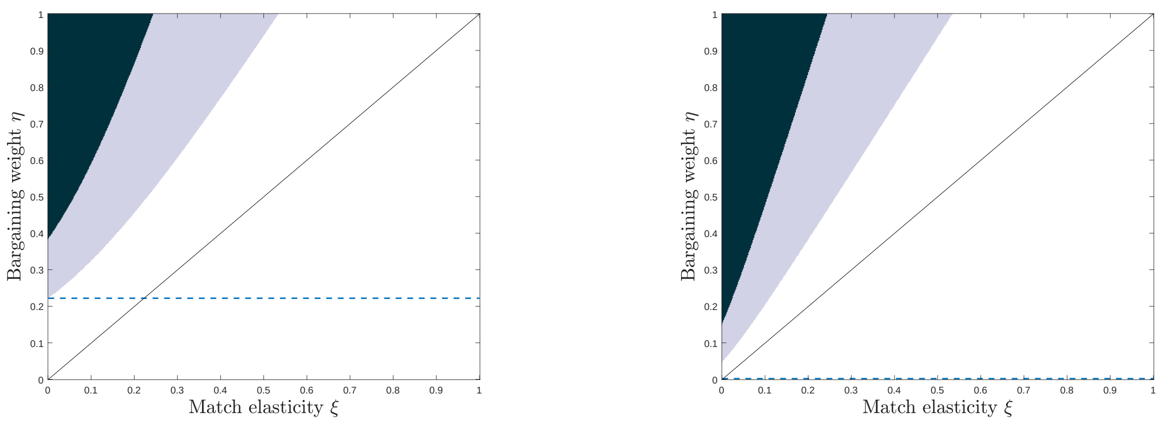
An additional model-implied restriction, which we have neglected so far, is that the leisure value b needs to be positive. Realistically, the unemployed receive welfare benefits rather than being taxed. We investigated under which parameter combinations b turns out to be negative. This happens if the worker's bargaining weight is rather low. The match elasticity has no effect on the implied leisure value. In Figure 1, the parameter combinations beneath the dashed line lead to a negative leisure value and are therefore not admissible. In the euro area, the implied leisure value is negative only for extremely low values of the worker's bargaining weight.

3.4 Labor market policies

How do government policies influence equilibrium indeterminacy? This is the question we turn to next. We consider unemployment benefits and labor taxes.

Unemployment benefits. In the labor search model, unemployment benefits are represented by the leisure value b, which amounts to the size of the worker's outside option in the wage bargaining process. The preceding discussion showed that, for a given (ξ, η)-pair, a higher responsiveness in labor market tightness boosts the wage channel and thereby increases the incidence of indeterminacy. This insight led us to the following thought experiment. Setting b/y close to 1, as in Hagedorn and Manovskii (2008), should make profits and hence tightness more volatile. Will this alternative calibration then lead to a larger region of indeterminacy? To answer this question, we change our calibration strategy; we fix the leisure value b and let the vacancy posting cost c be determined residually. Figure 2 shows the result of this exercise.

Figure 2: Standard search model: low vs. high leisure value b



Note: Indeterminacy regions are shaded black, uniqueness regions with a negative stable root in vacancy posting condition are shaded gray, uniqueness regions with a positive stable root are white. Below the dotted line, the implied vacancy posting cost exceeds firm output, $c > y$.

In the panel on the left, $b = 0.4$ as proposed by Shimer (2005); in the panel on the right, we follow Hagedorn and Manovskii (2008) and set $b = 0.955$. Recall that we normalize firm output to unity, $y = 1$. Indeed, as conjectured, a larger b expands the indeterminacy region. Note that the dashed line represents a threshold level for η below which the implied vacancy posting cost exceeds firm output, i.e. $c/y > 1$.

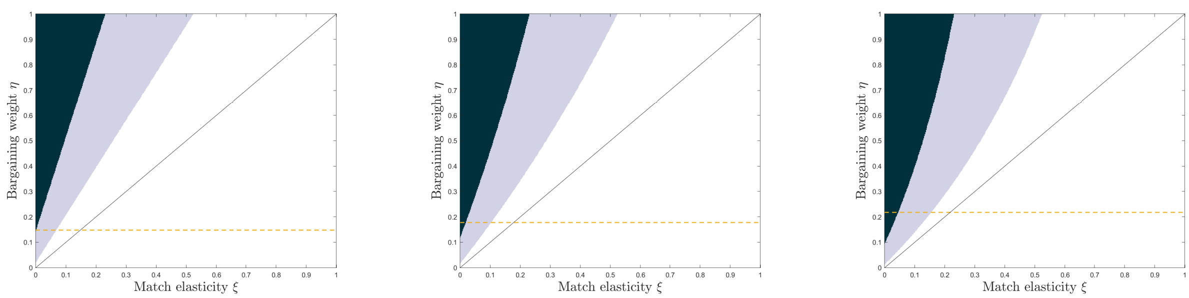
Labor income taxes. Introducing a proportional labor income tax $\tau \in (0, 1)$ on workers changes the determinacy frontier to:

$$\Psi\eta > \frac{1 + \beta(1-\rho)}{\beta(1-\rho)p}\xi, \tag{28}$$

where $\Psi = [1-(1-\eta)\tau]^{-1} \geq 1$. The left hand side of (28) can be interpreted as a tax-adjusted bargaining power of workers. Introducing labor taxes in the model, which implies $\Psi > 1$ and hence $\Psi\eta > \eta$, has the effect of making indeterminacy more likely. This is because the threshold value for the bargaining power parameter above which indeterminacy obtains is decreased.

In Figure 3, we verify numerically that a higher tax rate enlarges the indeterminacy region. The left hand panel displays the case without taxes. In the middle panel, we consider a tax rate of 20%, the average income tax in the US (Kliem and Kriwoluzky, 2014; Zubairy, 2014). In the right hand panel, we set a European-style tax rate of 38% as reported in European Commission (2018).

Figure 3: Standard search model: zero, low and high labor tax rate



Note: Indeterminacy regions are shaded black, uniqueness regions with a negative stable root in vacancy posting condition are shaded gray, uniqueness regions with a positive stable root are white. Below the dotted line, the implied leisure value b is negative.

With a tax on labor income, the vacancy posting condition in the standard DMP model reads[11]

$$(c/\chi)\theta_t^{\xi} = (1-\rho)\beta E_t\{(1-\Psi\eta)(y-b) - (\Psi-1)b - \Psi\eta c\theta_{t+1} + (c/\chi)\theta_{t+1}^{\xi}\}. \qquad (29)$$

An increase in the tax rate has the effect of raising the workers' effective bargaining power. Even though it is workers that pay the tax, Nash bargaining implies that the tax shrinks the match surplus. So, employers and workers split the tax burden according to the sharing rule. As a consequence, the Nash bargaining wage becomes more sensitive to labor market conditions, see (29). Then, through the increased importance of the wage channel, the indeterminacy region expands.

The labor tax makes working less attractive relative to non-work. Intuitively then, its effect is similar to that of an increase in unemployment benefits.

[11]The full derivation of the standard search model with labor income taxes is provided in the appendix.

4 Determinacy in the labor search model with effort

In the following, we characterize the conditions for local equilibrium existence and uniqueness for two model variants: 1) the model with variable labor effort and constant search effort, 2) the model with variable search effort and constant labor effort.

Labor effort and indeterminacy. Employment flows are not the only form of labor adjustment. In many countries, hours worked per employee are an important margin along which labor varies (see the evidence in e.g. Ohanian and Raffo, 2012; Dossche et al., 2019). Moreover, variable labor utilization, or effort, has been proposed as a third labor margin to help explain the observed procyclicality of labor productivity.[12] Burda et al. (2019) use the American Time Use Survey 2003-12 to show that 'non-work at work', which we might interpret as low effort per hour, is substantial and varies countercyclically. More specifically, they find that time spent in non-work conditional on any positive amount rises, while the fraction of workers reporting positive values declines with unemployment. Since the former effect dominates, there is a positive relationship between non-work and the unemployment rate. This evidence suggests that variable effort is a relevant labor adjustment margin in the US. In a business cycle model estimated for the euro area, Lewis et al. (2019) show that a model with labor effort outperforms one with variable capital utilization.

Benhabib and Farmer (1994) show that increasing returns can be a source of indeterminacy. In our model, hours and variable labor effort allow for increasing returns to hours in production. As a result, the worker's marginal product and therefore his asset value rises. This strengthens the wage channel, making indeterminacy more likely in the model with hours and effort than in the standard search model.

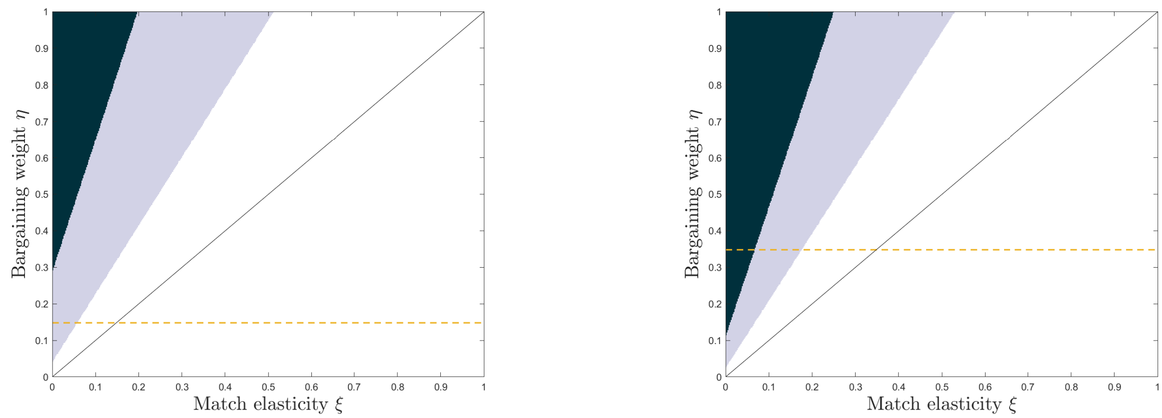
Figure 4 shows the determinacy regions in our model, setting the constant of relative risk aversion to $\sigma = 2$. The chart on the left is the standard labor search model without hours or effort; the one on the right is the model featuring both hours and effort. The figure shows that introducing hours and effort into the model expands the indeterminacy region somewhat.

The region where b is negative is larger in the model with two additional labor margins than they are in the standard labor search model. This is intuitive, since introducing hours and effort reduces the model-implied leisure value; we can write $b = b^s - \phi/(1 + \sigma_h)$, where b^s is the leisure value in the standard labor search model without hours and effort.

We conclude from this exercise that increasing returns due to variable labor utilization have a rather small effect on indeterminacy. This contrasts with Wen (1998), who argues that

[12] A non-exhaustive list includes Oi (1962), Bils and Cho (1994), Rotemberg and Summers (1990), Barnichon (2010), and Galí and van Rens (2014).

Figure 4: Labor effort and determinacy: constant vs. variable labor effort



Note: Indeterminacy regions are shaded black, uniqueness regions with negative stable root in vacancy posting condition shaded gray, uniqueness regions with positive stable root are white. Below the dashed line, the implied leisure value b is negative.

variable capital utilization can generate indeterminacy under empirically relevant parameter choices.

Search effort and indeterminacy. In the standard model we have found that a bargaining share of workers sufficiently above the value required for efficiency according to the Hosios condition results in an indeterminate equilibrium. Does this result depend on the (common) assumption of one-sided search — on the part of firms — which we have maintained thus far? In the standard labor search model, only the firms actively search by posting vacancies. Quite plausibly, though, unemployed workers could drive up their search intensity whenever it is advantageous to do so, i.e. whenever the expected return to searching more intensively exceeds the associated marginal cost.

Suppose that firms become optimistic and post many vacancies, leading to a tightening of the labor market. The job finding rate increases, which induces the unemployed to raise their search effort. This effect — in isolation — reduces labor market tightness, the job finding probability and the wage. In other words, the wage channel is weakened under variable search intensity. Models with two-sided search, on the part of both firms and unemployed workers, have been developed in e.g. Merz (1995) and Hashimzade and Ortigueira (2005), among others.[13] For simplicity, we abstract from on-the-job search in the present analysis; only unemployed workers engage in search.

Search intensity is an empirically relevant model ingredient. A large body of evidence, discussed in more detail below, suggests that search intensity by job-seekers varies over

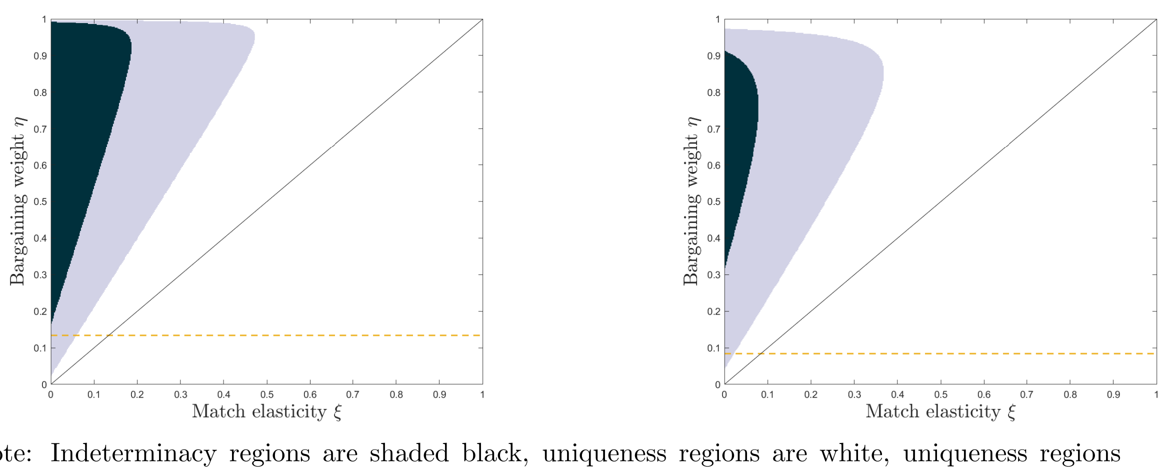
[13] Berentsen et al. (2007) present a two-sided search model of money.

the business cycle. The standard search model generates too little labor market volatility (Shimer, 2005). Gomme and Lkhagvasuren (2015) show that unemployment volatility is higher in a model with variable search intensity, thus bringing the search model closer to the data in this dimension. The mechanism is the following.

With constant search intensity, firms that expect a boost to profits (e.g. thanks to an expected technological improvement) post more vacancies, raising the job finding rate and thus the workers' outside option. The resulting rise in wages eats up much of the firm's expected rise in profits. Instead, with variable search intensity, the value of being unemployed rises by less — since exerting more search effort is costly –, and therefore the wage also rises by less. This leaves a larger surplus for the firm, which in turn amplifies the rise in vacancies. The mechanism is similar to the search complementarities in Fernández-Villaverde et al. (2019)'s model with inter-firm matching: as one party increases its search activities, it becomes advantageous for the other party to do the same.

Figure 5 shows the determinacy regions in the model with variable search effort, where the coefficient of risk aversion is set to $\sigma = 1$. A highly convex cost function with ζ set to a value above 10, shown in the left panel, brings the two-sided search model close to our baseline model with constant search intensity, see the right hand side of Figure 1. The indeterminacy region shrinks as we make the search cost function flatter, lowering ζ. In the panel on the right, search costs are quadratic following the evidence in Yashiv (2000), i.e. $\zeta = 1$. If we instead assume that the $\mathcal{G}(s_t)$-function is convex but fairly flat, following the argument in Shimer (2004), and set $\zeta = 0.1$, indeterminacy all but disappears (not shown).

Figure 5: Search effort and determinacy: highly convex vs. quadratic search cost function



Note: Indeterminacy regions are shaded black, uniqueness regions are white, uniqueness regions with negative stable root in vacancy posting condition shaded gray, uniqueness regions with positive stable root are white. Below the dashed line, the implied leisure value b is negative.

To summarize, a convex but fairly flat search cost function reduces or even eliminates indeterminacy. Below, we discuss the literature on search intensity and the empirical evidence on the size of ζ.

A look at the literature. A large empirical literature documents that search intensity by the unemployed varies across time, lending support to the idea of endogenizing s_t. However, there is no agreement on its cyclical properties, which is in part related to the fact that search intensity is not directly observable. A number of proxies have been proposed.

Shimer (2004) uses the number of search methods from the Current Population Survey (CPS) and points out that this measure is countercyclical. Pan (2019) constructs search activity indices for different sectors in the US, based on Internet search volumes, which are also countercyclical. These approaches do not differentiate between on-the-job search and search activity by the unemployed, which according to Faberman et al. (2017) might differ to a large extent. Mukoyama et al. (2018) combine responses from the American Time Use Survey (ATUS) with the CPS to show that job search effort by the non-employed is countercyclical.

Krueger and Mueller (2010) consider the time the unemployed spend on search activities. Using the ATUS, they show that job search time increases with the expected wage. Gomme and Lkhagvasuren (2015) argue that this is indirect evidence of procyclical search intensity, since expansions are times when expected wages are high, and individual wages are also highly procyclical as shown by Solon et al. (1994).[14] Moreover, Gomme and Lkhagvasuren (2015) find that labor market tightness is highly correlated with search intensity by the short-term unemployed. The two-sided search model here and in Pissarides (2000) is consistent with search effort being procyclical.

The other important issue for our analysis is the shape of the search cost function. How large is ζ? While Stiglitz (1987) considers both convex and concave search costs, many studies impose convexity on the search cost function: Merz (1995), Kaas (2010), Gomme and Lkhagvasuren (2015) all do this. First, to the extent that search activity is time-intensive, the natural constraint imposed by the time endowment makes every additional unit of search more and more costly. This reasoning for convexity in search costs is arguably more applicable to on-the-job search, where a searching worker is already close to his time constraint. Second, as explained above, Gomme and Lkhagvasuren (2015) argue that search costs help to generate employment volatility; this goes some way in solving the so-called 'Shimer puzzle' if ζ is not too high. Empirical evidence in Christensen et al. (2005) supports a specification of search costs that is quadratic, although we note that, here also, the authors

[14] The argument here is that the observed acyclicality of average wages is driven in part by a compositional bias.

analyze on-the-job search. Instead, in Yashiv (2000), only unemployed workers exert effort; that paper also presents evidence of quadratic search costs.

Yan (2013) provides empirical evidence of fixed job search costs. Fernández-Villaverde et al. (2019) employ a search cost function with a non-convexity. This feature is critical to the existence of multiple equilibria in their model. Since some agents will choose not to search at all, there exists an equilibrium with low output, low search and high unemployment in addition to an equilibrium with high output, high search activity and low unemployment. Cheron and Decreuse (2016) present evidence of postings by job-seekers (or recruiting firms) that testify from past search activity and remain online even after a match has taken place. Removing these postings is costly and they therefore live on as 'phantoms'. This evidence also suggests that searching entails some fixed costs. When we added a fixed cost component to our function $\mathcal{G}(s_t)$, however, this did not alter our findings qualitatively.

In conclusion, the evidence suggests that search costs are convex in search effort, which implies a smaller indeterminacy region compared with the standard DMP model.

5 An alternative model with hiring costs

The search-and-matching model fails to replicate certain salient features of the labor market (Shimer, 2005). In particular, it predicts a strong immediate response of vacancies to productivity shocks, whereas in the data, we instead observe hump-shaped dynamics.[15] As explained in Hertweck (2013), the reason for this counterfactual prediction lies in the assumption of linear vacancy posting costs, c, which a firm incurs each period, irrespective of whether or not the matching process is successful. Effective hiring costs are in this case given by c/q_t, i.e. vacancy posting costs multiplied by the expected duration of a vacancy. After a positive productivity shock, vacancy duration $1/q_t$ and hence effective hiring costs increase sharply and persistently; this is due to congestion externalities as explained above. The persistence in elevated hiring costs induces firms to post many vacancies immediately, giving rise to a convex-shaped impulse response in the number of vacancies.

Replacing linear vacancy posting costs with hiring costs, akin to Gertler and Trigari (2009), brings the model closer to the data in this dimension. Conceptually, an important difference between the two models is that in the hiring cost model, *filling* a vacancy entails a cost, rather than *posting* a vacancy.[16] Hence, a firm that wishes to hire a worker always

[15]See Pissarides (2000).

[16]We would like to stress that our modeling approach differs from Mortensen and Nagypál (2007) and Ljungqvist and Sargent (2017), who maintain the search framework setup with the vacancy posting costs and merely add post-bargaining hiring costs, which make the firm's surplus more elastic to productivity changes. In addition, pre-bargaining hiring costs as in Pissarides (2009) have an effect on the equilibrium wage.

finds one at the current cost. In the absence of search externalities, that cost is independent of search time and, thus, independent of labor market tightness. Hertweck (2013) has shown that, when the effective hiring cost is proportional to the aggregate hiring rate, firms have strong incentives to smooth hiring activities over several periods. As a result, the model closely replicates the hump-shaped impulse responses observed in the data. The hiring cost approach has become more popular recently; applications include Galí and van Rens (2014).

In the following, we investigate the determinacy properties of the hiring cost model. For expositional clarity, we abstract from endogenous labor and search effort.

5.1 Model setup

Let us introduce the hiring rate as the number of new matches over employment, $x_t = m_t/n_t$. Hiring costs to an individual firm depend on the aggregate hiring rate and are given by cx_t per newly matched worker. The aggregate resource constraint changes to $Y_t = y_t n_t - cx_t m_t$, and the vacancy posting condition of the standard labor search model (11) is replaced with

$$cx_t = (1-\rho)E_t\{\beta_{t,t+1}(y_{t+1} - w_{t+1} + cx_{t+1})\}. \tag{30}$$

Notice from (30) that the firm's surplus from hiring is different from the standard search model. As a consequence, the bargaining wage is also different,

$$w_t = \eta(y_t + p_t cx_t) + (1-\eta)b. \tag{31}$$

Equilibrium in the hiring cost model is defined as follows.

Definition 2. *A decentralized equilibrium in the hiring cost model is a set of infinite sequences for quantities $\{u_t, m_t, n_{t+1}, Y_t\}_{t=0}^{\infty}$, matching rates $\{p_t, x_t\}_{t=0}^{\infty}$ and wages $\{w_t\}_{t=0}^{\infty}$, satisfying the transversality condition, such that:*

1. *given aggregate matching rates and wages, the quantities solve the household's problem,*
2. *given aggregate matching rates and wages, the quantities solve the firm's problem,*
3. *employment is determined by the law of motion (2),*
4. *wages solve the Nash bargaining problem,*
5. *goods markets clear.*

Table 2: Equilibrium conditions: hiring cost model

Unemployment, u_t	$u_t = 1 - n_t$
Hiring rate, m_t	$x_t = m_t/n_t$
Job finding rate, p_t	$p_t = m_t/u_t$
Employment, n_t	$n_{t+1} = (1-\rho)(n_t + m_t)$
GDP, Y_t	$Y_t = y_t n_t - c x_t m_t$
Hiring, x_t	$cx_t = (1-\rho)\beta E_t \left\{(Y_t/Y_{t+1})^\sigma (y_{t+1} - w_{t+1} + cx_{t+1})\right\}$
Wages, w_t	$w_t = \eta(y_t + p_t c x_t) + (1-\eta)b$

The equilibrium conditions of the hiring cost model are presented in Table 2, where firm output y_t is exogenous. In linearized form, the system can be written compactly in two equations describing one control variable, the hiring rate x_t, and one state variable, employment n_{t+1},

$$\alpha E_t \left\{\hat{x}_{t+1}\right\} = \left[1 + \sigma\rho + 2\sigma\frac{cxm}{Y} + \beta(1-\rho)\eta p\frac{\rho}{u}\right]\hat{x}_t + \beta(1-\rho)\frac{\eta p}{u}\hat{n}_t, \tag{32}$$

$$\hat{n}_{t+1} = \rho\hat{x}_t + \hat{n}_t, \tag{33}$$

where we define α as follows,

$$\alpha = \beta(1-\rho)(1-2\eta p) + 2\sigma\frac{cxm}{Y},$$

The two-equation system of the hiring cost model, (32) and (33), can be written in matrix notation,

$$\begin{bmatrix} E_t\{\hat{x}_{t+1}\} \\ \hat{n}_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha}\left[1 + \sigma\rho + 2\sigma\frac{cxm}{Y} + \beta(1-\rho)\eta p\frac{\rho}{u}\right] & \frac{1}{\alpha}\beta(1-\rho)\frac{\eta p}{u} \\ \rho & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_t \\ \hat{n}_t \end{bmatrix}. \tag{34}$$

Notice that in this model, the hiring rate, rather than labor market tightness, is the relevant control variable.

5.2 Determinacy analysis

We first analyze numerically the determinacy properties of the more general model with risk averse households. To this end, we use the same calibration as in the standard search model. The steady state hiring rate x, employment n, number of vacancies v, and labor market tightness θ are identical in both models. However, the implied leisure value b becomes,

$$b = y - \frac{1-\beta(1-\rho)(1-\eta p)}{\beta(1-\rho)(1-\eta)} cx.$$

We find that the hiring cost model has a determinate solution, even when we allow for variable labor effort. No parameter combination (ξ, η) gives rise to indeterminacy. What explains this result?

To gain more intuition on the mechanics behind this result, let us replace the wage (31) in the hiring condition (30) to obtain

$$cx_t = (1-\rho)\beta E_t\{(1-\eta)(y-b)\underbrace{-\eta p_{t+1} cx_{t+1}}_{\text{wage channel}} \underbrace{+cx_{t+1}}_{\text{hiring cost channel}}\}. \tag{35}$$

As in the standard model with linear vacancy posting costs, hiring affects the worker's asset value through two channels: the wage channel and the hiring cost channel. The size of the consolidated coefficient on $E_t\{x_{t+1}\}$ on the right hand side of (35) is thus a critical determinant of equilibrium (in-)determinacy. As shown in Section 3.2, indeterminacy in the standard model with risk neutrality requires that the consolidated coefficient on $E_t\{\theta_{t+1}\}$ is smaller than -1. In the model with hiring costs, this case is ruled out, given that the bargaining share and the steady-state job finding rate — which both enhance the strength of the wage channel — are both restricted to the unit interval. In addition, the match elasticity ξ, which — if calibrated to low values — helps to dampen the impact of the hiring cost channel in the standard model, no longer appears in equation (35). The match elasticity ξ is no longer relevant for equilibrium determinacy, since it does not appear in the equilibrium conditions (see Table 2). Indeed, ξ in the hiring cost model affects only the number of vacancies needed to attain the hiring rate x_t for a given matching function.[17] In the absence of search externalities, low values of ξ no longer dampen the hiring cost channel. Therefore, indeterminacy cannot arise in this model.

To prove this statement formally, we again consider risk neutrality as a special case, where $\alpha = \beta(1-\rho)(1-2\eta p)$. The dynamic system simplifies to

$$\begin{bmatrix} E_t\{\hat{x}_{t+1}\} \\ \hat{n}_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\beta(1-\rho)(1-2\eta p)} + \frac{\eta p}{1-2\eta p}\frac{\rho}{u} & \frac{\eta p}{1-2\eta p}\frac{1}{u} \\ \rho & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_t \\ \hat{n}_t \end{bmatrix}. \tag{36}$$

Proposition 4. *Under risk neutrality, the model with hiring costs is determinate, i.e. it has*

[17] In other words, vacancies are residually determined in the hiring cost model and play no allocational role.

a unique stable equilibrium.

Proof. The trace and the determinant of the transition matrix are

$$\mathrm{tr} = 1 + \frac{1}{\beta(1-\rho)(1-2\eta p)} + \frac{\eta p}{1-2\eta p}\frac{\rho}{u},$$

$$\mathrm{D} = \frac{1}{\beta(1-\rho)(1-2\eta p)}.$$

The two roots of the system can be written in terms of the trace and the determinant,

$$\lambda_i = \frac{1}{2}(\mathrm{tr} \pm \sqrt{\mathrm{tr}^2 - 4\mathrm{D}}) \text{ with } i = 1, 2.$$

Using a proof by contradiction, we can show that the first root of the transition matrix is smaller than unity for $2\eta p < 1$, which rules out non-existence. By a similar argument, the second root is necessarily unstable. Therefore, the system has a unique solution.[18] □

6 Conclusion

We introduce two types of effort, worker effort and search effort by job-seekers, into a simple search-and-matching model. We analyze how these extensions affect the model's determinacy properties. As shown in the literature, indeterminacy can arise in the canonical labor search model when the price of a worker, i.e. the wage, increases strongly in response to firms' vacancy posting and the ensuing tightening of the labor market. This happens when a very high bargaining power of workers is combined with a matching function that is highly elastic to vacancies. Then, the price of a worker increases strongly through the thick-market effect by which any additional vacancy increases the job finding rate and, in turn, the wage.

In addition to clarifying the mechanism leading to indeterminacy in the standard search model, we show how different calibration strategies alter the regions of indeterminacy in the two-dimensional parameter space spanned by the match elasticity and the bargaining share. Our key insight here is that, ceteris paribus, a stronger wage channel (i.e. a larger effect of a labor market tightening on wages) is associated with larger regions of indeterminacy. One example is a more fluid labor market, another is a calibration with a high value of non-market activity.

The presence of variable labor effort expands the regions of indeterminacy compared to a model featuring employment only. This result is driven by the increasing returns to hours

[18] For details, see the appendix.

in production in the model with hours and effort as additional labor margins. The rise in a worker's marginal product in an expansion increases the Nash bargaining wage.

Variable search effort has the opposite effect; as long as search costs are convex in search intensity, the indeterminacy region shrinks in comparison to the standard search model. Vacancy posting by firms raises the job finding rate; this leads to greater search effort by the unemployed, which in turn dampens the tightening of the labor market and the associated rise in the wage.

We have also shown that indeterminacy is eliminated in a framework where labor market frictions are modeled as hiring costs rather than a search process with linear vacancy posting costs.

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Going the Extra Mile: Effort by Workers and Job-Seekers — Appendix —

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1 Standard Labor Search Model

One-worker firms, Cobb-Douglas matching function, linear vacancy posting costs, predetermined employment, exogenous separations, no participation margin and no hours margin (labor force and hours constant and normalized to unity), Nash wage bargaining.

1.1 Preliminaries

Matching function

$$m_t = \chi u_t^{\xi} v_t^{1-\xi}$$

Unemployment rate

$$u_t = 1 - n_t$$

Labor market tightness

$$\theta_t = \frac{v_t}{u_t}$$

Vacancy filling rate

$$q_t \equiv \frac{m_t}{v_t} = \chi \theta_t^{-\xi}$$

Job finding rate

$$p_t \equiv \frac{m_t}{u_t} = \theta_t q_t$$

Employment dynamics

$$n_t = (1-\rho)(n_{t-1} + m_{t-1})$$

Market clearing

$$Y_t = C_t$$

Aggregate accounting

$$Y_t = y_t n_t - c v_t$$

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1.2 Production, firm's match surplus, hiring

Production of output y_t takes place in one-worker firms with labor only (i.e., no capital) and constant hours. Firms' wage costs are w_t, such that period-t profits are $y_t - w_t$. Let $\mathcal{J}_t$ denote the firm's match surplus, i.e. the value of filling a vacancy. It is the sum of current profits and the firm's continuation value. The latter is the expected future match surplus in case the employment relationship continues, which happens with probability $(1-\rho)$. The firm's value is zero in case the worker and the firm separate, which happens with probability ρ. Thus,

$$\mathcal{J}_t = y_t - w_t + E_t\{\beta_{t,t+1}[(1-\rho)\mathcal{J}_{t+1} + \rho \cdot 0]\},$$

where $\beta_{t-1,t} = \beta\frac{\lambda_t}{\lambda_{t-1}}$ is the household's stochastic discount factor and $\lambda_t = C_t^{-\sigma}$ is the marginal utility of consumption. The firm's match surplus can be written as

$$\mathcal{J}_t = y_t - w_t + (1-\rho)E_t\{\beta_{t,t+1}\mathcal{J}_{t+1}\}. \tag{1}$$

The value of posting a vacancy is given by minus the vacancy posting cost $c > 0$, plus the expected future value of the vacancy. The latter is given by the weighted average of the value of filling the vacancy, i.e. the firm's match value in the next period, which has probability $q_t(1-\rho)$, and the future value of the unfilled vacancy, V_{t+1}, which has probability $(1 - q_t(1-\rho))$. Therefore,

$$V_t = -c + E_t\{\beta_{t,t+1}[q_t(1-\rho)\mathcal{J}_{t+1} + (1 - q_t(1-\rho))V_{t+1}]\}. \tag{2}$$

Free entry drives the value of a vacancy to zero at each point in time, such that $V_t = 0$ for all t and thus

$$\frac{c}{q_t} = (1-\rho)E_t\{\beta_{t,t+1}\mathcal{J}_{t+1}\}. \tag{3}$$

Combining the firm's asset value (1) and the free entry condition (3), we get the following expression for the firm's match surplus

$$\mathcal{J}_t = y_t - w_t + \frac{c}{q_t}. \tag{4}$$

The derivative of the firm's match surplus to the wage is

$$\frac{\partial \mathcal{J}_t}{\partial w_t} = -1.$$

Finally, using the firm's match surplus (4) to substitute out $\mathcal{J}_{t+1}$ in the free entry condition (3), we obtain the job creation condition

$$\frac{c}{q_t} = (1-\rho)E_t\left\{\beta_{t,t+1}\left[y_{t+1} - w_{t+1} + \frac{c}{q_{t+1}}\right]\right\}. \tag{5}$$

1.3 Utility and worker's match surplus

Utility maximization is given by

$$\max_{\{C_t\}_{t=0}^{\infty}} U = E_0 \sum_{t=0}^{\infty} \beta_{0,t}\left[\frac{C_t^{1-\sigma} - 1}{1-\sigma}\right],$$

where $\sigma \geq 0$ is the coefficient of risk aversion. Denote the value of being employed W_t and the value of being unemployed U_t. In period t, an employed worker receives the after-tax wage income $(1-\tau)w_t$. In the next period, he is either still employed with probability $(1-\rho)$, in which case he has an expected value of $E_t\{\beta_{t,t+1}W_{t+1}\}$, or the employment relation is dissolved with probability ρ, then his expected value is $E_t\{\beta_{t,t+1}U_{t+1}\}$. The worker's asset value therefore is

$$W_t = (1-\tau)w_t + E_t\{\beta_{t,t+1}[(1-\rho)W_{t+1} + \rho U_{t+1}]\}, \tag{6}$$

where $\tau \in (0,1)$ is a proportional labor income tax. The value of being unemployed U_t is in turn given by

$$U_t = b + E_t\{\beta_{t,t+1}[p_t(1-\rho)W_{t+1} + (1-p_t(1-\rho))U_{t+1}]\}. \tag{7}$$

An unemployed worker receives an income of b units of consumption goods in period t. In the next period, he faces a probability p_t of finding a new job, which turns active with probability $(1-\rho)$, and which has an expected value of $E_t\{\beta_{t,t+1}W_{t+1}\}$, and consequently a probability $[1-p_t(1-\rho)]$ of remaining unemployed, which has an expected value of $E_t\{\beta_{t,t+1}U_{t+1}\}$. Defining the worker's surplus as $\mathcal{W}_t = W_t - U_t$, we can subtract (7) from (6) to write the match surplus going to the worker as

$$\mathcal{W}_t = (1-\tau)w_t - b + (1-\rho)E_t\left\{\beta_{t,t+1}[(1-p_t)\mathcal{W}_{t+1}]\right\}. \tag{8}$$

The derivative of the worker's surplus with respect to the wage is $\partial\mathcal{W}_t/\partial w_t = 1$.

1.4 Wage bargaining

Under Nash bargaining, the equilibrium wage satisfies

$$\max_{w_t} \mathcal{W}_t^{\eta}\mathcal{J}_t^{1-\eta}.$$

The first order condition to this problem is

$$\eta\mathcal{W}_t^{\eta-1}\frac{\partial\mathcal{W}_t}{\partial w_t}\mathcal{J}_t^{1-\eta} + (1-\eta)\mathcal{J}_t^{-\eta}\frac{\partial\mathcal{J}_t}{\partial w_t}\mathcal{W}_t^{\eta} = 0,$$

which can be simplified to

$$\eta\frac{\mathcal{J}_t}{\mathcal{W}_t}\frac{\partial\mathcal{W}_t}{\partial w_t} + (1-\eta)\frac{\partial\mathcal{J}_t}{\partial w_t} = 0.$$

Put differently, the surplus sharing rule is

$$\mathcal{W}_t = \Upsilon_t\mathcal{J}_t, \tag{9}$$

where Υ_t denotes the effective bargaining power,

$$\Upsilon_t = \frac{\eta}{1-\eta}\frac{\frac{\partial\mathcal{W}_t}{\partial w_t}}{-\frac{\partial\mathcal{J}_t}{\partial w_t}}. \tag{10}$$

Plugging the derivatives of the worker's and the firm's surplus into (9), we find that $\Upsilon = \frac{\eta}{1-\eta}$ and so the sharing rule boils down to

$$\mathcal{W}_t = \frac{\eta}{1-\eta}\mathcal{J}_t. \tag{11}$$

Using the worker's and the firm's surplus, (8) and (1), to replace $\mathcal{W}_t$ and $\mathcal{J}_t$ in (11), we obtain

$$(1-\tau)w_t - b + (1-\rho)E_t\{\beta_{t,t+1}(1-p_t)\mathcal{W}_{t+1}\} = \frac{\eta}{1-\eta}\left[y_t - w_t + (1-\rho)E_t\{\beta_{t,t+1}\mathcal{J}_{t+1}\}\right].$$

Then, using the surplus sharing rule (9) to replace $\mathcal{W}_{t+1}$ with $\frac{\eta}{1-\eta}\mathcal{J}_{t+1}$ yields

$$(1-\tau)w_t - b + (1-\rho)E_t\left\{\beta_{t,t+1}(1-p_t)\frac{\eta}{1-\eta}\mathcal{J}_{t+1}\right\} = \frac{\eta}{1-\eta}\left[y_t - w_t + (1-\rho)E_t\{\beta_{t,t+1}\mathcal{J}_{t+1}\}\right].$$

Collecting terms in $\frac{\eta}{1-\eta}E_t\{\beta_{t,t+1}\mathcal{J}_{t+1}\}$ yields

$$(1-\tau)w_t - b = \frac{\eta}{1-\eta}\left[y_t - w_t + p_t(1-\rho)E_t\{\beta_{t,t+1}\mathcal{J}_{t+1}\}\right]$$

Using the free entry condition (3) to replace $(1-\rho)E_t\{\beta_{t,t+1}\mathcal{J}_{t+1}\}$ with $\frac{c}{q_t}$ yields

$$(1-\tau)w_t - b = \frac{\eta}{1-\eta}\left(y_t - w_t + p_t\frac{c}{q_t}\right).$$

Collecting the terms in w_t and using $p_t = \theta_t q_t$, we can write

$$\left(1-\tau+\frac{\eta}{1-\eta}\right)w_t - b = \frac{\eta}{1-\eta}\left(y_t + c\theta_t\right).$$

Simplify the term in brackets to get

$$\left(-\tau+\frac{1}{1-\eta}\right)w_t = b + \frac{\eta}{1-\eta}(y_t + c\theta_t).$$

We multiply by $(1-\eta)$ and rearrange to obtain,

$$[1-(1-\eta)\tau]w_t = \eta(y_t + c\theta_t) + (1-\eta)b. \tag{12}$$

We solve for the wage w_t as follows

$$w_t = \Psi[\eta(y_t + c\theta_t) + (1-\eta)b], \tag{13}$$

where $\Psi = [1-(1-\eta)\tau]^{-1} \geq 1$. Using the bargaining wage (13) in the job creation condition (5) yields

$$\frac{c}{q_t} = (1-\rho)E_t\left\{\beta_{t,t+1}\left[y_{t+1} - \Psi\left[\eta(y_{t+1}+c\theta_{t+1}) + (1-\eta)b\right] + \frac{c}{q_{t+1}}\right]\right\}.$$

We use $\beta_{t,t+1} = \frac{\lambda_{t+1}}{\lambda_t}$, $\lambda_t = C_t^{-\sigma}$ and $C_t = Y_t$ to replace $\beta_{t,t+1}$ with $\beta Y_t^{\sigma}/Y_{t+1}^{\sigma}$, and rearrange to obtain

$$\frac{c}{q_t} = \beta(1-\rho)E_t\left\{\frac{Y_t^{\sigma}}{Y_{t+1}^{\sigma}}\left[(1-\Psi\eta)(y_{t+1}-b) - (\Psi-1)b - \Psi\eta c\theta_{t+1} + \frac{c}{q_{t+1}}\right]\right\}.$$

Replace q_t with $\chi\theta_t^{-\xi}$ to get an alternative expression for the job creation condition (JCC)

$$\frac{c}{\chi}\theta_t^{\xi} = \beta(1-\rho)E_t\left\{\frac{Y_t^{\sigma}}{Y_{t+1}^{\sigma}}\left[(1-\Psi\eta)(y_{t+1}-b) - (\Psi-1)b - \Psi\eta c\theta_{t+1} + \frac{c}{\chi}\theta_{t+1}^{\xi}\right]\right\}. \tag{14}$$

In steady state, the JCC is

$$\frac{c}{\chi}\theta^{\xi} = \beta(1-\rho)\left[(1-\Psi\eta)(y-b)-(\Psi-1)b-\Psi\eta c\theta+\frac{c}{\chi}\theta^{\xi}\right].$$

Rearranging, we obtain

$$\frac{1-\beta(1-\rho)}{\beta(1-\rho)}\frac{c}{\chi}\theta^{\xi} = (1-\Psi\eta)y-(1-\eta)\Psi b-\Psi\eta c\theta. \tag{15}$$

We can solve the steady state JCC for b as follows

$$b = \frac{1-\Psi\eta}{(1-\eta)\Psi}y-\frac{c}{(1-\eta)\Psi}\left(\frac{1-\beta(1-\rho)}{\beta(1-\rho)q}+\Psi\eta\theta\right).$$

1.5 Equilibrium conditions

Endogenous variables u_t, θ_t, n_t, Y_t, v_t.

$$u_t = 1-n_t$$

$$\theta_t = \frac{v_t}{u_t}$$

$$n_t = (1-\rho)(n_{t-1}+\chi u_{t-1}^{\xi}v_{t-1}^{1-\xi})$$

$$Y_t = y_t n_t - c v_t$$

$$\frac{c}{\chi}\theta_t^{\xi} = \beta(1-\rho)E_t\left\{\frac{Y_t^{\sigma}}{Y_{t+1}^{\sigma}}\left[(1-\Psi\eta)(y_{t+1}-b)-(\Psi-1)b-\Psi\eta c\theta_{t+1}+\frac{c}{\chi}\theta_{t+1}^{\xi}\right]\right\}$$

Exogenous variable: y_t.

1.6 Recursive steady state

Normalize $y=1$. Calibrate β, σ, ρ, ξ, ρ, q, c, u. Implied steady state variables or parameters: n, v, θ, χ, b, Y.

Independent of ξ and η

$$n = 1-u$$

$$v = \frac{\rho}{1-\rho}\frac{n}{q}$$

$$\theta = \frac{v}{u}$$

$$Y = yn - cv$$

Not independent of ξ and η

$$\chi = q\theta^{\xi}$$

$$b = \frac{1-\Psi\eta}{(1-\eta)\Psi}y-\frac{c}{(1-\eta)\Psi}\left(\frac{1-\beta(1-\rho)}{\beta(1-\rho)q}+\Psi\eta\theta\right)$$

Notice that labor taxes ($\tau > 0$) decrease the implied leisure value b.

1.7 Alternative steady state formulation (1)

Normalize $y = 1$. Calibrate β, σ, ρ, ξ, ρ, q, b/y, u. However, we calibrate b/y and back out c, rather than the other way around. Implied steady state variables or parameters: n, v, θ, χ, c, Y.

Independent of ξ and η

$$n = 1 - u$$

$$v = \frac{\rho}{1-\rho}\frac{n}{q}$$

$$\theta = \frac{v}{u}$$

Not independent of ξ and η

$$\chi = q\theta^{\xi}$$

$$c = [(1-\Psi\eta)y - (1-\eta)\Psi b]\left(\frac{1-\beta(1-\rho)}{\beta(1-\rho)q} + \Psi\eta\theta\right)^{-1}$$

$$Y = yn - cv$$

1.8 Alternative steady state formulation (2)

Normalize $y = 1$. Calibrate β, σ, ρ, ξ, ρ. However, instead of calibrating q, c and u, we now normalize $\theta = 1$, and we calibrate p and cv/y. Implied steady state variables or parameters: x, u, n, v, q, c, Y; χ, b.

Independent of ξ and η

$$x = \frac{\rho}{1-\rho}$$

$$u = \left(1 + \frac{p}{x}\right)^{-1}$$

$$n = 1 - u$$

$$v = u\theta$$

$$q = \frac{p}{\theta}$$

$$c = \frac{cv}{y}\frac{y}{v}$$

$$Y = yn - cv$$

Not independent of ξ and η

$$\chi = q\theta^{\xi}$$

$$b = \frac{1-\Psi\eta}{(1-\eta)\Psi}y - \frac{c}{(1-\eta)\Psi}\left(\frac{1-\beta(1-\rho)}{\beta(1-\rho)q} + \Psi\eta\theta\right)$$

1.9 Linearization of equilibrium conditions

Unemployment

$$\hat{u}_t = -\frac{n}{u}\hat{n}_t = \frac{u-1}{u}\hat{n}_t. \tag{16}$$

Vacancies. We linearize the definition of labor market tightness,

$$\hat{\theta}_t = \hat{v}_t - \hat{u}_t, \tag{17}$$

and solve for vacancies,

$$\hat{v}_t = \hat{\theta}_t + \hat{u}_t.$$

Substitute $\hat{u}_t$ using (16), to obtain

$$\hat{v}_t = \hat{\theta}_t - \frac{n}{u}\hat{n}_t = \hat{\theta}_t + \frac{u-1}{u}\hat{n}_t. \tag{18}$$

Employment dynamics

$$\hat{n}_{t+1} = (1-\rho)\hat{n}_t + \rho[\xi\hat{u}_t + (1-\xi)\hat{v}_t]$$

Substitute $\hat{u}_t$ and $\hat{v}_t$ using (16) and (18), to obtain

$$\hat{n}_{t+1} = (1-\rho)\hat{n}_t + \rho\xi\frac{u-1}{u}\hat{n}_t + \rho(1-\xi)\left[\hat{\theta}_t + \frac{u-1}{u}\hat{n}_t\right].$$

Collect terms in $\hat{n}_t$,

$$\hat{n}_{t+1} = \rho(1-\xi)\hat{\theta}_t + \left[(1-\rho) + \rho\xi\frac{u-1}{u} + \rho(1-\xi)\frac{u-1}{u}\right]\hat{n}_t.$$

Simplify,

$$\hat{n}_{t+1} = \rho(1-\xi)\hat{\theta}_t + \left[(1-\rho) + \rho\frac{u-1}{u}\right]\hat{n}_t. \tag{19}$$

Finally, we have the linearized employment dynamics equation in terms of θ_t,

$$\hat{n}_{t+1} = \rho(1-\xi)\hat{\theta}_t + \frac{u-\rho}{u}\hat{n}_t. \tag{20}$$

Aggregate resource constraint

$$\hat{Y}_t = \frac{yn}{Y}(\hat{n}_t + \hat{y}_t) - \frac{cv}{Y}\hat{v}_t.$$

Use the linearized definition of labor market tightness (18) to replace vacancies $\hat{v}_t$,

$$\hat{Y}_t = \frac{yn}{Y}\hat{n}_t - \frac{cv}{Y}\left(\hat{\theta}_t - \frac{n}{u}\hat{n}_t\right) + \frac{yn}{Y}\hat{y}_t.$$

Collect terms in $\hat{n}_t$,

$$\hat{Y}_t = \frac{n}{Y}\left(y + \frac{cv}{u}\right)\hat{n}_t - \frac{cv}{Y}\hat{\theta}_t + \frac{yn}{Y}\hat{y}_t.$$

Use the identity $\frac{v}{u} = \theta$,

$$\hat{Y}_t = \frac{n}{Y}(y + c\theta)\hat{n}_t - \frac{cv}{Y}\hat{\theta}_t + \frac{yn}{Y}\hat{y}_t. \tag{21}$$

We can iterate the aggregate resource constraint (21) to get output in $t+1$,

$$\hat{Y}_{t+1} = \frac{n}{Y}(y + c\theta)\hat{n}_{t+1} - \frac{cv}{Y}\hat{\theta}_{t+1} + \frac{yn}{Y}\hat{y}_{t+1},$$

and then replace $\hat{n}_{t+1}$ using the linearized employment dynamics equation (20)

$$\hat{Y}_{t+1} = \frac{n}{Y}(y + c\theta)\left[\rho(1-\xi)\hat{\theta}_t + \frac{u-\rho}{u}\hat{n}_t\right] - \frac{cv}{Y}\hat{\theta}_{t+1} + \frac{yn}{Y}\hat{y}_{t+1}. \tag{22}$$

Job creation condition. Rewrite job creation condition (14) more conveniently as

$$\frac{c}{\chi}\theta_t^{\xi}Y_t^{-\sigma} = \beta(1-\rho)E_t\left\{Y_{t+1}^{-\sigma}\left[(1-\Psi\eta)(y_{t+1}-b)-(\Psi-1)b-\Psi\eta c\theta_{t+1}+\frac{c}{\chi}\theta_{t+1}^{\xi}\right]\right\}.$$

Linearizing this equation yields

$$\begin{aligned}\frac{c}{\chi}\theta^{\xi}Y^{-\sigma}(\xi\hat{\theta}_t-\sigma\hat{Y}_t) = -\beta(1-\rho)Y^{-\sigma-1}\left[(1-\Psi\eta)(y-b)-(\Psi-1)b-\Psi\eta c\theta+\frac{c}{\chi}\theta^{\xi}\right]\sigma Y E_t\{\hat{Y}_{t+1}\} \\ +\beta(1-\rho)Y^{-\sigma}E_t\left\{(1-\Psi\eta)\hat{y}_{t+1}+\frac{c}{\chi}\theta^{\xi-1}\xi\theta\hat{\theta}_{t+1}-\Psi\eta c\theta\hat{\theta}_{t+1}\right\}.\end{aligned}$$

Divide by $Y^{-\sigma}$ to obtain

$$\begin{aligned}\frac{c}{\chi}\theta^{\xi}(\xi\hat{\theta}_t-\sigma\hat{Y}_t) &= -\beta(1-\rho)\left[(1-\Psi\eta)(y-b)-(\Psi-1)b-\Psi\eta c\theta+\frac{c}{\chi}\theta^{\xi}\right]\sigma E_t\{\hat{Y}_{t+1}\} \\ &\quad +\beta(1-\rho)\left[(1-\Psi\eta)E_t\{\hat{y}_{t+1}\}+\left(\frac{c}{\chi}\theta^{\xi}\xi-\Psi\eta c\theta\right)E_t\{\hat{\theta}_{t+1}\}\right].\end{aligned}$$

Using $\beta(1-\rho)[(1-\Psi\eta)(y-b)+(1-\Psi)b-\Psi\eta c\theta+\frac{c}{\chi}\theta^{\xi}] = \frac{c}{\chi}\theta^{\xi}$, we can simplify

$$\frac{c}{\chi}\theta^{\xi}(\xi\hat{\theta}_t-\sigma\hat{Y}_t) = -\frac{c}{\chi}\theta^{\xi}\sigma E_t\{\hat{Y}_{t+1}\}+\beta(1-\rho)\left[(1-\Psi\eta)E_t\{\hat{y}_{t+1}\}+\left(\frac{c}{\chi}\theta^{\xi}\xi-\Psi\eta c\theta\right)E_t\{\hat{\theta}_{t+1}\}\right].$$

Dividing by $\frac{c}{\chi}\theta^{\xi}$, we get

$$\xi\hat{\theta}_t-\sigma\hat{Y}_t = -\sigma E_t\{\hat{Y}_{t+1}\}+\frac{\beta(1-\rho)}{\frac{c}{\chi}\theta^{\xi}}\left[(1-\Psi\eta)E_t\{\hat{y}_{t+1}\}+\left(\frac{c}{\chi}\theta^{\xi}\xi-\Psi\eta c\theta\right)E_t\{\hat{\theta}_{t+1}\}\right].$$

Rearranging, and assuming constant productivity $\hat{y}_t = 0$, we get an equation that is similar to the one in Krause-Lubik (2010),

$$\xi\hat{\theta}_t-\sigma\hat{Y}_t = -\sigma E_t\{\hat{Y}_{t+1}\}+\beta(1-\rho)(\xi-\Psi\eta\chi\theta^{1-\xi})E_t\{\hat{\theta}_{t+1}\}. \tag{23}$$

Replace $\hat{Y}_t$ and $E_t\{\hat{Y}_{t+1}\}$ using (21) and (22), respectively, to obtain,

$$\begin{aligned}\xi\hat{\theta}_t-\sigma\left[\frac{n}{Y}(y+c\theta)\hat{n}_t-\frac{cv}{Y}\hat{\theta}_t\right] &= -\sigma\left\{\frac{n}{Y}(y+c\theta)\left[\rho(1-\xi)\hat{\theta}_t+\frac{u-\rho}{u}\hat{n}_t\right]-\frac{cv}{Y}\hat{\theta}_{t+1}\right\} \\ &\quad +\beta(1-\rho)(\xi-\Psi\eta\chi\theta^{1-\xi})E_t\{\hat{\theta}_{t+1}\}.\end{aligned}$$

Multiply out

$$\begin{aligned}\xi\hat{\theta}_t-\sigma\frac{n}{Y}(y+c\theta)\hat{n}_t+\sigma\frac{cv}{Y}\hat{\theta}_t &= -\sigma\frac{n}{Y}(y+c\theta)\rho(1-\xi)\hat{\theta}_t-\sigma\frac{n}{Y}(y+c\theta)\frac{u-\rho}{u}\hat{n}_t+\sigma\frac{cv}{Y}E_t\{\hat{\theta}_{t+1}\} \\ &\quad +\beta(1-\rho)(\xi-\Psi\eta\chi\theta^{1-\xi})E_t\{\hat{\theta}_{t+1}\}.\end{aligned}$$

Collect terms in $\hat{\theta}_t$ and $\hat{\theta}_{t+1}$,

$$\begin{aligned}
&\left[\xi + \sigma\frac{cv}{Y} + \sigma\frac{n}{Y}(y+c\theta)\rho(1-\xi)\right]\hat{\theta}_t + \sigma\frac{n}{Y}(y+c\theta)\left(-1+\frac{u-\rho}{u}\right)\hat{n}_t \\
= \quad &\left[\beta(1-\rho)(\xi - \Psi\eta\chi\theta^{1-\xi}) + \sigma\frac{cv}{Y}\right] E_t\{\hat{\theta}_{t+1}\}.
\end{aligned}$$

Rearrange

$$\left[\xi + \sigma\frac{cv}{Y} + \sigma\frac{n}{Y}(y+c\theta)\rho(1-\xi)\right]\hat{\theta}_t - \sigma\frac{n}{Y}(y+c\theta)\frac{\rho}{u}\hat{n}_t = \left[\beta(1-\rho)(\xi - \Psi\eta\chi\theta^{1-\xi}) + \sigma\frac{cv}{Y}\right] E_t\{\hat{\theta}_{t+1}\}. \quad (24)$$

Defining

$$\alpha_1 = \beta(1-\rho)(\xi - \Psi\eta\chi\theta^{1-\xi}) + \sigma\frac{cv}{Y}, \quad (25)$$

$$\alpha_2 = \sigma\frac{n}{Y}(y+c\theta), \quad (26)$$

we can write (24) as

$$\alpha_1 E_t\{\hat{\theta}_{t+1}\} = \left[\xi + \sigma\frac{cv}{Y} + \alpha_2\rho(1-\xi)\right]\hat{\theta}_t - \alpha_2\frac{\rho}{u}\hat{n}_t.$$

Notice that, using $q = \chi\theta^{-\xi}$ and $p = q\theta$, we can write α_1 as follows

$$\alpha_1 = \beta(1-\rho)\xi\left(1 - \frac{\eta}{\xi}\Psi p\right) + \sigma\frac{cv}{Y}. \quad (27)$$

Under the Hosios condition, the composite parameter simplifies to

$$\alpha_1 = \beta(1-\rho)\xi(1-\Psi p) + \sigma\frac{cv}{Y}. \quad (28)$$

Rearranging, we finally obtain

$$E_t\{\hat{\theta}_{t+1}\} = \left[\frac{\xi + \sigma\frac{cv}{Y}}{\alpha_1} + \rho(1-\xi)\frac{\alpha_2}{\alpha_1}\right]\hat{\theta}_t - \frac{\alpha_2}{\alpha_1}\frac{\rho}{u}\hat{n}_t.$$

1.10 Model solution

We can write the system of two equations in matrix form:

$$\begin{bmatrix} E_t\{\hat{\theta}_{t+1}\} \\ \hat{n}_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{\xi+\sigma\frac{cv}{Y}}{\alpha_1} + \rho(1-\xi)\frac{\alpha_2}{\alpha_1} & -\frac{\alpha_2}{\alpha_1}\frac{\rho}{u} \\ \rho(1-\xi) & 1-\frac{\rho}{u} \end{bmatrix}\begin{bmatrix} \hat{\theta}_t \\ \hat{n}_t \end{bmatrix}. \quad (29)$$

There exists a unique model solution if and only if the number of unstable eigenvalues ($\lambda_{1/2} < 1$) is equal to the number of controls and if the number of stable eigenvalues ($\lambda_{1/2} > 1$) is equal to the number of states. If there are too many unstable eigenvalues the model solution is explosive, while if there are too few unstable eigenvalues there are multiple equilibria (indeterminacy). In this model where there is one control variable (θ_t) and one state variable (n_t),

1. the model solution is **unique** if and only if either $\mid \lambda_1 \mid < 1$ and $\mid \lambda_2 \mid > 1$, or $\mid \lambda_1 \mid > 1$ and $\mid \lambda_2 \mid < 1$.

2. the model solution is **indeterminate** if both roots lie inside the unit circle, $\mid \lambda_1 \mid < 1$ and $\mid \lambda_2 \mid < 1$.

3. the model solution is **non-existent** if both roots lie outside the unit circle, $\mid \lambda_1 \mid > 1$ and $\mid \lambda_2 \mid > 1$.

1.11 Special case: risk neutrality

Under risk neutrality, i.e. if $\sigma = 0$, then $\alpha_1 = \beta(1-\rho)\xi(1-\frac{\eta}{\xi}\Psi p)$ and $\alpha_2 = 0$ and thus

$$\begin{bmatrix} \hat{\theta}_{t+1} \\ \hat{n}_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\beta(1-\rho)(1-\frac{\eta}{\xi}\Psi p)} & 0 \\ \rho(1-\xi) & 1-\frac{\rho}{u} \end{bmatrix} \begin{bmatrix} \hat{\theta}_t \\ \hat{n}_t \end{bmatrix} \tag{30}$$

Notice that in this special case, the transition matrix is lower triangular, hence the eigenvalues are its diagonal elements. Therefore, under risk neutrality,

1. the model solution is **unique** if and only
   (a) either $|\frac{1}{\beta(1-\rho)(1-\frac{\eta}{\xi}\Psi p)}| < 1$ and $|1-\frac{\rho}{u} > 1|$, that is if $|\beta(1-\rho)(1-\frac{\eta}{\xi}\Psi p)| > 1$ and $\rho > 2u > 0$.
   (b) or $|\frac{1}{\beta(1-\rho)(1-\frac{\eta}{\xi}\Psi p)}| > 1$ and $|1-\frac{\rho}{u}| < 1$, that is if $|\beta(1-\rho)(1-\frac{\eta}{\xi}\Psi p)| < 1$ and $0 < \rho < 2u$.
2. the model solution is **indeterminate** if $|\beta(1-\rho)(1-\frac{\eta}{\xi}\Psi p)| > 1$ and $0 < \rho < 2u$.
3. the model solution is **non-existent** if $|\beta(1-\rho)(1-\frac{\eta}{\xi}\Psi p)| < 1$ and $\rho > 2u > 0$.

Regarding Case 2, we note that indeterminacy arises when the workers' bargaining power, which measures the share of the match surplus going to workers, exceeds by a sufficiently large amount the contribution of the workers to the match success, which is captured by the match elasticity to unemployment, i.e. when $\eta >> \xi$. The condition for indeterminacy is given by

$$|1-\frac{\eta}{\xi}\Psi p| > \frac{1}{\beta(1-\rho)}. \tag{31}$$

Mathematically, this inequality is satified for

$$-\frac{\eta}{\xi}\Psi p > \frac{1-\beta(1-\rho)}{\beta(1-\rho)}. \tag{32}$$

However, we can rule out this case given that the right hand side of (32) is positive, and the left hand side is necessarily negative since Ψ, η, ξ and p are all positive. Therefore, indeterminacy arises if and only if

$$\Psi\eta > \frac{1+\beta(1-\rho)}{\beta(1-\rho)p}\xi. \tag{33}$$

Standard search model without taxes. Suppose that the labor tax rate is zero, such that $\tau = 0$ and thus $\Psi = 1$. Replacing the job finding rate with the underlying deep parameters using $p = \frac{\rho}{1-\rho}\frac{1-u}{u}$, we can write the determinacy frontier as

$$\eta > \underbrace{\frac{1+\beta(1-\rho)}{\beta(1-u)\frac{\rho}{u}}}_{>1}\xi. \tag{34}$$

Condition (34) shows that, on the (ξ, η)-plane, the indeterminacy frontier is a straight line with slope greater than 1. All (ξ, η)-pairs above this line are associated with an indeterminate model solution.

Now, regardless of parameter choices, the model itself provides further restrictions that influence equilibrium existence and uniqueness.

1. The restriction that the job finding rate must be lower than 1, i.e. $p < 1$ requires that the separation rate must not exceed the steady state unemployment rate, i.r. $\rho < u$. Therefore, Cases 1(a) and 3 can be ruled out with the current calibration strategy that ensures a fixed u.

2. Given that β, ρ and p all lie between 0 and 1, it is clear that under the Hosios condition $\eta = \xi$, the first root of the transition matrix is unstable, $|\beta(1-\rho)(1-p)| < 1$. Therefore, the Hosios condition rules out indeterminacy, i.e. Case 3, as well as Case 1(a).

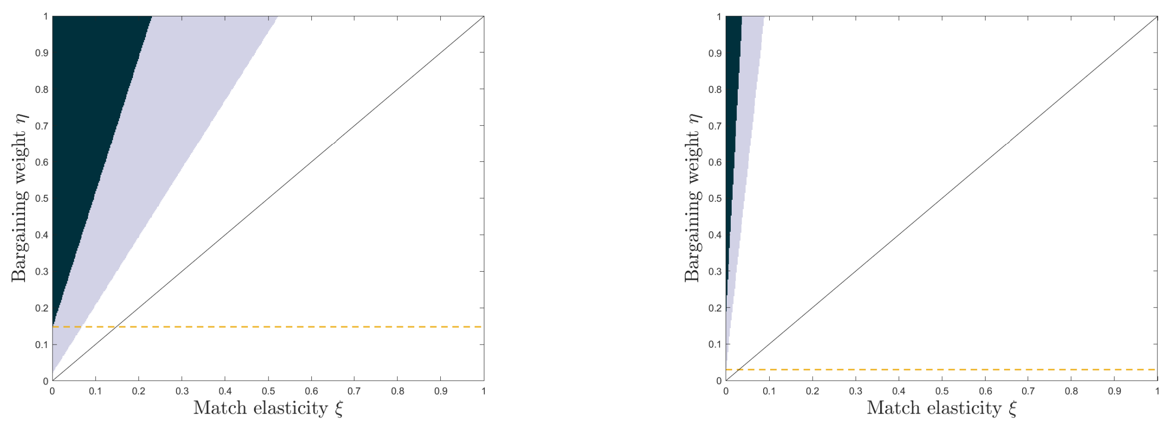
The effect of labor taxes. The left hand side of (33) can be interpreted as a tax-adjusted bargaining power of workers. Introducing labor taxes in the model, i.e. setting $0 < \tau < 1$, which implies $\Psi > 1$ and hence $\Psi\eta > \eta$, has the effect of making indeterminacy more likely. This is because the threshold value for the bargaining power parameter above which indeterminacy obtains, is decreased.

1.12 Labor market liquidity

To illustrate the importance of the liquidity of the labor market for determinacy, we present in Figure 1 two calibrations: one for the US, the other for the euro area. For the US, we set $u = 0.06$ and $\rho = 0.033$; for the euro area, we set $u = 0.1$ and $\rho = 0.0101$ in line with the data. The parameter on the horizontal axis is the elasticity of the matching function to unemployment, ξ; the parameter on the vertical axis is the workers' share of the surplus, which in the standard labor search model is equivalent to the Nash bargaining weight, η.

Figure 1 confirms that the Hosios condition (see Hosios, 1990) along the 45-degree line guarantees equilibrium uniqueness. The indeterminacy region, the combination of ξ and η that satisfies (33), shows up as a shaded triangle in the upper left corner of the figure, where workers appropriate a share of the wage bargain far above their contribution to the realization of the match surplus.

Figure 1: Standard search model: US vs. euro area calibration



Note: Indeterminacy regions are shaded black, uniqueness regions with a negative stable root in vacancy posting condition are shaded gray, uniqueness regions with a positive stable root are white. Below the dashed line, the implied leisure value b is negative.

The figure shows a noticeable difference in the size of the indeterminacy region between the US and the euro area. Labor market rigidities, which are greater in the euro area, reduce the risk of indeterminacy substantially. However, even in the US calibration, the parameter values for ξ and η that have been proposed in the empirical literature are associated with equilibrium uniqueness.

For the match elasticity to total search effort ξ, Petrongolo and Pissarides (2001) propose a range of values between 0.5 and 0.7. Hall (2005) estimates a lower match elasticity using the US Job Openings and Labor Turnover Survey (JOLTS). He first computes tightness θ by dividing the number of vacancies by the unemployed; second, he computes the job finding rate p as the ratio of new hires ('matches') to the unemployed; finally, he calculates the ratio of $\ln\Delta p$ to $\ln\Delta\theta$, where Δ measures the change over the period December 2000 to December 2002. This yields 0.765, such that the match elasticity ξ is 0.235. For different groups of unemployed job-seekers, Hall and Schulhofer-Wohl (2018) report a range of estimates of the match elasticity from 0.45 to 0.83.

When the worker's bargaining weight η is set equal to ξ, the Hosios condition is satisfied. Empirical evidence on the size of the bargaining weight is scant. In Hall and Milgrom (2008), the implied worker's share equals 0.54 as in Mortensen and Pissarides (1994), even though wage setting is rather different, resting on a bargaining model with different threat points than the ones assumed under Nash bargaining. In both models, this value is obtained by solving the zero profit condition to match the unemployment rate in the data.

In an alternative calibration exercise, we normalize steady state labor market tightness such that $\theta = 1$, and we fix the share of vacancy posting costs in firm output cv/y to 1%. Then, we set the job finding rate p to 0.45 in one calibration and to 0.27 in another. These values correspond to the average US job finding rates for the periods 1948-2007 and 2008-2015, respectively, according to Hall and Schulhofer-Wohl (2018). Notice that this is not a huge deviation from the baseline strategy, since at every point in the resulting figure, the steady state unemployment rate is the same; it is given by $u = (1 + p/x)^{-1}$, where x is the hiring rate. We find that a lower p, i.e. a less flexible labor market, is associated with a smaller indeterminacy region.[1] Also here, b turns negative for low values of η.

2 Model with Hours and Labor Effort

Preliminaries as before.

2.1 Production, firm's match surplus, hiring

The production function of the representative one-worker firm now reads

$$y_t = e_t h_t, \tag{35}$$

where h_t are hours per worker and e_t is effort per hour. The firms' wage costs are $w_t h_t$, such that period-t profits are $y_t - w_t h_t$. Let $\mathcal{J}_t$ denote the firm's match surplus, i.e. the value of filling a vacancy. It is the sum of current profits and the firm's continuation value. The latter is the expected future match surplus in case the employment relationship continues, which happens with probability $(1 - \rho)$. The firm's value is zero in case the worker and the firm separate, which happens with probability ρ. Thus,

$$\mathcal{J}_t = y_t - w_t h_t + E_t\{\beta_{t,t+1}[(1 - \rho)\mathcal{J}_{t+1} + \rho \cdot 0]\},$$

[1] For brevity, we do not show those figures here. They are available from the authors upon request.

where $\beta_{t-1,t} = \beta \frac{\lambda_t}{\Lambda_{t-1}}$ is the household's stochastic discount factor and $\lambda_t = C_t^{-\sigma}$ is the marginal utility of consumption. The firm's match surplus can be written as

$$\mathcal{J}_t = y_t - w_t h_t + (1-\rho) E_t\{\beta_{t,t+1} \mathcal{J}_{t+1}\}. \tag{36}$$

The value of posting a vacancy is given by minus the vacancy posting cost c, plus the expected future value of the vacancy. The latter is given by the weighted average of the value of filling the vacancy, i.e. the firm's match value in the next period, which has probability $q_t(1-\rho)$, and the future value of the unfilled vacancy, V_{t+1}, which has probability $(1 - q_t(1-\rho))$. Therefore,

$$V_t = -c + E_t\{\beta_{t,t+1}[q_t(1-\rho)\mathcal{J}_{t+1} + (1 - q_t(1-\rho))V_{t+1}]\}. \tag{37}$$

Free entry drives the value of a vacancy to zero at each point in time, such that $V_t = 0$ for all t and thus

$$\frac{c}{q_t} = (1-\rho) E_t\{\beta_{t,t+1} \mathcal{J}_{t+1}\}. \tag{38}$$

Combining the firm's asset value (36) and the free entry condition (38), we get the following expression for the firm's match surplus

$$\mathcal{J}_t = y_t - w_t h_t + \frac{c}{q_t}. \tag{39}$$

The derivative of the firm's match surplus to the wage is

$$\frac{\partial \mathcal{J}_t}{\partial w_t} = -1.$$

Finally, using the firm's match surplus (39) to substitute out $\mathcal{J}_{t+1}$ in the free entry condition (38), we obtain the job creation condition

$$\frac{c}{q_t} = (1-\rho) E_t \left\{ \beta_{t,t+1} \left[y_{t+1} - w_{t+1} h_{t+1} + \frac{c}{q_{t+1}} \right] \right\}. \tag{40}$$

2.2 Utility and worker's match surplus

Utility maximization is given by

$$\max_{\{C_t, h_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta_{0,t} \left[\frac{C_t^{1-\sigma} - 1}{1-\sigma} - n_t g(h_t, e_t) \right],$$

with labor disutility

$$g(h_t, e_t) = \frac{\lambda_h h_t^{1+\sigma_h}}{1+\sigma_h} + h_t \frac{\lambda_e e_t^{1+\sigma_e}}{1+\sigma_e}, \tag{41}$$

where λ_h (λ_e) > 0 is the weight on hours (effort) in labor disutility and σ_h (σ_e) ≥ 0 determines the degree of increasing marginal disutility of hours (effort). The first term in (41) captures disutility from spending h_{it} hours at work, rather than some best alternative, even when exerting no productive effort. The second term reflects disutility from exerting effort.

Denote the value of being employed W_t and the value of being unemployed U_t. In period t, an employed worker receives the wage income $w_t h_t$ and suffers the disutility $g(h_t, e_t)$. In the next period, he is either still employed with probability $1-\rho$, in which case he has an expected value of $E_t\{\beta_{t,t+1} W_{t+1}\}$, or the employment relation is dissolved with probability ρ, then his expected value is $E_t\{\beta_{t,t+1}U_{t+1}\}$. The worker's asset value of being matched to a firm is

$$W_t = w_t h_t - \frac{g(h_t, e_t)}{\Lambda_t} + E_t\{\beta_{t,t+1}[(1-\rho)W_{t+1} + \rho U_{t+1}]\}, \tag{42}$$

where we divide labor disutility $g(h_t)$ by the marginal utility of consumption Λ_t to convert utils into consumption units. The value of being unemployed U_t is in turn given by

$$U_t = b + E_t\{\beta_{t,t+1}[p_t(1-\rho)W_{t+1} + (1 - p_t(1-\rho))U_{t+1}]\}. \tag{43}$$

An unemployed worker receives an income b units of consumption goods in period t. In the next period, he faces a probability $p_t(1-\rho)$ of finding a new job and a probability $1 - p_t(1-\rho)$ of remaining unemployed. Defining the worker's surplus as $\mathcal{W}_t = W_t - U_t$, we can subtract (43) from (42) to write the match surplus going to the worker as

$$\mathcal{W}_t = w_t h_t - \frac{g(h_t, e_t)}{\lambda_t} - b + (1-\rho)E_t\left\{\beta_{t,t+1}[(1-p_t)\mathcal{W}_{t+1}]\right\}. \tag{44}$$

2.3 Effort

Effort is determined as in Bils and Cho (1994). Every period, the firm and the worker negotiate over hours and effort in order to minimize labor disutility (41), subject to the production function (35),

$$\min_{h_t, e_t} \frac{\lambda_h h_t^{1+\sigma_h}}{1+\sigma_h} + h_t \frac{\lambda_e e_t^{1+\sigma_e}}{1+\sigma_e} - \varphi_{yt}(y_t - e_t h_t).$$

The first order conditions for hours h_t and effort e_t are, respectively,

$$0 = \lambda_h h_t^{\sigma_h} + \frac{\lambda_e e_t^{1+\sigma_e}}{1+\sigma_e} + \varphi_{yt}\frac{y_t}{h_t}, \tag{45}$$

$$0 = \lambda_e h_t e_t^{\sigma_e} + \varphi_{yt}\frac{y_t}{e_t}. \tag{46}$$

Writing (46) as $-\lambda_e e_t^{1+\sigma_e} = \varphi_{yt} y_t / h_t$, and plugging this into (45) yields

$$\begin{aligned} 0 &= \lambda_h h_t^{\sigma_h} + \frac{\lambda_e e_t^{1+\sigma_e}}{1+\sigma_e} - \lambda_e e_t^{1+\sigma_e} \\ &= \lambda_h h_t^{\sigma_h} + \left(\frac{1}{1+\sigma_e} - 1\right)\lambda_e e_t^{1+\sigma_e} \\ &= \lambda_h h_t^{\sigma_h} - \frac{\sigma_e}{1+\sigma_e}\lambda_e e_t^{1+\sigma_e}. \end{aligned}$$

Solving for effort, we obtain

$$e_t^{1+\sigma_e} = \frac{1+\sigma_e}{\sigma_e}\frac{\lambda_h}{\lambda_e} h_t^{\sigma_h}, \tag{47}$$

such that equilibrium effort is an increasing function of hours per worker,

$$e_t = e_0 h_t^{\frac{\sigma_h}{1+\sigma_e}}, \tag{48}$$

where

$$e_0 = \left(\frac{1+\sigma_e}{\sigma_e} \frac{\lambda_h}{\lambda_e} \right)^{\frac{1}{1+\sigma_e}}. \tag{49}$$

The elasticity of effort with respect to hours is given by $\frac{\sigma_h}{1+\sigma_e} > 0$. Using the optimal effort choice (47), we can rewrite the production function (35) as

$$y_t = e_0 h_t^{\phi}, \tag{50}$$

with

$$\phi = 1 + \frac{\sigma_h}{1+\sigma_e}. \tag{51}$$

Consider that in the standard model with constant labor effort, the disutility of effort is extremely large, $\sigma_e \to \infty$, hence effort does not vary. If this is the case, then $\phi \to 1$. Thus the production function does not exhibit increasing return to scale anymore.

Substituting the combined first order conditions for hours and effort (47) in the labor disutility function (41) to eliminate $\frac{\lambda_e}{1+\sigma_e} e_t^{1+\sigma_e} = \frac{\lambda_h}{\sigma_e} h_t^{\sigma_h}$, the disutility of working is a function of hours only,

$$g(h_t) = \lambda_h \frac{1+\sigma_h+\sigma_e}{(1+\sigma_h)\sigma_e} h_t^{1+\sigma_h}. \tag{52}$$

Note that

$$g'(h_t) = \lambda_h \frac{1+\sigma_h+\sigma_e}{\sigma_e} h_t^{\sigma_h}, \tag{53}$$

$$g(h_t) = \frac{g'(h_t) h_t}{1+\sigma_h},$$

$$\lim_{\sigma_e \to \infty} \frac{1+\sigma_h+\sigma_e}{\sigma_e} = 1.$$

2.4 Hours

Hours are determined jointly by the firm and the worker to maximize the sum of the firm's surplus and the worker's surplus, (44) and (36). The first order condition for hours worked satisfies

$$\phi e_0 h_t^{\phi-1} = \frac{g'(h_t)}{\lambda_t}.$$

Using the relation $g(h_t) = \frac{g'(h_t)h_t}{1+\sigma_h}$, this can also be expressed as

$$e_0 h_t^{\phi} = \frac{1+\sigma_h}{\phi} \frac{g(h_t)}{\lambda_t} = \frac{1+\sigma_h}{\phi} mrs_t. \tag{54}$$

2.5 Wage bargaining

The surplus sharing rule is given by (11) as before. Inserting the worker's and firm's surplus, (44) and (36), we obtain

$$w_t h_t - \frac{g(h_t)}{\lambda_t} - b + (1-\rho) E_t\{\beta_{t,t+1}(1-p_t)\mathcal{W}_{t+1}\} = \frac{\eta}{1-\eta}[e_0 h_t^{\phi} - w_t h_t + (1-\rho) E_t\{\beta_{t,t+1}\mathcal{J}_{t+1}\}].$$

Using the sharing rule (11) to replace $\mathcal{W}_{t+1}$ with $\frac{\eta}{1-\eta}\mathcal{J}_{t+1}$ yields

$$w_t h_t - \frac{g(h_t)}{\lambda_t} - b + (1-\rho)E_t\{\beta_{t,t+1}(1-p_t)\frac{\eta}{1-\eta}\mathcal{J}_{t+1}\} = \frac{\eta}{1-\eta}[e_0 h_t^{\phi} - w_t h_t + (1-\rho)E_t\{\beta_{t,t+1}\mathcal{J}_{t+1}\}].$$

Cancelling terms in $(1-\rho)\frac{\eta}{1-\eta}E_t\{\beta_{t,t+1}\mathcal{J}_{t+1}\}$ yields

$$w_t h_t - \frac{g(h_t)}{\lambda_t} - b - \frac{\eta}{1-\eta}p_t(1-\rho)E_t\{\beta_{t,t+1}\mathcal{J}_{t+1}\} = \frac{\eta}{1-\eta}(e_0 h_t^{\phi} - w_t h_t).$$

Using the free entry condition (38) to replace $(1-\rho)E_t\{\beta_{t,t+1}\mathcal{J}_{t+1}\}$ with $\frac{c}{q_t}$ yields

$$w_t h_t - \frac{g(h_t)}{\lambda_t} - b = \frac{\eta}{1-\eta}\left(e_0 h_t^{\phi} - w_t h_t + p_t\frac{c}{q_t}\right).$$

Use $p_t/q_t = \theta_t$, multiply by $(1-\eta)$, collect terms in $w_t h_t$ and rearrange to get the per-person wage

$$w_t h_t = (1-\eta)\left(\frac{g(h_t)}{\lambda_t} + b\right) + \eta(e_0 h_t^{\phi} + c\theta_t). \tag{55}$$

Using the wage equation in the vacancy posting condition (40) to replace $w_t h_t$ yields:

$$\frac{c}{q_t} = (1-\rho)E_t\left\{\beta_{t,t+1}\left[e_0 h_{t+1}^{\phi} - [\eta(e_0 h_{t+1}^{\phi} + c\theta_{t+1}) + (1-\eta)(mrs_{t+1} + b)] + \frac{c}{q_{t+1}}\right]\right\}.$$

Write this more conveniently as

$$\frac{c}{q_t} = \beta(1-\rho)E_t\left\{\frac{Y_t^{\sigma}}{Y_{t+1}^{\sigma}}\left[(1-\eta)(e_0 h_{t+1}^{\phi} - mrs_{t+1} - b) - \eta c\theta_{t+1} + \frac{c}{q_{t+1}}\right]\right\}. \tag{56}$$

Plug the hours choice (54) into the job creation condition (56) to substitute mrs_t

$$\frac{c}{q_t} = \beta(1-\rho)E_t\left\{\frac{Y_t^{\sigma}}{Y_{t+1}^{\sigma}}\left[(1-\eta)\left[\left(1 - \frac{\phi}{1+\sigma_h}\right)e_0 h_{t+1}^{\phi} - b\right] - \eta c\theta_{t+1} + \frac{c}{q_{t+1}}\right]\right\}. \tag{57}$$

2.6 Equilibrium conditions

Variables u_t, θ_t, n_t, Y_t, v_t, h_t and mrs_t:

$$u_t = 1 - n_t$$

$$\theta_t = \frac{v_t}{u_t}$$

$$n_t = (1-\rho)(n_{t-1} + \chi u_{t-1}^{\xi} v_{t-1}^{1-\xi})$$

$$Y_t = e_0 h_t^{\phi} n_t - c v_t$$

$$\frac{c}{q_t} = \beta(1-\rho)E_t\left\{\frac{Y_t^{\sigma}}{Y_{t+1}^{\sigma}}\left[(1-\eta)\left[\left(1 - \frac{\phi}{1+\sigma_h}\right)e_0 h_{t+1}^{\phi} - b\right] - \eta c\theta_{t+1} + \frac{c}{q_{t+1}}\right]\right\}$$

$$e_0 h_t^{\phi} = \frac{1+\sigma_h}{\phi} mrs_t$$

$$mrs_t = \lambda_h \frac{1+\sigma_h+\sigma_e}{(1+\sigma_h)\sigma_e} h_t^{1+\sigma_h} Y_t^{\sigma}$$

2.7 Recursive steady state

Normalize $h = e_0 = 1$. Calibrate β, σ, ρ, ξ, ρ. Then, Krause and Lubik (2010) also calibrate q, c, u. In the model with hours and effort, we also need to set σ_h, σ_e. Implied steady state variables or parameters: n, v, θ, χ, mrs, b, Y.

$$n = 1 - u$$

$$v = \frac{\rho}{1-\rho}\frac{n}{q}$$

$$\theta = \frac{v}{u}$$

$$\chi = q\theta^{\xi}$$

$$mrs = \frac{\phi}{1+\sigma_h} e_0 h^{\phi}$$

$$b = \left(1 - \frac{\phi}{1+\sigma_h}\right) e_0 h^{\phi} - \frac{1}{1-\eta}\left(\frac{1-\beta(1-\rho)}{\beta(1-\rho)}\frac{c}{\chi}\theta^{\xi} + \eta c\theta\right)$$

$$Y = e_0 h^{\phi} n - cv$$

$$\lambda_h = \frac{\phi\sigma_e}{1+\sigma_h+\sigma_e} Y^{-\sigma} = \frac{\sigma_e}{1+\sigma_e} Y^{-\sigma}, \tag{58}$$

$$\lambda_e = \frac{1+\sigma_e}{\sigma_e}\lambda_h = Y^{-\sigma}. \tag{59}$$

2.8 Linearization of equilibrium conditions

Unemployment, tightness, employment law of motion are the same as before.

Marginal rate of substitution

$$\widehat{mrs}_t = (1+\sigma_h)\hat{h}_t + \sigma\hat{Y}_t$$

Hours

$$\phi\hat{h}_t = \widehat{mrs}_t \tag{60}$$

Combining the latter two equations yields

$$\widehat{mrs}_t = \frac{1+\sigma_h}{\phi}\widehat{mrs}_t + \sigma\hat{Y}_t.$$

Collecting terms,

$$\left(1 - \frac{1+\sigma_h}{\phi}\right)\widehat{mrs}_t = \sigma\hat{Y}_t,$$

and solving for $\hat{mrs}_t$, we obtain

$$\widehat{mrs}_t = \frac{\sigma\phi}{\phi - (1+\sigma_h)}\hat{Y}_t. \tag{61}$$

Aggregate accounting

$$\hat{Y}_t = \frac{e_0 n h^{\phi}}{Y}(\hat{n}_t + \phi\hat{h}_t) - \frac{cv}{Y}\hat{v}_t. \tag{62}$$

Substituting vacancies $\hat{v}_t$ using (18) yields

$$\hat{Y}_t = \frac{e_0 n h^{\phi}}{Y}(\hat{n}_t + \phi \hat{h}_t) - \frac{cv}{Y}\left(\hat{\theta}_t - \frac{n}{u}\hat{n}_t\right).$$

Collecting the terms in $\hat{n}_t$, we have

$$\hat{Y}_t = \left(\frac{e_0 n h^{\phi}}{Y} + \frac{cv}{Y}\frac{n}{u}\right)\hat{n}_t + \frac{e_0 n h^{\phi}}{Y}\phi \hat{h}_t - \frac{cv}{Y}\hat{\theta}_t.$$

Using $\left(\frac{e_0 n h^{\phi}}{Y} + \frac{cv}{Y}\frac{n}{u}\right) = 1 + \frac{cv}{Y}\frac{1}{u}$, we can simplify,

$$\hat{Y}_t = \left(1 + \frac{c\theta}{Y}\right)\hat{n}_t + \frac{e_0 n h^{\phi}}{Y}\phi \hat{h}_t - \frac{cv}{Y}\hat{\theta}_t,$$

where we have also used the definition of θ. Substitute hours $\hat{h}_t$ using (60) and (61),

$$\hat{Y}_t = \left(1 + \frac{c\theta}{Y}\right)\hat{n}_t + \frac{e_0 n h^{\phi}}{Y}\frac{\sigma\phi}{\phi - (1+\sigma_h)}\hat{Y}_t - \frac{cv}{Y}\hat{\theta}_t,$$

and collect terms in $\hat{Y}_t$ to obtain

$$\left(1 - \frac{e_0 n h^{\phi}}{Y}\frac{\sigma\phi}{\phi - (1+\sigma_h)}\right)\hat{Y}_t = \left(1 + \frac{c\theta}{Y}\right)\hat{n}_t - \frac{cv}{Y}\hat{\theta}_t.$$

This can be alternatively expressed as

$$\delta_1 \hat{Y}_t = \left(1 + \frac{c\theta}{Y}\right)\hat{n}_t - \frac{cv}{Y}\hat{\theta}_t. \tag{63}$$

where δ_1 is given by

$$\delta_1 = 1 - \frac{e_0 n h^{\phi}}{Y}\frac{\sigma\phi}{\phi - (1+\sigma_h)}. \tag{64}$$

Substituting ϕ, this can also be expressed as

$$\delta_1 = 1 + \frac{e_0 n h^{\phi}}{Y}\sigma\frac{1+\sigma_e+\sigma_h}{\sigma_e\sigma_h}. \tag{65}$$

Iterating by one period and using employment dynamics (20) to substitute n_{t+1} in (63), we obtain

$$\delta_1 \hat{Y}_{t+1} = \left(1 + \frac{c\theta}{Y}\right)\left(\rho(1-\xi)\hat{\theta}_t + \frac{u-\rho}{u}\hat{n}_t\right) - \frac{cv}{Y}\hat{\theta}_{t+1}. \tag{66}$$

Job creation condition

Rewrite job creation condition (57) more conveniently as

$$\frac{c}{\chi}\theta_t^{\xi}Y_t^{-\sigma} = \beta(1-\rho)E_t\left\{Y_{t+1}^{-\sigma}[(1-\eta)\left[\left(\frac{1+\sigma_h}{\phi} - 1\right)mrs_{t+1} - b\right] - \eta c\theta_{t+1} + \frac{c}{\chi}\theta_{t+1}^{\xi}]\right\}.$$

Linearizing this equation yields

$$\frac{c}{\chi}\theta^{\xi}Y^{-\sigma}(\xi\hat{\theta}_t - \sigma\hat{Y}_t) = -\beta(1-\rho)Y^{-\sigma-1}\left[(1-\eta)\left[\left(\frac{1+\sigma_h}{\phi}-1\right)mrs - b\right] - c\eta\theta + \frac{c}{\chi}\theta^{\xi}\right]\sigma Y\hat{Y}_{t+1}$$
$$+\beta(1-\rho)Y^{-\sigma}(1-\eta)\left(\frac{1+\sigma_h}{\phi}-1\right)(mrs)\widehat{mrs}_{t+1}$$
$$+\beta(1-\rho)Y^{-\sigma}\left[\frac{c}{\chi}\theta^{\xi-1}\xi\theta\hat{\theta}_{t+1} - \eta c\theta\hat{\theta}_{t+1}\right]$$

Divide by $Y^{-\sigma}$ to obtain

$$\frac{c}{\chi}\theta^{\xi}(\xi\hat{\theta}_t - \sigma\hat{Y}_t) = -\beta(1-\rho)\left[(1-\eta)\left[\left(\frac{1+\sigma_h}{\phi}-1\right)mrs - b\right] - c\eta\theta + \frac{c}{\chi}\theta^{\xi}\right]\sigma\hat{Y}_{t+1}$$
$$+\beta(1-\rho)(1-\eta)\left(\frac{1+\sigma_h}{\phi}-1\right)(mrs)\widehat{mrs}_{t+1}$$
$$+\beta(1-\rho)\left[\frac{c}{\chi}\theta^{\xi-1}\xi\theta\hat{\theta}_{t+1} - \eta c\theta\hat{\theta}_{t+1}\right].$$

Using $\beta(1-\rho)\left[(1-\eta)\left[\left(\frac{1+\sigma_h}{\phi}-1\right)mrs - b\right] - c\eta\theta + \frac{c}{\chi}\theta^{\xi}\right] = \frac{c}{\chi}\theta^{\xi}$, we can simplify

$$\frac{c}{\chi}\theta^{\xi}(\xi\hat{\theta}_t - \sigma\hat{Y}_t) = -\frac{c}{\chi}\theta^{\xi}\sigma\hat{Y}_{t+1} + \beta(1-\rho)(1-\eta)\left(\frac{1+\sigma_h}{\phi}-1\right)(mrs)\widehat{mrs}_{t+1}$$
$$+\beta(1-\rho)\left[\frac{c}{\chi}\theta^{\xi-1}\xi\theta\hat{\theta}_{t+1} - \eta c\theta\hat{\theta}_{t+1}\right].$$

Dividing by $\frac{c}{\chi}\theta^{\xi}$, we get

$$\xi\hat{\theta}_t - \sigma\hat{Y}_t = -\sigma\hat{Y}_{t+1} + \frac{\beta(1-\rho)}{\frac{c}{\chi}\theta^{\xi}}\left[\left(\frac{c}{\chi}\theta^{\xi}\xi - \eta c\theta\right)\hat{\theta}_{t+1} + (1-\eta)\left(\frac{1+\sigma_h}{\phi}-1\right)(mrs)\widehat{mrs}_{t+1}\right].$$

Rearranging, we get

$$\xi\hat{\theta}_t - \sigma\hat{Y}_t = -\sigma\hat{Y}_{t+1} + \beta(1-\rho)\left[\xi - \eta\chi\theta^{1-\xi}\right]\hat{\theta}_{t+1} - \frac{\beta(1-\rho)}{\frac{c}{\chi}\theta^{\xi}}(1-\eta)\left(1-\frac{1+\sigma_h}{\phi}\right)(mrs)\widehat{mrs}_{t+1}.$$

Now, we replace $m\hat{r}s_{t+1}$ with the expression in (61)

$$\xi\hat{\theta}_t - \sigma\hat{Y}_t = -\sigma\hat{Y}_{t+1} + \beta(1-\rho)\left[\xi - \eta\chi\theta^{1-\xi}\right]\hat{\theta}_{t+1} - \frac{\beta(1-\rho)}{\frac{c}{\chi}\theta^{\xi}}(1-\eta)\left(1-\frac{1+\sigma_h}{\phi}\right)(mrs)\frac{\phi}{\phi-(1+\sigma_h)}\sigma\hat{Y}_{t+1}.$$

which simplifies to

$$\xi\hat{\theta}_t - \sigma\hat{Y}_t = -\sigma\hat{Y}_{t+1} + \beta(1-\rho)\left[\xi - \eta\chi\theta^{1-\xi}\right]\hat{\theta}_{t+1} - \frac{\beta(1-\rho)}{\frac{c}{\chi}\theta^{\xi}}(1-\eta)(mrs)\sigma\hat{Y}_{t+1}.$$

Collecting terms in $\sigma\hat{Y}_{t+1}$ we get

$$\xi\hat{\theta}_t - \sigma\hat{Y}_t = \beta(1-\rho)\left[\xi - \eta\chi\theta^{1-\xi}\right]\hat{\theta}_{t+1} - \left[1 + \frac{\beta(1-\rho)}{\frac{c}{\chi}\theta^{\xi}}(1-\eta)(mrs)\right]\sigma\hat{Y}_{t+1},$$

Replace $\hat{Y}_t$ and $\hat{Y}_{t+1}$ using (63) and (66), respectively, to obtain,

$$\begin{aligned}\xi\hat{\theta}_t - \frac{\sigma}{\delta_1}\left[\left(1+\frac{c\theta}{Y}\right)\hat{n}_t - \frac{cv}{Y}\hat{\theta}_t\right] &= \beta(1-\rho)\left[\xi - \eta\chi\theta^{1-\xi}\right]\hat{\theta}_{t+1} \\ &\quad -\sigma\frac{\delta_2}{\delta_1}\left[\left(1+\frac{c\theta}{Y}\right)\left(\rho(1-\xi)\hat{\theta}_t + \frac{u-\rho}{u}\hat{n}_t\right) - \frac{cv}{Y}\hat{\theta}_{t+1}\right],\end{aligned}$$

where δ_2 is given by

$$\delta_2 = 1 + \frac{\beta(1-\rho)}{c/q}(1-\eta)(mrs).$$

Using $mrs = \frac{\phi}{1+\sigma_h}e_0 h^{\phi}$, this can also be written as

$$\delta_2 = 1 + \frac{\beta(1-\rho)}{c/q}(1-\eta)\frac{\phi}{1+\sigma_h}e_0 h^{\phi}. \tag{67}$$

Collect terms in $\hat{\theta}_t$ and $\hat{\theta}_{t+1}$,

$$\begin{aligned}&\left[\xi + \frac{\sigma}{\delta_1}\frac{cv}{Y} + \sigma\frac{\delta_2}{\delta_1}\left(1+\frac{c\theta}{Y}\right)\rho\left(1-\xi\right)\right]\hat{\theta}_t + \sigma\frac{\delta_2}{\delta_1}\left(1+\frac{c\theta}{Y}\right)\left(1-\frac{1}{\delta_2}-\frac{\rho}{u}\right)\hat{n}_t \\ = \;&\left[\beta(1-\rho)\left[\xi-\eta\chi\theta^{1-\xi}\right] + \sigma\frac{\delta_2}{\delta_1}\frac{cv}{Y}\right]\hat{\theta}_{t+1}.\end{aligned} \tag{68}$$

Defining

$$\alpha_1 = \beta(1-\rho)\xi\left(1-\frac{\eta}{\xi}p\right) + \sigma\frac{\delta_2}{\delta_1}\frac{cv}{Y}, \tag{69}$$

$$\alpha_2 = \sigma\frac{\delta_2}{\delta_1}\left(1+\frac{c\theta}{Y}\right), \tag{70}$$

we can write (68) as

$$\alpha_1\hat{\theta}_{t+1} = \left[\xi + \frac{\sigma}{\delta_1}\frac{cv}{Y} + \alpha_2\rho(1-\xi)\right]\hat{\theta}_t + \alpha_2\left(1-\frac{1}{\delta_2}-\frac{\rho}{u}\right)\hat{n}_t$$

Summary of linearized equilibrium conditions

$$\hat{n}_{t+1} = \rho(1-\xi)\hat{\theta}_t + \frac{u-\rho}{u}\hat{n}_t$$

$$\hat{\theta}_{t+1} = \frac{1}{\alpha_1}\left[\xi + \frac{\sigma}{\delta_1}\frac{cv}{Y} + \alpha_2\rho(1-\xi)\right]\hat{\theta}_t - \frac{\alpha_2}{\alpha_1}\left(\frac{\rho}{u} - \frac{\delta_2-1}{\delta_2}\right)\hat{n}_t$$

The no-effort model is nested in the benchmark model: when $\sigma_e \to \infty$, then $\phi = 1$. Then δ_1 reduces to

$$\delta_1 = 1 + \left(1+\frac{cv}{Y}\right)\frac{\sigma}{\sigma_h}, \tag{71}$$

We can write the system of equations in matrix form:

$$\begin{bmatrix}\hat{\theta}_{t+1} \\ \hat{n}_{t+1}\end{bmatrix} = \begin{bmatrix}\frac{\xi+\frac{\sigma}{\delta_1}\frac{cv}{Y}}{\alpha_1} + \rho(1-\xi)\frac{\alpha_2}{\alpha_1} & -\frac{\alpha_2}{\alpha_1}\left(\frac{\rho}{u} - \frac{\delta_2-1}{\delta_2}\right) \\ \rho(1-\xi) & \frac{u-\rho}{u}\end{bmatrix}\begin{bmatrix}\hat{\theta}_t \\ \hat{n}_t\end{bmatrix} \tag{72}$$

3 Model with Search Effort

One-worker firms, Cobb-Douglas matching function, linear vacancy posting costs, costs of varying search intensity of the unemployed, predetermined employment, exogenous separations, no participation margin and no hours margin (labor force as well as hours constant and normalized to unity), Nash wage bargaining.

3.1 Preliminaries

Matching function with variable search intensity

$$m_t = \chi(s_t u_t)^{\xi} v_t^{1-\xi}$$

Unemployment rate

$$u_t = 1 - n_t$$

Labor market tightness

$$\theta_t = \frac{v_t}{u_t}$$

Vacancy filling rate

$$q_t \equiv \frac{m_t}{v_t} = \chi\left(\frac{s_t}{\theta_t}\right)^{\xi}$$

Job finding rate

$$p_t \equiv \frac{m_t}{u_t} = \theta_t q_t$$

Employment dynamics

$$n_t = (1-\rho)(n_{t-1} + m_{t-1})$$

Market clearing

$$Y_t = C_t$$

Aggregate accounting

$$Y_t = y_t n_t - c v_t - \mathcal{G}(s_t) u_t,$$

where $\mathcal{G}(s_t)$ is the resource cost of exerting search effort s_t per unemployed worker, to be specified below.

3.2 Production, firm's match surplus, hiring

Production of output y_t takes place in one-worker firms with labor only (i.e., no capital) and constant hours. Firms' wage costs are w_t, such that period-t profits are $y_t - w_t$. Let $\mathcal{J}_t$ denote the firm's match surplus, i.e. the value of filling a vacancy. It is the sum of current profits and the firm's continuation value. The latter is the expected future match surplus in case the employment relationship continues, which happens with probability $(1-\rho)$. The firm's value is zero in case the worker and the firm separate, which happens with probability ρ. Thus,

$$\mathcal{J}_t = y_t - w_t + E_t\{\beta_{t,t+1}[(1-\rho)\mathcal{J}_{t+1} + \rho \cdot 0]\},$$

where $\beta_{t-1,t} = \beta\frac{\lambda_t}{\lambda_{t-1}}$ is the household's stochastic discount factor and $\lambda_t = C_t^{-\sigma}$ is the marginal utility of consumption. The firm's match surplus can be written as

$$\mathcal{J}_t = y_t - w_t + (1-\rho)E_t\{\beta_{t,t+1}\mathcal{J}_{t+1}\}. \tag{73}$$

The value of posting a vacancy is given by minus the vacancy posting cost c, plus the expected future value of the vacancy. The latter is given by the weighted average of the value of filling the vacancy, i.e. the firm's match value in the next period, which has probability $q_t(1-\rho)$, and the future value of the unfilled vacancy, V_{t+1}, which has probability $(1-q_t(1-\rho))$. Therefore,

$$V_t = -c + E_t\{\beta_{t,t+1}[q_t(1-\rho)\mathcal{J}_{t+1} + (1-q_t(1-\rho))V_{t+1}]\}. \tag{74}$$

Free entry drives the value of a vacancy to zero at each point in time, such that $V_t = 0$ for all t and thus

$$\frac{c}{q_t} = (1-\rho)E_t\{\beta_{t,t+1}\mathcal{J}_{t+1}\}. \tag{75}$$

Combining the firm's asset value (73) and the free entry condition (75), we get the following expression for the firm's match surplus

$$\mathcal{J}_t = y_t - w_t + \frac{c}{q_t}. \tag{76}$$

The derivative of the firm's match surplus to the wage is $\frac{\partial \mathcal{J}_t}{\partial w_t} = -1$. Finally, using the firm's match surplus (76) to substitute out $\mathcal{J}_{t+1}$ in the free entry condition (3), we obtain the vacancy posting condition

$$\frac{c}{q_t} = (1-\rho)E_t\left\{\beta_{t,t+1}\left(y_{t+1} - w_{t+1} + \frac{c}{q_{t+1}}\right)\right\}. \tag{77}$$

3.3 Utility and worker's match surplus

Utility maximization is given by

$$\max_{\{C_t\}_{t=0}^{\infty}} U = E_0\sum_{t=0}^{\infty}\beta_{0,t}\left[\frac{C_t^{1-\sigma}-1}{1-\sigma}\right]$$

where $\sigma \geq 0$ is the constant of relative risk aversion, subject to the budget constraint

$$C_t + T_t = n_t w_t + u_t(b - \mathcal{G}(s_t)), \tag{78}$$

where T_t are lump-sum taxes, b are unemployment benefits. The function $\mathcal{G}(s_t)$ captures the cost of searching for a job, where $s_t \geq 0$. Merz (1995) calls this a 'shoe-leather' cost of search.

Denote the value of being employed W_t and the value of being unemployed U_t. In period t, an employed worker receives the wage income w_t. In the next period, he is either still employed with probability $(1-\rho)$, in which case he has an expected value of $E_t\{\beta_{t,t+1}W_{t+1}\}$, or the employment relation is dissolved with probability ρ, then his expected value is $E_t\{\beta_{t,t+1}U_{t+1}\}$. The worker's asset value therefore is

$$W_t = w_t + E_t\{\beta_{t,t+1}[(1-\rho)W_{t+1} + \rho U_{t+1}]\}. \tag{79}$$

Defining the worker's surplus as $\mathcal{W}_t = W_t - U_t$, the employment value can be written more conveniently as:

$$W_t = w_t + E_t\{\beta_{t,t+1}(-\rho\mathcal{W}_{t+1} + W_{t+1})\}. \tag{80}$$

The value of being unemployed U_t is in turn given by

$$U_t = \max_{s_t \geq 0} \{b - \mathcal{G}(s_t) + E_t\{\beta_{t,t+1}[p_t(1-\rho)W_{t+1} + [1 - p_t(1-\rho)]U_{t+1}]\}\}.$$

An unemployed worker receives an income of b units of consumption goods in period t and spends resources on searching for a job, which are given by $\mathcal{G}(s_t)$. In the next period, he faces a probability p_t of finding a new job, which turns active with probability $(1-\rho)$ and has an expected value of $E_t\{\beta_{t,t+1}W_{t+1}\}$, and consequently a probability $[1 - p_t(1-\rho)]$ of remaining unemployed, which has an expected value of $E_t\{\beta_{t,t+1}U_{t+1}\}$. Using again the definition of the surplus from working, the unemployment value can be written as follows,

$$U_t = \max_{s_t \geq 0} \{b - \mathcal{G}(s_t) + E_t\{\beta_{t,t+1}[p_t(1-\rho)\mathcal{W}_{t+1} + U_{t+1}]\}\}. \tag{81}$$

We can subtract (81) from (80) to write the match surplus going to the worker as

$$\mathcal{W}_t = \max_{s_t \geq 0} \{w_t - (b - \mathcal{G}(s_t)) + (1-\rho)(1-p_t)E_t\{\beta_{t,t+1}\mathcal{W}_{t+1}\}\}. \tag{82}$$

The derivative of the worker's surplus with respect to the wage is $\partial\mathcal{W}_t/\partial w_t = 1$.

The optimal search intensity of worker i satisfies the following first order condition:

$$\mathcal{G}'(s_{it}) - \frac{\partial p_{it}}{\partial s_{it}}(1-\rho)E_t\{\beta_{t,t+1}\mathcal{W}_{t+1}\} = 0.$$

As explained in chapter 5 of Pissarides (2001), worker i chooses s_i, taking the aggregate job finding rate p_t and labor market tightness θ_t as given. His personal job finding rate does, however, depend upon his search intensity as follows,

$$p_{it} = p_t(s_{it}; s_t, \theta_t).$$

For each efficiency unit supplied in the search process, workers transition from unemployment to employment at rate $\frac{m_t}{s_t u_t}$. Therefore, the transition probability of worker i per period is given by

$$p_{it} = \frac{m_t}{s_t u_t} \cdot s_{it},$$

and the derivative is

$$\frac{\partial p_{it}}{\partial s_{it}} = \frac{p_t}{s_t}.$$

At the optimum, the marginal cost of searching equals the expected future value of searching. We can write the (symmetric) first order condition for search intensity as

$$\frac{s_t\mathcal{G}'(s_t)}{p_t} = (1-\rho)E_t\{\beta_{t,t+1}\mathcal{W}_{t+1}\}. \tag{83}$$

Combining the worker's asset value (82) and the optimal search condition (83), we get the following expression for the worker's surplus

$$\mathcal{W}_t = w_t - (b - \mathcal{G}(s_t)) + (1-p_t)\frac{s_t \mathcal{G}'(s_t)}{p_t}. \tag{84}$$

Finally, using this new expression for the worker's match surplus (84) to substitute out $\mathcal{W}_{t+1}$ in the equilibrium search condition (83), we obtain the optimal search intensity

$$\frac{s_t \mathcal{G}'(s_t)}{p_t} = (1-\rho)E_t\left\{\beta_{t,t+1}\left[w_{t+1} - (b - \mathcal{G}(s_{t+1})) + (1-p_{t+1})\frac{s_{t+1}\mathcal{G}'(s_{t+1})}{p_{t+1}}\right]\right\}. \tag{85}$$

3.4 Wage bargaining

Under Nash bargaining, the equilibrium wage satisfies

$$\max_{w_t} \mathcal{W}_t^{\eta} \mathcal{J}_t^{1-\eta}.$$

The first order condition to this problem is

$$\eta \mathcal{W}_t^{\eta-1}\frac{\partial \mathcal{W}_t}{\partial w_t}\mathcal{J}_t^{1-\eta} + (1-\eta)\mathcal{J}_t^{-\eta}\frac{\partial \mathcal{J}_t}{\partial w_t}\mathcal{W}_t^{\eta} = 0,$$

which can be simplified to

$$\eta \frac{\mathcal{J}_t}{\mathcal{W}_t}\frac{\partial \mathcal{W}_t}{\partial w_t} + (1-\eta)\frac{\partial \mathcal{J}_t}{\partial w_t} = 0.$$

Put differently, the surplus sharing rule is

$$\mathcal{W}_t = \Upsilon_t \mathcal{J}_t, \tag{86}$$

where Υ_t denotes the effective bargaining power,

$$\Upsilon_t = \frac{\eta}{1-\eta}\frac{\frac{\partial \mathcal{W}_t}{\partial w_t}}{-\frac{\partial \mathcal{J}_t}{\partial w_t}}. \tag{87}$$

Plugging the derivatives of the worker's and the firm's surplus into (9), we find that $\Upsilon = \frac{\eta}{1-\eta}$ and so the sharing rule boils down to

$$\mathcal{W}_t = \frac{\eta}{1-\eta}\mathcal{J}_t. \tag{88}$$

Using the worker's and the firm's surplus, (82) and (73), to replace $\mathcal{W}_t$ and $\mathcal{J}_t$ in (88), we obtain

$$w_t - (b - \mathcal{G}(s_t)) + (1-\rho)E_t\{\beta_{t,t+1}(1-p_t)\mathcal{W}_{t+1}\} = \frac{\eta}{1-\eta}\left[y_t - w_t + (1-\rho)E_t\{\beta_{t,t+1}\mathcal{J}_{t+1}\}\right].$$

Then, using the surplus sharing rule (86) to replace $\mathcal{W}_{t+1}$ with $\frac{\eta}{1-\eta}\mathcal{J}_{t+1}$ yields

$$w_t - (b - \mathcal{G}(s_t)) + (1-\rho)E_t\left\{\beta_{t,t+1}(1-p_t)\frac{\eta}{1-\eta}\mathcal{J}_{t+1}\right\} = \frac{\eta}{1-\eta}\left[y_t - w_t + (1-\rho)E_t\{\beta_{t,t+1}\mathcal{J}_{t+1}\}\right].$$

Collecting terms in $\frac{\eta}{1-\eta}(1-\rho)E_t\{\beta_{t,t+1}\mathcal{J}_{t+1}\}$ yields

$$w_t - (b - \mathcal{G}(s_t)) = \frac{\eta}{1-\eta}\left[y_t - w_t + p_t(1-\rho)E_t\{\beta_{t,t+1}\mathcal{J}_{t+1}\}\right]$$

Using the free entry condition (75) to replace $(1-\rho)E_t\{\beta_{t,t+1}\mathcal{J}_{t+1}\}$ with $\frac{c}{q_t}$ yields

$$\left(1+\frac{\eta}{1-\eta}\right)w_t-(b-\mathcal{G}(s_t))=\frac{\eta}{1-\eta}\left(y_t+p_t\frac{c}{q_t}\right).$$

Using $p_t=\theta_t q_t$, we can write

$$\frac{1}{1-\eta}w_t-(b-\mathcal{G}(s_t))=\frac{\eta}{1-\eta}\left(y_t+c\theta_t\right).$$

Rearrange and multiply by $(1-\eta)$ to solve for the wage w_t as follows

$$w_t=\eta\left(y_t+c\theta_t\right)+(1-\eta)(b-\mathcal{G}(s_t)). \tag{89}$$

In comparison with the standard model featuring constant search effort, the wage equation now has an additional term that reflects search costs which reduce the value of being unemployed.

Bargaining wage and vacancy posting. Using the bargaining wage (89) in the job creation condition (77) yields

$$\frac{c}{q_t}=(1-\rho)E_t\left\{\beta_{t,t+1}\left[y_{t+1}-\left[\underbrace{\eta\left(y_{t+1}+c\theta_{t+1}\right)+(1-\eta)(b-\mathcal{G}(s_{t+1}))}\right]+\frac{c}{q_{t+1}}\right]\right\}.$$

We use $\beta_{t,t+1}=\beta\frac{\lambda_{t+1}}{\lambda_t}$, $\lambda_t=C_t^{-\sigma}$ and $C_t=Y_t$ to replace $\beta_{t,t+1}$ with $\beta Y_t^{\sigma}/Y_{t+1}^{\sigma}$, and rearrange to obtain

$$\frac{c}{q_t}=\beta(1-\rho)E_t\left\{\frac{Y_t^{\sigma}}{Y_{t+1}^{\sigma}}\left[(1-\eta)\left[y_{t+1}-(b-\mathcal{G}(s_{t+1}))\right]-\eta c\theta_{t+1}+\frac{c}{q_{t+1}}\right]\right\}. \tag{90}$$

In steady state, the JCC is

$$\frac{c}{q}=\beta(1-\rho)\left[(1-\eta)\left(y-b+\mathcal{G}(s)\right)-\eta c\theta+\frac{c}{q}\right].$$

Collecting terms in $\frac{c}{q}$ yields

$$\frac{1-\beta(1-\rho)}{\beta(1-\rho)}\frac{c}{q}=(1-\eta)\left(y-b+\mathcal{G}(s)\right)-\eta c\theta.$$

We can introduce $\tilde{b}=b-\mathcal{G}(s)$ and obtain

$$\frac{1-\beta(1-\rho)}{\beta(1-\rho)}\frac{c}{q}=(1-\eta)(y-\tilde{b})-\eta c\theta.$$

Solve for $\tilde{b}$ as follows

$$\tilde{b}=y-\frac{1}{1-\eta}\left(\frac{1-\beta(1-\rho)}{\beta(1-\rho)}\frac{c}{q}+\eta c\theta\right). \tag{91}$$

Bargaining wage and search intensity. Using the bargaining wage (13) in the search intensity condition (85) yields

$$\frac{s_t \mathcal{G}'(s_t)}{p_t} = (1-\rho) E_t \left\{ \beta_{t,t+1} \left(\left[\underbrace{\eta\left(y_{t+1} + c\theta_{t+1}\right) + (1-\eta)(b - \mathcal{G}(s_{t+1}))} \right] - (b - \mathcal{G}(s_{t+1})) + (1 - p_{t+1}) \frac{s_{t+1} \mathcal{G}'(s_{t+1})}{p_{t+1}} \right) \right\}.$$

We use $\beta_{t,t+1} = \beta \frac{\lambda_{t+1}}{\lambda_t}$, $\lambda_t = C_t^{-\sigma}$ and $C_t = Y_t$ to replace $\beta_{t,t+1}$ with $\beta Y_t^{\sigma} / Y_{t+1}^{\sigma}$, and rearrange to obtain the search condition

$$\frac{s_t \mathcal{G}'(s_t)}{p_t} = \beta(1-\rho) E_t \left\{ \frac{Y_t^{\sigma}}{Y_{t+1}^{\sigma}} \left(\eta\left[y_{t+1} - (b - \mathcal{G}(s_{t+1})) + c\theta_{t+1}\right] + (1 - p_{t+1}) \frac{s_{t+1} \mathcal{G}'(s_{t+1})}{p_{t+1}} \right) \right\}.$$

We can express the optimal search condition in a simpler way. First, iterate the sharing rule (88) by one period,

$$\mathcal{W}_{t+1} = \frac{\eta}{1-\eta} \mathcal{J}_{t+1},$$

then plug this into the optimal search condition (83) to obtain:

$$\frac{s_t \mathcal{G}'(s_t)}{p_t} = (1-\rho) E_t \left\{ \beta_{t,t+1} \frac{\eta}{1-\eta} \mathcal{J}_{t+1} \right\}.$$

Finally, we replace $(1-\rho) E_t\{\beta_{t,t+1} \mathcal{J}_{t+1}\}$ with $\frac{c}{q_t}$ using the free entry condition (75),

$$\frac{s_t \mathcal{G}'(s_t)}{p_t} = \frac{\eta}{1-\eta} \frac{c}{q_t},$$

and rearrange to obtain search intensity as a (positive) function of labor market tightness,

$$s_t \mathcal{G}'(s_t) = \frac{\eta}{1-\eta} c\theta_t. \tag{92}$$

3.5 Functional form for search costs

Setting $\mathcal{G}(s_t) = \frac{\kappa}{1+\zeta} s_t^{1+\zeta}$, the first derivative is $\mathcal{G}'(s_t) = \kappa s_t^{\zeta}$ and we can rewrite the optimal search condition as follows,

$$\kappa s_t^{1+\zeta} = \frac{\eta}{1-\eta} c\theta_t. \tag{93}$$

We can rewrite the matching function by substituting search intensity s_t, i.e.

$$s_t = \left(\frac{\eta}{1-\eta} \frac{c\theta_t}{\kappa} \right)^{\frac{1}{1+\zeta}}.$$

to obtain

$$m_t = \chi \left(\frac{\eta}{1-\eta} \frac{c\theta_t}{\kappa} \right)^{\frac{\xi}{1+\zeta}} u_t^{\xi} v_t^{1-\xi}.$$

Replacing θ_t, we can rearrange this as

$$m_t = \chi \left(\frac{\eta}{1-\eta} \frac{c}{\kappa} \right)^{\frac{\xi}{1+\zeta}} u_t^{\xi - \frac{\xi}{1+\zeta}} v_t^{1-\left(\xi - \frac{\xi}{1+\zeta}\right)}.$$

Under variable search intensity, the matching function is constant returns to scale with respect to unemployment and vacancies, but only if the search cost function is convex, i.e. if $\zeta > 0$. If the search cost function is linear ($\zeta = 0$), matches are not affected by unemployment but only by vacancies,

$$m_t = \chi \left(\frac{\eta}{1-\eta} \frac{c}{\kappa} \right)^{\xi} v_t.$$

Notice that in steady state, the search cost function and its first derivative are:

$$\mathcal{G}(s) = \frac{\kappa}{1+\zeta} s^{1+\zeta}$$

$$\mathcal{G}'(s) = \kappa s^{\zeta}$$

3.6 Equilibrium conditions

Endogenous variables u_t, θ_t, p_t, q_t, n_t, Y_t, v_t, s_t, $\mathcal{G}(s_t)$, $\mathcal{G}'(s_t)$.

$$u_t = 1 - n_t$$

$$\theta_t = \frac{v_t}{u_t}$$

$$p_t = \theta_t q_t$$

$$q_t = \chi \left(\frac{s_t}{\theta_t} \right)^{\xi}$$

$$n_{t+1} = (1-\rho)[n_t + \chi (s_t u_t)^{\xi} v_t^{1-\xi}]$$

$$\frac{c}{q_t} = \beta(1-\rho) E_t \left\{ \frac{Y_t^{\sigma}}{Y_{t+1}^{\sigma}} \left[(1-\eta) \left[y_{t+1} - (b - \mathcal{G}(s_{t+1})) \right] - \eta c \theta_{t+1} + \frac{c}{q_{t+1}} \right] \right\}$$

$$s_t \mathcal{G}'(s_t) = \frac{\eta}{1-\eta} c \theta_t$$

$$Y_t = y_t n_t - c v_t - \mathcal{G}(s_t) u_t$$

$$\mathcal{G}(s_t) = \frac{\kappa}{1+\zeta} s_t^{1+\zeta}$$

$$\mathcal{G}'(s_t) = \kappa s_t^{\zeta}$$

Exogenous variable: y_t.

3.7 Recursive steady state

Normalize $y = s = 1$. Calibrate β, σ, ρ, ξ, ρ, q, c, u, ζ. Implied steady state variables or parameters: n, x, m, v, θ, p, $\tilde{b}$, $\mathcal{G}'(s)$, κ, ι, $\mathcal{G}(s)$, b, Y, χ.

$$n = 1 - u$$

$$x = \frac{\rho}{1-\rho}$$

$$m = xn$$

$$v = \frac{m}{q}$$

$$\theta = \frac{v}{u}$$

$$p = \frac{m}{u}$$

$$\tilde{b} = y - \frac{1}{1-\eta}\left(\frac{1-\beta(1-\rho)}{\beta(1-\rho)}\frac{c}{q} + \eta c\theta\right)$$

$$\mathcal{G}'(s) = \frac{\eta}{1-\eta}\frac{c\theta}{s}$$

$$\kappa = \frac{\mathcal{G}'(s)}{s^{\zeta}}$$

$$\iota = 1 + \zeta$$

$$\mathcal{G}(s) = \frac{\kappa}{1+\zeta}s^{1+\zeta}$$

$$b = \tilde{b} + \mathcal{G}(s)$$

$$Y = yn - cv - \mathcal{G}(s)u$$

$$\chi = q\left(\frac{s}{\theta}\right)^{-\xi}$$

3.8 Linearization of equilibrium conditions

Unemployment

$$\hat{u}_t = -\frac{n}{u}\hat{n}_t = \frac{u-1}{u}\hat{n}_t. \tag{94}$$

Vacancies

$$\hat{v}_t = \hat{\theta}_t + \hat{u}_t = \hat{\theta}_t + \frac{u-1}{u}\hat{n}_t. \tag{95}$$

Search cost function, first derivative

$$\mathcal{G}'(s)\hat{\mathcal{G}}'(s_t) = \zeta\kappa s^{\zeta}\hat{s}_t.$$

Write this as

$$\hat{\mathcal{G}}'(s_t) = \zeta\hat{s}_t. \tag{96}$$

Search intensity

$$\hat{s}_t + \hat{\mathcal{G}}'(s_t) = \hat{\theta}_t.$$

Combine with the derivative of the search cost function (96) to get

$$(1+\zeta)\,\hat{s}_t = \hat{\theta}_t,$$

or, more simply,

$$\hat{s}_t = \frac{1}{\iota}\hat{\theta}_t, \tag{97}$$

where ι is the (inverse) elasticity of search intensity to labor market tightness, $1+\zeta$.

Search cost function

$$\mathcal{G}(s)\hat{\mathcal{G}}(s_t) = \kappa s^{1+\zeta}\hat{s}_t.$$

Write this as

$$\hat{\mathcal{G}}(s_t) = (1+\zeta)\hat{s}_t.$$

Finally, using the linearized optimal search intensity equation (97), we can write search costs in terms of labor market tightness,

$$\hat{\mathcal{G}}(s_t) = \hat{\theta}_t. \tag{98}$$

Matching function

$$\hat{m}_{t+1} = \xi(\hat{u}_t + \hat{s}_t) + (1-\xi)\hat{v}_t. \tag{99}$$

Employment dynamics

$$\hat{n}_{t+1} = (1-\rho)\hat{n}_t + \rho\hat{m}_t$$

Plug in $\hat{m}_t$ from (99)

$$\hat{n}_{t+1} = (1-\rho)\hat{n}_t + \rho[\xi(\hat{u}_t + \hat{s}_t) + (1-\xi)\hat{v}_t]$$

Substitute $\hat{u}_t$ and $\hat{v}_t$ using (94) and (95), to obtain

$$\hat{n}_{t+1} = (1-\rho)\hat{n}_t + \rho\xi\frac{u-1}{u}\hat{n}_t + \rho\xi\hat{s}_t + \rho(1-\xi)\left[\hat{\theta}_t + \frac{u-1}{u}\hat{n}_t\right].$$

Collect terms in $\hat{n}_t$ and rearrange,

$$\hat{n}_{t+1} = \rho(1-\xi)\hat{\theta}_t + \left[(1-\rho) + \rho\xi\frac{u-1}{u} + \rho(1-\xi)\frac{u-1}{u}\right]\hat{n}_t + \rho\xi\hat{s}_t.$$

Simplify and replace search intensity using (97),

$$\hat{n}_{t+1} = \rho(1-\xi)\hat{\theta}_t + \left[(1-\rho) + \rho\frac{u-1}{u}\right]\hat{n}_t + \frac{\rho\xi}{\iota}\hat{\theta}_t.$$

Finally, we have the linearized employment dynamics equation in terms of tightness $\hat{\theta}_t$,

$$\hat{n}_{t+1} = \left(1 - \xi + \frac{\xi}{\iota}\right)\rho\hat{\theta}_t + \frac{u-\rho}{u}\hat{n}_t. \tag{100}$$

Aggregate resource constraint

$$Y_t = y_t n_t - cv_t - \mathcal{G}(s_t)u_t$$

Linearizing this equation yields

$$\hat{Y}_t = \frac{yn}{Y}(\hat{n}_t + \hat{y}_t) - \frac{cv}{Y}\hat{v}_t - \frac{\mathcal{G}(s)u}{Y}(\hat{\mathcal{G}}(s_t) + \hat{u}_t).$$

Use the linearized definition of labor market tightness (95) to replace vacancies $\hat{v}_t$; replace unemployment $\hat{u}_t$ using (94),

$$\hat{Y}_t = \frac{yn}{Y}\hat{n}_t + \frac{yn}{Y}\hat{y}_t - \frac{cv}{Y}\left(\hat{\theta}_t - \frac{n}{u}\hat{n}_t\right) - \frac{\mathcal{G}(s)u}{Y}\left(\hat{\mathcal{G}}(s_t) + \frac{u-1}{u}\hat{n}_t\right).$$

Collect terms in $\hat{n}_t$,

$$\hat{Y}_t = \left[\frac{n}{Y}\left(y + \frac{cv}{u}\right) - \frac{\mathcal{G}(s)u}{Y}\frac{u-1}{u}\right]\hat{n}_t - \frac{cv}{Y}\hat{\theta}_t + \frac{yn}{Y}\hat{y}_t - \frac{\mathcal{G}(s)u}{Y}\hat{\mathcal{G}}(s_t). \tag{101}$$

Notice that we can rewrite the term in square brackets,

$$\begin{aligned}
\frac{n}{Y}\left(y + \frac{cv}{u}\right) - \frac{\mathcal{G}(s)u}{Y}\frac{u-1}{u} &= \frac{ny}{Y} + \frac{cv}{Y}\frac{1-u}{u} - \frac{\mathcal{G}(s)u}{Y} + \frac{\mathcal{G}(s)u}{Y}\frac{1}{u} \\
&= \left(\underbrace{\frac{ny}{Y} - \frac{cv}{Y} - \frac{\mathcal{G}(s)u}{Y}}\right) + \frac{\mathcal{G}(s)u}{Y}\frac{1}{u} + \frac{cv}{Y}\frac{1}{u} \\
&= 1 + \frac{\mathcal{G}(s)u}{Y}\frac{1}{u} + \frac{cv}{Y}\frac{1}{u} \\
&= 1 + \frac{\mathcal{G}(s)}{Y} + \frac{c\theta}{Y}.
\end{aligned}$$

Therefore, we can write the resource constraint as

$$\hat{Y}_t = \Omega\hat{n}_t - \frac{cv}{Y}\hat{\theta}_t + \frac{yn}{Y}\hat{y}_t - \frac{\mathcal{G}(s)u}{Y}\hat{\mathcal{G}}(s_t),$$

where $\Omega = 1 + \frac{\mathcal{G}(s)}{Y} + \frac{c\theta}{Y}$. Replace the linearized search cost $\hat{\mathcal{G}}(s_t)$ using (98),

$$\hat{Y}_t = \Omega\hat{n}_t - \frac{cv}{Y}\hat{\theta}_t + \frac{yn}{Y}\hat{y}_t - \frac{\mathcal{G}(s)u}{Y}\hat{\theta}_t,$$

and collect terms in $\hat{\theta}_t$,

$$\hat{Y}_t = \Omega\hat{n}_t - \left(\frac{cv}{Y} + \frac{\mathcal{G}(s)u}{Y}\right)\hat{\theta}_t + \frac{yn}{Y}\hat{y}_t. \tag{102}$$

We can iterate the aggregate resource constraint (102) to get aggregate output in $t+1$,

$$\hat{Y}_{t+1} = \Omega\hat{n}_{t+1} - \left(\frac{cv}{Y} + \frac{\mathcal{G}(s)u}{Y}\right)\hat{\theta}_{t+1} + \frac{yn}{Y}\hat{y}_{t+1},$$

and then replace $\hat{n}_{t+1}$ using the linearized employment dynamics equation (100),

$$\hat{Y}_{t+1} = \Omega\left[\left(1 - \xi + \frac{\xi}{\iota}\right)\rho\hat{\theta}_t + \frac{u-\rho}{u}\hat{n}_t\right] - \left(\frac{cv}{Y} + \frac{\mathcal{G}(s)u}{Y}\right)\hat{\theta}_{t+1} + \frac{yn}{Y}\hat{y}_{t+1}. \tag{103}$$

Job creation condition. Using $q_t = \chi(\frac{s_t}{\theta_t})^{\xi}$, we can rewrite job creation condition (90) more conveniently as

$$\frac{c}{\chi}s_t^{-\xi}\theta_t^{\xi}Y_t^{-\sigma} = \beta(1-\rho)E_t\left\{Y_{t+1}^{-\sigma}\left[(1-\eta)(y_{t+1} - (b - \mathcal{G}(s_{t+1})) - \eta c\theta_{t+1} + \frac{c}{\chi}s_{t+1}^{-\xi}\theta_{t+1}^{\xi}\right]\right\}.$$

Linearizing this equation yields

$$\begin{aligned}
\frac{c}{\chi}s^{-\xi}\theta^{\xi}Y^{-\sigma}(\xi\hat{\theta}_t - \xi\hat{s}_t - \sigma\hat{Y}_t) = {}& -\beta(1-\rho)Y^{-\sigma}\left[(1-\eta)(y-\tilde{b}) - \eta c\theta + \frac{c}{\chi}s^{-\xi}\theta^{\xi}\right]\sigma E_t\{\hat{Y}_{t+1}\} \\
&+ \beta(1-\rho)Y^{-\sigma}E_t\left\{(1-\eta)(y\hat{y}_{t+1} + \mathcal{G}(s)\hat{\mathcal{G}}(s_{t+1})) - \eta c\theta\hat{\theta}_{t+1} + \frac{c}{\chi}s^{-\xi}\theta^{\xi}\xi(\hat{\theta}_{t+1} - \hat{s}_{t+1})\right\}
\end{aligned}$$

Divide by $Y^{-\sigma}$, set $\hat{s}_t = \frac{1}{\iota}\hat{\theta}_t$ and $\hat{g}_t = \frac{\varphi}{\iota}\hat{\theta}_t$ to obtain

$$\begin{aligned}\frac{c}{\chi}s^{-\xi}\theta^{\xi}\left(\left(1-\frac{1}{\iota}\right)\xi\hat{\theta}_t - \sigma\hat{Y}_t\right) &= -\beta(1-\rho)\left[(1-\eta)(y-\tilde{b}) - \eta c\theta + \frac{c}{\chi}s^{-\xi}\theta^{\xi}\right]\sigma E_t\{\hat{Y}_{t+1}\} \\ &\quad +\beta(1-\rho)\left[(1-\eta)yE_t\{\hat{y}_{t+1}\} + \left((1-\eta)\mathcal{G}(s) - \eta c\theta + \left(1-\frac{1}{\iota}\right)\frac{c}{\chi}s^{-\xi}\theta^{\xi}\xi\right)E_t\{\hat{\theta}_{t+1}\}\right]\end{aligned}$$

Using $\beta(1-\rho)[(1-\eta)(y-\tilde{b}) - \eta c\theta + \frac{c}{\chi}s^{-\xi}\theta^{\xi}] = \frac{c}{\chi}s^{-\xi}\theta^{\xi}$, we can simplify

$$\begin{aligned}\frac{c}{\chi}s^{-\xi}\theta^{\xi}\left(\frac{\iota-1}{\iota}\xi\hat{\theta}_t - \sigma\hat{Y}_t\right) &= -\frac{c}{\chi}s^{-\xi}\theta^{\xi}\sigma E_t\{\hat{Y}_{t+1}\} \\ &\quad +\beta(1-\rho)\left[(1-\eta)yE_t\{\hat{y}_{t+1}\} + \left((1-\eta)\mathcal{G}(s) - \eta c\theta + \frac{\iota-1}{\iota}\frac{c}{\chi}s^{-\xi}\theta^{\xi}\xi\right)E_t\{\hat{\theta}_{t+1}\}\right].\end{aligned}$$

Dividing by $\frac{c}{\chi}s^{-\xi}\theta^{\xi}$ and rearranging, we get

$$\begin{aligned}\frac{\iota-1}{\iota}\xi\hat{\theta}_t - \sigma\hat{Y}_t &= -\sigma E_t\{\hat{Y}_{t+1}\} + \beta(1-\rho)(1-\eta)\frac{\chi}{c}s^{\xi}\theta^{-\xi}yE_t\{\hat{y}_{t+1}\} \\ &\quad +\beta(1-\rho)\left[\left((1-\eta)\frac{\mathcal{G}(s)}{c\theta} - \eta\right)\chi s^{\xi}\theta^{1-\xi} + \frac{\iota-1}{\iota}\xi\right]E_t\{\hat{\theta}_{t+1}\}.\end{aligned}$$

Replace $\hat{Y}_t$ and $E_t\{\hat{Y}_{t+1}\}$ using (102) and (103), respectively, to obtain, on the left hand side,

$$LHS = \frac{\iota-1}{\iota}\xi\hat{\theta}_t - \sigma\left[\Omega\hat{n}_t - \left(\frac{cv}{Y} + \frac{\mathcal{G}(s)u}{Y}\right)\hat{\theta}_t + \frac{yn}{Y}\hat{y}_t\right],$$

and on the right hand side,

$$\begin{aligned}RHS &= -\sigma\left\{\Omega\left[\left(1-\xi+\frac{\xi}{\iota}\right)\rho\hat{\theta}_t + \frac{u-\rho}{u}\hat{n}_t\right] - \left(\frac{cv}{Y} + \frac{\mathcal{G}(s)u}{Y}\right)E_t\{\hat{\theta}_{t+1}\} + \frac{yn}{Y}E_t\{\hat{y}_{t+1}\}\right\} \\ &\quad +\beta(1-\rho)\left[\left((1-\eta)\frac{\mathcal{G}(s)}{c\theta} - \eta\right)\chi s^{\xi}\theta^{1-\xi} + \frac{\iota-1}{\iota}\xi\right]E_t\{\hat{\theta}_{t+1}\} \\ &\quad +\beta(1-\rho)(1-\eta)\frac{1}{c\theta}\chi s^{\xi}\theta^{1-\xi}yE_t\{\hat{y}_{t+1}\}.\end{aligned}$$

Multiply out the left hand side,

$$LHS = \frac{\iota-1}{\iota}\xi\hat{\theta}_t - \sigma\Omega\hat{n}_t + \sigma\left(\frac{cv}{Y} + \frac{\mathcal{G}(s)u}{Y}\right)\hat{\theta}_t - \sigma\frac{yn}{Y}\hat{y}_t,$$

and collect terms in $\hat{\theta}_t$,

$$LHS = \left[\frac{\iota-1}{\iota}\xi + \sigma\left(\frac{cv}{Y} + \frac{\mathcal{G}(s)u}{Y}\right)\right]\hat{\theta}_t - \sigma\Omega\hat{n}_t - \sigma\frac{yn}{Y}\hat{y}_t.$$

Then, multiply out the right hand side and use $\chi s^{\xi}\theta^{1-\xi} = p$,

$$\begin{aligned} RHS &= -\sigma\Omega\left(1-\xi+\frac{\xi}{\iota}\right)\rho\hat{\theta}_t - \sigma\Omega\frac{u-\rho}{u}\hat{n}_t \\ &+\beta(1-\rho)\left[\left((1-\eta)\frac{\mathcal{G}(s)}{c\theta}-\eta\right)\chi s^{\xi}\theta^{1-\xi}+\frac{\iota-1}{\iota}\xi\right]E_t\{\hat{\theta}_{t+1}\}+\sigma\left(\frac{cv}{Y}+\frac{\mathcal{G}(s)u}{Y}\right)E_t\{\hat{\theta}_{t+1}\} \\ &+\beta(1-\rho)(1-\eta)\frac{yp}{c}E_t\{\hat{y}_{t+1}\}-\sigma\Omega\frac{yn}{Y}E_t\{\hat{y}_{t+1}\}. \end{aligned}$$

and collect terms in $E_t\{\hat{\theta}_{t+1}\}$ and $E_t\{\hat{y}_{t+1}\}$ to obtain:

$$\begin{aligned} RHS &= -\sigma\Omega\left(1-\xi+\frac{\xi}{\iota}\right)\rho\hat{\theta}_t - \sigma\Omega\frac{u-\rho}{u}\hat{n}_t \\ &+\left\{\beta(1-\rho)\left[\left((1-\eta)\frac{\mathcal{G}(s)}{c\theta}-\eta\right)\chi s^{\xi}\theta^{1-\xi}+\frac{\iota-1}{\iota}\xi\right]+\sigma\left(\frac{cv}{Y}+\frac{\mathcal{G}(s)u}{Y}\right)\right\}E_t\{\hat{\theta}_{t+1}\} \\ &+\left[\beta(1-\rho)(1-\eta)\frac{yp}{c\theta}-\sigma\Omega\frac{yn}{Y}\right]E_t\{\hat{y}_{t+1}\}. \end{aligned}$$

Move terms in $\hat{\theta}_t$ and $\hat{n}_t$ to the left hand side,

$$LHS = \left[\frac{\iota-1}{\iota}\xi+\sigma\left(\frac{cv}{Y}+\frac{\mathcal{G}(s)u}{Y}\right)+\sigma\Omega\left(1-\xi+\frac{\xi}{\iota}\right)\rho\right]\hat{\theta}_t - \sigma\Omega\left(1-\frac{u-\rho}{u}\right)\hat{n}_t - \sigma\frac{yn}{Y}\hat{y}_t.$$

Simplify,

$$LHS = \left[\frac{\iota-1}{\iota}\xi+\sigma\left(\frac{cv}{Y}+\frac{\mathcal{G}(s)u}{Y}\right)+\sigma\Omega\left(1-\xi+\frac{\xi}{\iota}\right)\rho\right]\hat{\theta}_t - \sigma\Omega\frac{\rho}{u}\hat{n}_t - \sigma\frac{yn}{Y}\hat{y}_t.$$

Defining

$$\alpha_1 = \beta(1-\rho)\left[\frac{\iota-1}{\iota}\xi+\left(\frac{1-\eta}{\eta}\frac{\mathcal{G}(s)}{c\theta}-1\right)\eta\chi s^{\xi}\theta^{1-\xi}\right]+\sigma\left(\frac{cv}{Y}+\frac{\mathcal{G}(s)u}{Y}\frac{\varphi}{\iota}\right),$$

$$\alpha_2 = \Omega = 1+\frac{c\theta}{Y}+\frac{\mathcal{G}(s)}{Y}, \tag{104}$$

$$\alpha_3 = \beta(1-\rho)(1-\eta)\frac{yp}{c\theta}-\sigma\Omega\frac{yn}{Y}, \tag{105}$$

we can write the linearized job creation condition as

$$\begin{aligned} \alpha_1 E_t\{\hat{\theta}_{t+1}\}+\alpha_3 E_t\{\hat{y}_{t+1}\} &= \left[\frac{\iota-1}{\iota}\xi+\sigma\left(\frac{cv}{Y}+\frac{\mathcal{G}(s)u}{Y}\right)+\sigma\left(1-\xi+\frac{\xi}{\iota}\right)\alpha_2\rho\right]\hat{\theta}_t & (106)\\ &\quad -\sigma\alpha_2\frac{\rho}{u}\hat{n}_t-\sigma\frac{yn}{Y}\hat{y}_t. & (107) \end{aligned}$$

Notice that, setting $\chi s^{\xi}\theta^{1-\xi} = p$ and using $\frac{1-\eta}{\eta}\frac{1}{c\theta} = \frac{1}{s\mathcal{G}'(s)}$, we can write α_1 as follows

$$\alpha_1 = \beta(1-\rho)\left[\frac{\iota-1}{\iota}\xi+\left(\frac{1}{\iota}-1\right)\eta p\right]+\sigma\left(\frac{cv}{Y}+\frac{\mathcal{G}(s)u}{Y}\right).$$

or alternatively,

$$\alpha_1 = \beta(1-\rho)\frac{\iota-1}{\iota}\xi\left(1-\frac{\eta}{\xi}p\right)+\sigma\left(\frac{cv}{Y}+\frac{\mathcal{G}(s)u}{Y}\right). \tag{108}$$

3.9 Model solution

Assuming constant productivity ($\hat{y}_t = 0$ for all t), we can write the system of two equations in matrix form:

$$\begin{bmatrix} E_t\{\hat{\theta}_{t+1}\} \\ \hat{n}_{t+1} \end{bmatrix} = \begin{bmatrix} \left\{\frac{\iota-1}{\iota}\frac{\xi}{\alpha_1} + \frac{\sigma}{\alpha_1}\left(\frac{cv}{Y} + \frac{\mathcal{G}(s)u}{Y}\right) + \sigma\left(1-\xi+\frac{\xi}{\iota}\right)\frac{\alpha_2}{\alpha_1}\rho\right\} & -\frac{\alpha_2}{\alpha_1}\frac{\rho}{u}\sigma \\ \left(1-\xi+\frac{\xi}{\iota}\right)\rho & 1-\frac{\rho}{u} \end{bmatrix} \begin{bmatrix} \hat{\theta}_t \\ \hat{n}_t \end{bmatrix}. \tag{109}$$

There exists a unique model solution if and only if the number of unstable eigenvalues ($\lambda_{1/2} < 1$) is equal to the number of controls and if the number of stable eigenvalues ($\lambda_{1/2} > 1$) is equal to the number of states. If there are too many unstable eigenvalues, the model solution is explosive, while if there are too few unstable eigenvalues, there are multiple equilibria (indeterminacy). In this model where there is one control variable (θ_t) and one state variable (n_t),

1. the model solution is **unique** if either $\mid \lambda_1 \mid < 1$ and $\mid \lambda_2 \mid > 1$, or $\mid \lambda_1 \mid > 1$ and $\mid \lambda_2 \mid < 1$.
2. the model solution is **indeterminate** if both roots lie inside the unit circle, $\mid \lambda_1 \mid < 1$ and $\mid \lambda_2 \mid < 1$.
3. the model solution is **non-existent** if both roots lie outside the unit circle, $\mid \lambda_1 \mid > 1$ and $\mid \lambda_2 \mid > 1$.

3.10 Special case: risk neutrality

Under risk neutrality ($\sigma = 0$), we have that

$$\alpha_1 = \beta(1-\rho)\frac{\iota-1}{\iota}\xi\left(1-\frac{\eta}{\xi}p\right).$$

The equation system reduces to

$$\begin{bmatrix} E_t\{\hat{\theta}_{t+1}\} \\ \hat{n}_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\beta(1-\rho)(1-\frac{\eta}{\xi}p)} & 0 \\ (1-\xi+\frac{\xi}{\zeta})\rho & 1-\frac{\rho}{u} \end{bmatrix} \begin{bmatrix} \hat{\theta}_t \\ \hat{n}_t \end{bmatrix}. \tag{110}$$

Notice that in this special case, the transition matrix is lower triangular, hence the eigenvalues are its diagonal elements. Therefore, under risk neutrality,

1. the model solution is **unique** if and only
   (a) either $\left|\frac{1}{\beta(1-\rho)(1-\frac{\eta}{\xi}p)}\right| < 1$ and $\left|1-\frac{\rho}{u} > 1\right|$, that is if $\left|\beta(1-\rho)(1-\frac{\eta}{\xi}p)\right| > 1$ and $2u < \rho$.
   (b) or $\left|\frac{1}{\beta(1-\rho)(1-\frac{\eta}{\xi}p)}\right| > 1$ and $|1-\frac{\rho}{u}| < 1$, that is if $\left|\beta(1-\rho)(1-\frac{\eta}{\xi}p)\right| < 1$ and $\rho < 2u$.
2. the model solution is **indeterminate** if $\left|\beta(1-\rho)(1-\frac{\eta}{\xi}p)\right| > 1$ and $\rho < 2u$.
3. the model solution is **non-existent** if $\left|\beta(1-\rho)(1-\frac{\eta}{\xi}p)\right| < 1$ and $2u < \rho$.

The restriction that the job finding rate must be lower than 1, i.e. $p < 1$ requires that the separation rate must not exceed the steady state unemployment rate, i.e. $\rho < u$. Therefore, Cases 1(a) and 3 can be ruled out with our current calibration strategy that ensures a fixed u.

Regarding Case 2, we note that indeterminacy arises when the workers' bargaining power, which measures the share of the match surplus going to workers, exceeds by a sufficiently large amount the contribution of the workers to the match success, which is captured by the match elasticity to unemployment, i.e. when $\eta >> \xi$. More precisely, the condition for indeterminacy is given by

$$\left|1-\frac{\eta}{\xi}p\right| > \frac{1}{\beta(1-\rho)}.$$

There are two conditions under which this inequality is satisfied. The first case is:

$$1-\frac{\eta}{\xi}p > \frac{1}{\beta(1-\rho)}.$$

This inequality cannot be satisfied for positive values of η, ξ and p; the left hand side of the inequality is positive but below unity. Therefore, the left hand side cannot exceed the right hand side, which is necessarily above unity. Let us now consider the second case. Indeterminacy arises if

$$-\left(1-\frac{\eta}{\xi}p\right) > \frac{1}{\beta(1-\rho)}.$$

Adding 1 on both sides and rearranging, we can rewrite this as

$$\frac{\eta}{\xi} > \underbrace{\frac{1+\beta(1-\rho)}{\beta(1-\rho)p}}_{>1}. \tag{111}$$

Condition (111) is identical to the indeterminacy frontier in the one-sided search model.

4 Model with Hiring Costs

4.1 Preliminaries

Matching function

$$m_t = \chi u_t^{\xi} v_t^{1-\xi}$$

Unemployment rate

$$u_t = 1 - n_t$$

Hiring rate

$$x_t = \frac{m_t}{n_t}$$

Employment dynamics

$$n_t = (1-\rho)\left(n_{t-1} + m_{t-1}\right)$$

Market clearing

$$Y_t = C_t$$

Aggregate accounting

$$Y_t = y_t n_t - c x_t m_t$$

4.2 Production, firm's match surplus, hiring

Production of output y_t takes place in one-worker firms with labor only (i.e., no capital) and constant hours. Firms' wage costs are w_t, such that period-t profits are $y_t - w_t$. Let $\mathcal{J}_t$ denote the firm's match surplus, i.e. the value of hiring a worker. It is the sum of current profits and the firm's continuation value. The latter is the expected future match surplus in case the employment relationship continues, which happens with probability $(1-\rho)$. The firm's value is zero in case the worker and the firm separate, which happens with probability ρ. Thus,

$$\mathcal{J}_t = y_t - w_t + E_t\{\beta_{t,t+1}[(1-\rho)\mathcal{J}_{t+1} + \rho \cdot 0]\},$$

where $\beta_{t-1,t} = \beta\frac{\lambda_t}{\Lambda_{t-1}}$ is the household's stochastic discount factor and $\lambda_t = C_t^{-\sigma}$ is the marginal utility of consumption. The firm's match surplus can be written as

$$\mathcal{J}_t = y_t - w_t + (1-\rho)E_t\{\beta_{t,t+1}\mathcal{J}_{t+1}\}. \tag{112}$$

Meeting a worker entails a cost, c, which is proportional to the aggregate hiring rate x_t and taken as given by the firm. Free entry into the labor market ensures that the following no-arbitrage condition must hold for all t,

$$cx_t = (1-\rho)E_t\{\beta_{t,t+1}\mathcal{J}_{t+1}\}. \tag{113}$$

Combining the firm's asset value (112) and the free entry condition (113), we get the following expression for the firm's match surplus

$$\mathcal{J}_t = y_t - w_t + cx_t. \tag{114}$$

The derivative of the firm's match surplus to the wage is $\frac{\partial \mathcal{J}_t}{\partial w_t} = -1$. Finally, using the firm's match surplus (114), iterated by one period, to substitute out $\mathcal{J}_{t+1}$ in the free entry condition (113), we obtain

$$cx_t = (1-\rho)E_t\left\{\beta_{t,t+1}\left[y_{t+1} - w_{t+1} + cx_{t+1}\right]\right\}. \tag{115}$$

4.3 Utility and worker's match surplus

Utility maximization is given by

$$\max_{\{C_t\}_{t=0}^{\infty}} U = E_0 \sum_{t=0}^{\infty} \beta_{0,t}\left[\frac{C_t^{1-\sigma} - 1}{1-\sigma}\right]$$

where $\sigma \geq 0$ is the coefficient of risk aversion. Denote the value of being employed W_t and the value of being unemployed U_t. In period t, an employed worker receives the wage income w_t. In the next period, he is either still employed with probability $(1-\rho)$, in which case he has an expected value of $E_t\{\beta_{t,t+1}W_{t+1}\}$, or the employment relation is dissolved with probability ρ, then his expected value is $E_t\{\beta_{t,t+1}U_{t+1}\}$. The worker's asset value therefore is

$$W_t = w_t + E_t\{\beta_{t,t+1}[(1-\rho)W_{t+1} + \rho U_{t+1}]\}. \tag{116}$$

The value of being unemployed U_t is in turn given by

$$U_t = b + E_t\{\beta_{t,t+1}[p_t(1-\rho)W_{t+1} + (1 - p_t(1-\rho))U_{t+1}]\}. \tag{117}$$

An unemployed worker receives an income of b units of consumption goods in period t. In the next period, he faces a probability $p_t(1-\rho)$ of finding a new job, which has an expected value of $E_t\{\beta_{t,t+1}W_{t+1}\}$, and a probability $1-p_t(1-\rho)$ of remaining unemployed, which has an expected value of $E_t\{\beta_{t,t+1}U_{t+1}\}$. Defining the worker's surplus as $\mathcal{W}_t = W_t - U_t$, we can subtract (7) from (6) to write the match surplus going to the worker as

$$\mathcal{W}_t = w_t - b + (1-\rho)E_t\left\{\beta_{t,t+1}[(1-p_t)\mathcal{W}_{t+1}]\right\}. \tag{118}$$

The derivative of the worker's surplus with respect to the wage is $\partial\mathcal{W}_t/\partial w_t = 1$.

4.4 Wage bargaining

Under Nash bargaining, the equilibrium wage satisfies

$$\max_{w_t} \mathcal{W}_t^{\eta}\mathcal{J}_t^{1-\eta}.$$

The first order condition to this problem is

$$\eta\mathcal{W}_t^{\eta-1}\frac{\partial\mathcal{W}_t}{\partial w_t}\mathcal{J}_t^{1-\eta} + (1-\eta)\mathcal{J}_t^{-\eta}\frac{\partial\mathcal{J}_t}{\partial w_t}\mathcal{W}_t^{\eta} = 0,$$

which can be simplified to

$$\eta\frac{\mathcal{J}_t}{\mathcal{W}_t}\frac{\partial\mathcal{W}_t}{\partial w_t} + (1-\eta)\frac{\partial\mathcal{J}_t}{\partial w_t} = 0.$$

Put differently, the surplus sharing rule is

$$\mathcal{W}_t = \Upsilon_t\mathcal{J}_t, \tag{119}$$

where Υ_t denotes the effective bargaining power,

$$\Upsilon_t = \frac{\eta}{1-\eta}\frac{\frac{\partial\mathcal{W}_t}{\partial w_t}}{-\frac{\partial\mathcal{J}_t}{\partial w_t}}. \tag{120}$$

Plugging the derivatives of the worker's and the firm's surplus into (119), we find that $\Upsilon = \frac{\eta}{1-\eta}$ and so the sharing rule boils down to

$$\mathcal{W}_t = \frac{\eta}{1-\eta}\mathcal{J}_t. \tag{121}$$

Using the worker's and the firm's surplus, (118) and (112), to replace $\mathcal{W}_t$ and $\mathcal{J}_t$ in (121), we obtain

$$w_t - b + (1-\rho)E_t\{\beta_{t,t+1}(1-p_t)\mathcal{W}_{t+1}\} = \frac{\eta}{1-\eta}\left[y_t - w_t + (1-\rho)E_t\{\beta_{t,t+1}\mathcal{J}_{t+1}\}\right].$$

Then, using the surplus sharing rule (121) to replace $\mathcal{W}_{t+1}$ with $\frac{\eta}{1-\eta}\mathcal{J}_{t+1}$ yields

$$w_t - b + (1-\rho)E_t\left\{\beta_{t,t+1}(1-p_t)\frac{\eta}{1-\eta}\mathcal{J}_{t+1}\right\} = \frac{\eta}{1-\eta}\left[y_t - w_t + (1-\rho)E_t\{\beta_{t,t+1}\mathcal{J}_{t+1}\}\right].$$

Collecting terms in $(1-\rho)\frac{\eta}{1-\eta}E_t\{\beta_{t,t+1}\mathcal{J}_{t+1}\}$ yields

$$w_t - b = \frac{\eta}{1-\eta}\left[y_t - w_t + p_t(1-\rho)E_t\{\beta_{t,t+1}\mathcal{J}_{t+1}\}\right]$$

Using the free entry condition (113) to replace $(1-\rho)E_t\{\beta_{t,t+1}\mathcal{J}_{t+1}\}$ with cx_t yields

$$w_t - b = \frac{\eta}{1-\eta}\left(y_t - w_t + p_t c x_t\right).$$

Finally, we can solve for the wage w_t as follows:

$$\left(1+\frac{\eta}{1-\eta}\right)w_t - b = \frac{\eta}{1-\eta}(y_t + p_t c x_t),$$

$$\frac{1}{1-\eta}w_t = \frac{\eta}{1-\eta}(y_t + p_t c x_t) + b,$$

$$w_t = \eta(y_t + p_t c x_t) + (1-\eta)b. \tag{122}$$

Using the wage (122) in the job creation condition (115) yields

$$cx_t = (1-\rho)E_t\left\{\beta_{t,t+1}\left[(y_{t+1} - \eta(y_{t+1} + p_{t+1}cx_{t+1}) - (1-\eta)b) + cx_{t+1}\right]\right\}.$$

We use $\beta_{t,t+1} = \frac{\lambda_{t+1}}{\lambda_t}$, $\lambda_t = C_t^{-\sigma}$ and $C_t = Y_t$ to replace $\beta_{t,t+1}$ with $\beta Y_t^\sigma / Y_{t+1}^\sigma$, and rearrange to obtain

$$cx_t = \beta(1-\rho)E_t\left\{\frac{Y_t^\sigma}{Y_{t+1}^\sigma}\left[(1-\eta)(y_{t+1}-b) - \eta p_{t+1}cx_{t+1} + cx_{t+1}\right]\right\}. \tag{123}$$

In steady state, the JCC is

$$cx = \beta(1-\rho)\left[(1-\eta)(y-b) - \eta p c x + cx\right].$$

Rearranging, we obtain the leisure value

$$b = y - \frac{1-\beta(1-\rho)(1-\eta p)}{\beta(1-\eta)(1-\rho)}cx. \tag{124}$$

4.5 Equilibrium conditions

Endogenous variables u_t, m_t, p_t, n_t, x_t, Y_t,

$$u_t = 1 - n_t$$

$$x_t = \frac{m_t}{n_t}$$

$$p_t = \frac{m_t}{u_t}$$

$$n_t = (1-\rho)(n_{t-1} + m_{t-1})$$

$$cx_t = \beta(1-\rho)E_t\left\{\frac{Y_t^\sigma}{Y_{t+1}^\sigma}\left[(1-\eta)(y_{t+1}-b) + (1-\eta p_{t+1})cx_{t+1}\right]\right\}$$

$$Y_t = y_t n_t - cx_t m_t$$

Exogenous variable: y_t.

4.6 Recursive steady state

Normalize $y = 1$. Calibrate β, σ, ρ, ξ, q, c, u. Implied steady state variables or parameters: n, x, m, p, Y, b. Employment:

$$n = 1 - u$$

From the employment dynamics equation in steady state, we have the hiring rate is given by

$$x = \frac{\rho}{1-\rho}.$$

Then we can solve the definition of the hiring rate for the steady state number of matches,

$$m = xn.$$

The steady state number of matches determines the job finding rate as follows,

$$p = \frac{m}{u}.$$

The steady state JCC gives us the leisure value,

$$b = y - \frac{1-\beta(1-\rho)(1-\eta p)}{\beta(1-\eta)(1-\rho)}cx.$$

Finally, we can compute aggregate output as

$$Y = yn - cxm.$$

4.7 Linearization of equilibrium conditions

Unemployment

$$\hat{u}_t = -\frac{n}{u}\hat{n}_t = \frac{u-1}{u}\hat{n}_t \tag{125}$$

Hiring Rate

$$\hat{x}_t = \hat{m}_t - \hat{n}_t. \tag{126}$$

Job Finding Rate

$$\hat{p}_t = \hat{m}_t - \hat{u}_t = \hat{m}_t + \frac{1-u}{u}\hat{n}_t.$$

Replacing matches $\hat{m}_t$ using (126), we obtain

$$\hat{p}_t = \hat{x}_t + \hat{n}_t + \frac{1-u}{u}\hat{n}_t = \hat{x}_t + \frac{1}{u}\hat{n}_t. \tag{127}$$

Employment dynamics

$$\hat{n}_{t+1} = (1-\rho)\hat{n}_t + \rho\hat{m}_t = \hat{n}_t + \rho\hat{x}_t \tag{128}$$

Aggregate resource constraint

$$\hat{Y}_t = \frac{yn}{Y}(\hat{y}_t + \hat{n}_t) - \frac{cxm}{Y}(\hat{x}_t + \hat{m}_t).$$

Replace the number of matches $\hat{m}_t$ with $\hat{x}_t + \hat{n}_t$ to get

$$\hat{Y}_t = \frac{yn}{Y}(\hat{y}_t + \hat{n}_t) - \frac{cxm}{Y}(2\hat{x}_t + \hat{n}_t), \tag{129}$$

or, collecting terms in $\hat{n}_t$,

$$\hat{Y}_t = \frac{yn}{Y}\hat{y}_t + \left(\frac{yn}{Y} - \frac{cxm}{Y}\right)\hat{n}_t - \frac{cxm}{Y}2\hat{x}_t. \tag{130}$$

Using the steady state aggregate accounting equation $\frac{yn}{Y} - \frac{cxm}{Y} = 1$, this simplifies to

$$\hat{Y}_t = \frac{yn}{Y}\hat{y}_t + \hat{n}_t - 2\frac{cxm}{Y}\hat{x}_t. \tag{131}$$

We can iterate the aggregate resource constraint (131) to get output in $t+1$,

$$\hat{Y}_{t+1} = \frac{yn}{Y}\hat{y}_{t+1} + \hat{n}_{t+1} - 2\frac{cxm}{Y}\hat{x}_{t+1},$$

and then replace $\hat{n}_{t+1}$ using the linearized employment dynamics equation (128)

$$\hat{Y}_{t+1} = \frac{yn}{Y}\hat{y}_{t+1} + (\hat{n}_t + \rho\hat{x}_t) - 2\frac{cxm}{Y}\hat{x}_{t+1}. \tag{132}$$

Job creation condition. Rewrite the job creation condition (123) more conveniently as

$$cx_t Y_t^{-\sigma} = \beta(1-\rho)E_t\left\{Y_{t+1}^{-\sigma}\left[(1-\eta)(y_{t+1} - b) + (1-\eta p_{t+1})cx_{t+1}\right]\right\}.$$

Linearizing this equation yields

$$\begin{aligned} cY^{-\sigma}x\hat{x}_t - cxY^{-\sigma}\sigma\hat{Y}_t = &-\beta(1-\rho)Y^{-\sigma}\left[(1-\eta)(y-b) + (1-\eta p)cx\right]\sigma E_t\{\hat{Y}_{t+1}\} \\ &+ \beta(1-\rho)Y^{-\sigma}E_t\left\{(1-\eta)y\hat{y}_{t+1} + (1-\eta p)cx\hat{x}_{t+1} - \eta pcx\hat{p}_{t+1}\right\} \end{aligned}$$

Divide by $Y^{-\sigma}$ to obtain

$$\begin{aligned} cx\hat{x}_t - cx\sigma\hat{Y}_t = &-\beta(1-\rho)\left[(1-\eta)(y-b) + (1-\eta p)cx\right]\sigma E_t\{\hat{Y}_{t+1}\} \\ &+ \beta(1-\rho)E_t\left\{(1-\eta)y\hat{y}_{t+1} + (1-\eta p)cx\hat{x}_{t+1} - \eta pcx\hat{p}_{t+1}\right\} \end{aligned}$$

Using $\beta(1-\rho)[(1-\eta)(y-b) + (1-\eta p)cx] = cx$, we can simplify

$$cx\hat{x}_t - cx\sigma\hat{Y}_t = -cx\sigma E_t\{\hat{Y}_{t+1}\} + \beta(1-\rho)E_t\left\{(1-\eta)y\hat{y}_{t+1} + (1-\eta p)cx\hat{x}_{t+1} - \eta pcx\hat{p}_{t+1}\right\}.$$

Dividing by cx, we get

$$\hat{x}_t - \sigma\hat{Y}_t = -\sigma E_t\{\hat{Y}_{t+1}\} + \beta(1-\rho)E_t\left\{\frac{1-\eta}{cx}y\hat{y}_{t+1} + (1-\eta p)\hat{x}_{t+1} - \eta p\hat{p}_{t+1}\right\}.$$

Assuming constant productivity $\hat{y}_t = 0$, we get

$$\hat{x}_t - \sigma \hat{Y}_t = -\sigma E_t\{\hat{Y}_{t+1}\} + \beta(1-\rho)E_t\left\{(1-\eta p)\hat{x}_{t+1} - \eta p \hat{p}_{t+1}\right\}. \tag{133}$$

Replace $\hat{Y}_t$ and $E_t\{\hat{Y}_{t+1}\}$, using (131) and (132) respectively, and impose again constant productivity to obtain,

$$\hat{x}_t - \sigma\left(\hat{n}_t - 2\frac{cxm}{Y}\hat{x}_t\right) = -\sigma E_t\left\{(\hat{n}_t + \rho\hat{x}_t) - 2\frac{cxm}{Y}\hat{x}_{t+1}\right\} + \beta(1-\rho)E_t\left\{(1-\eta p)\hat{x}_{t+1} - \eta p\hat{p}_{t+1}\right\}.$$

Cancel $-\sigma\hat{n}_t$ from both sides, and assume constant productivity, $\hat{y}_t = 0$,

$$\left(1 + 2\sigma\frac{cxm}{Y}\right)\hat{x}_t = -\sigma\rho\hat{x}_t + 2\sigma\frac{cxm}{Y}E_t\left\{\hat{x}_{t+1}\right\} + \beta(1-\rho)E_t\left\{(1-\eta p)\hat{x}_{t+1} - \eta p\hat{p}_{t+1}\right\}.$$

Replace the job finding rate using $\hat{p}_{t+1} = \hat{x}_{t+1} + \frac{1}{u}\hat{n}_{t+1}$,

$$\left(1 + 2\sigma\frac{cxm}{Y}\right)\hat{x}_t = -\sigma\rho\hat{x}_t + 2\sigma\frac{cxm}{Y}E_t\left\{\hat{x}_{t+1}\right\} + \beta(1-\rho)E_t\left\{(1-\eta p)\hat{x}_{t+1} - \eta p\left(\hat{x}_{t+1} + \frac{1}{u}\hat{n}_{t+1}\right)\right\}.$$

Collect terms in $\hat{x}_t$ and $\hat{x}_{t+1}$,

$$\left(1 + 2\sigma\frac{cxm}{Y} + \sigma\rho\right)\hat{x}_t = \beta(1-\rho)E_t\left\{\left(1 - 2\eta p + \frac{2\sigma\frac{cxm}{Y}}{\beta(1-\rho)}\right)\hat{x}_{t+1} - \frac{\eta p}{u}\hat{n}_{t+1}\right\}.$$

Replace $\hat{n}_{t+1}$ with $\hat{n}_t + \rho\hat{x}_t$, to get

$$\left(1 + \sigma\rho + 2\sigma\frac{cxm}{Y}\right)\hat{x}_t = \beta(1-\rho)E_t\left\{\left(1 - 2\eta p + \frac{2\sigma\frac{cxm}{Y}}{\beta(1-\rho)}\right)\hat{x}_{t+1} - \frac{\eta p}{u}(\hat{n}_t + \rho\hat{x}_t)\right\}.$$

Rearrange to get

$$\beta(1-\rho)E_t\left\{\left(1 - 2\eta p + \frac{2\sigma\frac{cxm}{Y}}{\beta(1-\rho)}\right)\hat{x}_{t+1}\right\} = \left(1 + \sigma\rho + 2\sigma\frac{cxm}{Y} + \beta(1-\rho)\frac{\eta p}{u}\rho\right)\hat{x}_t + \beta(1-\rho)\frac{\eta p}{u}\hat{n}_t. \tag{134}$$

Divide by $\beta(1-\rho)$,

$$\left(1 - 2\eta p + \frac{2\sigma\frac{cxm}{Y}}{\beta(1-\rho)}\right)E_t\left\{\hat{x}_{t+1}\right\} = \left(\frac{1 + \sigma\rho + 2\sigma\frac{cxm}{Y}}{\beta(1-\rho)} + \eta p\frac{\rho}{u}\right)\hat{x}_t + \eta p\frac{1}{u}\hat{n}_t. \tag{135}$$

Defining

$$\alpha_1^{hc} = 1 - 2\eta p + \frac{2\sigma\frac{cxm}{Y}}{\beta(1-\rho)}, \tag{136}$$

we can write (135) as

$$\alpha_1^{hc}E_t\left\{\hat{x}_{t+1}\right\} = \left(\frac{1 + \sigma\rho + 2\sigma\frac{cxm}{Y}}{\beta(1-\rho)} + \eta p\frac{\rho}{u}\right)\hat{x}_t + \eta p\frac{1}{u}\hat{n}_t. \tag{137}$$

We can write the system of equations in matrix form:

$$\begin{bmatrix} E_t\left\{\hat{x}_{t+1}\right\} \\ \hat{n}_{t+1} \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{\alpha_1^{hc}}\frac{1+\sigma\rho+2\sigma\frac{cxm}{Y}}{\beta(1-\rho)} + \frac{\eta p}{\alpha_1^{hc}}\frac{\rho}{u}\right) & \frac{\eta p}{\alpha_1^{hc}}\frac{1}{u} \\ \rho & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_t \\ \hat{n}_t \end{bmatrix}$$

We can write the two-equation system in a more compact way:

$$\begin{bmatrix} E_t\{\hat{x}_{t+1}\} \\ \hat{n}_{t+1} \end{bmatrix} = \Phi^{-1} \begin{bmatrix} \hat{x}_t \\ \hat{n}_t \end{bmatrix}, \tag{138}$$

where the transition matrix is

$$\Phi^{-1} = \begin{bmatrix} \left(\frac{1}{\alpha_1^{hc}} \frac{1+\sigma\rho+2\sigma\frac{cxm}{Y}}{\beta(1-\rho)} + \frac{\eta p}{\alpha_1^{hc}} \frac{\rho}{u} \right) & \frac{\eta p}{\alpha_1^{hc}} \frac{1}{u} \\ \rho & 1 \end{bmatrix} \tag{139}$$

The trace and the determinant of Φ^{-1} are given by

$$\mathrm{tr} = \left(1 + \frac{\eta p}{\alpha_1^{hc}} \frac{\rho}{u} \right) + \frac{1}{\alpha_1^{hc}} \frac{1+\sigma\rho+2\sigma\frac{cxm}{Y}}{\beta(1-\rho)}, \tag{140}$$

$$\mathrm{D} = \frac{1}{\alpha_1^{hc}} \frac{1+\sigma\rho+2\sigma\frac{cxm}{Y}}{\beta(1-\rho)}. \tag{141}$$

The two roots of the system can be written in terms of the trace and the determinant,

$$\lambda_i = \frac{1}{2} \left(\mathrm{tr} \pm \sqrt{\mathrm{tr}^2 - 4\mathrm{D}} \right) \text{ with } i = 1, 2.$$

4.8 Special case: risk neutrality

Let us again consider risk neutrality as a special case, where $\alpha_1^{hc} = \beta(1-\rho)(1-2\eta p)$. The transition matrix reads

$$\Phi_{hc}^{-1} = \begin{bmatrix} \left(\frac{1}{\beta(1-\rho)(1-2\eta p)} + \frac{\frac{\rho}{u}\eta p}{1-2\eta p} \right) & \frac{\frac{1}{u}\eta p}{1-2\eta p} \\ \rho & 1 \end{bmatrix} \tag{142}$$

Notice that the trace and the determinant of Φ_{hc}^{-1} are given by

$$\mathrm{tr} = 1 + \frac{1}{\beta(1-\rho)(1-2\eta p)} + \frac{\frac{\rho}{u}\eta p}{1-2\eta p},$$

$$\mathrm{D} = \frac{1}{\beta(1-\rho)(1-2\eta p)}.$$

The two roots of the system can be written in terms of the trace and the determinant,

$$\lambda_i = \frac{1}{2}(\mathrm{tr} \pm \sqrt{\mathrm{tr}^2 - 4\mathrm{D}}) \text{ with } i = 1, 2.$$

We can show that the first root of matrix Φ_{hc}^{-1} is smaller than 1, which rules out non-existence. We use a proof by contradiction. Suppose that the first root is unstable, such that

$$\frac{1}{2}(\mathrm{tr} + \sqrt{\mathrm{tr}^2 - 4\mathrm{D}}) > 1.$$

Multiply both sides of this inequality by 2 and subtract the trace from both sides to get

$$\sqrt{\mathrm{tr}^2 - 4\mathrm{D}} > 2 - \mathrm{tr}.$$

Taking the square, we have the following condition,

$$\mathrm{tr}^2 - 4\mathrm{D} > 4 - 2\mathrm{tr} + \mathrm{tr}^2$$

The square of the trace cancels out and we divide by -2 to obtain

$$2\mathrm{D} < \mathrm{tr} - 2$$

Plugging in our expressions for the trace and the determinant, we obtain

$$\frac{2}{\beta(1-\rho)(1-2\eta p)} < \frac{1}{\beta(1-\rho)(1-2\eta p)} + \frac{\frac{\rho}{u}\eta p}{1-2\eta p} - 1$$

which can be written as

$$\frac{2}{\beta(1-\rho)(1-2\eta p)} < \frac{1+\beta(1-\rho)\frac{\rho}{u}\eta p - \beta(1-\rho)(1-2\eta p)}{\beta(1-\rho)(1-2\eta p)}$$

Or alternatively,

$$\frac{2}{\beta(1-\rho)(1-2\eta p)} < \frac{1-\beta(1-\rho)(1-2\eta p-\frac{\rho}{u}\eta p)}{\beta(1-\rho)(1-2\eta p)}$$

We multiply by the common denominator $\beta(1-\rho)(1-2\eta p)$, which is positive under the assumption that $\eta p < \frac{1}{2}$,

$$2 < 1 - \beta(1-\rho)\left(1 - 2\eta p - \frac{\rho}{u}\eta p\right)$$

Subtracting 1 from both sides and dividing by $\beta(1-\rho)$, this inequality simplifies to

$$\frac{1}{\beta(1-\rho)} < -\left(1 - 2\eta p - \frac{\rho}{u}\eta p\right)$$

Rearranging again, we obtain

$$1 + \frac{1}{\beta(1-\rho)} < 2\eta p + \frac{\rho}{u}\eta p$$

The left hand side of this inequality is larger than 2, but the right hand of the equation is the sum of two positive numbers that are smaller than 1. Therefore, we have a contradiction and the first root cannot be unstable.

Next, we show that the second root of matrix Φ_{hc}^{-1} is greater than 1, which rules out indeterminacy. Again the proof is by contradiction. Suppose that the second root is stable, such that

$$\frac{1}{2}(\mathrm{tr} - \sqrt{\mathrm{tr}^2 - 4\mathrm{D}}) < 1.$$

Multiply both sides of this inequality by -2 and add the trace to both sides to get

$$\sqrt{\mathrm{tr}^2 - 4\mathrm{D}} > -2 + \mathrm{tr}.$$

Squaring both sides yields

$$\mathrm{tr}^2 - 4\mathrm{D} > 4 - 2\mathrm{tr} + \mathrm{tr}^2$$

Notice that this is the same inequality as above, which we have shown to be a contradiction. Therefore, the second root must be unstable. We have one stable root and one unstable root, which implies that the system has a unique solution.

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