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TEMPERED STABLE ORNSTEIN-UHLENBECK PROCESSES: A PRACTICAL VIEW

by Michele Leonardo Bianchi,* Svetlozar T. Rachev† and Frank J. Fabozzi‡

Abstract

We study the one-dimensional Ornstein-Uhlenbeck (OU) processes with marginal law given by the tempered stable and tempered infinitely divisible distributions proposed by Rosiński (2007) and Bianchi et al. (2010b), respectively. In general, the use of non-Gaussian OU processes is impeded by difficulty in calibration and simulation. Accordingly, we investigate the law of transition between consecutive observations of OU processes and – with a view to practical applications – evaluate the characteristic function of integrated tempered OU processes in three cases: classical tempered stable, variance gamma, and rapidly decreasing tempered stable. Then we analyze how one can draw a random sample from this class of processes using both the classical inverse transform algorithm and an acceptance-rejection method based on the simulation of a stable random sample. Finally, with a maximum likelihood estimation method based on the fast Fourier transform, we assess the performance of the simulation algorithm empirically.

JEL Classification: C02, C46.

Keywords: Ornstein-Uhlenbeck processes, tempered stable distributions, tempered infinitely divisible distributions, integrated processes, acceptance-rejection sampling, maximum likelihood estimation.

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1 Introduction[1]

Over the past decade non-Gaussian Ornstein-Uhlenbeck (OU) processes introduced by Barndorff-Nielsen and Shephard (2001) have been widely studied from both empirical and theoretical points of view and applied to finance, economics, engineering and other applied sciences. This family of processes can capture important distributional properties observed in real data and offer a more flexible structure with respect to Gaussian-based models. This flexibility, the possibility to explain certain *stylized facts* of financial time series for example, and a suitable degree of computational tractability have increased the number of applications in finance. In particular, they have been applied to stochastic volatility, interest rate and credit risk models (see Nicolato and Venardos (2003), Kokholm and Nicolato (2010) and Keller-Ressel et al. (to appear)).

It is within this context that we study tempered OU processes. The formal and elegant definition of tempered stable (TS) distributions and processes has been proposed in the work of Rosiński (2007) where a completely monotone function is chosen to transform the Lévy measure of a stable distribution (various parametric classes have been discussed by Terdik and Woyczyński (2006)). Subsequently, Bianchi et al. (2010b) introduced the class of tempered infinitely divisible (TID) distributions by multiplying the Lévy measure of a stable distribution with a positive definite radial function (see Schoenberg (1938)) instead of with a completely monotone function as in Rosiński (2007). The TID class of distributions has the same desirable properties as the tempered stable class, but with the advantage that it may admit exponential moments of any order. More precisely in some cases, the characteristic function of a TID random variable is extendible to an entire function on $\mathbb{C}$, that is, it admits any exponential moment. Recently, two classes of distributions broader than the TS class were proposed: Rosiński and Sinclair (2010) introduced the generalized tempered stable class and Grabchak (2012) proposed the p-tempered stable class.

In this paper, we study OU processes with marginal law given by TS and TID distributions. In particular, we analyze three different parametric examples belonging to these classes: the *classical tempered stable* (CTS) distribution, the *variance gamma* (VG) distribution which can be seen as a limiting case of the previous distribution, and the *rapidly decreasing tempered stable* (RDTS) distribution. First, we evaluate the characteristic function of integrated tempered OU processes for these three cases with a view toward practical applications. Then, by taking into consideration the two recent works of Kawai and Masuda (2011a) and Kawai and Masuda (2012) we investigate the transition law between consecutive observations of RDTS processes. Finally, in the CTS case we apply the acceptance-rejection method based on the simulation of stable random numbers studied in Kawai and Masuda (2011b) and then compare it to the classical inverse transform algorithm (both random number generation methods will be briefly described in Section 8.1).

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In the RDTS case, only the inverse transform algorithm is analyzed. Furthermore, by considering a maximum likelihood estimation method based on the fast Fourier transform, we perform an empirical analysis to assess the algorithms performance.

The remainder of this paper is organized as follows. In Section 2, we briefly review well-known results on OU processes. In Sections 4, 5, and 6 we evaluate the characteristic function of the integrated tempered stable processes (CTS, VG, and RDTS) and find the transition law of consecutive discrete observations of OU processes with the RDTS marginal law. In Section 7, we briefly describe the maximum likelihood estimation method. We discuss two different random sample methods in Section 8. The first is based on the inversion of the cumulative distribution function; the second uses an acceptance-rejection method based on the simulation of stable random numbers. While in the CTS case we compare these two random sample methods in terms of computing time and of simulation error, in the RDTS case we analyze only the inverse transform algorithm. Then, we report and discuss the results. Section 9 concludes the paper. In Appendix A we briefly review and analyze the fast Fourier transform approach for evaluating the density function given a closed-form expression for the characteristic function.

2 Ornstein-Uhlenbeck processes

We begin this section with some useful definitions. The characteristic function of the one-dimensional infinitely divisible distribution X is given by the Lévy-Khinchine formula:

$$E[\exp(iuX)] = \exp\left(i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{\infty}(e^{iux} - 1 - iux1_{|x|\leq 1})\nu(dx)\right). \qquad (2.1)$$

In the formula, the measure ν is referred to as the *Lévy measure*. The measure is a Borel measure satisfying the conditions that $\nu(0) = 0$ and $\int_{\mathbb{R}}(1 \wedge |x^2|)\nu(dx) < \infty$. The parameters γ and σ are real numbers. The variable γ is referred to as the *center* or *drift* and determines the location. This triplet (σ^2, ν, γ), referred to as the Lévy triplet, is uniquely defined for each infinitely divisible distribution. If $\sigma = 0$, then the distribution X is referred to as a *purely non-Gaussian distribution*, and in the case when its mean is finite the characteristic function is computed by

$$E[\exp(iuX)] = \exp\left(i\gamma u + \int_{-\infty}^{\infty}(e^{iux} - 1 - iux1_{|x|\leq 1})\nu(dx)\right).$$

In the following, we will consider only purely non-Gaussian infinitely divisible distributions. Given a purely non-Gaussian distribution D one can define a pure jump process X_t with $t \geq 0$ such that the distribution of the increment $X_1 - X_0$ is D.[2] We say that a stochastic process is of finite variation if the sample paths are of finite variation with probability 1, that is if

$$\int_{-1}^{1} |x|\nu(dx) < \infty.$$

[2] Assuming that the process X_t starts from 0, X_1 represents the law of the increment with $t = 1$.

If this is not the case, we say that the process is of infinite variation.

Consider the stochastic process λ_t defined as

$$d\lambda_t = -\theta\lambda_t dt + dZ_{\theta t}, \tag{2.2}$$

where $\theta > 0$ and Z_t is a Lévy process staring from 0, or equivalently

$$\lambda_t = e^{-\theta t}\lambda_0 + e^{-\theta t}\int_0^t e^{\theta s} dZ_{\theta s}. \tag{2.3}$$

We refer to processes of this family as OU processes. We refer to the process Z_t used to drive the OU process as the *background driving Lévy process* (BDLP). This family of stochastic processes has been widely investigated in the literature (see Barndorff-Nielsen and Shephard (2001), Schoutens (2003), and Cont and Tankov (2004) for a detailed introduction on this topic).

In financial applications, one is interested in the integrated process A_t defined as

$$A_t = \int_0^t \lambda_s ds, \tag{2.4}$$

that can be rewritten as

$$A_t = \theta^{-1}(1 - e^{-\theta t})\lambda_0 + \theta^{-1}\int_0^t (1 - e^{-\theta(t-s)}) dZ_{\theta s},$$

and in defining λ_t such that the characteristic function of A_t

$$\phi_{A_t}(u) = E[\exp(iuA_t)] \tag{2.5}$$

has a closed-form solution. OU processes have been applied to the modeling of stochastic volatility to price equity options and the modeling of default intensity to price credit default swaps by Nicolato and Venardos (2003), Cariboni and Schoutens (2009), Kokholm and Nicolato (2010), Wyłomańska (2011), and Bianchi (2012).

Given a one-dimensional distribution D (not necessarily restricted to the positive half-line), there exists a (stationary) OU process whose marginal law is D (i.e. a D-OU process) if and only if D is self-decomposable (see Schoutens (2003)). The cumulant function of Z_1,[3] that is

$$k_Z(u) = \log E[\exp(-uZ_1)] \tag{2.6}$$

can be derived by the cumulant function of the law D, since the following equality holds

$$k_Z(u) = uk'_D(u). \tag{2.7}$$

Equivalently, the cumulant characteristic function of Z_1, that is

$$\psi_Z(u) = \log E[\exp(iuZ_1)], \tag{2.8}$$

[3] Assuming that the process Z_t starts from 0, Z_1 represents the law of the increment with $t = 1$.

can be derived by the cumulant characteristic function of the law D, since the following equality holds

$$\psi_Z(u) = u\psi'_D(u). \tag{2.9}$$

Let $\nu(z)$ be the Lévy density of the marginal law D, then the Lévy density $w(z)$ of Z_1 can be computed by the following equality

$$\mu(z) = -\nu(z) - z\nu'(z). \tag{2.10}$$

An important equality useful for computing the maximum likelihood estimate is given by

$$\psi_{Z^*(\Delta)}(u) = \psi_D(ue^{\theta\Delta}) - \psi_D(u), \tag{2.11}$$

where the random variable $Z^*(\Delta)$ is defined as

$$Z^*(\Delta) = \int_0^{\Delta} e^{\theta s} dZ(\theta s) = \int_0^{\theta\Delta} e^{s} dZ(s) = e^{-\theta t} \int_t^{t+\Delta} e^{\theta s} dZ(\theta s). \tag{2.12}$$

Given the logarithm of the moment-generating function of a random variable L_1

$$\vartheta(u) = \log E[\exp(uL_1)]$$

and by considering the Lemma proven in Eberlein and Raible (1999), the following equality holds

$$E\left[\exp\left(\int_0^t f(s)dL_s\right)\right] = \exp\left(\int_0^t \vartheta(f(s))ds\right), \tag{2.13}$$

where L is a Lévy process with some finite exponential moments and f is a bounded function $\mathbb{R} \to \mathbb{C}$, that is $|\Re(f)| < M$. It follows that

$$E[\exp(iuA_t)] = \exp\left(iu\lambda_0\theta^{-1}(1 - e^{-\theta t}) + \int_0^t \theta\vartheta_Z(iu\theta^{-1}(1 - e^{-\theta(t-s)}))ds\right). \tag{2.14}$$

A closed-form solution for equation (2.14) is known in the Gamma-OU and IG-OU cases. In the present paper, we want to explicitly compute (2.14) in the tempered stable (TS-OU), in the variance gamma (VG-OU), and in rapidly decreasing tempered stable (RDTS-OU) case. This family of processes was introduced by Rosiński (2007) and Terdik and Woyczyński (2006). The distributional assumptions considered here are of interest in practical applications. Moreover, there exist algorithms to draw random variates from these OU processes, as described by Zhang and Zhang (2009) and Kawai and Masuda (2011a).

3 The totally skewed stable distribution

By following the approach of Kawai and Masuda (2011b), the law X is said to have a totally positively skewed stable distribution with parameter C and α (β is fixed and equal to 1) if the characteristic function of X is given by

$$\begin{aligned} E[\exp(iuX)] &= exp\left(a\Gamma(-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)|u|^{\alpha}\left(1 - i\tan\frac{\pi\alpha}{2}sgn(u)\right)\right) \\ &= \begin{cases} \exp\left(\int_0^{\infty}(e^{iuz} - 1)\frac{C}{z^{\alpha+1}}dz\right), & \text{if } \alpha \in (0,1), \\ \exp\left(\int_0^{\infty}(e^{iuz} - 1 - iuz)\frac{C}{z^{\alpha+1}}dz\right), & \text{if } \alpha \in (1,2), \end{cases} \end{aligned}$$

which is defined in $\mathbb{R}_+$ if $\alpha \in (0,1)$ and in $\mathbb{R}$ if $\alpha \in (1,2)$, and its Lévy measure can be written as

$$\nu_{S_+}(z) = \frac{C}{z^{1+\alpha}} 1_{z>0}.$$

We refer to the totally positively skewed stable distribution as $S_+(C,\alpha)$. It can be simulated through the well-known representation proposed in Chambers et al. (1976),

$$S_+(C,\alpha) = \left(-C\Gamma(-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)\right)^{1/\alpha} \frac{\alpha U + \theta}{(\cos U \cos\theta)^{1/\alpha}} \left(\frac{\cos((1-\alpha)U - \theta)}{E}\right)^{\frac{1}{\alpha}-1} \tag{3.1}$$

where $\theta = \arctan(\tan(\pi\alpha/2))$, U is a uniform random variable on $(-\pi/2, \pi/2)$, and E is a standard exponential random variable independent of U. The totally positively skewed stable law may be used to generate random samples from CTS and RDTS laws by taking into consideration an acceptance-rejection method. In this section, we have only dealt with a unilateral setting. The more general bilateral setting can be treated simply by considering the difference between two independent processes totally skewed in opposite directions.

4 The classical tempered stable case

4.1 Basic definitions

Let α, C, λ_+, and λ_- be positive constants, $m \in \mathbb{R}$, and $\alpha \in (0,2)\backslash\{1\}$.[4] The law X is said to have a CTS distribution with parameters (α, C, λ_+, λ_-, m) if the characteristic function of X is given by

$$\begin{aligned}\phi_X(u) = E[exp(iuX)] = \exp(&iu(m - C\Gamma(1-\alpha)(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1})) \\ &+ C\Gamma(-\alpha)((\lambda_+ - iu)^\alpha - \lambda_+^\alpha + (\lambda_- + iu)^\alpha - \lambda_-^\alpha))\end{aligned} \tag{4.1}$$

and its Lévy measure can be written as

$$\begin{aligned}\nu(z) &= q(z)\nu_{S_\alpha}(z) = \\ &= \frac{Ce^{-\lambda_+ z}}{z^{1+\alpha}} 1_{z>0} + \frac{Ce^{-\lambda_- |z|}}{|z|^{1+\alpha}} 1_{z<0},\end{aligned} \tag{4.2}$$

where $q(z)$ is the tempering function of the CTS law and ν_{S_α} is the Lévy measure of the stable law, that is,

$$\nu_{S_\alpha} = \frac{C}{z^{1+\alpha}} 1_{z>0} + \frac{C}{|z|^{1+\alpha}} 1_{z<0}. \tag{4.3}$$

Given a CTS law X_1, one can define a Lévy process X_t which we refer to as a CTS process. It can be proven that if $\alpha < 1$, the process has finite variation. By simple

[4] In both CTS and RDTS cases, we do not discuss the case $\alpha = 1$.

calculations, one can write the cumulant function of Z_1 that is given by[5]

$$\begin{aligned} k_{Z_{CTS}}(u) = uk'_{CTS}(u) = &-u(m - C\Gamma(1-\alpha)(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1})) \\ &- uC\Gamma(1-\alpha)((\lambda_+ + u)^{\alpha-1} - (\lambda_- - u)^{\alpha-1}) \end{aligned} \tag{4.4}$$

and the moment-generating function is

$$\begin{aligned} \vartheta_{Z_{CTS}}(u) = &u(m - C\Gamma(1-\alpha)(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1})) \\ &+ uC\Gamma(1-\alpha)((\lambda_+ - u)^{\alpha-1} - (\lambda_- + u)^{\alpha-1}). \end{aligned} \tag{4.5}$$

By equation (2.11) it follows that the cumulant characteristic function of Z^*_{CTS} is

$$\begin{aligned} \psi_{Z^*_{CTS}(\Delta)}(u) &= \psi_{CTS}(ue^{\theta\Delta}) - \psi_{CTS}(u) \\ &= iu(e^{\theta\Delta} - 1)(m - C\Gamma(1-\alpha)(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1})) \\ &\quad + C\Gamma(-\alpha)((\lambda_+ - iue^{\theta\Delta})^\alpha + (\lambda_- + iue^{\theta\Delta})^\alpha \\ &\quad - (\lambda_+ - iu)^\alpha - (\lambda_- + iu)^\alpha). \end{aligned} \tag{4.6}$$

The law X^+ is said to be a CTS subordinator with parameters (α, C, λ_+) if $0 < \alpha < 1$ and the characteristic function of X^+ is given by

$$\begin{aligned} \phi_{X^+}(u) = E[exp(iuX^+)] = \exp(&-iuC\Gamma(1-\alpha)\lambda_+^{\alpha-1} \\ &+ C\Gamma(-\alpha)((\lambda_+ - iu)^\alpha - \lambda_+^\alpha)) \end{aligned} \tag{4.7}$$

and similarly one can define X^-. It is simple to prove that a CTS law with $0 < \alpha < 1$ is the convolution, up to a constant term m, of X^+ and X^-.

The law $\widetilde{X}^+$ is said to have a spectrally positive CTS law with parameters (α, C, λ_+) if $\alpha \in (0,2)\backslash\{1\}$ and the characteristic function of $\widetilde{X}^+$ is given by

$$\begin{aligned} \phi_{\widetilde{X}^+}(u) = E[exp(iu\widetilde{X}^+)] = \exp(&-iuC\Gamma(1-\alpha)\lambda_+^{\alpha-1} \\ &+ C\Gamma(-\alpha)((\lambda_+ - iu)^\alpha - \lambda_+^\alpha)) \end{aligned} \tag{4.8}$$

and similarly one can define a spectrally negative CTS process $\widetilde{X}^-$. It is simple to prove that a CTS law with $1 < \alpha < 2$ is the convolution, up to a constant term m, of a spectrally positive and a spectrally negative CTS process.

4.2 The integrated process

In this section, we provide a closed-form solution for equation (2.14) in the CTS case.

Theorem 4.1. *Let X be a CTS law with parameters (α, C, λ_+, λ_-, m), and λ_t a OU process with marginal law X. Then, the cumulant characteristic function of*

[5] The relation between characteristic function, cumulant function, moment-generating function, and cumulant characteristic function applied in this study are in given in Table 2.1 at page 16 of Schoutens (2003).

the integrated process A_t defined in equation (2.4) is given by

$$\begin{aligned}\psi_{A_t}(u) = &\log E[\exp(iuA_t)] = iu\theta^{-1}\Bigg(\lambda_0(1-e^{-\theta t}) \\ &- (m - C\Gamma(1-\alpha)(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}))(1 - e^{-\theta t} - \theta t) + C\Gamma(1-\alpha)\Bigg(\\ &\frac{1}{\lambda_+^{\alpha-1}k_+(k_+-1)}\Bigg(\frac{k_+\big(1-k_+(1-e^{-\theta t})\big)^{\alpha+1}\,{}_2F_1\big(1,\alpha+1,\alpha+2;1+\frac{k_+e^{-\theta t}}{1-k_+}\big)}{(\alpha+1)(k_+-1)} \\ &- \frac{\big(1-k_+\big(1-e^{-\theta t}\big)\big)^{\alpha}}{\alpha} - \frac{k_{+2}F_1\big(1,\alpha+1,\alpha+2;\frac{1}{1-k_+}\big)}{(\alpha+1)(k_+-1)} + \frac{1}{\alpha}\Bigg) \\ &- \frac{1}{\lambda_-^{\alpha-1}k_+(k_-+1)}\Bigg(\frac{k_-\big(1+k_-(1-e^{-\theta t})\big)^{\alpha+1}\,{}_2F_1\big(1,\alpha+1,\alpha+2;1-\frac{k_-e^{-\theta t}}{1+k_-}\big)}{(\alpha+1)(k_-+1)} \\ &- \frac{\big(1+k_-\big(1-e^{-\theta t}\big)\big)^{\alpha}}{\alpha} - \frac{k_{-2}F_1\big(1,\alpha+1,\alpha+2;\frac{1}{1+k_-}\big)}{(\alpha+1)(k_-+1)} + \frac{1}{\alpha}\Bigg)\Bigg)\Bigg),\end{aligned}$$

with $k_+ = \frac{iu}{\theta\lambda_+}$ and $k_- = \frac{iu}{\theta\lambda_-}$.

Proof. See Section A.1 in the Appendix. □

4.3 Transition law of CTS-OU processes

In this section, we remind some results proved in Kawai and Masuda (2011a) and Kawai and Masuda (2012) on the transition law of CTS-OU processes of finite and infinite variation. Although here we consider only the one-sided case as defined in equations (4.7) and (4.8), using similar arguments the results can be extended to the bilateral case.

Theorem 4.2. *Let λ_t be a CTS-OU process. Given $\lambda_{t\Delta}$ and $\Delta > 0$, if $\alpha < 1$, for each $t \in \mathbb{N}$ it holds that*

$$\lambda_{(t+1)\Delta} = e^{-\theta\Delta}\lambda_{t\Delta} + \eta_0(\Delta) + \sum_{k=1}^{N(\Delta)} \zeta_k(\Delta) \tag{4.9}$$

where the equality is in distribution and all the random elements are mutually independent and specified as follows

- *$\eta_0(\Delta)$ has a one-sided CTS distribution with parameters ($C(1-e^{-\alpha\theta\Delta})$, λ_+, α);*
- *$N(\Delta)$ is a Poisson random variable with intensity $-C(1-e^{-\alpha\theta\Delta})\Gamma(-\alpha)\lambda_+^{\alpha}$;*
- *$\{\zeta_k(\Delta)\}_{k\in\mathbb{N}}$ is a sequence of i.i.d. random variables with common probability density*

$$v_\Delta(x) = \frac{1}{(1-e^{-\alpha\theta\Delta})\Gamma(-\alpha)\lambda_+^{\alpha}} x^{-1-\alpha}\left(e^{-\lambda_+ x} - e^{-\lambda_+ e^{\theta\Delta} x}\right).$$

If $1 < \alpha < 2$, *then*

$$\lambda_{(t+1)\Delta} = e^{-\theta\Delta}\lambda_{t\Delta} + \eta_0(\Delta) + \eta_1(\Delta) + \left(\sum_{k=1}^{N(\Delta)} \xi_k(\Delta) - \gamma\right), \tag{4.10}$$

with

$$\gamma = C\lambda_+^{\alpha-1}\left(e^{-\alpha\theta\Delta} - e^{-\theta\Delta} - (e^{-\theta\Delta} - e^{-2\theta\Delta})(1-\alpha)\right),$$

where the equality in (4.10) is in distribution, all the random elements are mutually independent and specified as follows

- $\eta_0(\Delta)$ *has a one-sided CTS distribution with parameters* $(C(1-e^{-\alpha\theta\Delta}),\ \lambda_+,\ \alpha)$;
- $\eta_1(\Delta)$ *has a one-sided CTS distribution with parameters* $(C\lambda_+ e^{-\alpha\theta\Delta}(e^{\theta\Delta}-1),\ \lambda_+ e^{\theta\Delta},\ \alpha-1)$;
- $N(\Delta)$ *is a Poisson random variable with intensity*

$$\kappa(\Delta) = C\lambda_+^{\alpha}\Gamma(-\alpha)(\alpha(1-e^{-\theta\Delta}) + e^{-\alpha\theta\Delta} - 1);$$

- $\{\xi_k(\Delta)\}_{k\in\mathbb{N}}$ *is a sequence of i.i.d. random variables with common probability density*

$$v_\Delta(x) = \kappa(\Delta)^{-1}Ce^{-\alpha\theta\Delta}\frac{e^{-\lambda_+ x} - e^{-\lambda_+ e^{\theta\Delta}x} - \lambda_+(e^{\theta\Delta}-1)xe^{-\lambda_+ e^{\theta\Delta}x}}{x^{\alpha+1}}.$$

Proof. See Theorem 3.1 in Kawai and Masuda (2011a) and Theorem 3.1 in Kawai and Masuda (2012). □

5 The variance gamma case

5.1 Basic definitions

In the limiting case $\alpha = 0$, equation (4.1) becomes

$$\begin{aligned}\phi_X(u) = E[exp(iuX)] = \exp(&iu(m - C(\lambda_+ - \lambda_-)\lambda_+^{-1}\lambda_-^{-1}) \\ &- C\log(\lambda_+\lambda_- + (\lambda_+ - \lambda_-)iu + u^2) + C\log(\lambda_+\lambda_-))\end{aligned} \tag{5.1}$$

and the law of X is said to have a VG distribution. Its Lévy measure can be written as

$$\nu(z) = \frac{Ce^{-\lambda_+ z}}{z}1_{z>0} + \frac{Ce^{-\lambda_-|z|}}{|z|}1_{z<0}. \tag{5.2}$$

As in the CTS case, one can define a Lévy process X_t which we refer to as a VG process. By simple calculations, the cumulant function of Z_1 can be shown to be

$$\begin{aligned}k_{Z_{VG}}(u) = uk'_{CTS}(u) = &-u(m - C(\lambda_+ - \lambda_-)\lambda_+^{-1}\lambda_-^{-1}) \\ &+ \frac{Cu}{\lambda_- - u} - \frac{Cu}{\lambda_+ + u}.\end{aligned} \tag{5.3}$$

Then,

$$\begin{aligned}\vartheta_{Z_{VG}}(u) = u(m - C(\lambda_+ - \lambda_-)\lambda_+^{-1}\lambda_-^{-1}) \\ + \frac{Cu}{\lambda_+ - u} - \frac{Cu}{\lambda_- + u}.\end{aligned} \tag{5.4}$$

By equation (2.11) it follows that the cumulant characteristic function of Z^*_{VG} is

$$\begin{aligned}\psi_{Z^*_{VG}(\Delta)}(u) &= \psi_{VG}(ue^{\theta\Delta}) - \psi_{VG}(u) \\ &= iu(e^{\theta\Delta} - 1)(m - C(\lambda_+ - \lambda_-)\lambda_+^{-1}\lambda_-^{-1}) \\ &\quad - C\log(\lambda_+\lambda_- + (\lambda_+ - \lambda_-)iue^{\theta\Delta} + u^2e^{2\theta\Delta}) \\ &\quad + C\log(\lambda_+\lambda_- + (\lambda_+ - \lambda_-)iu + u^2).\end{aligned} \tag{5.5}$$

5.2 The integrated process

In this section we provide a closed-form solution for equation (2.14) in the VG case.

Theorem 5.1. *Let X be a VG law with parameters (C, λ_+, λ_-, m), and λ_t a OU process with marginal law X. Then, the cumulant characteristic function of the integrated process A_t defined in equation (2.4) is given by*

$$\begin{aligned}\psi_{A_t}(u) = &\log E[\exp(iuA_t)] = iu\theta^{-1}\lambda_0(1 - e^{-\theta t}) \\ &- iu\theta^{-1}(m - C(\lambda_+ - \lambda_-)\lambda_+^{-1}\lambda_-^{-1})(1 - e^{-\theta t} - \theta t) \\ &+ \frac{\theta C}{iu - \theta\lambda_+}\left(\lambda_+\log\left(\frac{\lambda_+}{|\lambda_+ - iu\theta^{-1}(1 - e^{-\theta t})|}\right) - iut\right) \\ &- \frac{\theta C}{iu + \theta\lambda_-}\left(\lambda_-\log\left(\frac{\lambda_-}{|\lambda_- + iu\theta^{-1}(1 - e^{-\theta t})|}\right) + iut\right).\end{aligned}$$

Proof. See Section A.2 in the Appendix. □

5.3 Transition law of VG-OU processes

The transition law of VG-OU processes can be easily derived by its BDLP which is a compound Poisson process, as shown in Schoutens (2003). It is not difficult to prove that

$$\lambda_{(t+1)\Delta} = e^{-\theta\Delta}\lambda_{t\Delta} + \sum_{k=1}^{N(\Delta)}\zeta_k^+(\Delta) - \sum_{k=1}^{N(\Delta)}\zeta_k^-(\Delta) \tag{5.6}$$

where $N(\Delta)$ is a Poisson random variable with intensity C and ζ_k^+ and ζ_k^- follow a $\Gamma(1, \lambda_+)$ and $\Gamma(1, \lambda_-)$ law, respectively. However, this processes can be simulated by considering the series representation via the inverse mass function as described in Barndorff-Nielsen and Shephard (2001) and Schoutens (2003). This last method is useful when the simulation is inserted into an optimization algorithm because it allows one to keep fixed the simulated draws, as the parameters are varied during the optimization algorithm.

6 The rapidly decreasing tempered stable case

6.1 Basic definitions

Let α, C, λ_+, λ_- positive constants, $m \in \mathbb{R}$ and $\alpha \in (0,2)\backslash\{1\}$. The law X is said to have a RDTS distribution if the characteristic function of X is given by

$$\phi_X(u) = E[exp(iuX)] = \exp(ium + CG(iu;\alpha,\lambda_+) + CG(-iu;\alpha,\lambda_-) \tag{6.1}$$

with

$$\begin{aligned} G(x;\alpha,\lambda) = 2^{-\alpha/2-1}\lambda^\alpha \Bigg(& \Gamma\Big(-\frac{\alpha}{2}\Big) M\Big(-\frac{\alpha}{2},\frac{1}{2};\Big(\frac{\sqrt{2}x}{2\lambda}\Big)^2\Big) \\ & + \frac{\sqrt{2}x}{\lambda}\Gamma\Big(\frac{1-\alpha}{2}\Big) M\Big(\frac{1}{2}-\frac{\alpha}{2},\frac{3}{2};\Big(\frac{\sqrt{2}x}{2\lambda}\Big)^2\Big) \\ & - \frac{\sqrt{2}x}{\lambda}\Gamma\Big(\frac{1}{2}-\frac{\alpha}{2}\Big) - \Gamma\Big(-\frac{\alpha}{2}\Big)\Bigg), \end{aligned} \tag{6.2}$$

where $M(a,c;z)$ is the Kummer's or confluent hypergeometric function of the first kind as defined in equation (13.1.2) in Abramowitz and Stegun (1974) (see also Tricomi (1954)). An efficient algorithm to compute the characteristic function in equation (6.1) can be constructed (see Gil et al. (2007), and Bianchi et al. (2010a)). For a more simple derivatives computation, by the definition of the function M, the function G can be written as

$$G(x;\alpha,\lambda) = \frac{1}{2}\sum_{n=2}^{\infty}\frac{x^n}{n!}\left(\frac{\lambda}{\sqrt{2}}\right)^{\alpha-n}\Gamma\left(\frac{n-\alpha}{2}\right). \tag{6.3}$$

Its Lévy measure can be written as

$$\begin{aligned} \nu(z) &= q(z)\nu_{S_\alpha}(z) \\ &= \frac{Ce^{-\lambda_+^2 z^2/2}}{z^{1+\alpha}}1_{z>0} + \frac{Ce^{-\lambda_-^2 z^2/2}}{|z|^{1+\alpha}}1_{z<0}, \end{aligned} \tag{6.4}$$

where $q(z)$ is the tempering function of the RDTS law and $\nu_{S_\alpha}(z)$ is the Lévy measure of the stable law described in equation (4.3). Given a RDTS law X_1, one can define a Lévy process X_t which we refer to as a RDTS process. If $\alpha < 1$, it can be demonstrated that the process has finite variation. By simple calculations, one can write the cumulant function of Z_1 as

$$k_{Z_{RDTS}}(u) = uk'_{RDTS}(u) = -um + uCG'(-u;\alpha,\lambda_+) - uCG'(u;\alpha,\lambda_-) \tag{6.5}$$

where

$$G'(x;\alpha,\lambda) = \frac{1}{2}\sum_{n=2}^{\infty}\frac{x^{n-1}}{n!n}\left(\frac{\lambda}{\sqrt{2}}\right)^{\alpha-n}\Gamma\left(\frac{n-\alpha}{2}\right), \tag{6.6}$$

and the moment-generating function is

$$\vartheta_{Z_{RDTS}}(u) = um - uCG'(u;\alpha,\lambda_+) + uCG'(-u;\alpha,\lambda_-). \tag{6.7}$$

The series $xG'(x;\alpha,\lambda)$ converges since the following inequalities holds

$$xG'(x;\alpha,\lambda) < G(x;\alpha,\lambda).$$

By equation (2.11) it follows that the cumulant characteristic function of Z^*_{RDTS} is

$$\begin{aligned}\psi_{Z^*_{RDTS}(\Delta)}(u) &= \psi_{RDTS}(ue^{\theta\Delta}) - \psi_{RDTS}(u)\\ &= ium(e^{\theta\Delta}-1) + CG(iue^{\theta\Delta};\alpha,\lambda_+) + CG(-iue^{\theta\Delta};\alpha,\lambda_-)\\ &\quad - CG(iu;\alpha,\lambda_+) - CG(-iu;\alpha,\lambda_-).\end{aligned} \tag{6.8}$$

The law X^+ is said to be a one-sided RDTS with parameters (α, C, λ_+) if $\alpha \in (0,2)\backslash\{1\}$ and the characteristic function of X^+ is given by

$$\phi_{X^+}(u) = E[exp(iuX^+)] = CG(iu;\alpha,\lambda_+), \tag{6.9}$$

and similarly one can define X^-. It is simple to prove that a RDTS law is the convolution, up to a constant term m, of X^+ and X^-.

6.2 The integrated process

In this section we provide a closed-form solution (up to an infinite summation) for equation (2.14) in the RDTS case.

Theorem 6.1. *Let X be a RDTS law with parameters (α, C, λ_+, λ_-, m), and λ_t a OU process with marginal law X. Then, the cumulant characteristic function of the integrated process A_t defined in equation (2.4) is given by*

$$\begin{aligned}\psi_{A_t}(u) =& \log E[\exp(iuA_t)] = iu\theta^{-1}\lambda_0(1-e^{-\theta t}) - iu\theta^{-1}m(1-e^{-\theta t}-\theta t)\\ &- C\sum_{n=2}^{\infty}\frac{(iu)^n}{\theta^n n!2n}\left(\frac{\lambda_+}{\sqrt{2}}\right)^{\alpha-n}\Gamma\left(\frac{n-\alpha}{2}\right)B_{1-e^{-\theta t}}(n+1,0)\\ &+ C\sum_{n=2}^{\infty}\frac{(-iu)^n}{\theta^n n!2n}\left(\frac{\lambda_-}{\sqrt{2}}\right)^{\alpha-n}\Gamma\left(\frac{n-\alpha}{2}\right)B_{1-e^{-\theta t}}(n+1,0).\end{aligned}$$

Proof. See Section A.3 in the Appendix. □

6.3 Transition law of RDTS-OU processes

In this section we will only deal with the one-sided setting. The bilateral setting can be treated simply by considering the difference between two independent processes. Recall that the Lévy measure of a RDTS distribution is given by equation (6.4)

$$\nu(z) = C\frac{e^{-\lambda_+^2 z^2/2}}{z^{\alpha+1}}$$

therefore by considering equation (2.10) one can write

$$\mu(z) = C\alpha\frac{e^{-\lambda_+^2 z^2/2}}{z^{\alpha+1}} + C\lambda_+^2\frac{e^{-\lambda_+^2 z^2/2}}{z^{\alpha-1}}. \tag{6.10}$$

where by definition $\alpha \in (0,2)$. As shown in Theorem 4.2, in the CTS case the transition law depends on the value of α. In the RDTS case we have a unique transition law for each value of $\alpha \in (0,2)$.

Theorem 6.2. *Let λ_t be a RDTS-OU process. Given $\lambda_{t\Delta}$ and $\Delta > 0$, for each $t \in \mathbb{N}$ it holds that*

$$\lambda_{(t+1)\Delta} = e^{-\theta\Delta}\lambda_{t\Delta} + \eta_0(\Delta) + \sum_{k=1}^{N(\Delta)} \zeta_k(\Delta) \tag{6.11}$$

where the equality is in distribution and all the random elements are mutually independent and specified as follows

- *$\eta_0(\Delta)$ has a one-sided RDTS distribution with parameters ($C(1-e^{-\alpha\theta\Delta})$, λ_+, α);*
- *$N(\Delta)$ is a Poisson random variable with intensity $C\frac{\lambda_+^\alpha}{2^{\frac{\alpha}{2}+1}}(e^{-\alpha\theta\Delta}-1)\Gamma\left(-\frac{\alpha}{2}\right)$;*
- *$\{\zeta_k(\Delta)\}_{k\in\mathbb{N}}$ is a sequence of i.i.d. random variables with common probability density*

$$v_\Delta(x) = \frac{1}{\frac{\lambda_+^\alpha}{\alpha}(e^{\alpha\theta\Delta}-1)\Gamma\left(1-\frac{\alpha}{2}\right)} x^{-\alpha-1}\left(e^{-\lambda_+^2 x^2/2} - e^{-\lambda_+^2 e^{2\theta\Delta}x^2/2}\right).$$

Proof. By following Theorem 3.1 of Kawai and Masuda (2012), the Lévy measure of the random variable $\int_0^{\theta\Delta} e^{-\theta\Delta+s}dZ_s$ is given by

$$\begin{aligned}
\mu_\Delta(z) &= \int_0^{\theta\Delta} e^s \mu(e^s z)ds \\
&= \frac{C}{z^{\alpha+1}}\int_0^{\theta\Delta}(\alpha+\lambda_+^2 e^{2s}z^2)e^{-\alpha s}e^{-\lambda_+^2 e^{2s}z^2/2}ds \\
&= -\frac{C}{z^{\alpha+1}}\int_0^{\theta\Delta}\frac{\partial}{\partial s}e^{-\alpha s}e^{-\lambda_+^2 e^{2s}z^2/2}ds \\
&= \frac{C}{z^{\alpha+1}}\left(e^{-\lambda_+^2 z^2/2} - e^{-\alpha\theta\Delta-\lambda_+^2 e^{2\theta\Delta}z^2/2}\right) \\
&= \frac{C}{z^{\alpha+1}}\left(e^{-\lambda_+^2 z^2/2}(1-e^{-\alpha\theta\Delta})\right) + \frac{Ce^{-\alpha\theta\Delta}}{z^{\alpha+1}}\left(e^{-\lambda_+^2 z^2/2} - e^{-\lambda_+^2 e^{2\theta\Delta}z^2/2}\right). \\
&= \mu_{1,\Delta}(z) + \mu_{2,\Delta}(z).
\end{aligned} \tag{6.12}$$

Both $\mu_{1,\Delta}(z)$ and $\mu_{2,\Delta}(z)$ are positive functions on $\mathbb{R}_+$. The term $\mu_{1,\Delta}(z)$ corresponds to the Lévy measure of a RDTS random variable with parameters α, $C(1-e^{-\alpha\theta\Delta})$ and λ_+. Then, for each $\alpha < 2$ the term $\mu_{2,\Delta}(z)$ satisfies the following equality

$$\int_0^\infty \mu_{2,\Delta}(z) = C\frac{\lambda_+^\alpha}{2^{\frac{\alpha}{2}+1}}(e^{-\alpha\theta\Delta}-1)\Gamma\left(-\frac{\alpha}{2}\right) < \infty.$$

The integral has been computed by considering the equality (3.343) in Gradshteĭn and Ryzhik (2007)

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{z^{\rho+1}}dx = \frac{b^\rho - a^\rho}{\rho}\Gamma(1-\rho)$$

with $a > 0$, $b > 0$, and $\rho < 1$, and by considering the change of variable $z^2 = x$, $a = \lambda_+^2/2$, $b = \lambda_+^2 e^{2\theta\Delta}/2$, and $\rho = \alpha/2$. This proves that $\mu_{2,\Delta}(z)$ is the Lévy measure of a compound Poisson process. □

In order to simulate ζ_k we consider the inequality proved in Lemma 1 of Zhang and Zhang (2008), that is, for every $w > 0$

$$e^{-\lambda_+^2 z^2/2} - e^{-\lambda_+^2 e^{2\theta\Delta} z^2/2} = \frac{\lambda_+^2}{2} \int_{z^2}^{e^{2\theta\Delta} z^2} e^{-\frac{\lambda_+^2}{2} x} dx \leq \frac{\lambda_+^2}{2} e^{-\lambda_+^2 z^2/2} z^2 (e^{2\theta\Delta} - 1)$$

thus

$$\begin{aligned} v_\Delta(x) &\leq \frac{\alpha}{2^{1+\frac{\alpha}{2}}} \frac{e^{2\theta\Delta} - 1}{e^{\alpha\theta\Delta} - 1} \left(\frac{2^{\frac{\alpha}{2}} \lambda_+^{2-\alpha}}{\Gamma\left(1 - \frac{\alpha}{2}\right)} x^{(2-\alpha)-1} e^{-\lambda_+^2 x^2/2} \right) \\ &\leq \frac{\alpha}{2^{\frac{\alpha}{2}}} \frac{e^{2\theta\Delta} - 1}{e^{\alpha\theta\Delta} - 1} \left(\frac{2^{\frac{\alpha}{2}} \lambda_+^{2-\alpha}}{\Gamma\left(1 - \frac{\alpha}{2}\right)} x^{(2-\alpha)-1} e^{-\lambda_+^2 x^2/2} \right) \end{aligned} \tag{6.13}$$

where the factor in the parentheses is the density of a generalized gamma law

$$f_\Gamma(x) = \frac{c}{\beta^{c\alpha'} \Gamma(\alpha')} x^{c\alpha'-1} e^{-\left(\frac{x}{\beta}\right)^c}$$

as defined in Johnson et al. (1994) and with $\alpha' = 1 - \alpha/2$, $\beta = \sqrt{2}/\lambda_+$, and $c = 2$ positive constants. Then the following inequality holds

$$h(\Delta, \alpha, \theta) = \frac{\alpha}{2^{\frac{\alpha}{2}}} \frac{e^{2\theta\Delta} - 1}{e^{\alpha\theta\Delta} - 1} \geq 1,$$

and this means that the acceptance rate[6] depends on Δ, α, and θ. Then, it holds that

$$g_\Delta(x) := \frac{v_\Delta(x)}{h(\Delta, \alpha, \theta) f_\Gamma(x)} = \frac{2^{\frac{\alpha}{2}-1}}{\lambda_+^2 x^2 (e^{2\theta\Delta} - 1)} \left(1 - e^{-\frac{\lambda_+^2}{2} x^2 \left(e^{2\theta\Delta} - 1\right)} \right),$$

and as proposed by Kawai and Masuda (2011a), the following acceptance-rejection algorithm can be used to simulate $\zeta_k(\Delta)$. A generalized gamma law Z with parameters (α', β, c) can be simulated by considering the following equality that

$$Z = \beta X^{\frac{1}{c}}$$

where X is a standard gamma law with parameter α' as also proposed by Gentle (2003). Thus, in order to simulate $\zeta_k(\Delta)$, we consider the following acceptance-rejection algorithm. In the first step, we generate a uniform law U on $(0, 1)$ and a generalized gamma law V with parameters $(1 - \alpha/2, \sqrt{2}/\lambda_+, 2)$. In the second step, if $U \leq g_\Delta(V)$, then we set $\zeta_k(\Delta) := V$; otherwise, we return to the first step.

[6] See Section 8.1 for a brief description of the acceptance-rejection method.

7 Maximum likelihood estimation

In this section, we show how to estimate parameters using the maximum likelihood estimation (MLE) method. Given a sample from a D-OU process with observations $\{\lambda_i\}_{0\leq i\leq N}$, the MLE estimate is a solution $\hat{\Theta}$ of the optimization problem

$$\hat{\Theta} = \max_{\theta} \log L(\theta; \{\lambda_i\}_{0\leq i\leq N})$$

where the likelihood function is given by

$$L(\Theta) = f_D(\lambda_0) e^{\theta N \Delta} \prod_{i=1}^{N} f_{Z^*(\Delta)}(e^{\theta\Delta}\lambda_i - \lambda_{i-1}),$$

the law D depends on the model analyzed and the law $Z^*(\Delta)$ is derived by the law D, as shown in Section 2. When the density functions f_D and $f_{Z^*(\Delta)}$ can be computed in closed-form, the problem is less computational demanding compared to when the density function has to be computed via the discrete fast Fourier transform (FFT). In Appendix A.4, the FFT approach is described.

8 The empirical study

In this section, we conduct an empirical investigation to assess the performance of the simulation algorithms. In particular, we generate random numbers from the three OU processes analyzed in this paper. Thus, in order to assess the simulation algorithm, we perform on each simulated sample a MLE. First, we consider the VG-OU process. Then the CTS-OU driven by a CTS subordinator (i.e. $\alpha < 1$) and the RDTS-OU process driven by a one-sided RDTS law. As shown in Sections 4.3 and 6.3, while in the CTS-OU case the transition law has a different representation depending if α is less than or greater than 1, in the RDTS-OU case the transition law has the same representation for each value of α. In practice this means that the simulation of a RDTS-OU process is simpler compared to the CTS-OU process, since when $\alpha > 1$ the CTS-OU is represented by a convolution of three random components. In fact, as shown in Theorem 6.2, a RDTS-OU process is represented by a convolution of two random components (the same is true in the CTS-OU case when $\alpha < 1$). Before studying these processes, we empirically investigate the algorithms used to simulate a CTS subordinator and a one-sided RDTS law.

8.1 Simulate CTS random numbers

The simulation of a TS random draw is not a simple task. A series representation algorithm has been proposed in Rosiński (2007) and empirically studied in Bianchi et al. (2010a) and Imai and Kawai (2011). Since there exists an efficient algorithm to draw random samples from stable distributions (see Chambers et al. (1976)), the problem of generating random numbers from a TS law X can be solved by using a stable law Y possessing a probability density g similar to the probability density f of X. One can generate a value for Y and accept (reject) this

value if a given condition is satisfied (not satisfied).[7] We refer to the probability of the acceptance event as the *acceptance rate*. This acceptance-rejection simulation method has been widely studied in the literature (see Kawai and Masuda (2011a), Kawai and Masuda (2011b), Kawai and Masuda (2012), Jelonek (2012), and references therein). By applying this algorithm, one can sample TS random numbers in an exact (or approximate) way if the tail index α is less (or greater) than 1. The computational cost strictly depends on the parameters. In the case when $\alpha < 1$, a double rejection sampling algorithm that does not depend upon the model parameters has been proposed by Devroye (2009).

Furthermore, one can use the method proposed by Rosiński (2001) where the ratio between the Lévy measures of a TS and a stable law is used to construct an acceptance-rejection algorithm. Also in this case the probability of the acceptance event depends on the parameters of the TS distribution to simulate. However, this method does not seem to work efficiently and further investigation is needed to use it in practice.

Alternatively, one can consider the inverse transform algorithm. That is, given the cumulative distribution function F, one can use the following three-step procedure:

1. generate a sequence $U_1, \ldots, U_n$ of i.i.d. uniform variables;
2. find a root X_i of the equation $F(X_i) - U_i = 0$ for each $i = 1, \ldots, n$;
3. return the sequence $X_1, \ldots, X_n$.

In the TS case, it is easy to see that the function F is not available in closed form. To find values of F, first we have to invert the characteristic function,[8] to find both the density (f) and the cumulative distribution function (F), and then find the value X_i satisfying the equality $F(X_i) - U_i = 0$. Even if this method may seem computational demanding, an efficient procedure can be written in order to increase the speed and make the time necessary for the simulation of large matrices of minor concern. This approach can be efficiently applied for all values of the parameters and can compete with the acceptance-rejection method when a huge matrix of TS random numbers has to be drawn. See also the recent work of Ballotta and Kyriakou (2011) and Bianchi et al. (2013).

However, it sometimes happens that for certain values of the parameters the Fourier inversion does not work properly due to numerical errors. In these cases, to obtain the right simulation algorithm, one has to increase the integration limits or decrease the discretization step into the FFT algorithm in order to reduce the error in evaluating the density function and the cumulative distribution function. However, the error in evaluating these two functions decreases at the expense of computational time (see Appendix A.4). Similarly, for other values of the parameters it may happen that, if using the acceptance-rejection algorithm, the acceptance rate is so low that it cannot be used in real applications (as observed above, it is only for $\alpha < 1$ that a double rejection sampling algorithm that does not depend

[7] More details on this method can be found in Gentle (2003) or Rachev et al. (2011).

[8] Here the term inversion means Fourier inversion. More details on this method can be found in Appendix A.4.

upon the model parameters has been developed). This in practice means that the optimal choice of the algorithm may depend on the parameters. We test both the acceptance-rejection and the inverse transform algorithm by drawing stdCTS[9] random numbers considering three parameter sets (α, λ_+, λ_-): (0.75, 0.5, 0.5), (1.5, 1, 0.5), and (1.75, 0.1, 0.05). The computational time needed for generating a matrix with dimension 20,000 $\times$ 1,000 is 15 (56, and 11) seconds with the acceptance-rejection algorithm,[10] slightly more than 5 seconds with the inverse algorithm in all the cases considered. According to the notation introduced in Appendix A.4, we consider $\log_{10} h = -3$ and $q = 17$, representing the grid spacing and the grid steps in the FFT inversion. We point out that, if using the acceptance-rejection algorithm, the computational time can be much greater if one selects other parameters. Conversely, the computational time of the inverse algorithm depends only on h and q.

8.2 The simulation study

In addition to simulating and estimating OU processes with VG, CTS, and RDTS marginal laws, we empirically study the algorithms to simulate the CTS subordinator and the one-sided RDTS law. As already observed in Section 8.1 for the bilateral stdCTS law, if one considers the simulation of a CTS subordinator for certain values of the parameters, the acceptance-rejection algorithm is better than the inverse transform one. This is principally due to the fact that the evaluation of the cumulative distribution function (F) needed in the inverse transform algorithm may become computationally demanding for some parameter set (i.e., $\alpha \leq 0.4$).[11] The converse is true for other parameter sets, as the acceptance rate becomes small. We note that the inverse transform algorithm also works properly when $\alpha > 1$ (i.e., in the spectrally positive case) while the acceptance-rejection algorithm has to be slightly modified as proposed by Kawai and Masuda (2011b), and Kawai and Masuda (2012).

In order to test both the acceptance-rejection and the inverse transform algorithm we simulate samples from various CTS one-sided laws by considering three different parameter sets (C, λ_+, α): (1, 1, 0.4), (1, 1, 0.8), and (1, 1, 1.2) with $\Delta = 0.1$. Under this distributional assumption, as well as for all other considered distributions, after generating random numbers we perform a MLE to assess the simulation algorithm performance. The likelihood function is computed by inverting the characteristic function (see Appendix A.4). We repeat the exercise 1,000 times by considering samples with 500, 1,000, 2,000, 5,000, and 10,000 (also, 50,000 in the OU cases) observations.

In Table 1 we report the relative error of the estimators $\hat{C}$, $\hat{\lambda}_+$, and $\hat{\alpha}$ and in Figure 4 we show the boxplots of their empirical distribution. On each box, the central mark is the median, the edges of the box are the 25-th and 75-th percentiles, the whiskers extend to the most extreme data points not considered outliers, and

[9] A stdCTS law has a distribution with zero mean and unit variance (see Scherer et al. (2012)). In the following we will refer to it as a stdCTS with parameters (λ_-, λ_+, α).

[10] The procedure was run on an 8 cores AMD FX processor with 16GB of Ram with a Linux based 64-bit operating system.

[11] See Appendix A.4 for a detail discussion on this point.

outliers are plotted individually. The relative error of the parameter p is defined by

$$e_{rel}^{1-\delta}(p) = \max\left\{\left|\frac{q_{\delta/2}(\hat{p}) - p}{p}\right|, \left|\frac{q_{1-\delta/2}(\hat{p}) - p}{p}\right|\right\}. \tag{8.1}$$

and it means the the relative error of the estimator is smaller than $e_{rel}^{1-\delta}$ with probability $1-\delta$ measured by the empirical distribution of the estimator ($q_\beta(\hat{p})$ represents the β-th quantile of the estimator of p).

In the RDTS case, we test only the classical inverse transform algorithm, because an acceptance-rejection method has not been studied yet.[12] In order to test the inverse transform algorithm, we simulate samples from various RDTS one-sided laws by considering three parameter sets (C, λ_+, α): (1, $\sqrt{2}$, 0.8), (1, $\sqrt{2}$, 1.2), and (1, $\sqrt{2}$, 1.6) with $\Delta = 0.1$.[13] After generating random numbers we perform a parameter estimation (MLE) as in the CTS subordinator case. In Table 2 we report the relative error of the estimators $\hat{C}$, $\hat{\lambda}_+$, and $\hat{\alpha}$ and in Figure 5 we show the boxplots of their empirical distribution.

In Figure 4 we observe that medians (the central mark in the box) are close to the corresponding true values (except when we simulate by the inverse transform algorithm the CTS subordinator with parameters (1, 1, 0.4)). The empirical distributions of $\hat{C}$, $\hat{\lambda}_+$, and $\hat{\alpha}$ are slightly skewed for small sample sizes and become more symmetric for large sample sizes per estimate. These properties of the estimators may be due to the constraints set in the optimization algorithm.[14] The accuracy of the estimator $\hat{\alpha}$ is higher than the one for $\hat{C}$ and $\hat{\lambda}_+$, and the one for $\hat{C}$ is higher than the one for $\hat{\lambda}_+$ (compare the relative errors in Table 1). In particular, the accuracy of $\hat{C}$ and $\hat{\lambda}_+$ is very poor for small sample sizes per estimate (i.e., sample size less than $5,000$). For reliable estimates, we need at least a sample size of $5,000$ because of the relative error $e_{rel}^{0.9}$ reported in Table 1. The accuracy depends on the parameters selected and on both the simulation and estimation algorithms. Because these algorithms are based on the FFT inversion, they also depend on the choice of the grid spacing and of the grid steps determined by the parameters h and q, as discussed in Appendix A.4. The acceptance-rejection method performs better when $\alpha = 0.4$. Conversely, the inverse transform method is slightly better when $\alpha = 0.8$ and $\alpha = 1.2$. As shown in Figure 5 and Table 2, similar conclusions are also true in the RDTS case.

We conduct a similar study for OU processes. By equation (5.6) and Theorems 4.2 and 6.2, we can generate OU processes driven by a VG law, a CTS subordinator, and a one-sided RDTS law. More specifically, we empirically investigate the VG-OU process and consider three parameter sets (C, λ_+, λ_-, θ): (10, 50, 100, 0.5), (20, 100, 200, 0.5), and (40, 200, 400, 0.5) with $\Delta = 1/250$. The boxplots of their empirical distribution are shown Figure 1. Then, we study an OU process driven by a CTS subordinator and consider three parameter sets (C, λ_+, α, θ): (1, 1, 0.4,

[12] The acceptance-rejection algorithm proposed by Rosiński (2001) does not work properly, even if $\alpha > 1$.

[13] We choose $\sqrt{2}$ instead of 1 in order to take into consideration the difference in the definition of the Lévy measure of a CTS law (see equation (4.2)) and of a RDTS law (see equation (6.4)).

[14] We use the Matlab r2012a function *fmincon* for the optimization routine.

0.5), (1, 1, 0.6, 0.5), and (1, 1, 0.8, 0.5) with $\Delta = 0.1$. In Tables 1 we report the relative error of the estimators $\hat{C}$, $\hat{\lambda}_+$, $\hat{\alpha}$, $\hat{\theta}$ and in Figure 2 we show the boxplots of their empirical distribution.

Finally, we study an OU process driven by a one-sided RDTS subordinator and consider three parameter sets $(C, \lambda_+, \alpha, \theta)$: $(1, \sqrt{2}, 0.8, 0.5)$, $(1, \sqrt{2}, 1.2, 0.5)$, and $(1, \sqrt{2}, 1.6, 0.5)$ with $\Delta = 0.1$. In the RDTS-OU case, we estimate the parameter θ by performing a linear regression on the time-series of observations and, if the estimated $\hat{\theta}$ is less than 0.45 or greater than 0.55, we set it to 0.5. This choice affects not only the estimation of θ but also the estimation of all the other parameters. As reported in Table 2, due to this particular constraint the error does not always decrease as of the number of observations increases.

In Figures 1, 2, and 3 we observe that medians (the central mark in the box) are close to the corresponding true values. The VG-OU model is the best performing model in terms of accuracy in the estimation of the parameter θ. In both the CTS-OU and RDTS-OU cases, the accuracy of the estimator $\hat{\alpha}$ is higher than the ones for $\hat{C}$ and $\hat{\lambda}_+$, and the accuracy of $\hat{C}$ is higher than that for $\hat{\lambda}_+$ (compare the relative errors in Tables 1 and 2). In the CTS-OU case, the accuracy of the estimator $\hat{\alpha}$ is higher than the one for $\hat{\theta}$ only when $\alpha = 0.8$. In contrast, in the RDTS-OU case, the accuracy of the estimator $\hat{\alpha}$ is higher than the accuracy for $\hat{\theta}$. This occurs also because the parameter θ is estimated by linear regression and not by MLE. As observed in both the CTS and RDTS cases, for reliable estimates we need at least a sample size of $5,000$ (10,000 in the VG-OU case) because of the relative error $e_{rel}^{0.9}$ shown in Tables 1 and 2. As already observed above, the accuracy depends on the parameters selected. This is principally due to the fact that the accuracy in the computation of the density function via FFT depends on the parameter choice (see Appendix A.4).

Using the notation introduced in Appendix A.4, in the FFT inversion we consider $\log_{10} h = -3$ and $q = 17$ in the CTS and RDTS cases, $\log_{10} h = -4$ and $q = 18$ in the CTS-OU and RDTS-OU cases, and $\log_{10} h = -5$ and $q = 18$ in the VG-OU case. This choice partially motivates the larger error observed in the CTS random number generation via inverse transformation with $\alpha = 0.4$, as reported in Table 1. However, these choices show a satisfactory balance between efficiency and accuracy. We note that, if one selects different parameters or a different value for Δ, these conclusion may no longer hold or may be inefficient.

9 Conclusions

In this paper, we review the literature on tempered stable Ornstein-Uhlenbeck processes and then set forth a framework for their application to real-world problems. For example, the theoretical results derived in this paper can be applied in finance to model the term structure of interest rates, credit default swaps prices, and stochastic volatilities. Moreover, the simulation algorithms that we describe can be useful in model calibration in finance such as calibrating dynamic pricing models given observed market prices. With increased frequency, the calibration of the models used to price financial assets and derivative products is obtained through methods that simultaneously utilize simulation and optimization. These

methods are usually used to estimate models expressed in state-space form.

More in detail, the objective of this paper is threefold. First, we compute the characteristic function of integrated OU processes driven by three different processes: CTS, VG, and RDTS processes. Second, after having reviewed the transition law of CTS-OU processes, we derive the transition law between consecutive observations of RDTS-OU processes as a convolution of two random components: a RDTS distribution and a compound Poisson distribution. The RDTS component can be simulated by the inverse transform technique and the compound Poisson component can be simulated by an acceptance-rejection sampling techniques based on generalized gamma random numbers. In the CTS-OU case, the transition law between consecutive observations differ if α is less than or greater than 1: if $\alpha < 1$ ($\alpha > 1$) the transition law between consecutive observations of CTS-OU processes is a convolution of two random components (three random components). In contrast, in the RDTS-OU case the transition law between consecutive observations has the same distributional form for all values of α. Finally, we derive some useful formulas and we empirically investigated how one can efficiently simulate and estimate OU processes of this kind.

A Appendix

A.1 Proof of Theorem 4.1

Proof. In order to evaluate equation (2.14) in the CTS case, we compute the integral

$$\int_0^t \theta \vartheta_{Z_{CTS}}(iu\theta^{-1}(1-e^{-\theta(t-s)}))ds. \tag{A.1}$$

By changing variable $x = 1 - e^{-\theta(t-s)}$ in the integral above, we write

$$\int_0^{1-e^{-\theta t}} \frac{\vartheta_{Z_{CTS}}(iu\theta^{-1}x)}{(1-x)}dx$$

therefore we have the equality

$$\begin{aligned}\int_0^{1-e^{-\theta t}} \Big(&\frac{iu\theta^{-1}x(m - C\Gamma(1-\alpha)(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}))}{(1-x)} \\ &+ \frac{iu\theta^{-1}xC\Gamma(1-\alpha)((\lambda_+ - iu\theta^{-1}x)^{\alpha-1} - (\lambda_- + iu\theta^{-1}x)^{\alpha-1})}{(1-x)}\Big)dx.\end{aligned} \tag{A.2}$$

The first part of the integral (A.2) can be easily evaluated

$$\begin{aligned}&\int_0^{1-e^{-\theta t}} \frac{iu\theta^{-1}x(m - C\Gamma(1-\alpha)(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}))}{(1-x)}dx \\ =\;& iu\theta^{-1}(m - C\Gamma(1-\alpha)(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}))\int_0^{1-e^{-\theta t}} \Big(-1 + \frac{1}{1-x}\Big)dx \\ =\;& -iu\theta^{-1}(m - C\Gamma(1-\alpha)(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}))(1 - e^{-\theta t} - \theta t).\end{aligned} \tag{A.3}$$

The second part of the integral (A.2) is more challenging and can be rewritten as the difference between two similar integrals

$$\begin{aligned}&\int_0^{1-e^{-\theta t}} iu\theta^{-1}C\Gamma(1-\alpha)\Big(\frac{x(\lambda_+ - iu\theta^{-1}x)^{\alpha-1}}{1-x} - \frac{x(\lambda_- + iu\theta^{-1}x)^{\alpha-1}}{1-x}\Big)dx \\ =\;& iu\theta^{-1}C\Gamma(1-\alpha)\Big(\int_0^{1-e^{-\theta t}} \frac{x(\lambda_+ - iu\theta^{-1}x)^{\alpha-1}}{1-x}dx - \int_0^{1-e^{-\theta t}} \frac{x(\lambda_- + iu\theta^{-1}x)^{\alpha-1}}{1-x}dx\Big).\end{aligned} \tag{A.4}$$

Therefore, equation (A.4) becomes

$$iu\theta^{-1}C\Gamma(1-\alpha)\Big(\int_0^{1-e^{-\theta t}} \frac{x(\lambda_+ - iu\theta^{-1}x)^{\alpha-1}}{1-x}dx - \int_0^{1-e^{-\theta t}} \frac{x(\lambda_- + iu\theta^{-1}x)^{\alpha-1}}{1-x}dx\Big). \tag{A.5}$$

By defining $k_+ = \frac{iu}{\theta\lambda_+}$ and by changing variable $y = 1 - k_+x$, we rewrite the first integral of equation (A.4) as

$$\begin{aligned}\int_0^{1-e^{-\theta t}} \frac{x(\lambda_+ - iu\theta^{-1}x)^{\alpha-1}}{1-x}dx &= \lambda_+^{\alpha-1}\int_0^{1-e^{-\theta t}} \frac{x(1-k_+x)^{\alpha-1}}{1-x}dx \\ &= \frac{1}{\lambda_+^{\alpha-1}k_+(k_+-1)}\int_1^{1-k_+(1-e^{-\theta t})} \frac{(y-1)y^{\alpha-1}}{1+\frac{y}{k_+-1}}dy.\end{aligned} \tag{A.6}$$

The last integral can be computed as follows

$$\int \frac{(y-1)y^{\alpha-1}}{1+\frac{y}{k_+-1}}dy = \int \frac{y^\alpha}{1+\frac{y}{k_+-1}}dy - \int \frac{y^{\alpha-1}}{1+\frac{y}{k_+-1}}dy, \tag{A.7}$$

thus, by considering the equality

$${}_2F_1(1,\alpha+1,\alpha+2;w) = \sum_{n=0}^{+\infty} \frac{\alpha+1}{\alpha+n+1} w^n,$$

one can write the first integral of equation (A.7)

$$\begin{aligned}\int \frac{y^\alpha}{1+\frac{y}{k_+-1}}dy &= \int y^\alpha \sum_{n=0}^{+\infty} \frac{(-1)^n y^n}{(k_+-1)^n} = \sum_{n=0}^{+\infty} \frac{(-1)^n y^{\alpha+n+1}}{(\alpha+n+1)(k_+-1)^n} \\ &= y^{\alpha+1} \sum_{n=0}^{+\infty} \frac{(-1)^n y^n}{(\alpha+n+1)(k_+-1)^n} = \frac{y^{\alpha+1}\,{}_2F_1(1,\alpha+1,\alpha+2;-\frac{y}{k_+-1})}{\alpha+1}\end{aligned}$$

where the interchange of sum and integral can be justified by analytical continuation in $\mathbb{C}$. Similarly, one can compute the second integral of equation (A.7)

$$\int \frac{y^{\alpha-1}}{1+\frac{y}{k_+-1}}dy = -\frac{\alpha y^{\alpha+1}\,{}_2F_1(1,\alpha+1,\alpha+2;-\frac{y}{k_+-1})}{(\alpha+1)(k_+-1)} + \frac{y^\alpha}{\alpha}.$$

Finally, we obtain

$$\int \frac{(y-1)y^{\alpha-1}}{1+\frac{y}{k_+-1}}dy = \frac{k_+ y^{\alpha+1}\,{}_2F_1(1,\alpha+1,\alpha+2;-\frac{y}{k_+-1})}{(\alpha+1)(k_+-1)} - \frac{y^\alpha}{\alpha}.$$

By defining $k_- = \frac{iu}{\theta\lambda_-}$ and by changing variable $y = 1+k_-x$, we rewrite the second integral of equation (A.4) as

$$\int_0^{1-e^{-\theta t}} \frac{x(\lambda_- + iu\theta^{-1}x)^{\alpha-1}}{1-x}dx = \frac{1}{\lambda_-^{\alpha-1}k_-(k_-+1)} \int_1^{1+k_-(1-e^{-\theta t})} \frac{(y-1)y^{\alpha-1}}{1-\frac{y}{k_-+1}}dy, \tag{A.8}$$

and by considering a similar argument to the previous calculation, we obtain

$$\int \frac{(y-1)y^{\alpha-1}}{1-\frac{y}{k_-+1}}dy = \frac{k_- y^{\alpha+1}\,{}_2F_1(1,\alpha+1,\alpha+2;\frac{y}{k_-+1})}{(\alpha+1)(k_-+1)} - \frac{y^\alpha}{\alpha}.$$

Thus, equation (A.5) becomes

$$\begin{aligned}&iu\theta^{-1}C\Gamma(1-\alpha)\bigg(\int_0^{1-e^{-\theta t}} \frac{x(\lambda_+ + iu\theta^{-1}x)^{\alpha-1}}{1-x}dx - \int_0^{1-e^{-\theta t}} \frac{x(\lambda_- - iu\theta^{-1}x)^{\alpha-1}}{1-x}dx\bigg) \\ &= iu\theta^{-1}C\Gamma(1-\alpha)\bigg(\frac{1}{\lambda_+^{\alpha-1}k_+(k_+-1)}\bigg[\frac{k_+ y^{\alpha+1}\,{}_2F_1(1,\alpha+1,\alpha+2;-\frac{y}{k_+-1})}{(\alpha+1)(k_+-1)} - \frac{y^\alpha}{\alpha}\bigg]_1^{1-k_+(1-e^{-\theta t})} \\ &\quad - \frac{1}{\lambda_-^{\alpha-1}k_-(k_-+1)}\bigg[\frac{k_- y^{\alpha+1}\,{}_2F_1(1,\alpha+1,\alpha+2;\frac{y}{k_-+1})}{(\alpha+1)(k_-+1)} - \frac{y^\alpha}{\alpha}\bigg]_1^{1+k_-(1-e^{-\theta t})}\bigg).\end{aligned} \tag{A.9}$$

The integral (A.1) can be computed as follows

$$
\begin{aligned}
&\int_0^t \theta \vartheta_{Z_{CTS}}(iu\theta^{-1}(1-e^{-\theta(t-s)}))ds \\
&= -iu\theta^{-1}\Bigg((m - C\Gamma(1-\alpha)(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}))(1 - e^{-\theta t} - \theta t) + C\Gamma(1-\alpha)\Bigg(\\
&\quad \frac{1}{\lambda_+^{\alpha-1} k_+(k_+-1)}\Bigg(\frac{k_+\big(1-k_+(1-e^{-\theta t})\big)^{\alpha+1}\,{}_2F_1\big(1,\alpha+1,\alpha+2;1+\frac{k_+e^{-\theta t}}{1-k_+}\big)}{(\alpha+1)(k_+-1)} \\
&\quad - \frac{\big(1-k_+\big(1-e^{-\theta t}\big)\big)^{\alpha}}{\alpha} - \frac{k_+\,{}_2F_1\big(1,\alpha+1,\alpha+2;\frac{1}{1-k_+}\big)}{(\alpha+1)(k_+-1)} + \frac{1}{\alpha}\Bigg) \\
&\quad - \frac{1}{\lambda_-^{\alpha-1} k_+(k_-+1)}\Bigg(\frac{k_-\big(1+k_-(1-e^{-\theta t})\big)^{\alpha+1}\,{}_2F_1\big(1,\alpha+1,\alpha+2;1-\frac{k_-e^{-\theta t}}{1+k_-}\big)}{(\alpha+1)(k_-+1)} \\
&\quad - \frac{\big(1+k_-\big(1-e^{-\theta t}\big)\big)^{\alpha}}{\alpha} - \frac{k_-\,{}_2F_1\big(1,\alpha+1,\alpha+2;\frac{1}{1+k_-}\big)}{(\alpha+1)(k_-+1)} + \frac{1}{\alpha}\Bigg)\Bigg)\Bigg),
\end{aligned}
$$

with $k_+ = \frac{iu}{\theta\lambda_+}$ and $k_- = \frac{iu}{\theta\lambda_-}$. □

A.2 Proof of Theorem 5.1

Proof. In order to evaluate equation (2.14) in the VG case, we compute the integral

$$\int_0^{1-e^{-\theta t}} \frac{\vartheta_{Z_{VG}}(iu\theta^{-1}x)}{(1-x)}dx. \tag{A.10}$$

Therefore, we have the equality

$$
\begin{aligned}
\int_0^{1-e^{-\theta t}} \Big(&\frac{iu\theta^{-1}x(m - C(\lambda_+ - \lambda_-)\lambda_+^{-1}\lambda_-^{-1})}{(1-x)} \\
&+ \quad \frac{Ciu\theta^{-1}x}{(\lambda_+ - iu\theta^{-1}x)(1-x)} - \frac{Ciu\theta^{-1}x}{(\lambda_- + iu\theta^{-1}x)(1-x)} - \Big)dx.
\end{aligned} \tag{A.11}
$$

The first part of the integral (A.11), similar to the integral (A.3), can be easily evaluated

$$
\begin{aligned}
&\int_0^{1-e^{-\theta t}} \frac{iu\theta^{-1}x(m - C(\lambda_+ - \lambda_-)\lambda_+^{-1}\lambda_-^{-1})}{(1-x)}dx \\
= \quad &- iu\theta^{-1}(m - C(\lambda_+ - \lambda_-)\lambda_+^{-1}\lambda_-^{-1})(1 - e^{-\theta t} - \theta t).
\end{aligned} \tag{A.12}
$$

By setting $k = iu\theta^{-1}$, the second part of the integral (A.11) can be rewritten as

$$\begin{aligned}
&\int_0^{1-e^{-\theta t}} \Big(\frac{Ckx}{(\lambda_+ - kx)(1-x)} - \frac{Ckx}{(\lambda_- + kx)(1-x)}\Big)dx \\
= \;& Ck \int_0^{1-e^{-\theta t}} \Big(- \frac{\lambda_+}{(\lambda_+ - k)(\lambda_+ - kx)} + \frac{1}{(\lambda_+ - k)(1-x)} \\
&\qquad + \frac{\lambda_-}{(\lambda_- + k)(\lambda_- + kx)} - \frac{1}{(\lambda_- + k)(1-x)}\Big)dx \\
= \;& Ck\Big[\frac{\lambda_+}{k(\lambda_+ - k)}\log(|\lambda_+ - kx|) + \frac{\lambda_-}{k(\lambda_- + k)}\log(|\lambda_- + kx|) \\
&\quad - \Big(\frac{1}{\lambda_+ - k} - \frac{1}{\lambda_- + k}\Big)\log(|1-x|)\Big]_0^{1-e^{-\theta t}} \\
= \;& C\Big[\frac{\lambda_+}{\lambda_+ - k}\log(|\lambda_+ - k(1-e^{-\theta t})|) + \frac{\lambda_-}{\lambda_- + k}\log(|\lambda_- + k(1-e^{-\theta t})|) \\
&\quad + \Big(\frac{1}{\lambda_+ - k} - \frac{1}{\lambda_- + k}\Big)k\theta t - \frac{\lambda_+ \log \lambda_+}{\lambda_+ - k} - \frac{\lambda_- \log \lambda_-}{\lambda_- + k}\Big].
\end{aligned} \tag{A.13}$$

Thus, the integral has the following solution

$$\begin{aligned}
&\int_0^{1-e^{-\theta t}} \Big(\frac{Ckx}{(\lambda_+ - kx)(1-x)} - \frac{Ckx}{(\lambda_- + kx)(1-x)}\Big)dx \\
= \;& \frac{C}{\lambda_+ - iu\theta^{-1}}\Big(\lambda_+ \log\Big(\frac{|\lambda_+ - iu\theta^{-1}(1-e^{-\theta t})|}{\lambda_+}\Big) + iut\Big) \\
&+ \frac{C}{\lambda_- + iu\theta^{-1}}\Big(\lambda_- \log\Big(\frac{|\lambda_- + iu\theta^{-1}(1-e^{-\theta t})|}{\lambda_-}\Big) - iut\Big) \\
= \;& \frac{\theta C}{iu - \theta\lambda_+}\Big(\lambda_+ \log\Big(\frac{\lambda_+}{|\lambda_+ - iu\theta^{-1}(1-e^{-\theta t})|}\Big) - iut\Big) \\
&- \frac{\theta C}{iu + \theta\lambda_-}\Big(\lambda_- \log\Big(\frac{\lambda_-}{|\lambda_- + iu\theta^{-1}(1-e^{-\theta t})|}\Big) + iut\Big).
\end{aligned}$$

The integral (A.10) can be computed as follows

$$\begin{aligned}
\int_0^{1-e^{-\theta t}} \frac{\vartheta_{Z_{VG}}(iu\theta^{-1}x)}{(1-x)}dx = \;& -iu\theta^{-1}(m - C(\lambda_+ - \lambda_-)\lambda_+^{-1}\lambda_-^{-1})(1 - e^{-\theta t} - \theta t) \\
&+ \frac{\theta C}{iu - \theta\lambda_+}\Big(\lambda_+ \log\Big(\frac{\lambda_+}{|\lambda_+ - iu\theta^{-1}(1-e^{-\theta t})|}\Big) - iut\Big) \\
&- \frac{\theta C}{iu + \theta\lambda_-}\Big(\lambda_- \log\Big(\frac{\lambda_-}{|\lambda_- + iu\theta^{-1}(1-e^{-\theta t})|}\Big) + iut\Big).
\end{aligned}$$

□

A.3 Proof of Theorem 6.1

Proof. In order to evaluate equation (2.14) in the RDTS case, we compute the integral

$$\int_0^{1-e^{-\theta t}} \frac{\vartheta_{Z_{RDTS}}(iu\theta^{-1}x)}{(1-x)}dx. \tag{A.14}$$

Therefore, we have

$$\int_0^{1-e^{-\theta t}} \Big(\frac{iu\theta^{-1}xm}{(1-x)} - \frac{Ciu\theta^{-1}xG'(iu\theta^{-1}x;\alpha,\lambda_+)}{(1-x)} + \frac{Ciu\theta^{-1}xG'(-iu\theta^{-1}x;\alpha,\lambda_-)}{(1-x)} \Big) dx. \tag{A.15}$$

In order to evaluate the integral in equation (A.15), we compute the following integral

$$\begin{aligned}
\int_0^{1-e^{-\theta t}} \frac{xG'(x;\alpha,\lambda)}{1-x} dx &= \int_0^{1-e^{-\theta t}} \sum_{n=2}^{\infty} \frac{1}{n!2n} \frac{x^n}{1-x} \left(\frac{\lambda}{\sqrt{2}}\right)^{\alpha-n} \Gamma\left(\frac{n-\alpha}{2}\right) dx \\
&= \sum_{n=2}^{\infty} \frac{1}{n!2n} \left(\frac{\lambda}{\sqrt{2}}\right)^{\alpha-n} \Gamma\left(\frac{n-\alpha}{2}\right) \int_0^{1-e^{-\theta t}} \frac{x^n}{1-x} dx \\
&= \sum_{n=2}^{\infty} \frac{1}{n!2n} \left(\frac{\lambda}{\sqrt{2}}\right)^{\alpha-n} \Gamma\left(\frac{n-\alpha}{2}\right) B_{1-e^{-\theta t}}(n+1,0)
\end{aligned}$$

where $B_z(a,b)$ is the incomplete beta function.

The integral (A.14) can be computed as follows

$$\begin{aligned}
\int_0^{1-e^{-\theta t}} \frac{\vartheta_{Z_{RDTS}}(iu\theta^{-1}x)}{(1-x)} dx &= -iu\theta^{-1}m(1-e^{-\theta t}-\theta t) \\
&\quad - C\sum_{n=2}^{\infty} \frac{(iu)^n}{\theta^n n!2n} \left(\frac{\lambda_+}{\sqrt{2}}\right)^{\alpha-n} \Gamma\left(\frac{n-\alpha}{2}\right) B_{1-e^{-\theta t}}(n+1,0) \\
&\quad + C\sum_{n=2}^{\infty} \frac{(-iu)^n}{\theta^n n!2n} \left(\frac{\lambda_-}{\sqrt{2}}\right)^{\alpha-n} \Gamma\left(\frac{n-\alpha}{2}\right) B_{1-e^{-\theta t}}(n+1,0).
\end{aligned}$$

□

A.4 The FFT approach

In theory, given the characteristic function $\Phi(u)$ of a law X, density and cumulative distribution functions can be derived via the Fourier inversion formula, that is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \Phi(u) du \tag{A.16}$$

or

$$F(x+\frac{h}{2}) - F(x-\frac{h}{2}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \Phi(u) \frac{2}{u} \sin\frac{hu}{2} du, \tag{A.17}$$

where $h > 0$. Unfortunately, it is not always possible to find a closed-form solution for equations (A.16) and (A.17) and one has to then employ numerical integration of the inversion formulas. The algorithm described here draws from Rachev and Mittnik (2000) and Stoyanov and Racheva-Iotova (2004) (see Scherer et al. (2012) for further details on this topic).

To compute the density function for a large number of x values, the FFT algorithm can be efficiently employed, particularly when the likelihood function has to be computed and inserted into an optimization routine to perform a MLE. It is well known that the FFT approach is computationally efficient with only the disadvantage that the density function has to be evaluated on an equally spaced grid. As a consequence, one has to interpolate for intermediate values and when the arguments are outside the grid (this occurs rarely in practical applications).

The main idea is to calculate the integral in (A.16) for the following grid of equally spaced x values

$$x_k = (k - 1 - \frac{N}{2})h,$$

where $k = 1, ..., N$. Then equation (A.16) can be rewritten as

$$f(x_k) = \int_{-\infty}^{\infty} e^{-i2\pi\omega(k-1-\frac{N}{2})h}\varphi(2\pi\omega)d\omega. \tag{A.18}$$

Since the integral in (A.18) is convergent, we can choose a large enough upper and a small enough lower bound to approximately compute (A.18) through the Riemann sum, that is

$$f(x_k) \approx s\sum_{n=1}^{N} \varphi\Big(2\pi s\Big(n - 1 - \frac{N}{2}\Big)\Big)e^{-i2\pi(k-1-\frac{N}{2})(n-1-\frac{N}{2})sh}. \tag{A.19}$$

In this particular example, the upper and the lower bounds equal $\frac{sN}{2}$ and $-\frac{sN}{2}$, respectively. The integrand is evaluated for the equally spaced grid $n - 1 - \frac{N}{2}$, $n = 1, ..., N$ with distance s between them. The choice of the integral bounds is not arbitrary. If $s = (hN)^{-1}$, we have the following expression for the density

$$f(x_k) \approx \frac{1}{hN}\sum_{n=1}^{N} \varphi\Big(2\pi\frac{1}{hN}\Big(n - 1 - \frac{N}{2}\Big)\Big)e^{-i2\pi(k-1-\frac{N}{2})(n-1-\frac{N}{2})\frac{1}{N}}.$$

Having rearranged the terms in the exponent, the following approximation holds

$$f(x_k) \approx \frac{(-1)^{k-1+\frac{N}{2}}}{hN}\sum_{n=1}^{N}(-1)^{n-1}\varphi\Big(\frac{2\pi}{hN}\Big(n - 1 - \frac{N}{2}\Big)\Big)e^{\frac{-i2\pi(n-1)(k-1)}{N}}. \tag{A.20}$$

The discrete FFT is a numerical method developed for calculation of sequences such as $f(x_k)$ in (A.20) given the sequence

$$(-1)^{n-1}\varphi\Big(\frac{2\pi}{hN}\Big(n - 1 - \frac{N}{2}\Big)\Big). \tag{A.21}$$

So by applying the discrete Fourier transform for the sequence (A.21) we approximately compute the density values $f(x_k)$. The benefit of this approach is that the FFT algorithm needs $N \log_2 N$ arithmetical operations. In comparison, the direct computation of the integral needs N^2. Obviously, the FFT approach reduces the computational burden enormously when N is a large number. It should be noted that the approximation error has three different sources: (1) the interchange

of the infinite integral bounds in (A.18) with finite ones; (2) the approximation of (A.18) with the Riemann sum (A.19), and; (3) the interpolation for intermediate values if the function argument is not a grid node.

The parameters of the FFT method N (the number of summands in the Riemann sum) and h (the grid spacing) should be carefully chosen since there is a trade-off between accuracy and computational burden. From the construction of (A.19), it follows that, to reduce the approximation error, N should be as large as possible. A peculiarity of the numerical method is that efficiency is gained if N is expressed as 2^q. In addition the length of the integration region in terms of the original variable u is $2\pi/h$. Hence h should be as small as possible to increase accuracy.

Additionally, by following the approach of Scherer et al. (2012), in both the CTS and RDTS cases, one can always approximate the density function in the standardized case (with zero mean and unit variance) with parameter (α, λ_+, λ_-) and then estimate the more general case with parameter (C, λ_+, λ_-, α, m) by properly scaling and shifting.

Because of the symmetry of the grid nodes, further efficiency can be achieved in the computations of (A.21) if we use the following relationship valid for any characteristic function

$$\Phi(-u) = \overline{\Phi(u)} \tag{A.22}$$

where $\overline{z}$ means the complex conjugate of z. Due to (A.22), it is possible to compute the $\Phi(u)$ only for positive grid nodes, then change the sign of their imaginary parts and achieve the characteristic function values for negative grid nodes. This approach, is particularly useful in the approximation of the RDTS and the RDTS-OU density, since in these cases a special function has to be evaluated.

Concerning the cumulative distribution function, there are two potential approaches, that is

1. it is possible to use (A.17) and derive an expression similar to (A.20);

2. one can work directly with the evaluated density according to (A.20) and:

$$F(x) = \int_{-\infty}^{x} f(u)du = h \sum_{\{k:u_k \leq x\}} f(u_k)$$

In our numerical experiment we consider the second approach.

In Table 3 we report the error and the computational burden of evaluating the density of one-sided CTS, RDTS, CTS-OU, and RDTS-OU[15] according to the FFT method with different choices of q and h and various parameter sets. The procedure was run on an 8 cores AMD FX processor with 16GB of RAM with a Linux based 64-bit operating system. The computational time necessary to interpolate intermediate values is negligible with respect to the computational time for the discrete FFT. The error is defined by

$$e^{FFT}(h, q; X(\Theta)) = \max_{x \in [x_{min}, x_{max}]} |f^{FFT}(x, h, q; X(\Theta)) - f^{exact}(x)| \tag{A.23}$$

[15] In the OU case we evaluate the density of the random variable $Z^*(\Delta)$.

where $N = 2^q$ is the number of summands in the Riemann sum, h is the grid spacing, X is a one-dimensional law with parameters Θ (that includes Δ), and f^{exact} represents the exact valuation of the density function of the law X. Since in all cases of interest, the density f cannot be computed exactly, we approximate it as follows

$$f^{exact}(x) \approx f^{FFT}(x, h^*, q^*; X(\Theta)$$

where $\log_{10} h^* = -5$ and $q^* = 25$ (in the algorithm considered in this study the greatest possible number of integration steps depends on the maximum available RAM). As expected, the optimal balance between accuracy and efficiency (i.e., the choice of the parameters h and q) depends on the model and the parameter choice. Specifically, one has to be careful about the choice of h and q for small values of α. Finally, we note that in the case of standardized laws, it is possible to improve the efficiency and the quality of the approximation by following the approach proposed in Section IV in Scherer et al. (2012).

Relative error												
CTS – inverse transfrom												
observations	$C=1$	$\lambda_+=1$	$\alpha=0.4$		$C=1$	$\lambda_+=1$	$\alpha=0.8$		$C=1$	$\lambda_+=1$	$\alpha=1.2$	
500	44.83	70.38	20.49		16.93	44.22	2.70		88.44	86.85	17.30	
1,000	36.45	52.66	17.48		12.19	31.70	1.91		77.57	66.31	16.67	
2,000	29.94	43.78	15.40		8.93	20.55	1.34		57.68	51.37	13.09	
5,000	25.95	36.93	13.59		5.78	12.96	0.91		31.48	31.08	7.80	
10,000	23.51	32.37	12.82		3.56	8.44	0.58		20.70	20.50	5.26	
CTS – acceptance-rejection												
observations	$C=1$	$\lambda_+=1$	$\alpha=0.4$		$C=1$	$\lambda_+=1$	$\alpha=0.8$		$C=1$	$\lambda_+=1$	$\alpha=1.2$	
500	21.08	39.65	8.87		18.52	45.48	2.75		89.75	90.14	17.98	
1,000	14.58	27.60	6.75		12.53	28.61	1.86		77.91	70.80	16.67	
2,000	9.85	17.49	4.82		8.33	19.30	1.32		60.45	51.80	13.16	
5,000	6.14	10.80	2.99		5.14	12.32	0.81		33.78	31.22	8.16	
10,000	4.34	7.82	2.10		3.86	8.52	0.61		21.20	21.70	5.57	
observations	CTS-OU – acceptance-rejection											
	$C=1$	$\lambda_+=1$	$\alpha=0.4$	$\theta=0.5$	$C=1$	$\lambda_+=1$	$\alpha=0.6$	$\theta=0.5$	$C=1$	$\lambda_+=1$	$\alpha=0.8$	$\theta=0.5$
500	23.41	90.53	6.23	0.04	26.06	62.22	5.95	0.92	33.87	69.05	5.49	8.17
1,000	16.78	51.15	5.07	0.03	17.56	41.31	4.07	0.61	22.99	48.91	3.87	5.93
2,000	12.90	31.10	3.66	0.02	11.87	27.64	2.91	0.43	17.04	33.30	2.86	4.49
5,000	9.84	16.55	2.97	0.02	7.95	16.43	1.90	0.27	12.06	19.98	2.11	3.32
10,000	8.32	11.19	2.56	0.01	5.47	11.34	1.29	0.21	10.00	15.03	1.74	2.85
50,000	5.84	6.75	1.92	0.01	2.69	5.25	0.70	0.11	7.28	8.63	1.32	2.24

Table 1: Relative errors $e_{rel}^{1-\delta}$ with $\delta = 0.1$ for different sample sizes per estimate (percentage values). We analyze CTS subordinators and CTS-OU processes driven by a CTS subordinator. Two simulation methods are considered: the inverse transform method and the acceptance-rejection method. In the CTS-OU case, the acceptance-rejection method is used to generate the CTS component.

Relative error												
RDTS – inverse transfrom												
observations	$C=1$	$\lambda_+=\sqrt{2}$	$\alpha=0.8$		$C=1$	$\lambda_+=\sqrt{2}$	$\alpha=1.2$		$C=1$	$\lambda_+=\sqrt{2}$	$\alpha=1.6$	
500	14.21	34.24	2.36		7.54	38.94	2.19		15.32	49.05	5.08	
1,000	9.98	25.74	1.72		5.98	32.22	1.72		11.29	37.25	4.23	
2,000	7.91	19.85	1.32		4.35	25.67	1.31		9.01	30.11	3.13	
5,000	5.64	15.10	0.93		3.17	19.02	0.99		6.88	22.24	2.45	
10,000	4.52	12.52	0.71		2.46	16.71	0.82		5.59	19.59	2.03	
observations	RDTS-OU – inverse transform											
	$C=1$	$\lambda_+=\sqrt{2}$	$\alpha=0.8$	$\theta=0.5$	$C=1$	$\lambda_+=\sqrt{2}$	$\alpha=1.2$	$\theta=0.5$	$C=1$	$\lambda_+=\sqrt{2}$	$\alpha=1.6$	$\theta=0.5$
500	21.23	30.71	3.54	5.76	5.02	32.12	1.75	5.63	18.65	42.55	4.45	7.23
1,000	19.15	20.60	3.59	7.64	4.75	20.80	1.33	7.54	13.50	27.92	3.08	7.22
2,000	22.71	12.84	4.50	8.20	4.65	14.02	1.02	7.81	9.25	18.58	2.02	8.23
5,000	27.03	9.48	5.11	8.34	5.09	9.28	0.92	8.56	7.48	11.57	1.52	8.17
10,000	22.12	6.71	4.21	8.24	5.03	7.20	0.88	8.39	5.29	9.02	1.23	7.92
50,000	13.32	3.71	2.61	5.30	3.14	3.96	0.51	4.99	3.12	5.21	0.69	4.97
observations	VG-OU – direct simulation											
	$C=10$	$\lambda_+=50$	$\lambda_-=100$	$\theta=0.5$	$C=20$	$\lambda_+=100$	$\lambda_-=200$	$\theta=0.5$	$C=40$	$\lambda_+=200$	$\lambda_-=400$	$\theta=0.5$
500	39.13	94.70	101.66	0.00	27.99	53.93	52.20	0.00	19.01	30.65	35.50	0.01
1,000	28.36	49.72	57.05	0.00	19.44	32.20	33.79	0.00	13.97	21.80	20.91	0.01
2,000	19.64	31.93	34.01	0.00	15.26	20.94	20.81	0.00	9.90	15.03	14.87	0.01
5,000	12.51	18.61	20.37	0.00	8.50	13.97	11.72	0.00	6.06	8.89	9.65	0.01
10,000	8.59	12.61	14.28	0.00	5.90	8.54	9.36	0.00	4.64	5.81	6.13	0.01
50,000	3.71	4.83	4.98	0.00	2.62	3.78	3.56	0.00	1.98	2.88	2.90	0.00

Table 2: Relative errors $e_{rel}^{1-\delta}$ with $\delta = 0.1$ for different sample sizes per estimate (percentage values). We analyze RDTS subordinators, RDTS-OU processes driven by a RDTS subordinator, and VG-OU processes. In both the RDTS and RDTS-OU cases, the inverse transform method is used to generate RDTS random numbers.

FFT density error													
$\log_{10}(h)$	-2	-2	-2	-2	-3	-3	-3	-3	-4	-4	-4	-4	
q	15	16	17	18	15	16	17	18	15	16	17	18	
	CTS												time*
$\alpha = 0.4$	7.4059	7.4059	7.4059	7.4059	0.8309	0.8309	0.8309	0.8309	1.1826	0.0061	0.0017	0.0017	28.09
$\alpha = 0.6$	0.1550	0.1550	0.1550	0.1550	0.0003	0.0003	0.0003	0.0003	1.6849	0.0059	0.0000	0.0000	29.17
$\alpha = 0.8$	0.0045	0.0045	0.0045	0.0045	0.0000	0.0000	0.0000	0.0000	4.4982	0.0090	0.0000	0.0000	28.71
$\alpha = 1.2$	0.0002	0.0002	0.0002	0.0002	0.0000	0.0000	0.0000	0.0000	1.4963	0.0034	0.0000	0.0000	29.91
$\alpha = 1.4$	0.0001	0.0001	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	2.0824	0.0034	0.0000	0.0000	41.38
$\alpha = 1.6$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	4.2677	0.0040	0.0000	0.0000	44.46
average time	0.14	0.16	0.24	0.32	0.13	0.16	0.24	0.33	0.14	0.18	0.25	0.36	
	RDTS												time*
$\alpha = 0.4$	6.8260	6.8260	6.8260	6.8260	0.7669	0.7669	0.7669	0.7669	0.1031	0.0280	0.0072	0.0018	49.24
$\alpha = 0.6$	0.1358	0.1358	0.1358	0.1358	0.0017	0.0005	0.0004	0.0004	0.1587	0.0419	0.0106	0.0027	46.41
$\alpha = 0.8$	0.0022	0.0021	0.0021	0.0021	0.0062	0.0015	0.0003	0.0001	0.5939	0.1537	0.0387	0.0097	46.31
$\alpha = 1.2$	0.0003	0.0003	0.0003	0.0003	0.0163	0.0040	0.0009	0.0001	1.3570	0.3929	0.1018	0.0256	47.31
$\alpha = 1.4$	0.0001	0.0001	0.0001	0.0001	0.0069	0.0017	0.0004	0.0000	0.6123	0.1687	0.0432	0.0108	49.18
$\alpha = 1.6$	0.0000	0.0000	0.0000	0.0001	0.0053	0.0013	0.0003	0.0000	0.4521	0.1274	0.0328	0.0082	49.26
average time	0.20	0.26	0.39	0.59	0.18	0.23	0.32	0.46	0.19	0.24	0.32	0.47	
	CTS-OU												time*
$\alpha = 0.4$	7.16e+2	7.16e+2	7.16e+2	7.16e+2	5.17e+2	5.17e+2	5.17e+2	5.17e+2	2.75e+7	1.64e+5	1.57e+2	1.57e+2	30.49
$\alpha = 0.6$	9.5240	9.5240	9.5240	9.5240	0.3044	0.3044	0.3044	0.3044	0.7084	0.0038	0.0009	0.0009	35.19
$\alpha = 0.8$	0.1247	0.1247	0.1247	0.1247	0.0007	0.0007	0.0007	0.0007	1.3043	0.0048	0.0000	0.0000	34.96
$\alpha = 1.2$	0.0009	0.0009	0.0009	0.0009	0.0000	0.0000	0.0000	0.0000	0.4638	0.0017	0.0000	0.0000	34.98
$\alpha = 1.4$	0.0002	0.0002	0.0002	0.0002	0.0000	0.0000	0.0000	0.0000	0.8477	0.0021	0.0000	0.0000	37.76
$\alpha = 1.6$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	5.4823	0.0024	0.0000	0.0000	39.41
average time	0.17	0.21	0.28	0.39	0.17	0.21	0.27	0.39	0.18	0.22	0.30	0.43	
	RDTS-OU												time*
$\alpha = 0.4$	7.16e+2	7.16e+2	7.16e+2	7.16e+2	5.17e+2	5.17e+2	5.17e+2	5.17e+2	1.57e+2	1.57e+2	1.57e+2	1.57e+2	74.53
$\alpha = 0.6$	9.5240	9.5240	9.5240	9.5240	0.3033	0.3033	0.3033	0.3033	0.0720	0.0224	0.0060	0.0015	76.29
$\alpha = 0.8$	0.1234	0.1234	0.1234	0.1234	0.0024	0.0014	0.0012	0.0011	0.1703	0.0428	0.0107	0.0027	76.93
$\alpha = 1.2$	0.0010	0.0010	0.0009	0.0009	0.0094	0.0023	0.0005	0.0001	0.8703	0.2322	0.0589	0.0147	76.35
$\alpha = 1.4$	0.0001	0.0001	0.0001	0.0001	0.0047	0.0012	0.0003	0.0000	0.4416	0.1174	0.0298	0.0074	79.09
$\alpha = 1.6$	0.0000	0.0000	0.0000	0.0000	0.0041	0.0010	0.0002	0.0000	0.3644	0.1001	0.0256	0.0064	79.00
average time	0.22	0.32	0.50	0.86	0.20	0.27	0.39	0.64	0.21	0.27	0.39	0.62	

Table 3: Maximum absolute difference $e^{FFT}(h, q; X(\Theta))$ between the density evaluated via the FFT method and the exact density (with $\log_{10} h^* = -5$ and $q^* = 25$). We consider the interval $[x_{min}, x_{max}]$ = [-5,5] with $\Delta x = 10^{-5}$ for a total of 1 million values. In all cases $\Delta = 0.1$, the parameter $C = 1$, and the parameter $\lambda_+ = 1$ in the CTS and CTS-OU cases, and $\lambda_+ = \sqrt{2}$ in the RDTS and RDTS-OU cases. The time is expressed in seconds.

Figure 1: Boxplots of the estimates of $\hat{C}$, $\hat{\lambda}_+$, $\hat{\lambda}_-$, $\hat{\theta}$ for VG-OU processes. Each boxplot consists of $1,000$ estimates. Three parameters sets (C, λ_+,λ_+, θ) are considered: 1 – (10, 50, 100. 0.5), 2 – (20, 100, 200. 0.5), and 2 – (40, 200, 400. 0.5), with $\Delta = 1/250$.

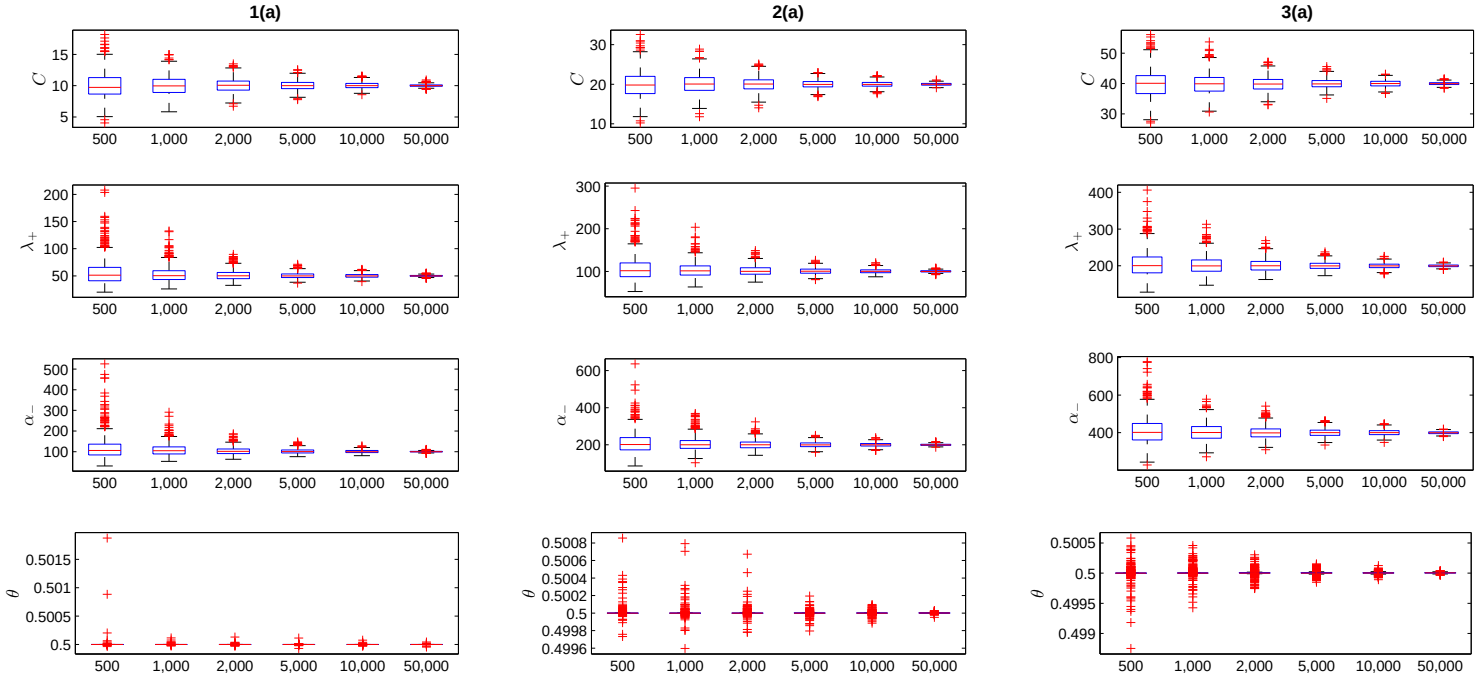


Figure 2: Boxplots of the estimates of $\hat{C}$, $\hat{\lambda}_+$, $\hat{\alpha}$, $\hat{\theta}$ for CTS-OU processes driven by a CTS subordinator. Each boxplot consists of $1,000$ estimates. Three parameters sets (C, λ_+,α) are considered: 1 – (1, 1, 0.4), 2 – (1, 1, 0.6), and 3 – (1, 1, 0.8), with $\Delta = 0.1$. The acceptance-rejection method is used to generate the CTS component.

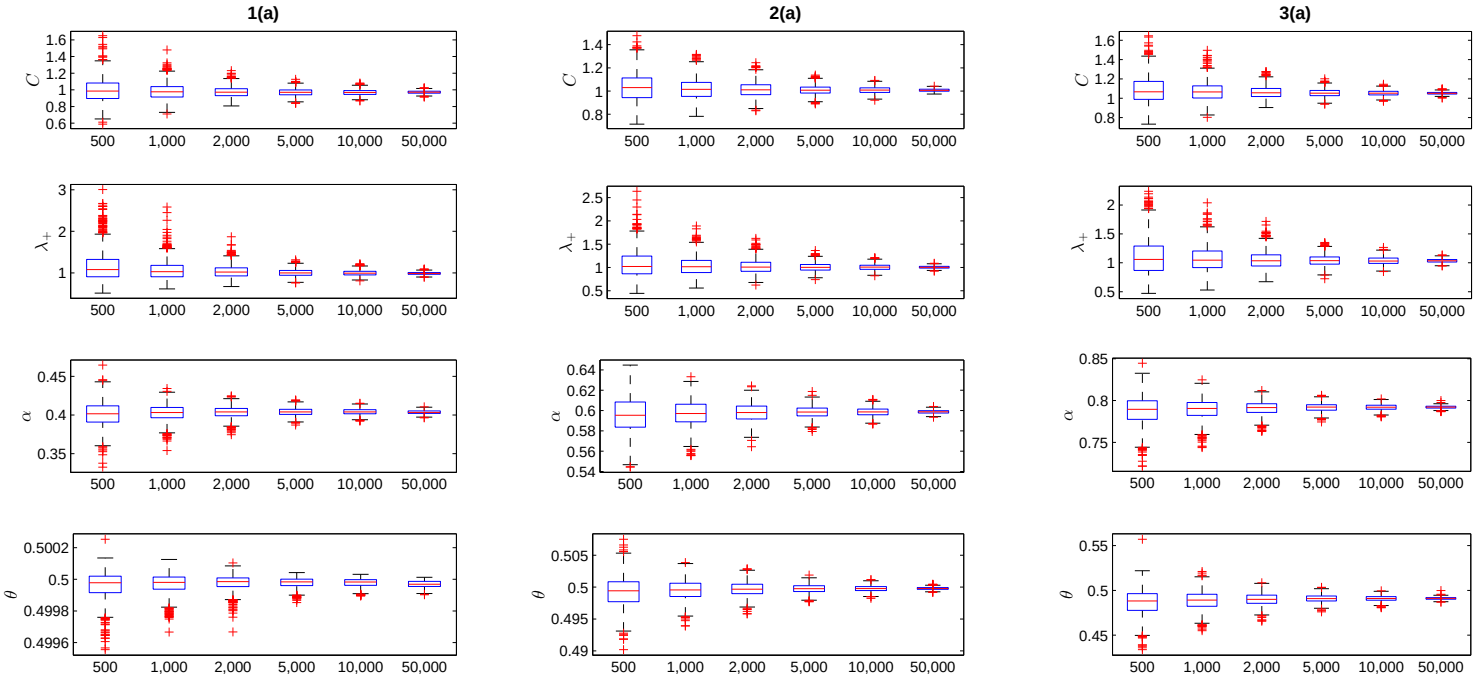


Figure 3: Boxplots of the estimates of $\hat{C}$, $\hat{\lambda}_+$, $\hat{\alpha}$, $\hat{\theta}$ for the RDTS-OU processes driven by a one-side RDTS process. Each boxplot consists of $1,000$ estimates. Three parameters sets (C, λ_+,α) are considered: 1 – (1, $\sqrt{2}$, 0.8), 2 – (1, $\sqrt{2}$, 1.2), and 3 – (1, $\sqrt{2}$, 1.6), with $\Delta = 0.1$. The inverse transform method is used to generate the RDTS component.

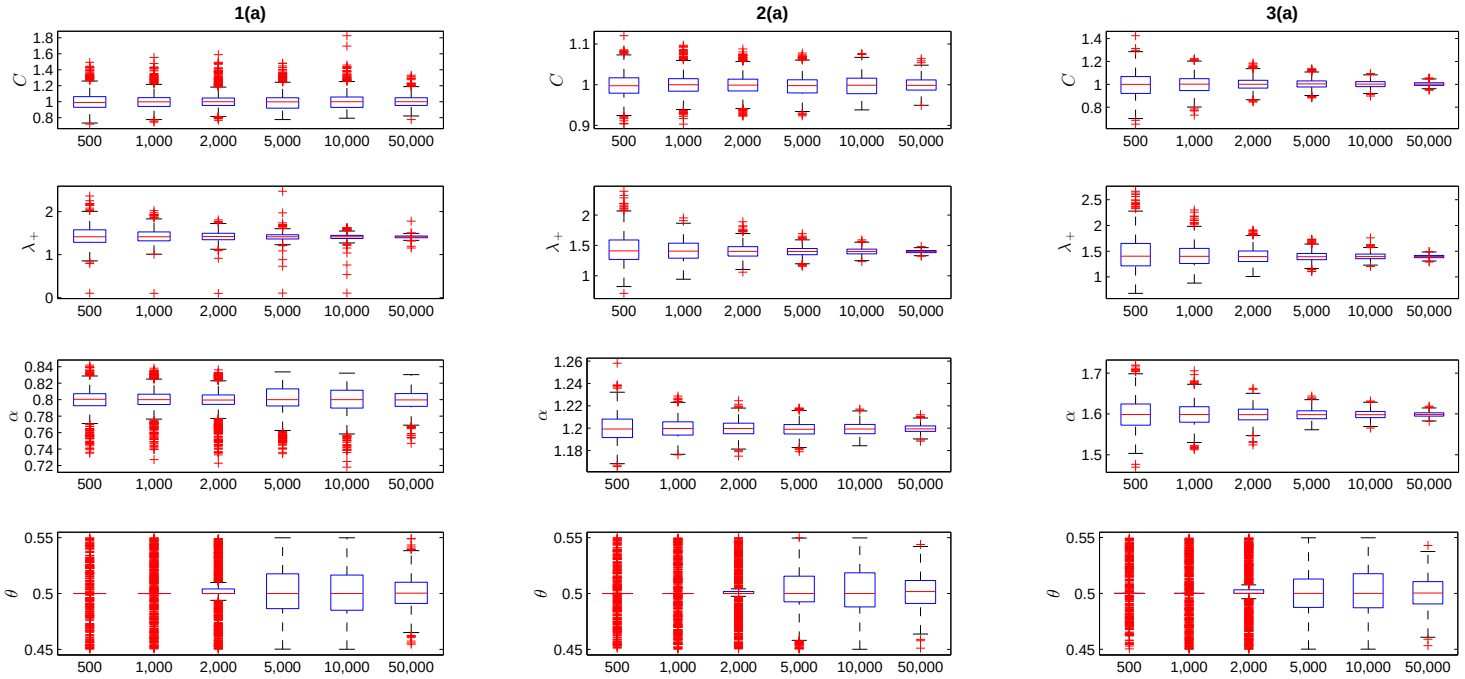


Figure 4: Boxplots of the estimates of $\hat{C}$, $\hat{\lambda}_+$, and $\hat{\alpha}$ for the one-side CTS law. Each boxplot consists of $1,000$ estimates. Three parameters sets (C, λ_+,α) are considered: 1 – (1, 1, 0.4), 2 – (1, 1, 0.8), and 3 – (1, 1, 1.2), with $\Delta = 0.1$. Two simulation methods are used: (a) the inverse transform method and (b) the acceptance-rejection method.

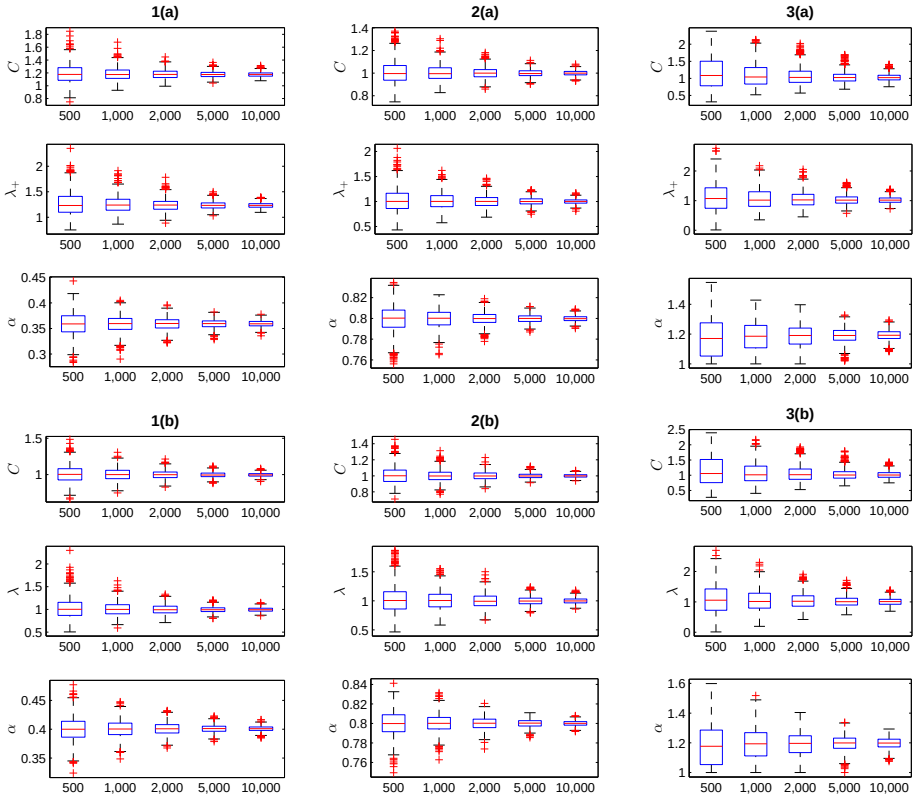
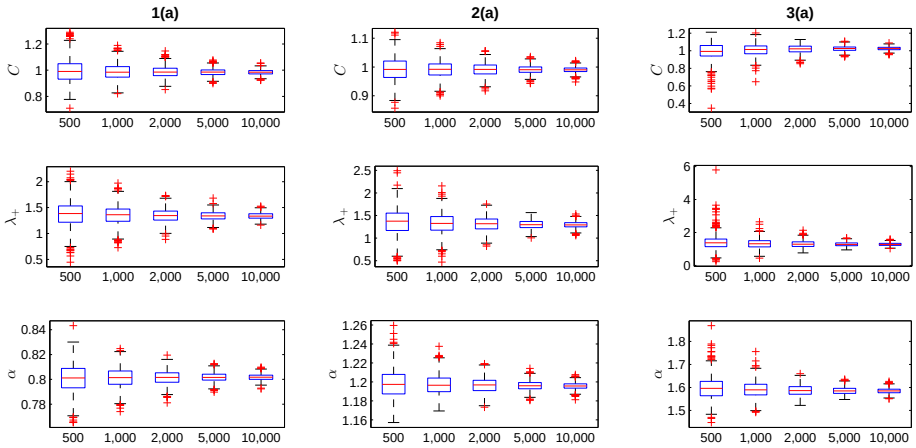


Figure 5: Boxplots of the estimates of $\hat{C}$, $\hat{\lambda}_+$, and $\hat{\alpha}$ for the one-side RDTS law. Each boxplot consists of $1,000$ estimates. Three parameters sets (C, λ_+,α) are considered: 1 – (1, $\sqrt{2}$, 0.8), 2 – (1, $\sqrt{2}$, 1.2), and 3 – (1, $\sqrt{2}$, 1.6), with $\Delta = 0.1$. The inverse transform method is used to generate the random samples.



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