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A NOTE ON RATIONALIZABILITY AND RESTRICTIONS ON BELIEF

by Giuseppe Cappelletti*

Abstract

Rationalizability is a widely accepted solution concept in the study of strategic form game with complete information and is fully characterized in terms of assumptions on the rationality of the players and common certainty of rationality. Battigalli and Siniscalchi extend rationalizability and derive the solution concept called $\Delta$-rationalizability. Their analysis is based on the following assumptions: (a) players are rational; (b) their first-order beliefs satisfy some restrictions; and (c) there is common belief of (a) and (b). In this note I focus on games with complete information and I characterize $\Delta$-rationalizability with a new notion of iterative dominance which is able to capture the additional hypothesis on players' beliefs.

JEL Classification: C72.
Keywords: rationalizability, strategic form game, complete information.

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1 Introduction

The solution concept called rationalizability, which was introduced by Bernheim (1984) and Pearce (1984), is widely accepted in the study of strategic form game with complete information and is fully characterized in term of rationality and common belief in rationality. There are settings where it is plausible to assume that players’ beliefs satisfy some restrictions that are not implied by assumptions concerning rationality or belief in rationality, or beliefs about such beliefs. Such restrictions may be related to some structural properties of the situation analyzed. For example, in a bargaining situation players can believe that their opponents have some preference for fair division, in an auction bidders can expect positive bids to win with positive probability (Battigalli and Siniscalchi, 2003b) or in a communication game players can believe that their opponents trust their messages (Crawford, 2003). Based on this observation, Battigalli and Siniscalchi (1999) investigate the implications of the following assumptions: (a) players are rational; (b) their first-order beliefs satisfy an exogenous restriction; and (c) there is common belief of (a) and (b). Their analysis extends rationalizability taking as given some exogenous restrictions on players’ belief and derives the solution concept called $\Delta - rationalizability$, which does not hinge on any assumption on equilibrium or correctness of beliefs. They apply this new solution concept to games with incomplete information (Battigalli, 2003; Battigalli and Siniscalchi, 2003a; Battigalli et al., 2008) and dynamic games (Battigalli, 1997, 2003; Battigalli and Siniscalchi, 2002, 2007).

Despite its great potential $\Delta - rationalizability$ has not received as many applications as has its unconstrained counterpart. One reason for this lack of attention is that many practitioners find $\Delta - rationalizability$ difficult to operationalize. In fact, it requires the iterative deletion of strategies that cannot be justified by beliefs consistent with progressively higher degrees of strategic sophistication. This procedure could be analytically cumbersome and numerically intractable.

A connection between $\Delta - rationalizability$ and dominance would be valuable on both practical and conceptual levels. I introduce a new dominance concept, called $\Delta - dominance$, and prove that, under appropriate conditions, rationalizability with exogenous restrictions on players’ belief and iterated $\Delta - dominance$ are equivalent in strategic form games with complete information. This extends the classical iterated dominance characterization of rationalizability and simplifies computation of the $\Delta - rationalizable$ in this type of games.

2 Rationalizability and restrictions on belief in strategic form games of complete information

To simplify the analysis I focus on strategic form game of complete information, a model of interactive decision-making in which each agent chooses his strategies once and for all, and these choices are made simultaneously. The model is a structure:

$$G = \langle N, \{S_i, u_i\}_{i \in N} \rangle$$

where for each player $i$, belonging to the set $N = \{1, 2, \ldots, n\}$, $S_i$ is a finite set of possible strategies. The payoff function $u_i$ is defined on the Cartesian product of players’ possible strategies. The views expressed in the article are those of the authors and do not involve the responsibility of the Bank of Italy. I would like to thank Pierpaolo Battigalli, Sandro Brusco, Giorgio Gobbi and two anonymous referees for their helpful comments and suggestions. All remaining errors are my own.
strategies, \( \prod_{i \in N} S_i \), and it assumes real values. The set of mixed strategies of player \( i \) is denoted as \( \Sigma_i \) and it coincides with the set of all probability measures defined on \( S_i \), a generic element of \( \Sigma_i \) is denoted as \( \sigma_i \). In order to shorten the notation I denote the opponents of player \( i \) with \(-i\).

Player \( i \)'s first-order beliefs are represented by a probability measure on the set of his opponents’ strategies, i.e. a generic first-order belief, \( \mu_i \), belongs to the set of probability measures with support contained in \( S_{-i} := \prod_{j \neq i} S_j \), the set of first-order beliefs is denoted as \( \Delta (S_{-i}) \). Players’ belief may be assumed to satisfy some restrictions which are justified or related to some structural properties of the game. Let me denote with \( \Delta_i \) any subset of \( \Delta (S_{-i}) \) and with \( \Delta \) the Cartesian product of all the players’ restrictions, \( \Delta := \Delta_1 \times \Delta_2 \times \ldots \times \Delta_n \).

Given a belief \( \mu_i \) and an action \( s_i \) let

\[
    u_i (s_i, \mu_i) := \sum_{s_{-i} \in S_{-i}} u_i (s_i, s_{-i}) \mu_i (s_{-i})
\]

be the expected utility for player \( i \) from playing \( s_i \) based on his belief \( \mu_i \).

**Definition 1** A strategy \( s_i \) is rational for player \( i \) with respect to \( \mu_i \) if and only if for every \( s'_i \in S_i \) the following inequality is satisfied:

\[
    u_i (s_i, \mu_i) \geq u_i (s'_i, \mu_i)
\]

A strategy \( s_i \) is rational for player \( i \) if the strategy \( s_i \) maximizes his expected utility, that is, \( s_i \) is a best response to the belief \( \mu_i \). I denote with \( \rho_i (\mu_i) \) the set of best responses to \( \mu_i \). In other words, a strategy \( s \) is rational for a player with first-order beliefs \( \mu \), when it is justifiable based on belief \( \mu \).

If the set of admissible beliefs is exogenously constrained the set of rational strategies is smaller and this leads to the definition of \( \Delta - \) rationalizability. A strategy profile \( (s_i, s_{-i}) \) is \( \Delta - \) rationalizable if and only if for each player \( i \) the strategy \( s_i \) belongs to \( S_i (k, \Delta) \) for any natural number \( k \), where \( S_i (k, \Delta) \) is defined as follow: for \( k \) equal to 0 the set \( S_i (0, \Delta) \) is equal to \( S_i \) and for every natural number \( k \) strictly greater than 0

\[
    S_i (k, \Delta) := \{ s_i \in S_i (k-1, \Delta) : \exists \mu_i \in \Delta_i \text{ such that } s_i \in \rho_i (\mu_i) \text{ and } \mu_i (S_{-i} (k-1, \Delta)) = 1 \}
\]

where \( S_{-i} (k-1, \Delta) \) is defined as \( \prod_{j \neq i} S_j (k-1, \Delta) \).

**Definition 2** Given a strategic form game \( G = \langle N, \{ S_i, u_i \}_{i \in N} \rangle \) and a set of restrictions on players’ beliefs \( \Delta \), the strategy \( s_i \) is \( (k, \Delta) - \) rationalizable if and only if \( s_i \) belongs to \( S_i (k, \Delta) \). The strategy \( s_i \) is \( \Delta - \) rationalizable if and only if \( s_i \) belongs to \( S_i (\infty, \Delta) \) where \( S_i (\infty, \Delta) := \bigcap_{k \geq 1} S_i (k, \Delta) \).

Let me denote the set of strategy profiles that are \( (k, \Delta) - \) rationalizable as \( S (k, \Delta) \) where \( S (k, \Delta) \) is equal to the Cartesian product of \( S_i (k, \Delta) \), that is \( S (k, \Delta) := \prod_{i \in N} S_i (k, \Delta) \).
2.1 \(\Delta\)-rationalizability and dominance

A strategy \(s_i\) is strictly dominated for player \(i\) by a mixed strategy \(\sigma_i\) on a subset of his opponents’ strategies \(B_{-i} \subseteq S_{-i}\) if and only if for every strategy profile of his opponents \(s_{-i}\)

\[
u_i(s_i, s_{-i}) < \sum_{s_i' \in S_i} \sigma_i(s_i') u_i(s_i', s_{-i})
\]

(5)

where \(\sigma_i(s_i')\) is the probability assigned to strategy \(s_i'\) by mixed strategy \(\sigma_i\). For a given rectangular subset \(B\) of \(S\), let \(S(B)\) be the set of strategy profiles \((s_i, s_{-i})\) such that, for each player \(s_i\) is not strictly dominated on \(B_{-i}\) by any mixed strategy \(\sigma_i\) which assigns positive probability only to strategies belonging to \(B_i\).

According to the first Pearce’s lemma (Pearce, 1984), a strategy is strictly dominated if and only if it is not a best response to any conceivable belief. Therefore if \(\Delta_i = \Delta(S_{-i})\) for every \(i \in N\) the set of \(\Delta - \text{rationalizable}\) strategies coincides with the set of iteratively undominated strategies. If \(\Delta_i^1\) is a strict subset of \(\Delta(S_{-i})\) the set of \(\Delta - \text{rationalizable}\) and iteratively undominated strategies do not coincide. Hence, if I want to characterize the set of \(\Delta - \text{rationalizable}\) strategies in terms of being iteratively undominated I need to generalize the concept of dominance in order to take into account the exogenous restrictions on players’ beliefs. Let \(p(s_i, s_{-i}; \sigma_i)\) be the set of beliefs that justifies choosing \(s_i\) instead of \(\sigma_i\) given that \(i\)’s opponents choice \(s_{-i}\) has a positive probability of being played:

\[
p(s_i, s_{-i}; \sigma_i) = \{\mu \in \Delta(S_{-i}) : \mu(s_{-i}) > 0 \text{ and } u_i(s_i, \mu_i) \geq \mu_i(\sigma_i, \mu_i)\}
\]

(6)

Now, I can state a definition of dominance that includes restrictions on players’ beliefs.

**Definition 3** A strategy \(s_i\) is strictly \(\Delta -\text{dominated}\) by \(\sigma_i\) on \(B_{-i} \subseteq S_{-i}\) if and only if for every \(s_{-i}\) belonging to \(B_{-i}\) either \(u_i(s_i, s_{-i}) < u_i(\sigma_i, s_{-i})\), or \(u_i(s_i, s_{-i}) \geq u_i(\sigma_i, s_{-i})\) implies \(p(s_i, s_{-i}; \sigma_i) \cap \{\mu_i \in \Delta_i : \mu_i(B_{-i}) = 1\} = \emptyset\).

This definition differs from the definition of dominance because a strategy \(s_i\) could be justified by some belief \(\mu_i\) but this belief is not admitted given the restrictions on players’ beliefs. As a result the set of strictly \(\Delta -\text{dominated}\) strategies is larger than the set of strictly dominated strategies. Suppose that for some \(s_{-i}\) it holds that \(u_i(s_i, s_{-i}) > u_i(\sigma_i, s_{-i})\) then it may be the case that \(p(s_i, s_{-i}; \sigma_i) \cap \{\mu_i \in \Delta_i : \mu_i(B_{-i}) = 1\} = \emptyset\). Take a strategic form game \(G\) with two players labelled 1 and 2. Player 1 has two possible strategies \(\{u, d\}\) and player 2 has two possible strategies \(\{L, R\}\). The payoffs of player 1 are summarized in the following table.

<table>
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<tr>
<td>(d)</td>
<td>1</td>
<td>2</td>
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If player 1 is certain that player 2 will choose action \(R\) then \(u_i(u, L) > u_i(d, L)\) but \(p(u, L; d) \cap \{\mu_i \in \Delta_1 : \mu_1(\{L, R\}) = 1\} = \emptyset\).²

**Lemma 1** Given a pure strategy \(s_i\), a mixed strategy \(\sigma_i\) and a subset \(B_{-i}\) of \(S_{-i}\) the following conditions are equivalent:

²In fact \(p(u, L; d) = \{\mu_i \in \Delta(\{L, R\}) : \mu_1(L) > \frac{1}{2}\} = \{\mu_i : \mu_i(R) = 1\}\).
1. \( s_i \) is strictly \( \Delta \)-dominated by \( \sigma_i \) on \( B_{-i} \);

2. for every \( \mu_i \in \{ \mu_i \in \Delta_i : \mu_i (B_{-i}) = 1 \} \) the expected utility associated to strategy \( s_i \) is strictly less then the one associated to \( \sigma_i \) (that is \( u_i (s_i, \mu_i) < u_i (\sigma_i, \mu_i) \)).

**Proof.** The proof is by contradiction.

(1) \( \Rightarrow \) (2) Suppose that \( s_i \) is strictly \( \Delta \)-dominated by \( \sigma_i \) on \( B_{-i} \) and assume that there exists \( \mu_i \) belonging to \( \{ \mu_i \in \Delta_i : \mu_i (B_{-i}) = 1 \} \) such that \( u_i (s_i, \mu_i) \geq u_i (\sigma_i, \mu_i) \), then there must be a strategy profile \( s_{-i} \) belonging to \( B_{-i} \) such that \( u_i (s_i, s_{-i}) \geq u_i (\sigma_i, s_{-i}) \) and \( \mu (s_{-i}) > 0 \). This contradicts the initial assumption that \( s_i \) is \( \Delta \)-dominated by \( \sigma_i \).

(2) \( \Rightarrow \) (1) Assume that \( u_i (s_i, \mu_i) < u_i (\sigma_i, \mu_i) \) for every \( \mu_i \in \{ \mu_i \in \Delta_i : \mu_i (B_{-i}) = 1 \} \) and that there exists a strategy profile \( s_{-i} \) such that \( u_i (s_i, s_{-i}) \geq u_i (\sigma_i, s_{-i}) \) and \( G (s_i, s_{-i}; \sigma_i) \cap \{ \mu_i \in \Delta_i : \mu_i (B_{-i}) = 1 \} \neq \emptyset \). Then there exists \( \mu_i \) belonging to \( \{ \mu_i \in \Delta_i : \mu_i (B_{-i}) = 1 \} \) such that \( u_i (s_i, \mu_i) \geq u_i (\sigma_i, \mu_i) \), this contradicts assumption (2). \( \blacksquare \)

**Definition 4** A strategy \( s_i \) is not strictly \( \Delta \)-dominated on \( B \) if and only if for every mixed strategy \( \sigma_i \) with support included in \( B_i \) there exists \( s_{-i} \in B_{-i} \) such that \( u_i (s_i, s_{-i}) \geq u_i (\sigma_i, s_{-i}) \) and \( p (s_i, s_{-i}; \sigma_i) \cap \Delta_i \neq \emptyset \).

This definition differs from the traditional one because it requires the existence of an acceptable belief that justifies \( s_i \) with respect to the any candidate alternative strategy \( \sigma_i \). That is, a strategy \( s_i \in S_i \) is not strictly \( \Delta \)-dominated by any mixed strategy for player \( i \) if and only if for every \( \sigma_i \) there exists \( \mu_i \in \Delta_i \) such that \( u_i (s_i, \mu_i) \geq u_i (\sigma_i, \mu_i) \). For a given rectangular subset \( B \subseteq S \), let \( S (B, \Delta) \) denote the set of strategy profiles \( (s_i, s_{-i}) \in S \) such that, for each \( i \), \( s_i \) is \( \Delta \)-undominated on \( B \).

### 2.2 Main result

It is possible to generalize the first Pearce’s lemma characterizing \( \Delta \)-rationalizability in terms of iterative elimination of strictly \( \Delta \)-dominated strategies. First, I need a preliminary result which relates strict \( \Delta \)-dominance and best responses with respect to a set of admissible beliefs. Let the set of all the players’ restrictions, \( \Delta_i \), be closed and convex if all its components, \( \Delta_i \), are closed and convex subsets of \( \Delta (S_{-i}) \).

**Lemma 2** Let \( G = \langle N, \{ B_i, u_i \}_{i \in N} \rangle \) be a strategic form game and \( \Delta \) is a closed and convex set of restrictions on belief, a strategy \( s_i \) is not strictly \( \Delta \)-dominated on \( B \) if and only if there exists \( \mu_i \) belonging to to the set of admissible beliefs \( \{ \mu \in \Delta_i : \mu (B_{-i}) = 1 \} \) and such that \( s_i \in \rho_i (\mu_i) \).

**Proof.** First I prove by contradiction that being \( \Delta \)-undominated implies being justifiable by some admissible belief. Assume that \( s_i \) is not strictly \( \Delta \)-dominated and there is no belief \( \mu_i \in \Delta_i \) such that \( s_i \in \rho_i (\mu_i) \). Then, the following system of inequalities has no solution in \( \{ \mu \in \Delta_i : \mu (B_{-i}) = 1 \} \):

\[
\sum_{s_{-i} \in S_{-i}} \mu_i (s_{-i}) \left[ u_i (s_i', s_{-i}) - u_i (s_i, s_{-i}) \right] \leq 0 \quad \text{for every } s_i' \neq s_i \in S_i \tag{7}
\]

\( ^3 \)Note that \( \Delta_i \cap \{ \mu \in \Delta (S_{-i}) : \mu (B_{-i}) = 1 \} = \{ \mu \in \Delta_i : \mu (B_{-i}) = 1 \} \).

This implies that if \( \{ \mu \in \Delta_i : \mu (B_{-i}) = 1 \} \) is the intersection of two convex sets then it is convex.
We have a collection of closed proper convex (linear) functions on \( \mathbb{R}^{S_i} \) indexed by \( s_i' \), that is for each \( s_i' \) in \( S_i \) we have a linear function \( \mu_i \rightarrow \sum_{s_{i-1} \in S_{i-1}} \mu_i (s_{i-1}) [u_i (s_i', s_{i-1}) - u_i (s_{i-1})] \).

\( \Delta_i \) is a non-empty closed, convex set in \( \mathbb{R}^{S_i} \) and since \( \Delta_i \) is bounded it has no direction of recession\(^4\) (see Rockafellar, 1996). Hence, the linear functions \( \mu_i \rightarrow \sum_{s_{i-1} \in S_{i-1}} \mu_i (s_{i-1}) [u_i (s_i', s_{i-1}) - u_i (s_{i-1})] \) have no common direction of recession which is also direction of recession of \( \Delta_i \). Based on these considerations, I can apply Theorem 21.3 in Rockafellar (1996) which states that if system (7) has no solution then there exists a non-negative real vector \( \lambda \), belonging to \( \mathbb{R}^{|S_i| - 1} \), and \( \varepsilon > 0 \) such that

\[
\sum_{s_i' \in S_i \setminus \{s_i\}} \sum_{s_{i-1} \in S_{i-1}} \lambda (s_i') \left[ u_i (s_i', s_{i-1}) - u_i (s_{i-1}) \right] \mu_i (s_{i-1}) \geq \varepsilon
\]

for every \( \mu_i \in \Delta_i \). Therefore,

\[
\sum_{s_i' \in S_i \setminus \{s_i\}} \sum_{s_{i-1} \in S_{i-1}} \frac{\lambda (s_i')}{\lambda (s_i)} \left[ u_i (s_i', s_{i-1}) - u_i (s_{i-1}) \right] \mu_i (s_{i-1}) \geq \frac{\varepsilon}{\sum_{s_i' \in S_i \setminus \{s_i\}} \lambda (s_i')}
\]

or equivalently,

\[
\sum_{s_{i-1} \in S_{i-1}} \left[ u_i (\sigma_i, s_{i-1}) - u_i (s_i, s_{i-1}) \right] \mu_i (s_{i-1}) \geq \varepsilon'
\]

where \( \varepsilon' := \frac{\varepsilon}{\sum_{s_i' \in S_i \setminus \{s_i\}} \lambda (s_i')} \geq 0 \) and \( \sigma_i \) is a mixed strategy assigning to each strategy \( s_i' \in S_i \setminus \{s_i\} \) probability equal to \( \frac{\lambda (s_i')}{\sum_{s_i' \in S_i \setminus \{s_i\}} \lambda (s_i')} \). Inequality (9) states that \( \sigma_i \) is strictly better than \( s_i \) for every conjecture \( \mu_i \in \Delta_i \), contradicting the initial assumption that \( s_i \) is not strictly \( \Delta - \text{dominated} \) (see Lemma 1).

In order to prove the opposite it is sufficient to notice that if a \( \mu_i \in \Delta_i \) exists such that \( s_i \in \rho (\mu_i) \) then \( s_i \) is not strictly \( \Delta - \text{dominated} \) by definition.

In order to relate \( \Delta - \text{rationalizability} \) and \( \Delta - \text{dominance} \) I have to consider that the set of feasible strategy and the set of admissible beliefs changes along the iterative procedure that define \( \Delta - \text{rationalizability} \). For an arbitrary natural number \( k \) the set of admissible strategy is \( S(k - 1, \Delta) \) and the set of relevant restrictions on beliefs is the projection of \( \Delta \) on \( S(k - 1, \Delta) \), therefore the set of not strictly \( \Delta - \text{dominated} \) strategies has to be computed on \( S(k - 1, \Delta) \) taking as relevant restrictions \( \Delta^k \) defined as the Cartesian product of \( \Delta^k_i := \{ \mu \in \Delta_i : \mu(S_{i-1}(k - 1, \Delta)) = 1 \} \).

**Lemma 3** Let \( G = \langle N, \{S_i, u_i\}_{i \in N} \rangle \) be a strategic form game and \( \Delta \) is a closed and convex set of restrictions on belief, for every natural number \( k \geq 1 \), the set of \( (k, \Delta) - \text{rationalizable} \) strategy coincides with the set of not strictly \( \Delta - \text{dominated} \) strategies on \( B^k \), \( S(k, \Delta) = S(B^k, \Delta^k) \), where \( B^k := S(k - 1, \Delta) \) and the set of restrictions is \( \Delta^k = \prod_{i \in N} \Delta^k_i \) with \( \Delta^k_i := \{ \mu \in \Delta_i : \mu(S_{i-1}(k - 1, \Delta)) = 1 \} \).

\(^4\)Let \( \Delta \) be a non-empty convex set in \( \mathbb{R}^n \). \( \Delta \) recedes in the direction \( d \) if and only if \( \Delta \) includes all the half-lines in the direction \( d \) which start at points of \( \Delta \). In other words, \( \Delta \) recedes in the direction \( d \), where \( d \neq 0 \), if and only if \( x + \lambda d \in \Delta \) for every \( \lambda \geq 0 \) and \( x \in \Delta \).
Proof. Lemma (2) implies that the set of \((1, \Delta)\)-rationalizable strategy profiles is equal to the set of not strictly \(\Delta\)-dominated strategy profiles, namely \(S(1, \Delta) = S(S, \Delta)\). For every natural number \(k\) let me consider the strategic form game 

\[ G^k := \left( N, \{ B^k_i, u_i \}_{i \in N} \right) \]

where the set of strategy for player \(i\) is defined as 

\[ B^k_i := S_i(B_{k-1}, \Delta) \]

The set of restrictions on beliefs for each player is the projection of the initial restriction on the set of \((k - 1, \Delta)\)-rationalizable strategy profiles, formally it is the set \(\Delta^k_i\) defined as 

\[ \Delta^k_i := \{ \mu \in \Delta_i : \mu(S_{-i}(k - 1, \Delta)) = 1 \} \]

Since \(\Delta^k_i\) is the intersection of two closed and convex sets, it is a closed and convex set. Given this observation, I can apply Lemma (2) and conclude that a strategy \(s_i\) is not strictly \(\Delta\)-dominated on \(B^k\) if and only if there exists \(\mu_i\) belonging to \(\{ \mu \in \Delta^k_i : \mu(B_{-i}) = 1 \}\) and such that 

\[ s_i \in \rho_i(\mu_i) \]

This means that \(S(k, \Delta) = S(B^k, \Delta^k)\). □

The previous lemma states that a strategy \(s_i\) is \((k, \Delta)\)-rationalizable if and only if it survives \(k\) step of iterative elimination of strictly \(\Delta\)-dominated strategies. As a result I have a full characterization of \(\Delta\)-rationalizability.

The following example shows that the requirement for \(\Delta\) to be convex is necessary in order for Lemma 2 to hold. Take a strategic form game \(G\) with two players labelled 1 and 2. Player 1 has three possible strategies \(\{u, m, d\}\) and player 2 has two possible strategies \(\{L, R\}\). The payoffs of Player 1 are summarized by the following table.

<table>
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<th>(R)</th>
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<tr>
<td>(u)</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>(m)</td>
<td>1.5</td>
<td>1.5</td>
</tr>
<tr>
<td>(d)</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

Suppose that player 1 has just two admissible beliefs about his opponent’s choice, labelled \(\mu'_1\) and \(\mu''_1\). In particular, he believes that player 2 chooses \(L\) either with probability \(\frac{4}{5}\) (\(\mu'_1\)) or with probability \(\frac{1}{5}\) (\(\mu''_1\)). Let me consider the set of \((1, \Delta)\)-rationalizable strategy for player 1, that is the set of all strategies that are the best response to one of the two acceptable beliefs that is the set of \((1, \Delta)\)-rationalizable strategies for player 1

\[ S_1(1, \Delta) = \bigcup_{\mu_1 \in \{ \mu'_1, \mu''_1 \}} \rho(\mu_1) = \{ u, d \} \]

Now, let me focus on the set of strategies that are not \(\Delta\)-dominated for player 1. If \(u\) is the best response to \(\mu'_1\) and \(d\) is the best response to \(\mu''_1\) then \(u\) and \(d\) are not \(\Delta\)-dominated. A mixed strategy \(\sigma_1\) \(\Delta\)-dominates strategy \(m\) if and only if it satisfies the following inequalities:

\[
\begin{align*}
\mu'_1(L) \left[ 2\sigma + (1 - \sigma - \lambda) \frac{3}{2} \right] + \mu'_1(R) \left[ 2\lambda + (1 - \sigma - \lambda) \frac{3}{2} \right] &> \frac{3}{2} & (10) \\
\mu''_1(L) \left[ 2\sigma + (1 - \sigma - \lambda) \frac{3}{2} \right] + \mu'_1(R) \left[ 2\lambda + (1 - \sigma - \lambda) \frac{3}{2} \right] &> \frac{3}{2} & (11)
\end{align*}
\]

where \(\sigma\) is the probability that the mixed strategy \(\sigma_1\) assigns to strategy \(u\) and \(\lambda\) is the probability that the mixed strategy \(\sigma_1\) assigns to strategy \(d\). Substituting the probability

\[ \sigma = \frac{4}{5}, \quad \lambda = \frac{1}{5} \]

Formally, I assume that the set of feasible beliefs has just two elements \(\mu'_1\) and \(\mu''_1\) which are such that 

\[ \mu'_1(L) = \frac{4}{5} \quad \text{and} \quad \mu''_1(L) = \frac{1}{5}. \]
assigned by the two acceptable beliefs:

\[ \sigma > 11\lambda \]  \hspace{1cm} (12) \\
\[ \lambda > 11\sigma \]  \hspace{1cm} (13)

which are mutually incompatible. Then, there is no mixed strategy that \( \Delta - \text{dominates} \) \( m \) and the set of \( \Delta - \text{undominated} \) strategies for Player 1, denoted as \( S_1(S, \Delta) \), is equal to \( \{u, m, d\} \) and is different from \( S_1(1, \Delta) \).

Figure 1: An example of the necessity of the convexity assumption

\[ u_1(s_1, L) \]
\[ U_1(s_1, \mu_1) \]
\[ u \]
\[ m \]
\[ d \]
\[ U_1(s_1, \mu'_1) \]
\[ u_1(s_1, R) \]

3 Conclusion

There are situations where it is plausible to assume that players’ beliefs satisfy some restrictions that are not implied by assumptions concerning rationality or belief in rationality, or beliefs about such beliefs. Based on this observation, Battigalli and Siniscalchi (2003a) introduce a new solution concept, called \( \Delta - \text{rationalizability} \), based on the assumptions that agents are rational, players’ beliefs satisfy some exogenous restrictions and there is common belief of the previous two hypothesis.

I characterize \( \Delta - \text{rationalizability} \) in term of iterated \( \Delta - \text{dominance} \), which generalizes the well-known relationship between rationalizability and iterated dominance in standard settings. \( \Delta - \text{dominance} \) differs from the traditional definition of dominance, because a strategy could be justified by some beliefs but this belief is not admissible given the assumed restrictions on players beliefs. This characterization simplifies the application of \( \Delta - \text{rationalizability} \) and broadens my understanding of this solution concept.

As a result, this research can facilitate the use of this kind of non-equilibrium analysis
that could shed new light on economic behavior. This characterization can offer some clarification of the concept of rationalizability for those interested in the foundation of game theory (Harsanyi, 1967; Mertens and Zamir, 1985; Brandenburger and Dekel, 1987; Bergemann and Morris, 2005, 2007; Ely and Peski, 2006). It would then be possible to generalize Lemma 2 to games with incomplete information (Battigalli and Siniscalchi, 2003; Dekel, Fudenberg, and Morris, 2005) and dynamic games (Shimoji and Watson, 1998; Battigalli, 2003).
References


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