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Maxmin portfolio choice

by Marco Taboga

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Abstract

We solve two robust portfolio selection problems, where a maxmin criterion is adopted to deal with parameter uncertainty. The two models, which yield closed formulae for the optimal allocation, lend themselves to be thoroughly analyzed both from a geometric and a game-theoretic point of view.

JEL classification: G11.

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Contents

*Bank of Italy, Economic Research Department.

1. Introduction[1]

Traditional static portfolio selection models, such as Markowitz's (1952), assume that the distribution of asset returns is objectively known to the decision maker. We extend Markowitz's model by relaxing this assumption and taking into account the possibility that the decision maker's beliefs about future asset returns cannot be summarized by a unique probability distribution, but there is a whole set of probability distributions which are deemed plausible. Markowitz's portfolio selection problem is:

$$\sup_{x \in \mathbb{R}^n} \mathrm{E}^Q [W(x)] - \frac{\gamma}{2} \mathrm{Var}^Q [W(x)]$$

where x is a vector of portfolio weights, W is stochastic future wealth (depending on the portfolio choice x), E^Q and Var^Q denote expectation and variance with respect to a probability measure Q and γ is a constant parametrizing risk aversion. A decision maker à la Markowitz chooses a portfolio in order to maximize a mean-variance functional and she is able to uniquely determine the mean and the variance of her future wealth, because her beliefs are represented by a unique proabablity measure Q. In our model, instead, there is not a unique distribution of asset returns Q, but a whole set of possible distributions Δ and, for each choice of the portfolio x, the investor evaluates the consequence of her choice under the worst possible scenario, i.e. under the probability measure which minimizes the value of the mean-variance functional. Thus, the investor solves the following maxmin problem:

$$\sup_{x \in \mathbb{R}^n} \inf_{Q \in \Delta} \mathrm{E}^Q [W(x)] - \frac{\gamma}{2} \mathrm{Var}^Q [W(x)]$$

The above problem is conceptually very simple. The only difficulty lies in the fact that it is not obvious how to specify the set Δ. We propose two different specifications, which allow to derive closed-form solutions of the problem. As we will later detail reviewing the related literature, our portfolio selection model is not the

[1] I wish to thank Erio Castagnoli, Fabio Maccheroni, Massimo Marinacci and two anonymous referees for helpful comments and suggestions. The views expressed in the article are those of the author and do not involve the responsibility of the Bank of Italy. E-mail: marco.taboga@bancaditalia.it .

first to adopt a maxmin rule. However, besides delivering easily interpretable closed formulae for the optimal portfolio under different specifications of Δ, we show that conclusions drawn by previous models are highly dependent on the assumptions made about Δ. In particular, existing models assume that Δ is a neighborhood of a reference model and that relative entropy measures the distance of alternative models from the reference model. In models based on relative entropy an "observational equivalence" property holds, whereby an increase in uncertainty is observationally equivalent to an increase in risk aversion. We show that observational equivalence ceseas to hold when measures of distance between models alternative to the relative entropy are used.

Parameter uncertainty has long been recognized as a problem in portfolio selection. Frankfurter, Phillips and Seagle (1971) are among the first researchers to stress the importance of parameter uncertainty in portfolio selection: they present the results of some simulations where Markowitz's portfolios are likely to be not more efficient than an equally weighted portfolio, because of errors in parameter estimation. A wealth of studies confirms their findings: among them Barry (1974), Bawa and Klein (1976) and Jobson and Korkie (1980). Also, a widespread opinion came about that portfolios constructed using sample moments of returns often involve very extreme positions, which are far from being truly optimal: Green and Hollifield (1992) provide a rigorous analysis of this claim. As a follow up to these critiques some proposals were made to improve upon Markowitz's (1952) model: for example, Frost and Savarino (1988) and Black and Litterman (1992) propose Bayesian procedures to improve the performance of estimated optimal portfolios. A very recent work by Jagannathan and Ma (2003) also gives a contribution to the debate: they discuss how imposing constraints on portfolio weights can sometimes improve the performance of estimated optimal portfolios. The main idea behind these studies is that the optimal portfolio can only be estimated by the investor, since the true distribution of asset returns is unknown, and straightforward substitution of estimated moments into Markowitz's formula for the optimal portfolio yields an estimator of the truly optimal portfolio, which, although consistent, has rather undesirable statistical properties. In sinthesis, what these models do is to devise better estimators of the optimal portfolio.

A different approach to robust portfolio selection is to incorporate parameter uncertainty directly into the financial decision problem, by specifying the investor's attitudes both towards investment risk and towards parameter uncertainty. A taxonomy of such models has been proposed by Uppal and Wang (2003), who identify two main classes of models.

A first class of models is characterized by the use of Gilboa and Schmeidler's (1989) multiple priors preferences. Roughly speaking, multiple priors preferences are based on the adoption of a minimax rule akin to the one we use in this paper: a decision maker with multiple priors evaluates acts by jointly determining a utility function and a set of probability measures and taking the infimum of the function over the set of probability measures. Epstein and Wang (1994), Dow and Werlang (1992) and Chen and Epstein (2002) develop portfolio selection models based on multiple priors preferences. Among them, only Chen and Epstein, whose model is set in continuous time, explicitly derive the optimal portfolio allocation.

A second class of models makes use of some tools borrowed from the robust control literature: Maenhout (2004) adapts a framework developed by Anderson, Hansen and Sargent (2003) to develop a continuous-time model where the decision maker has got a preference for robustness; Maenhout's (2004) model, which extends the classical Merton's (1990) model, assumes that the decision maker has got a reference probability measure over asset returns, but she considers also alternative probability measures, equivalent to the reference measure (in the probabilistic sense of equivalence), and she chooses among these measures according to a penalty function based on the relative entropy between the probability measures. Uppal and Wang (2003) extend Maenhout's model to take into account multiple sources of uncertainty and shed some light on real world phenomena such as underdiversification and the home bias. Both Maenhout and Uppal and Wang explicitly characterize the optimal trading strategies.

Among the models we have cited, none provides an asset allocation rule for the simplest case of a one-period horizon and a single investor. There are, however, some studies aimed at this direction: Krasker (1982), for example, analyzes the implications of minimax behavior for portfolio choice; in his model the agent minimizes over a set of probability measures obtained by ε-contamination of a reference measure; Krasker shows how some commonly held portfolios (portfolios without short positions or portfolios replicating the market portfolio) can be rationalized by individual minimax behavior, rather than on equilibrium considerations. Also Becker, Marty and Rustem (2000) analyze minimax portfolio strategies: the focus of their research is on computational issues and they limit their attention to the case where the investor is able to identify a finite number of scenarios, where by scenario they mean a possible combination of means and variances for the assets to be included in the portfolio.

This paper presents two static portfolio selection models, based on a maxmin criterion, which deliver easily interpretable portfolio allocation rules. The two models are similar: they both confine attention to mean-variance expected utility and to the case where the investor is uncertain about expected returns; the only difference between the two models is the specification of the set of probability measures over which the decision maker minimizes; we show how a slightly different specification of this set can lead to substantially different portfolio allocation rules. We try to improve our understanding of the two models by analyzing both their geometric properties, in line with a long standing tradition of analyzing Markowitz's-like optimal portfolios from a geometric standpoint, and by stressing the game-theoretic aspects of the minimax setting.

The first of our two models roughly confirms what has been found by Maenhout (2004) for the continuous-time case: the presence of parameter uncertainty (and aversion to it) is observationally equivalent to an increase in risk aversion; the investor behaves as if he corrected his initial estimate of the vector of expected excess returns by reducing all expected excess returns by the same proportion; as a consequence, the structure of the optimal portfolio is the same obtained in the classical Markowitz's framework, but everything is scaled down. This partially contradicts the literature questioning the validity of Markowitz's portfolios, since mean-variance efficient portfolios seem to be robust to the kind of misspecification introduced here.

The second model delivers an asset allocation rule which, to our knowledge, is completely novel. The investor's optimal behavior seems to be more pessimistic than in the first model: not only he reduces his initial estimate of excess expected returns, but the reduction is larger for those assets in which he has invested more heavily; he acts as though nature were so malignant to see to which assets the investor is more exposed and choose the worse scenarios exactly for those assets. In the second model, using a different measure of distance between probabilistic models (not based on relative entropy), observational equivalence between an increase in uncertainty and an increase in risk aversion no longer holds, which suggests that conclusions drawn by previous works might be rather specialized.

The paper is organized as follows: Section 2 describes the common features of the two portfolio selection models and some of their general properties. Section 3 describes the first model and Section 4 the second one. Section 5 contains some concluding remarks.

2. The framework

We consider the one-period allocation problem of an agent who has to decide how to invest a unit of wealth for one period, dividing it among $n+1$ assets. The gross return on the i-th asset after one period is a square integrable random variable denoted by R_i. The $(n \times 1)$ vector of the returns on the first n assets is denoted by R and the $(n \times 1)$ vector of portfolio weights (henceforth called a portfolio), indicating the fraction of wealth invested in each of the first n assets, is denoted by x. We assume the return on the $(n+1)$-th asset is riskfree, that is a.s. equal to a constant R_f.

The end-of-period wealth is denoted by $W(x)$, to highlight its dependence on the portfolio chosen by the investor, and is equal to:

$$W(x) = R_f + x' \left(R - \overrightarrow{1} R_f \right) \tag{1}$$

where $\overrightarrow{1}$ is a column vector of ones of dimension n. The above definition of $W(x)$ implicitly accomodates the requirement that the portfolio weights sum up to unity.

We assume that there are no frictions of any kind: securities are perfectly divisible; there are no transaction costs or taxes; the agent is a price-taker, in that she believes that her choices do not affect the distribution of asset returns; there are no institutional restrictions, so that the agent is allowed to buy, sell or short sell any desired amount of any security[2].

According to Markowitz's model, an investor with risk aversion γ chooses the portfolio weights x so as to solve:

$$\sup_{x\in\mathbb{R}^n} \mathrm{E}^Q\left[W(x)\right] - \frac{\gamma}{2}\mathrm{Var}^Q\left[W(x)\right]$$

where Q is the unique probability measure describing the investor's beliefs about future asset returns.

Instead, we assume that the investor solves the following problem:

$$\sup_{x\in\mathbb{R}^n} \inf_{Q\in\Delta} \mathrm{E}^Q\left[W(x)\right] - \frac{\gamma}{2}\mathrm{Var}^Q\left[W(x)\right]$$

where Δ is a set of probability measures. The investor's beliefs are not represented by a unique probability measure, but there are many such measures. A choice x is evaluated by considering the probability measure which minimizes the value of that choice, that is the decision maker evaluates her choices in the most conservative way, by choosing the worst-case scenario given her choice.

In what follows we will specify the set Δ in such a way that the following properties hold:

$$\begin{aligned} \forall Q,P &\in \Delta,\quad P\neq Q \Rightarrow E^P\left[R\right]\neq E^Q\left[R\right] \\ \forall Q &\in \Delta,\quad Var^Q\left[R\right]=\Sigma \end{aligned}$$

[2]This assumption can be weakened, by simply requiring that at an optimum institutional restrictions are not binding.

where Σ is a positive definite and symmetric $(n \times n)$ matrix. To avoid trivialities, we also assume that the eigenvalues of Σ are distinct and that $\mathrm{E}^Q[R] - \overrightarrow{1} R_f \neq 0$ for any $Q \in \Delta$. The meaning of the two properties above is straightforward: to two different probability measures in the set Δ are associated two different vectors of means, but the same covariance matrix.

A fundamental assumption of the mean-variance model is that the investor perfectly knows the vector of expected returns. Unfortunately, the assumption is hardly realistic, because, at best, the investor has got an imprecise estimate of it. Our model captures the investor's lack of confidence about her estimate of the vector of expected returns, by considering more than one value for it. As far as the covariance matrix Σ is concerned, we assume that there is no uncertainty about it. This is in line with the majority of the literature (see, for example, Chen and Epstein (2002) and Uppal and Wang(2003)) and has both practical and theoretical reasons. On the one hand, concentrating on the vector of means keeps the model tractable. On the other hand, it is often possible to sample the price processes generating the returns R at frequencies higher than the frequency considered for calculating the returns. If sampling at higher frequencies is possible and the price increments are serially independent, more precise estimates of Σ can be obtained without adding any precision to the estimate of μ; for a discussion of this point see for example Gourieroux and Jasiak (2001), Campbell and Viceira (2002) or Merton (1990).

Given the above assumptions, the portfolio problem becomes:

$$\sup_{x \in \mathbb{R}^n} \inf_{Q \in \Delta} R_f + x' \left(\mathrm{E}^Q[R] - \overrightarrow{1} R_f \right) - \frac{\gamma}{2} x' \Sigma x$$

We propose two different specifications of the set of probability measures Δ. In both cases, we take a reference measure P, under which R is normally distributed with mean μ and covariance matrix Σ and we specify the set Δ as a subset of the set

$$\Gamma = \left\{ Q(\delta) : \frac{dQ(\delta)}{dP} = \exp\left(\delta' \Sigma^{-1} (R - \mu) - \frac{1}{2} \delta' \Sigma^{-1} \delta \right), \delta \in \mathbb{R}^n \right\}$$

where by $\frac{dQ(\delta)}{dP}$ we have denoted the Radon-Nikodym derivative with respect to

P of the measure $Q(\delta)$ corresponding to a $\delta \in \mathbb{R}^n$. Γ contains all the probability measures $Q(\delta)$ which are equivalent to the reference measure P and such that under the measure $Q(\delta)$ the vector R of returns is multivariate normal with:

$$\begin{aligned} \mathrm{E}^{Q(\delta)}[R] &= \mu + \delta \\ \mathrm{Var}^{Q(\delta)}[R] &= \Sigma \end{aligned}$$

The first specification of Δ we propose is:

$$\Delta = \{Q(\delta) \in \Gamma : \mathcal{R}[Q(\delta) || P] \leq \varepsilon\}$$

where $\mathcal{R}[Q(\delta) || P]$ denotes the relative entropy of the measure $Q(\delta)$ with respect to the measure P and ε is a positive constant. The set Δ contains all those probability measures belonging to Γ whose relative entropy with respect to the reference measure does not exceed a prespecified constant. Calculating the relative entropy is a straightforward matter:

$$\mathcal{R}[Q(\delta) || P] = \mathrm{E}^{Q(\delta)}\left[\ln\left(\frac{dQ(\delta)}{dP}\right)\right] = \frac{1}{2}\delta'\Sigma^{-1}\delta$$

Taking as given this specification of the set Δ and noting that there is a one-to-one correspondence between Γ and $\mathbb{R}^n$, we can rewrite the portfolio selection problem as follows:

$$\sup_{x \in \mathbb{R}^n} \inf_{\delta \in \Delta_1} R_f + x'\left(\mu + \delta - \overrightarrow{1} R_f\right) - \frac{\gamma}{2} x'\Sigma x$$

where

$$\Delta_1 = \left\{\delta \in \mathbb{R}^n : \frac{1}{2}\delta'\Sigma^{-1}\delta \leq \varepsilon\right\}$$

Since Σ is positive definite, we can define an inner product on $\mathbb{R}^n$ as follows:

$$\langle \delta_1, \delta_2 \rangle_{\Sigma^{-1}} = \delta_1'\Sigma^{-1}\delta_2$$

and write:

$$\Delta_1 = \{\delta \in \mathbb{R}^n : ||\delta||_{\Sigma^{-1}} \leq \eta\}$$

where $\eta = \sqrt{2\varepsilon}$, so that it is clear that Δ_1 is just a closed ball in $\mathbb{R}^n$ centered about the zero vector: this interpretation will turn out to be useful in the ensuing discussion.

By analogy with the portfolio selection problem just outlined, we propose to analyze also the problem

$$\sup_{x\in\mathbb{R}^n}\inf_{\delta\in\Delta_2} R_f + x'\left(\mu + \delta - \overrightarrow{1}R_f\right) - \frac{\gamma}{2}x'\Sigma x$$

where

$$\Delta_2 = \{\delta \in \mathbb{R}^n : \|\delta\| \leq \eta\}$$

and $\|.\|$ is the usual Euclidean norm on $\mathbb{R}^n$.

Both problems have some features in common, which we discuss before analyzing them separately. They both are minimax problems: for a highly readable discussion of minimax problems we refer the reader to Rockafellar (1970). Define

$$V(x,\delta) = R_f + x'\left(\mu + \delta - \overrightarrow{1}R_f\right) - \frac{\gamma}{2}x'\Sigma x$$

A saddle point of $V(x,\delta)$ with respect to maximizing over $\mathbb{R}^n$ and minimizing over Δ_i ($\Delta_i = 1,2$) is a point $(x^*,\delta^*) \in \mathbb{R}^n \times \Delta_i$ satisfying:

$$V(x,\delta^*) \leq V(x^*,\delta^*) \leq V(x^*,\delta), \quad \forall x \in \mathbb{R}^n, \quad \forall \delta \in \Delta_i$$

A basic theorem from minimax theory (see Rockafellar (1996), Lemma 36.2) states that a saddle point has the property that:

$$\inf_{\delta\in\Delta_i}\sup_{x\in\mathbb{R}^n} V(x,\delta) = V(x^*,\delta^*) = \sup_{x\in\mathbb{R}^n}\inf_{\delta\in\Delta i} V(x,\delta)$$

As a consequence, if we are able to find a saddle point of $V(x,\delta)$, we have found a solution to the portfolio allocation problem, that is:

$$V(x^*,\delta^*) = \max_{x\in\mathbb{R}^n}\min_{\delta\in\Delta_i} R_f + x'\left(\mu + \delta - \overrightarrow{1}R_f\right) - \frac{\gamma}{2}x'\Sigma x$$

This characterization of a solution lends itself to a nice game-theoretic interpretation. We can think of our portfolio selection problem as a simultaneous two players game, where one player chooses the portfolio weights so as to maximize his own utility and the other player changes the vector of expected returns so as to make the first player as worse off as possible. A saddle point (x^*, δ^*) of the minimax problem is a Nash equilibrium of the game: the inequality

$$V\left(x, \delta^*\right) \leq V\left(x^*, \delta^*\right), \quad \forall x \in \mathbb{R}^n$$

means that, given player 2's decision to displace the vector of means by δ^*, player 1 cannot do any better by deviating from his choice x^*; likewise, the inequality

$$V\left(x^*, \delta^*\right) \leq V\left(x^*, \delta\right), \quad \forall \delta \in \Delta_i$$

implies that, given player 1's decision to choose the portfolio x^*, player 2 cannot do any better by deviating from his choice δ^*.

In the next two sections we will discuss the geometry of the two portfolio problems and their solutions: section 3 discusses the problem where minimisation is over Δ_1, which we dub "minimum entropy" minimax portfolio selection problem, while section 4 discusses the problem where minimization is over Δ_2, which we dub "minimum norm" minimax portfolio selection problem.

3. Minimum entropy portfolio selection

In this section we discuss how to solve the "minimum entropy" problem

$$\sup_{x \in \mathbb{R}^n} \inf_{\delta \in \Delta_1} R_f + x' \left(\mu + \delta - \overrightarrow{1} R_f\right) - \frac{\gamma}{2} x' \Sigma x \tag{2}$$

where

$$\Delta_1 = \{\delta \in \mathbb{R}^n : ||\delta||_{\Sigma^{-1}} \leq \eta\}$$

and

$$||\delta||_{\Sigma^{-1}} = \sqrt{\langle \delta, \delta \rangle_{\Sigma^{-1}}} = \sqrt{\delta' \Sigma^{-1} \delta}$$

In line with a long standing tradition of analyzing the geometric properties of portfolio selection models, we first present some facts about the geometry of our problem:

Theorem 1 *Let* (x^*, δ^*) *be a solution to (2), with* $x^* \neq 0$*, then* $x^* \in X^*$ *and* $\delta^* \in \Delta^*$ *where*

$$\begin{aligned} X^* &= \left\{ x \in \mathbb{R}^n : x = \lambda_1 \Sigma^{-1} \left(\mu + \delta^* - \overrightarrow{1} R_f \right), \lambda_1 \in \mathbb{R} \right\} \\ \Delta^* &= \{ \delta \in \Delta_1 : \delta = \lambda_2 \Sigma x^*, \lambda_2 \in \mathbb{R} \} \end{aligned}$$

Furthermore, for any $x \in \mathbb{R}^n$ *there exists* $\overline{x} \in X^*$ *such that:*

$$\begin{aligned} \overline{x}' \left(\mu + \delta^* - \overrightarrow{1} R_f \right) &= x' \left(\mu + \delta^* - \overrightarrow{1} R_f \right) \\ \overline{x}' \Sigma \overline{x} &\leq x' \Sigma x \end{aligned}$$

and for any $\delta \in \mathbb{R}^n$ *there exists* $\overline{\delta} \in \mathtt{span}\left(\Sigma x^*\right)$ *such that:*

$$\begin{aligned} x^{*\prime} \left(\mu + \overline{\delta} - \overrightarrow{1} R_f \right) &= x^{*\prime} \left(\mu + \delta - \overrightarrow{1} R_f \right) \\ \left\| \overline{\delta} \right\|_{\Sigma^{-1}} &\leq \left\| \delta \right\|_{\Sigma^{-1}} \end{aligned}$$

Given any displacement δ^* of the vector of expected returns, the optimal portfolio x^* lies in the one-dimensional subspace of $\mathbb{R}^n$ spanned by the vector $\Sigma^{-1} \left(\mu + \delta^* - \overrightarrow{1} R_f \right)$. Similarly, given any vector of portfolio weights x^*, the optimal displacement δ^* of the vector of expected returns belongs to the one-dimensional subspace of $\mathbb{R}^n$ spanned by the vector Σx^*. Furthermore, X^* is the minimum variance frontier, i.e. the set of all those portfolios which have the lowest possible variance for a given level of expected return and a given choice of δ^*. Likewise, the span of Σx^* can be interpreted as a "minimum entropy frontier", in the sense that it contains all those displacements of the vector of means which generate the lowest possible entropy for a given reduction of the portfolio expected return and a given choice of x^*. In terms of the game-theoretic interpretation given in the previous section, these optimality conditions read as follows: whatever his risk aversion coefficient γ, player 1 will never find it convenient to choose a portfolio out of the set X^*, because, for any portfolio not belonging to X^*, he is able to find a portfolio belonging to X^*, with the same expected return, but a lower variance; as far as player 2 is concerned, whatever the maximum level of entropy he can generate by changing the vector of expected returns, he will never find it convenient to choose a displacement to the vector of means out of the set Δ^*, because, for any displacement not belonging to Δ^*, he is able to find a displacement belonging to Δ^* which does not change player 1's utility, but generates less entropy; with the entropy thus saved, he is able to provoke a further displacement to the vector of means and make player 1 worse off.

Having investigated some basic aspects of the geometry of the portfolio problem, we now give its solution:

Theorem 2 *Let*

$$c = \eta / \sqrt{\left(\mu - \overrightarrow{1} R_f\right)' \Sigma^{-1} \left(\mu - \overrightarrow{1} R_f\right)}$$

If $c < 1$ problem (2) is solved by

$$\begin{aligned} x^* &= \frac{1-c}{\gamma} \Sigma^{-1} \left(\mu - \overrightarrow{1} R_f\right) \\ \delta^* &= -c \left(\mu - \overrightarrow{1} R_f\right) \end{aligned}$$

if $c \geq 1$ it is solved by:

$$\begin{aligned} x^* &= 0 \\ \delta^* &= -\left(\mu - \overrightarrow{1} R_f\right) \end{aligned}$$

Remember that the solution to the classical portfolio optimization problem (when there is no uncertainty about μ) is:

$$x^* = \frac{1}{\gamma} \Sigma^{-1} \left(\mu - \overrightarrow{1} R_f\right)$$

Introducing some uncertainty (letting $\eta \neq 0$) has the effect of inducing a proportional reduction of the weights given to the risky assets in the optimal portfolio, with the same constant of proportionality for all assets. Hence, introducing some ambiguity (or increasing it) is observationally equivalent to increasing the coefficient of absolute risk aversion γ. The proportions between portfolio weights remain unaltered: the structure of the portfolio remains essentially the same, but everything is scaled down and leverage is reduced. The reason why this happens is that the optimal perturbation to the vector of excess expected returns over the riskfree asset is just a proportional reduction of it: the agent behaves as if her initial estimate of the vector of excess expected returns were too optimistic and she reduces it proportionally, as if her optimism had to be corrected to the same degree for every single asset. This behavior is a consequence of the way we have specified the set Δ_1 over which minimization takes place and, in particular, it depends on the norm we have chosen on $\mathbb{R}^n$. In the next section we will show how changing the norm leads to a substantial modification of the agent's behavior.

4. Minimum norm portfolio selection

In this section we discuss how to solve the "minimum norm" problem

$$\sup_{x\in\mathbb{R}^n} \inf_{\delta\in\Delta_2} R_f + x'\left(\mu + \delta - \overrightarrow{1}R_f\right) - \frac{\gamma}{2}x'\Sigma x \tag{3}$$

where

$$\Delta_2 = \{\delta \in \mathbb{R}^n : \|\delta\| \leq \eta\}$$

and $\|.\|$ is the usual Euclidean norm on $\mathbb{R}^n$.

To parallel the discussion made in the previous section, we first present some facts about the geometry of the problem:

Theorem 3 *Let (x^*, δ^*) be a solution to (3), with $x^* \neq 0$, then $x^* \in X^*$ and $\delta^* \in \Delta^*$ where*

$$\begin{aligned} X^* &= \left\{x \in \mathbb{R}^n : x = \lambda_1 \Sigma^{-1}\left(\mu + \delta^* - \overrightarrow{1}R_f\right), \lambda_1 \in \mathbb{R}\right\} \\ \Delta^* &= \{\delta \in \Delta_2 : \delta = \lambda_2 x^*, \lambda_2 \in \mathbb{R}\} \end{aligned}$$

Furthermore, for any $x \in \mathbb{R}^n$ there exists $\overline{x} \in X^*$ such that:

$$\begin{aligned} \overline{x}'\left(\mu + \delta^* - \overrightarrow{1}R_f\right) &= x'\left(\mu + \delta^* - \overrightarrow{1}R_f\right) \\ \overline{x}'\Sigma\overline{x} &\leq x'\Sigma x \end{aligned}$$

and for any $\delta \in \mathbb{R}^n$ *there exists* $\overline{\delta} \in \mathtt{span}(x^*)$ *such that:*

$$\begin{aligned} x^{*\prime}\left(\mu + \overline{\delta} - \overrightarrow{1}R_f\right) &= x^{*\prime}\left(\mu + \delta - \overrightarrow{1}R_f\right) \\ \left\|\overline{\delta}\right\| &\leq \|\delta\| \end{aligned}$$

The interpretation of this theorem is virtually the same given to the analogous theorem in the previous section. The only relevant difference is in the geometry of Δ^*. Given any vector of portfolio weights x^*, the optimal displacement δ^* of the vector of expected returns belongs to the one-dimensional subspace of $\mathbb{R}^n$ spanned by the vector x^* (remember that in the previous section it belonged to the span of Σx^*). This can be thought of as a frontier of minimum norm displacements of the vector of expected returns, in the sense that it contains all those displacements which have the minimum possible norm for a given reduction of the portfolio expected return and a given choice of x^*. The geometry of X^* deserves no further comments, since it is the same as in the previous section.

The solution to (3) contains a constant, which is itself a solution to an equation whose properties are investigated by the following:

Lemma 4 *The function*

$$f(c) = \left(\mu - \overrightarrow{1}R_f\right)' \left[\frac{\gamma}{c}\Sigma + I\right]^{-2} \left(\mu - \overrightarrow{1}R_f\right)$$

is well-defined, continuous and strictly increasing on the interval $(0, \infty)$. *Furthermore,*

$$\begin{aligned} \lim_{c \to 0} f(c) &= 0 \\ \lim_{c \to \infty} f(c) &= \left(\mu - \overrightarrow{1}R_f\right)' \left(\mu - \overrightarrow{1}R_f\right) \end{aligned}$$

so that the equation

$$\left(\mu - \overrightarrow{1}R_f\right)' \left[\frac{\gamma}{c}\Sigma + I\right]^{-2} \left(\mu - \overrightarrow{1}R_f\right) = \eta^2 \tag{4}$$

admits a unique and positive solution whenever

$$\eta < \sqrt{\left(\mu - \overrightarrow{1}R_f\right)' \left(\mu - \overrightarrow{1}R_f\right)}$$

Since $f(c) - \eta^2$ is a strictly increasing and continuous function, the unique solution of Equation (4) can be easily found numerically, either by bisection or by a Newton algorithm employing the analytical derivatives reported in the proof in the Appendix. Note also that the solution c to the above equation, if seen as a function of the parameter η, is strictly increasing.

We are now ready to give the solution to the portfolio selection problem:

Theorem 5 *Let c be the unique positive value which solves the equation:*

$$\left(\mu - \overrightarrow{1}R_f\right)' \left[\frac{\gamma}{c}\Sigma + I\right]^{-2} \left(\mu - \overrightarrow{1}R_f\right) = \eta^2$$

Then problem (3) is solved by:

$$\begin{aligned} x^* &= [\gamma\Sigma + cI]^{-1}\left(\mu - \overrightarrow{1}R_f\right) \\ \delta^* &= -cx^* = -\left[\frac{\gamma}{c}\Sigma + I\right]^{-1}\left(\mu - \overrightarrow{1}R_f\right) \end{aligned}$$

if $\eta < \sqrt{\left(\mu - \overrightarrow{1}R_f\right)'\left(\mu - \overrightarrow{1}R_f\right)}$ and by

$$\begin{aligned} x^* &= 0 \\ \delta^* &= -\left(\mu - \overrightarrow{1}R_f\right) \end{aligned}$$

if $\eta \geq \sqrt{\left(\mu - \overrightarrow{1}R_f\right)'\left(\mu - \overrightarrow{1}R_f\right)}$.

The solution to the classical portfolio optimization problem (when there is no ambiguity about μ) can also be written as:

$$x^* = [\gamma\Sigma]^{-1}\left(\mu - \overrightarrow{1}R_f\right)$$

Introducing parameter uncertainty (letting $\eta \neq 0$), the matrix $\gamma\Sigma$ is augmented by the matrix cI. Hence, introducing parameter uncertainty (or increasing it) is equivalent to increasing all the variances of the asset returns by a constant, while leaving the covariances unchanged. Unlike in the case analysed in the previous section, the proportions between portfolio weights are altered and the structure of the portfolio changes. In Section 3 the investor behaved as if her initial estimate of the vector of excess expected returns were too optimistic, here the investor behaves as if the riskiness of the assets included in her portfolio were initially underestimated. Analysing the way δ^* is set provides further insights into the agent's behavior: δ^* is proportional to x^*; roughly speaking, this means that the reduction of the excess expected returns is not proportional, but the investor expects that her initial estimate of excess expected returns is more biased (and hence needs a greater correction) for those assets in which she is more heavily invested.

5. Concluding remarks

We have analyzed two portfolio selection models which extend Markowitz's model taking into account parameter uncertainty. When the probabilistic distribution of asset returns is not objectively known to the decision maker, there seems to be room for modelling devices which allow for the impossibility of forming a unique probability distribution. In our models optimal portfolios are chosen according to a maxmin criterion: the investor chooses a portfolio so as to maximize a mean-variance utility function and, at the same time, she selects a probability distribution of asset returns, among a set of plausible distributions, evaluating the consequences of her choice under the worst possible scenario. The specification of the set of probability measures over which minimization takes place turns out to be crucial: different specifications give substantially different results. Both of our models assume that uncertainty affects the vector of expected returns: the decision maker considers all the vectors in a neighborhood of a reference vector, which, roughly speaking, can be considered as an initial estimate made by the investor. In one case we obtain an optimal allocation rule which is not different from the rule we would obtain in a framework without parameter uncertainty: introducing uncertainty (and a maxmin behavior) is observationally equivalent to an increase in risk aversion. The implication of this model is that Markowitz's portfolios, calculated disregarding parameter uncertainty, seem to be robust: this partially contradicts a rich literature questioning the validity of Markowitz's portfolios when parameters are uncertain. However, with a second model, we obtain quite different implications: the structure of the optimal portfolio is different from that of a Markowitz's portfolio; the investor acts as though she was altering the covariance matrix of the returns, increasing all the variances and leaving the covariances unchanged. In this latter case the introduction of parameter uncertainty is observationally equivalent to an increase in the riskiness of the assets to be included in the optimal portfolio. Further insights into the differences among the two models can be derived by analyzing how the investor selects the vector of expected returns: in the first case she reduces proportionally all her initial estimates of the excess expected returns; in the second case she reduces more the estimates of the excess expected returns of those assets where she has more heavily invested, apparently displaying more pessimism than in the first case.

The models we have presented deliver easily interpretable closed formulae for the optimal portfolios: thus, they can be useful devices for analyzing both the normative and the positive implications of minimax behavior in portfolio selection. On the positive side they allow the researcher to easily perform some comparative statics and understand the behavioral implications of taking into account parameter uncertainty. On the normative side, they can be used as guidance in those situations where an investor wants to design portfolios which are optimal in the minimax sense.

Appendix

Proof of Theorem 1. We first prove that $\delta^* \in \Delta^*$ where

$$\Delta^* = \{\delta \in \Delta_1 : \delta = \lambda_2 \Sigma x^*, \lambda_2 \in \mathbb{R}\}$$

Remember that we have defined an inner product on $\mathbb{R}^n$ as follows:

$$\langle \delta_1, \delta_2 \rangle_{\Sigma^{-1}} = \delta_1' \Sigma^{-1} \delta_2$$

and that

$$\Delta_1 = \{\delta \in \mathbb{R}^n : \|\delta\|_{\Sigma^{-1}} \leq \eta\}$$

where $\|.\|_{\Sigma^{-1}}$ is the norm induced by the above inner product. We first observe that $\|\delta^*\|_{\Sigma^{-1}} = \eta$. Suppose instead that $\|\delta^*\|_{\Sigma^{-1}} < \eta$, then, by continuity of the norm, it is possible to find a strictly positive scalar ν such that $\|\delta^* - \nu x^*\|_{\Sigma^{-1}} < \eta$. $(\delta^* - \nu x^*)$ is an admissible choice since it belongs to Δ_1 and, unless $x^* = 0$, $x^{*\prime}(\delta^* - \nu x^*) < x^{*'}\delta^*$, so that δ^* cannot yield a minimum of $x^{*'}\delta^*$. Having established that if δ^* is an optimum then $\|\delta^*\|_{\Sigma^{-1}} = \eta$, it is easy to prove that δ^* must belong to the set Δ^*. Suppose δ^* does not belong to Δ^*. Take the projection $\lambda_* \Sigma x^*$ (λ_* is a scalar) of δ^* on the subspace spanned by the vector Σx^* and note that:

$$\begin{aligned} x^{*'}\delta^* &= \langle \Sigma x^*, \delta^* \rangle_{\Sigma^{-1}} = \\ &= \langle \Sigma x^*, \lambda_* \Sigma x^* + (\delta^* - \lambda_* \Sigma x^*) \rangle_{\Sigma^{-1}} = \\ &= \langle \Sigma x^*, \lambda_* \Sigma x^* \rangle_{\Sigma^{-1}} + \langle \Sigma x^*, \delta^* - \lambda_* \Sigma x^* \rangle_{\Sigma^{-1}} = \\ &= \langle \Sigma x^*, \lambda_* \Sigma x^* \rangle_{\Sigma^{-1}} = x^{*\prime} \lambda_* \Sigma x^* \end{aligned}$$

So, $\lambda_* \Sigma x^*$ yields the same value of the objective as δ^*. Furthermore, by Pithagoras' theorem:

$$\|\delta^*\|^2_{\Sigma^{-1}} = \|\lambda_* \Sigma x^*\|^2_{\Sigma^{-1}} + \|\delta^* - \lambda_* \Sigma x^*\|^2_{\Sigma^{-1}} > \|\lambda_* \Sigma x^*\|^2_{\Sigma^{-1}}$$

where the last inequality is strict because we are assuming that $\delta^* \notin \Delta^*$ and hence

$$\delta^* - \lambda_* \Sigma x^* \neq 0$$

The above inequality implies that $\|\lambda_* \Sigma x^*\|_{\Sigma^{-1}} < \eta$ and $\lambda_* \Sigma x^* \in \Delta_1$, but this is a contradiction: $x^{*'}\delta^* = x^{*\prime}\lambda_* \Sigma x^*$ implies that $\lambda_* \Sigma x^*$ is optimal and $\|\lambda_* \Sigma x^*\|_{\Sigma^{-1}} < \eta$ implies that it cannot be optimal. We now prove that $x^* \in X^*$, where

$$X^* = \left\{ x \in \mathbb{R}^n : x = \lambda_1 \Sigma^{-1} \left(\mu + \delta^* - \overrightarrow{1} R_f \right), \lambda_1 \in \mathbb{R} \right\}$$

Being an optimum, x^* solves:

$$\max_{x \in \mathbb{R}^n} \left\{ x'z - \frac{\gamma}{2} x' \Sigma x \right\}$$

where $z = \mu + \delta^* - \overrightarrow{1} R_f$. Define an inner product on $\mathbb{R}^n$ as follows:

$$\langle x_1, x_2 \rangle_\Sigma = x_1' \Sigma x_2$$

Suppose x^* does not belong to X^*.Take the projection $\lambda_* \Sigma^{-1} z$ (λ_* is a scalar) of x^* on X^* and note that:

$$\begin{aligned} x^{*'} z &= \left\langle \Sigma^{-1} z, x^* \right\rangle_\Sigma = \\ &= \left\langle \Sigma^{-1} z, \lambda_* \Sigma^{-1} z + \left(x^* - \lambda_* \Sigma^{-1} z \right) \right\rangle_\Sigma = \\ &= \left\langle \Sigma^{-1} z, \lambda_* \Sigma^{-1} z \right\rangle_\Sigma + \left\langle \Sigma^{-1} z, x^* - \lambda_* \Sigma^{-1} z \right\rangle_\Sigma = \\ &= \left\langle \Sigma^{-1} z, \lambda_* \Sigma^{-1} z \right\rangle_\Sigma = \left(\lambda_* \Sigma^{-1} z \right)' z \end{aligned} \tag{5}$$

Furthermore, by Pithagoras' theorem:

$$\|x^*\|_\Sigma^2 = \left\| \lambda_* \Sigma^{-1} z \right\|_\Sigma^2 + \left\| x^* - \lambda_* \Sigma^{-1} z \right\|_\Sigma^2 > \left\| \lambda_* \Sigma^{-1} z \right\|_\Sigma^2$$

where the last inequality is strict because we are assuming that $x^* \notin X^*$ and hence

$$x^* - \lambda_* \Sigma^{-1} z \neq 0$$

The above inequality implies that:

$$x^{*\prime} \Sigma x^* > \left(\lambda_* \Sigma^{-1} z \right)' \Sigma \left(\lambda_* \Sigma^{-1} z \right)$$

and, recalling (5):

$$x^{*\prime} z - \frac{\gamma}{2} x^{*\prime} \Sigma x^* < \left(\lambda_* \Sigma^{-1} z \right)' z - \frac{\gamma}{2} \left(\lambda_* \Sigma^{-1} z \right)' \Sigma \left(\lambda_* \Sigma^{-1} z \right)$$

so that x^* cannot be an optimum. ■

Proof of Theorem 2. Define the function

$$V(x,\delta) = R_f + x'\left(\mu + \delta - \overrightarrow{1} R_f\right) - \frac{\gamma}{2} x' \Sigma x$$

As we have explained in section 1, if there exists a couple (x^*, δ^*) satisfying

$$V(x,\delta^*) \leq V(x^*,\delta^*) \leq V(x^*,\delta), \quad \forall x \in \mathbb{R}^n, \quad \forall \delta \in \Delta_1 \tag{6}$$

then the problem (2) is solved by (x^*, δ^*). Let us start from the case

$$c = \eta / \sqrt{\left(\mu - \overrightarrow{1} R_f\right)' \Sigma^{-1} \left(\mu - \overrightarrow{1} R_f\right)} < 1$$

We will show that the couple

$$\begin{aligned} x^* &= \frac{1-c}{\gamma} \Sigma^{-1} \left(\mu - \overrightarrow{1} R_f\right) \\ \delta^* &= -c\left(\mu - \overrightarrow{1} R_f\right) \end{aligned}$$

satisfies the double inequality (6). Note that

$$V(x,\delta^*) = R_f + (1-c)\, x'\left(\mu - \overrightarrow{1} R_f\right) - \frac{\gamma}{2} x' \Sigma x$$

$V(x,\delta^*)$ is a strictly concave function of x, hence a first order condition is sufficient to locate a global maximum of $V(x,\delta^*)$ with respect to x. The first order condition yields:

$$x^* = \frac{1-c}{\gamma} \Sigma^{-1} \left(\mu - \overrightarrow{1} R_f\right)$$

so that indeed x^* satisfies the inequality

$$V(x,\delta^*) \leq V(x^*,\delta^*), \qquad \forall x \in \mathbb{R}^n$$

From the proof of Theorem 1 we know that, for any $x^* \neq 0$, δ^* can be a solution to the problem

$$\min_{\delta \in \Delta_1} V(x^*,\delta)$$

only if $\delta^* = \lambda_2 \Sigma x^*$ for some $\lambda_2 \in \mathbb{R}^n$ and if $\delta^{*\prime}\Sigma^{-1}\delta^* = \eta^2$. Substituting for δ^* in the last equality, we obtain:

$$\lambda_2^2 = \frac{\eta^2}{x^{*\prime}\Sigma x^*}$$

and substituting for x^*:

$$\lambda_2^2 = \left(\frac{\gamma}{1-c}\right)^2 \frac{\eta^2}{\left(\mu - \overrightarrow{1} R_f\right)' \Sigma^{-1} \left(\mu - \overrightarrow{1} R_f\right)}$$

As a consequence, if δ^* solves the problem, it must be either

$$\lambda_2 = \frac{\gamma c}{1-c}$$

or

$$\lambda_2 = -\frac{\gamma c}{1-c}$$

Since

$$V\left(x^*, \lambda_2 \Sigma x^*\right) = R_f + x^{*\prime}\left(\mu - \overrightarrow{1} R_f\right) - \frac{\gamma}{2} x^{*\prime}\Sigma x^* + \lambda_2 x^{*\prime}\Sigma x^*$$

and Σ is positive definite, it is clear that, between the two possible values of λ_2, $\lambda_2 = -\frac{\gamma c}{1-c}$ is the one which yields the lowest value of $V\left(x^*, \lambda_2 \Sigma x^*\right)$. As a consequence, the only point in Δ_1 which satisfies all the necessary conditions for a solution to (2) is:

$$\delta^* = -\frac{\gamma c}{1-c} \Sigma x^*$$

But $V\left(x^*, \delta\right)$ is continuous in δ and Δ_1 is a compactum, so that δ^* must indeed yield a minimum. Substituting for x^* gives:

$$\delta^* = -c\left(\mu - \overrightarrow{1} R_f\right)$$

and hence we have proved the second part of the inequality, i.e.:

$$V\left(x^*, \delta^*\right) \leq V\left(x^*, \delta\right), \quad \forall \delta \in \Delta_1$$

Let us now analyze the case in which $c \geq 1$. We have to show that the couple:

$$\begin{aligned} x^* &= 0 \\ \delta^* &= -\left(\mu - \overrightarrow{1} R_f\right) \end{aligned}$$

satisfies the double inequality (6). δ^* obviously satisfies $\|\delta\|_{\Sigma^{-1}} \leq \eta$. Observe that

$$V(x,\delta^*) = R_f - \frac{\gamma}{2}x'\Sigma x$$

since the quadratic form $x'\Sigma x$ is positive definite $V(x,\delta^*)$ obviously attains a global maximum at $x^* = 0$, so that indeed x^* satisfies the inequality

$$V(x,\delta^*) \leq V(x^*,\delta^*), \qquad \forall x \in \mathbb{R}^n$$

Furthermore,

$$V(x^*,\delta) = R_f, \qquad \forall \delta \in \Delta_1$$

so that $\delta^* = -\left(\mu - \overrightarrow{1} R_f\right) \in \Delta_1$ does satisfy the inequality:

$$V(x^*,\delta^*) \leq V(x^*,\delta), \qquad \forall \delta \in \Delta_1$$

■

Proof of Theorem 3. The proof of this theorem closely parallels the proof of Theorem 1. The proof that $x^* \in X^*$ is identical, so we only prove $\delta^* \in \Delta^*$, where

$$\Delta^* = \{\delta \in \Delta_2 : \delta = \lambda_2 x^*, \lambda_2 \in \mathbb{R}\}$$

Remember that

$$\Delta_2 = \{\delta \in \mathbb{R}^n : \|\delta\| \leq \eta\}$$

where $\|.\|$ is the usual euclidean norm. We first observe that $\|\delta^*\| = \eta$. Suppose instead that $\|\delta^*\|_\Sigma < \eta$, then, by continuity of the norm, it is possible to find a strictly positive scalar ν such that $\|\delta^* - \nu x^*\| < \eta$. $(\delta^* - \nu x^*)$ is an admissible choice since it belongs to Δ_2 and, unless $x^* = 0$, $x^{*\prime}(\delta^* - \nu x^*) < x^{*'}\delta^*$, so that δ^* cannot yield a minimum of $x^{*'}\delta^*$. Having established that if δ^* is an optimum then $\|\delta^*\| = \eta$, it is easy to prove that δ^* must belong to the set Δ^*. Suppose δ^* does not belong to Δ^*. Take the projection $\lambda_* x^*$ (λ_* is a scalar) of δ^* on the subspace spanned by the vector x^* and note that:

$$\begin{aligned}
x^{*'}\delta^* &= \langle x^*, \delta^* \rangle = \\
&= \langle x^*, \lambda_* x^* + (\delta^* - \lambda_* x^*) \rangle = \\
&= \langle x^*, \lambda_* x^* \rangle + \langle x^*, \delta^* - \lambda_* x^* \rangle = \\
&= \langle x^*, \lambda_* x^* \rangle = x^{*\prime}\lambda_* x^*
\end{aligned}$$

So, $\lambda_* x^*$ yields the same value of the objective as δ^*. Furthermore, by Pithagoras' theorem:

$$\|\delta^*\|^2 = \|\lambda_* x^*\|^2 + \|\delta^* - \lambda_* x^*\|^2 > \|\lambda_* \Sigma x^*\|^2$$

where the last inequality is strict because we are assuming that $\delta^* \notin \Delta^*$ and hence

$$\delta^* - \lambda_* x^* \neq 0$$

The above inequality implies that $\|\lambda_* x^*\| < \eta$ and $\lambda_* x^* \in \Delta_1$, but this is a contradiction: $x^{*'} \delta^* = x^{*\prime} \lambda_* x^*$ implies that $\lambda_* x^*$ is optimal and $\|\lambda_* x^*\| < \eta$ implies that it cannot be optimal. ■

Proof of Lemma 4. We can write

$$f(c) = c^2 z' [\gamma \Sigma + cI]^{-2} z$$

where we have set $z = \mu - \overrightarrow{1} R_f$. Since Σ is positive definite and c is strictly positive, the matrix $[\gamma \Sigma + cI]$ is positive definite and hence invertible, so that the function is well defined. Furthermore, if the eigenvalues of Σ are distinct also the eigenvalues of $[\gamma \Sigma + cI]$ are distinct, for any value of c. This is a consequence of the fact that, if the eigenvalues of $\gamma \Sigma$ are λ_1, $\lambda_2, \ldots$, λ_n, then the eigenvalues of $[\gamma \Sigma + cI]$ are $\lambda_1 + c$, $\lambda_2 + c, \ldots$, $\lambda_n + c$. Hence also the eigenvalues of the matrix

$$\frac{\gamma}{c} \Sigma + I$$

are distinct. We will now use the following lemma, which can be found in van Bossum (2002) and descends from a more general theorem in Torki (2001):

Let A be a symmetric and positive definite $(n \times n)$ matrix which can be written as:

$$A = y_1 B_1 + y_2 B_2 + \ldots + y_k B_k$$

where B_1, B_2,..., B_k are positive semidefinite matrices and y_1, y_2,..., y_k are positive scalars. Let A have n distinct eigenvalues λ_1, λ_2,..., λ_n (they are real and strictly *positive) and denote by u_1, u_2,..., u_n their respective eigenvectors (chosen so as to be orthonormal). Then:*

$$\begin{aligned}\frac{\partial \lambda_i}{\partial y_j} &= u_i' B_j u_i \\ \frac{\partial^2 \lambda_i}{\partial y_l \partial y_j} &= 2\sum_{r\neq i} \frac{u_i' B_l u_r u_i' B_j u_r}{\lambda_i - \lambda_r}\end{aligned}$$

Define:

$$A(c,t) = \frac{\gamma}{c}\Sigma + I + tzz'$$

We can diagonalize $A(c,0)$ as follows:

$$A(c,0) = U\Lambda U'$$

where U is the matrix whose columns u_1, u_2,..., u_n are the orthonormal eigenvectors of $A(c,0)$ and Λ is the diagonal matrix of eigenvalues. The function g can be written as:

$$f(c) = z'U\Lambda^{-2}U'z$$

Expanding the product:

$$f(c) = \frac{z'u_1u_1'z}{\lambda_1^2} + \frac{z'u_2u_2'z}{\lambda_2^2} + \ldots + \frac{z'u_nu_n'z}{\lambda_n^2}$$

or:

$$f(c) = \frac{u_1'zz'u_1}{\lambda_1^2} + \frac{u_2'zz'u_2}{\lambda_2^2} + \ldots + \frac{u_n'zz'u_n}{\lambda_n^2}$$

which, applying the above lemma, becomes:

$$f(c) = \sum_{i=1}^{n} \frac{1}{\lambda_i^2}\frac{\partial \lambda_i}{\partial t}$$

Taking the derivative with respect to c, we get:

$$\begin{aligned}f'(c) &= \frac{df}{d(\gamma/c)}\frac{d(\gamma/c)}{dc} = \\ &= \left(\sum_{i=1}^{n}\frac{1}{\lambda_i^2}\frac{\partial^2\lambda_i}{\partial(\gamma/c)\,\partial t} - 2\sum_{i=1}^{n}\frac{1}{\lambda_i^3}\frac{\partial\lambda_i}{\partial t}\frac{\partial\lambda_i}{\partial(\gamma/c)}\right)\left(-\frac{\gamma}{c^2}\right) = \\ &= -\frac{\gamma}{c^2}\left(\sum_{i=1}^{n}\frac{2}{\lambda_i^2}\sum_{j\neq i}\frac{u_i'\Sigma u_j u_i'zz'u_j}{\lambda_i-\lambda_j} - 2\sum_{i=1}^{n}\frac{u_i'\Sigma u_i u_i'zz'u_i}{\lambda_i^3}\right)\end{aligned}$$

Setting $Q_{ij} = u_i' \Sigma u_j u_j' z z' u_i$ and noting that $Q_{ij} = Q_{ji}$, we can write:

$$\begin{aligned} f'(c) &= -\frac{\gamma}{c^2} \sum_{i=1}^{n} \left[2 \sum_{j=1}^{i-1} \left(\frac{Q_{ij}}{\lambda_i^2 (\lambda_i - \lambda_j)} + \frac{Q_{ji}}{\lambda_j^2 (\lambda_j - \lambda_i)} \right) - 2 \frac{Q_{ii}}{\lambda_i^3} \right] = \\ &= -\frac{\gamma}{c^2} \sum_{i=1}^{n} \left[2 \sum_{j=1}^{i-1} \left(\frac{\left(\lambda_j^2 - \lambda_i^2\right) Q_{ij}}{\lambda_i^2 \lambda_j^2 (\lambda_i - \lambda_j)} \right) - 2 \frac{Q_{ii}}{\lambda_i^3} \right] = \\ &= -\frac{\gamma}{c^2} \sum_{i=1}^{n} \left[-2 \sum_{j=1}^{i-1} \left(\frac{(\lambda_i + \lambda_j) Q_{ij}}{\lambda_i^2 \lambda_j^2} \right) - \frac{Q_{ii} (\lambda_i + \lambda_i)}{\lambda_i^2 \lambda_i^2} \right] = \\ &= \frac{\gamma}{c^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{Q_{ij} (\lambda_i + \lambda_j)}{\lambda_i^2 \lambda_j^2} \end{aligned}$$

The Q_{ij}s are positive, since the matrix Σ is positive definite and the matrix zz' is semipositive definite; furthermore, at least one of the Q_{ij}s is strictly positive; this can be proved by contradiction: if the Q_{ij}s are all zero, then $z' u_i = 0$ for every i, but the eigenvectors are orthogonal to each other and span $\mathbb{R}^n$, so that $z = 0$, which we have excluded a priori. Since at least one of the Q_{ij}s is strictly positive and all the eigenvalues, c and γ are strictly positive, $f'(c) > 0$, and we have proved the claim that $f(c)$ is strictly increasing on the interval $(0, \infty)$. We now prove the two claims that

$$\begin{aligned} \lim_{c \to 0} f(c) &= 0 \\ \lim_{c \to \infty} f(c) &= \left(\mu - \overrightarrow{1} R_f\right)' \left(\mu - \overrightarrow{1} R_f\right) \end{aligned}$$

The first one is easily proved as follows:

$$\begin{aligned} \lim_{c \to 0} f(c) &= \lim_{c \to 0} \left(c^2 z' \left[\gamma \Sigma + cI\right]^{-2} z \right) = \\ &= z' \left[\gamma \Sigma\right]^{-2} z \lim_{c \to 0} c^2 = 0 \end{aligned}$$

The second one is a consequence of the continuity of $f(c)$:

$$\begin{aligned} \lim_{c \to \infty} f(c) &= z' \left[\lim_{c \to \infty} \frac{\gamma}{c} \Sigma + I \right]^{-2} z = \\ &= z' I^{-2} z = z' z \end{aligned}$$

Combining these two facts with the fact that $f(c)$ is continuous and strictly increasing on the interval $(0, \infty)$, we deduce that $f(c)$ assumes any value in the interval

$(0, z'z)$ and that indeed the equation

$$\left(\mu - \overrightarrow{1} R_f\right)' \left[\frac{\gamma}{c}\Sigma + I\right]^{-2} \left(\mu - \overrightarrow{1} R_f\right) = \eta^2$$

admits a unique and positive solution whenever

$$\eta < \sqrt{\left(\mu - \overrightarrow{1} R_f\right)' \left(\mu - \overrightarrow{1} R_f\right)}$$

■

Proof of Theorem 5. Define the function

$$V(x, \delta) = R_f + x' \left(\mu + \delta - \overrightarrow{1} R_f\right) - \frac{\gamma}{2} x' \Sigma x$$

We have to show that the couple (x^*, δ^*) satisfies

$$V(x, \delta^*) \leq V(x^*, \delta^*) \leq V(x^*, \delta), \quad \forall x \in \mathbb{R}^n, \quad \forall \delta \in \Delta_2 \tag{7}$$

Let's start from the case

$$\eta < \sqrt{\left(\mu - \overrightarrow{1} R_f\right)' \left(\mu - \overrightarrow{1} R_f\right)}$$

We will show that the couple

$$\begin{aligned} x^* &= [\gamma\Sigma + cI]^{-1} \left(\mu - \overrightarrow{1} R_f\right) \\ \delta^* &= -\left[\frac{\gamma}{c}\Sigma + I\right]^{-1} \left(\mu - \overrightarrow{1} R_f\right) \end{aligned}$$

satisfies the double inequality (7). Note that

$$V(x, \delta^*) = R_f + x' \left(\mu - \overrightarrow{1} R_f\right) - x' \left[\frac{\gamma}{c}\Sigma + I\right]^{-1} \left(\mu - \overrightarrow{1} R_f\right) - \frac{\gamma}{2} x' \Sigma x$$

$V(x, \delta^*)$ is a strictly concave function of x, hence a first order condition is sufficient to locate a global maximum of $V(x, \delta^*)$ with respect to x. The first order condition is:

$$\left(\mu - \overrightarrow{1} R_f\right) - \left[\frac{\gamma}{c}\Sigma + I\right]^{-1} \left(\mu - \overrightarrow{1} R_f\right) - \gamma\Sigma x = 0$$

Premultiplying everything by $\frac{c}{\gamma}\Sigma^{-1}\left[\frac{\gamma}{c}\Sigma + I\right]$, one easily gets the solution:

$$x^* = [\gamma\Sigma + cI]^{-1}\left(\mu - \overrightarrow{1}R_f\right)$$

so that indeed x^* satisfies the inequality

$$V(x, \delta^*) \leq V(x^*, \delta^*), \qquad \forall x \in \mathbb{R}^n$$

From the proof of Theorem 3 we know that, for any $x^* \neq 0$, δ^* can be a solution to the problem

$$\min_{\delta \in \Delta_1} V(x^*, \delta)$$

only if $\delta^* = \lambda x^*$ for some $\lambda \in \mathbb{R}^n$ and if $\delta^{*\prime}\delta^* = \eta^2$. Substituting for δ^* in the last equality, we obtain:

$$\lambda^2\left(\mu - \overrightarrow{1}R_f\right)' [\gamma\Sigma + cI]^{-2}\left(\mu - \overrightarrow{1}R_f\right) = \eta^2$$

or:

$$\lambda^2\left(\mu - \overrightarrow{1}R_f\right)' \left[\frac{\gamma}{c}\Sigma + I\right]^{-2}\left(\mu - \overrightarrow{1}R_f\right) = c^2\eta^2 \tag{8}$$

Since by assumption c is such that:

$$\left(\mu - \overrightarrow{1}R_f\right)' \left[\frac{\gamma}{c}\Sigma + I\right]^{-2}\left(\mu - \overrightarrow{1}R_f\right) = \eta^2$$

equation (8) becomes:

$$\lambda^2 = c^2$$

As a consequence, if δ^* solves the problem, it must be either $\lambda = c$ or $\lambda = -c$. Since

$$V(x^*, \lambda x^*) = R_f + x^{*\prime}\left(\mu - \overrightarrow{1}R_f\right) - \frac{\gamma}{2}x'\Sigma x + \lambda x^{*\prime}x^*$$

and the inner product $x^{*\prime}x^*$ is strictly postitive (we are assuming $x^* \neq 0$), it is clear that, between the two possible values of λ, $\lambda = -c$ is the one which yields the lowest value of $V(x^*, \lambda x^*)$. So, the only point in Δ_1 which satisfies all the necessary conditions for a solution to (3) is:

$$\delta^* = -cx^*$$

But $V(x^*, \delta)$ is continuous in δ and Δ_1 is a compactum, so that δ^* must indeed yield a minimum. Substituting for x^* gives:

$$\delta^* = -\left[\frac{\gamma}{c}\Sigma + I\right]^{-1}\left(\mu - \overrightarrow{1}R_f\right)$$

and hence we have proved the second part of the inequality, i.e.:

$$V(x^*, \delta^*) \leq V(x^*, \delta), \qquad \forall \delta \in \Delta_2$$

■

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