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Monetary policy and the transition to rational expectations

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MONETARY POLICY AND THE TRANSITION TO RATIONAL EXPECTATIONS

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Abstract

Under the assumption of bounded rationality, economic agents learn from their past mistaken predictions by combining new and old information to form new beliefs. The purpose of this paper is to examine how the policy-maker, by affecting private agents' learning process, determines the speed at which the economy converges to the rational expectation equilibrium. I find that by reacting strongly to private agents' expected inflation, a central bank would increase the speed of convergence.

I assess the relevance of the transition period from the learning to the rational expectations equilibrium when looking at a criterion for evaluating monetary policy decisions and suggest that a fast convergence is not always suitable.

JEL classification: E52, C62, D83, D84.

Keywords: Interest rate setting, adaptive learning, rational expectations, stability, speed of convergence.

Contents

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1. Introduction[1]

There is wide consensus on the fact that monetary policy may affect real variables in the short run. One recent strand of research has obtained this result explicitly incorporating frictions, such as nominal price rigidities, in a dynamic general equilibrium framework under the rational expectations hypothesis.

Recently, this issue has been analyzed questioning the assumption that agents are able to make unbiased predictions of the future course of the economy. Such predictions, it has been said, would be possible if people had observed the reactions of the policy-maker to various economic conditions over a long period of time. However, this would not always be the case. It can be argued, for example, that in the presence of policy regime shifts the public needs to learn about the new regime: in the early stages of this learning process, previously held public beliefs could lead to biased predictions. In order to avoid asymptotic instability in the economy, Evans and Honkapohja (2002a, 2002b, 2003) and Bullard and Mitra (2002) suggest that economic policies should be designed to be conducive to long-run convergence of private expectations to rational expectations (*E-Stability*)[2]. These papers and, in general, the literature on monetary policy and bounded rationality are extensively devoted to the analysis of the asymptotic properties of the equilibrium attainable under learning. There is, however, very little literature that studies the dynamic properties along the convergence process.

The purpose of this work is to examine how the policy-maker, by affecting private agents' learning process, can influence the transition to the rational expectations equilibrium (REE). I show that policies driving the economy to the same asymptotic REE could imply very different transitional dynamics in the real economy. By reacting strongly to expected inflation, a central bank would shorten the transition and increase the speed of convergence from the learning equilibrium to the REE.

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[1] I am particularly grateful to my advisor Albert Marcet for many helpful comments and discussions. I also thank Seppo Honkaphja, Kosuke Aoki, Ramon Marimon, Jordi Galì, Antonio Ciccone, Jose Rodriguez Vicente-Mora and Alfonso Rosolia for their comments. All remaining errors are mine. The opinions expressed here do not necessarily reflect those of the Bank of Italy.

[2] An earlier paper by Howitt (1992) had already shown that under some interest rate rules the rational expectation equilibrium is not learnable.

This is particularly relevant when policy decisions aim to influence social welfare: if policy-makers know that after a regime change private agents' perceived inflation would be higher than the REE, by choosing a policy that reacts strongly to expected inflation they could substantially increase social welfare. If, instead, perceived inflation is initially lower than the REE, a weak response to expected inflation and a slow transition might be preferable.

In order to study the transition in the learning process to the REE I adapt arguments described by Marcet and Sargent (1995), which in turn are based on the theoretical results of Benveniste, Metivier and Priouret (1990).

The paper is organized as follows. Section 2 presents the monetary policy design problem, describing the learning equilibrium under two different policy rules. The section ends showing that under the optimal RE discretionary policy, transition to the REE is very long. In section 3 I show how the policy-maker could characterize policies, evaluating the speed at which the learning equilibrium converges to the REE. In section 4 I study policies that allow the central bank to increase (or reduce) the speed of convergence without affecting the long-run equilibrium (i.e., the REE equilibrium) and in section 5 I analyze how these policies influence social welfare.

2. The framework

2.1 *The baseline model*

Much of the recent theoretical analysis on monetary policy has been conducted under the "New Phillips curve" paradigm reviewed in Clarida, Galí and Gertler (1999) and Woodford (1999). The baseline framework is a dynamic general equilibrium model with money and temporary nominal price rigidities. I consider the linearized reduced form of the economy with competitive monopolistic firms, staggered prices and private agents that maximize intertemporal utility. From the private agents' point of view there is an intertemporal *IS curve*

$$x_t = E_t^* x_{t+1} - \varphi \left(i_t - E_t^* \pi_{t+1} \right) + g_t \qquad (2.1)$$

and an aggregate supply (AS) modeled by an expectations-augmented *Phillips curve*:

$$\pi_t = \alpha x_t + \beta E_t^* \pi_{t+1}, \qquad (2.2)$$

where x_t is the output gap, measured as the log deviation of actual output (y_t) from potential output (z_t) (i.e., the level of output that would arise if wages and prices were perfectly competitive and flexible), π_t is actual inflation at time t, $E_t^* \pi_{t+1}$ is the level of inflation expected by private agents for period $t+1$, given the information at time t. Similarly $E_t^* x_{t+1}$ is the level of the output gap that private agents expect for period $t+1$, given the information at time t. I write E_t^* to indicate that expectations need not be rational (E_t without $*$ denotes RE); i_t is the short-term nominal interest rate and is taken to be the instrument for monetary policy; g_t is an i.i.d demand shock, with $g_t \sim N\left(0, \sigma_g^2\right)$.

The IS relationship approximates the Euler equation characterizing optimal aggregate consumption choices and the parameter φ can be interpreted as the rate of intertemporal substitution. The AS relation[3] approximates aggregate pricing emerging from monopolistically competitive firms' optimal behaviour in Calvo's model of staggered price determination[4].

In order to complete the model, it is necessary to specify how the interest rate is settled and how agents form beliefs. Choosing between policies based on simple rules or derived as a solution of a specified optimization problem is the starting point for the analysis of monetary policies. In the literature, there is no consensus on the terminology for rules and optimal policies. Here I consider the nominal interest rate as the policy instrument and model it by means of a reaction function, that is, a functional relationship between a dependent variable (the interest rate) and some endogenous (expected inflation and output gap) and exogenous (shocks) variables. I consider three cases. I start with a simple expectations-based policy rule that helps me to introduce in a very simple and intuitive way the concept of speed of convergence. Second, I describe the optimal RE policy under discretion derived in Evans and Honkapohja (2002)[5]. Finally, I introduce a set of expectations-based policy rules and show how to discriminate between the elements of this set, using a measure of speed of convergence.

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[3] Here we are not considering cost-push shocks. Introducing cost-push shocks in the Phillips curve would not change substantially the results on speed of convergence and the role of policy decisions along the transition. In section 5 I also analyze briefly results in terms of welfare also when I introduce cost-push shocks in the AS.

[4] Inflation is increasing with the output gap as price are set as a markup over real marginal costs, which are increasing with the output gap. Higher expected inflation raises current inflation, as price setters cannot fully adjust to current shocks.

[5] I leave for future research a general study of the transition of learning process for monetary policy problem under commitment

Concerning beliefs, I start each analysis by considering the rational expectations hypothesis in order to focus and discuss subsequently the implications of bounded rationality.

2.2 *A simple expectations-based reaction function*

It has long been recognized that monetary policy needs a forward-looking dimension. Let us assume that the central bank, in order to set the current interest rate, uses simple policy rules that feed back from expected values of future inflation and output gap

(2.3) $$i_t = -\gamma + \gamma_x E_t^* x_{t+1} + \gamma_\pi E_t^* \pi_{t+1}.$$

The class of *expectations-based reaction functions* that I first consider has $\gamma_x = \frac{1}{\varphi}$ in order to simplify the interaction between actual and expected variables. Under (2.3), in fact, the economy evolves according to the following system of equations:

(2.4) $$\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \begin{bmatrix} \alpha\varphi\gamma \\ \varphi\gamma \end{bmatrix} + \begin{bmatrix} \beta + \alpha\varphi\left(1 - \gamma_\pi\right) & 0 \\ \varphi\left(1 - \gamma_\pi\right) & 0 \end{bmatrix} \begin{bmatrix} E_t^* \pi_{t+1} \\ E_t^* x_{t+1} \end{bmatrix} + \begin{bmatrix} 0 & \alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ g_t \end{bmatrix},$$

where neither the IS nor the AS are affected by expectations on output gap[6].

2.2.1 *The rational expectations equilibrium*

Under rational expectations (i.e. $E_t^* x_{t+1} = E_t x_{t+1}$ and $E_t^* \pi_{t+1} = E_t \pi_{t+1}$) it has been shown that the dynamic system defined by (2.4) has a unique non-explosive equilibrium (Bullard and Mitra, 2002). The equilibrium can be written as a linear function of a constant and the shocks[7]:

(2.5) $$\pi_t = \overline{a}_\pi + \alpha g_t \quad \text{and} \quad x_t = \overline{a}_x + g_t,$$

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[6] For a more general class of expectations-based policy rules without restrictions on γ_x I refer to section 3.

[7] The solution (2.5) is often referred to as the minimal state variable (MSV) solution, following McCallum (1983), who introduced the concept for linear rational expectations models. This is a solution which depends linearly on a set of variables (here g_t and the intercept) and which is such that there does not exist a solution that depends linearly on a smaller set.

where[8]

(2.6) $$\overline{a}_\pi = \frac{\alpha\varphi\gamma}{(1-\beta-\alpha\varphi(1-\gamma_\pi))} \quad \text{and} \quad \overline{a}_x = \varphi\gamma + \varphi(1-\gamma_\pi)\, a_\pi.$$

Note that expression (2.5) describes the actual law of motion (ALM) of inflation and the output gap under RE hypothesis and policy (2.3). Private agents' perceived law of motion (PLM) of inflation and the output gap under RE are constant and equal to

(2.7) $$E_t\pi_{t+1} = \overline{a}_\pi \quad \text{and} \quad E_t x_{t+1} = \overline{a}_x.$$

Lemma 1 *Under the simple expectations-based reaction function (2.3) the necessary and sufficient condition for the rational expectations equilibrium to be unique is*

$$1 - \frac{1-\beta}{\alpha\varphi} < \gamma_\pi < 1 + \frac{1+\beta}{\alpha\varphi}$$

Proof. See Appendix 2.A. ■

While the previous lemma provides a characterization of the REE under the *expectations-based reaction function* (2.3), the following lemma underlines the relevance of policy-maker decisions in characterizing the REE and shows how the parameter γ_π influences output gap and inflation equilibrium.

Lemma 2 *Under the simple expectations-based reaction function (2.3) and under RE: if* $\gamma \neq 0$, *for* $\gamma_\pi > 1$, *the higher the value of* γ_π *the higher the inflation and output gap levels, for* $\gamma_\pi < 1$ *the higher the value of* γ_π *the lower the inflation and output gap levels.*

Proof. See Appendix 2.B. ■

Let us now characterize the equilibrium relaxing the hypothesis of rational expectations.

2.3 *Learning models and policy analysis*

The current standard hypothesis about expectations in monetary policy design is the rational expectations hypothesis, meaning that agents do not make systematic forecasting

[8] See Appendix 1 for a derivation of the REE.

errors and their guesses about the future are on average correct. In this paper I focus on a different approach to modeling expectations. I assume that households and firms make forecasts using adaptive learning algorithms. Under this approach, the rational expectations equilibrium may become a limit of the temporary learning equilibrium. By contrast, the rational expectations approach retains rational expectations equilibrium continuously over time.

In the literature, economic models with adaptive learning hypothesis have been used for two different purposes. First, because they provide an asymptotic justification for RE and a selection device in the presence of multiple REE, they have been used to offer a rationale for rational expectations. Second, they offer a description of the behaviour of the economy not only asymptotically, but also along the transition to the REE, showing dynamics that are not available under perfect rationality and that could be empirically relevant. Papers more focused on the first aspect of bounded rationality are, for example, those of Evans and Honkapohja (2002), Bullard and Mitra (2001) and Bullard and Mitra (2002); papers centred on the analysis of equilibrium along the transition to the REE include Timmerman (1996), Sargent (1999) and Marcet and Nicolini (2001). I follow this second approach and show that considering learning in a model of monetary policy design is particularly relevant in order to describe not only the rational expectations equilibrium to which we could converge under "plausible" learning schemes, but also the dynamics along the transition to that equilibrium.

2.3.1 *The learning mechanism*

Let us assume that private agents form expectations by learning from past experiences and update their forecasts through recursive least squares estimates[9].

Since, under the *simple expectations-based reaction function* (2.3), neither the IS nor the AS relations depend on expected output gap, the learning equilibrium can be described by focusing on beliefs regarding expected inflation[10]. I assume that agents do not know the effective value of $\overline{a}_{\pi}$ in equation (2.5), but estimate it using past information. In this case,

[9] See Marcet and Sargent (1989 a, b) or Evans and Honkapohja (2001) for a detailed analysis of least squares learning.

[10] In the next section I show formally that this does not affect the results.

private agents' expected inflation is given by:

(2.8) $$E_t^* \pi_{t+1} = a_{\pi,t},$$

where $a_{\pi,t}$ is a statistic inferred recursively from past data according to

(2.9) $$a_{\pi,t} = a_{\pi,t-1} + t^{-1} \left(\pi_{t-1} - a_{\pi,t-1} \right).$$

The perceived law of motion of inflation is updated by a term that depends on the last prediction error[11] weighted by the *gain sequence* t^{-1}. It is well known that in this case the adaptive procedure is the result of a least squares regression of inflation on a constant, and perceived inflation is just equal to the sample mean of past inflations (Marcet, Nicolini, 1997):

(2.10) $$a_{\pi,t} = \frac{1}{t} \sum_{i=1}^{t} \pi_{i-1}.$$

By substituting (2.8) into (2.4) I obtain the actual law of motion of inflation under adaptive learning:

(2.11) $$\pi_t = \alpha\varphi\gamma + \left[\beta + \alpha\varphi \left(1 - \gamma_\pi \right) \right] a_{\pi,t} + \alpha g_t.$$

An important aspect of recursive learning is that the learning equilibrium may converge to the REE, i.e., the estimated parameters $a_{\pi,t}$ converge asymptotically to $\overline{a}_\pi$. In order to provide the conditions for asymptotic stability of the REE under least squares learning, I follow the strand of the literature that uses the E-Stability principle (Marcet and Sargent, 1989a and Evans and Honkapohja, 2001).

2.3.2 *E-stability of the REE*

The stability under learning (E-stability) of a particular equilibrium, is addressed by studying the mapping from the estimated parameters, i.e., the perceived law of motion (PLM), to the true data generating process, i.e., the actual law of motion (ALM).

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[11] This formula implies that private agents do not use today's inflation to formulate their forecasts. This assumption is made purely for convenience and it is often made in models of learning as it simplifies solving the model. The dynamics of the model are unlikely to change.

When expectations in system (2.4) evolve according to expression (2.8), the inflation's ALM is

$$\pi_t = T(a_{\pi,t}) + \alpha g_t, \tag{2.12}$$

where

$$T(a_{\pi,t}) = \alpha\varphi\gamma + [\beta + \alpha\varphi(1 - \gamma_\pi)] a_{\pi,t} \tag{2.13}$$

is the mapping from PLM to ALM of inflation.

Let us define the asymptotic mean prediction error, the mean distance between the ALM and the PLM:

$$h(a_\pi) = \lim_{t\to\infty} [T(a_{\pi,t}) - a_{\pi,t}]. \tag{2.14}$$

As shown in Marcet and Sargent (1999a,b) and Evans and Honkapohja (2001), it turns out that the dynamic system described by equations (2.9), (2.12) and (2.13) can be studied in terms of the associated *ordinary differential equation* (ODE)

$$\frac{da_\pi}{d\tau} = h(a_\pi) = T(a_\pi) - a_\pi, \tag{2.15}$$

where τ denotes "notional" or "artificial" time. Note that the unique fixed point of the T-map, the a_π that makes $\frac{da_\pi}{d\tau} = 0$, is the level of inflation described in equation (2.6), i.e., the REE, $\overline{a}_\pi$. The REE is said to be E-stable if it is locally asymptotically stable under equation (2.14) and under some regularity conditions. As stressed by Evans and Honkapohja (2001), "E-stability determines the stability of the REE under a stylized learning rule in which the PLM parameters ($a_{\pi,t}$, *in our case*) are adjusted slowly in the direction of the implied ALM". Stability under learning, or learnability of the REE, is desirable because it indicates that if agents are learning from past data, their forecasts will converge over time to the REE.

The following lemma describes the necessary and sufficient conditions under which the REE (2.5) is E-stable, pointing out the role of policy decisions.

Lemma 3 *Under the simple expectations-based reaction function (2.3), the REE of (2.4) is E-stable if and only if*

$$\gamma_\pi > 1 - \frac{1-\beta}{\alpha\varphi}$$

Proof. See Appendix 2.C. ■

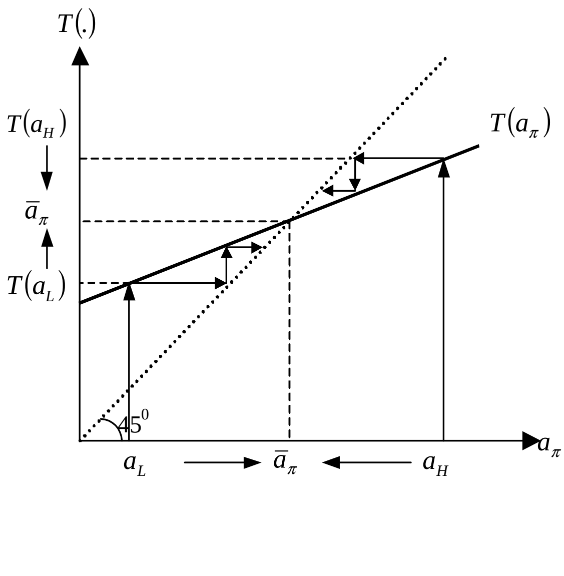
I now show that policy decisions (i.e., the value γ_π) are important not only to obtain an improvement in private agents' forecasts, but also to determine the speed at which the distance between PLM and ALM shorten over time and the learning equilibrium converges to the REE.

2.3.3 *Speed of convergence to the REE*

Figure 1 plots the mapping from PLM to ALM (2.13) and shows how private agents' estimates affect actual inflation along the transition to the REE.

Fig 1

Mapping from PLM to ALM



Lemma 3 states that if the slope of the mapping is smaller than 1, the REE is E-stable, that is, if we start from a perceived level of inflation $a_L < \overline{a}_\pi$ or $a_H > \overline{a}_\pi$, the mean of the prediction error (i.e., the distance between the ALM and the $T(.)$ mapping), $T\left(a_{\pi,t}\right) - a_{\pi,t}$, decreases over time and asymptotically converges to zero (i.e., it converges to the point $\overline{a}_\pi$).

Is there any difference between a policy that results in the slope of $T(.)$, $[\beta + \alpha\varphi(1-\gamma_\pi)] = 0.00001$ and one with $[\beta + \alpha\varphi(1-\gamma_\pi)] = 0.99999$? Looking at the recent literature on monetary policy and learning (Evans and Honkapohja 2001, 2002a, 2002b, 2003 and Bullard and Mitra 2002), the answer is negative. Since in both cases the REE is unique and E-stable, both policies are "good". However, looking at Figure 1 it is clear that the learning equilibrium along the transition is very different under the two policies. By choosing the γ_π, in fact, the policy-maker not only determines the level of inflation in the REE, but also the speed at which the distance between perceived and actual inflation narrows.

In the literature, the problem of the speed of convergence of recursive least square learning algorithms is analyzed mainly through numerical procedures and simulations. The few analytical results are obtained by using a theorem of Benveniste, Metiver and Priouret (1990) that relates the speed of convergence of the learning process to the eigenvalues of the associated ordinary differential equation (ODE) at the fixed point[12]. In the present case, the ODE to be analyzed is the one described in expression (2.15) and the associated eigenvalue is $\beta + \alpha\varphi(1-\gamma_\pi)$, i.e., the slope of the mapping from PLM to ALM (2.13).

The following propositions, adapting arguments in Marcet and Sargent (1995), show that the closer the slope of the mapping from PLM to ALM to 0.5, the slower the learning process.

Proposition 4 *Let us define*

$$S_1 = \left\{ \gamma_\pi : \gamma_\pi > \frac{\alpha\varphi + \beta - 1/2}{\alpha\varphi} \right\}$$

Under the simple expectations-based reaction function (2.3), if $\gamma_\pi \in S_1$, then there is Root-t convergence, i.e.,

$$\sqrt{t}\left(a_{\pi,t} - \overline{a}_\pi\right) \xrightarrow{D} N\left(0, \sigma_a^2\right)$$

with

$$\sigma_a^2 = \frac{\alpha^2 \sigma_g^2}{\left[1 - \beta - \alpha\varphi(1-\gamma_\pi)\right]} \tag{2.16}$$

Proof. See Appendix 2.D. ■

[12] See see for example Marcet and Sargent (1995) for an interpretation of the ODE.

If the conditions of proposition 4 are satisfied, the estimates $a_{\pi,t}$ converge to the RE value $\overline{a}_{\pi}$ at root-t speed, that is, the speed at which in classical econometrics the mean of the distribution of the *least square estimates* converges to the true value of the parameters estimated. Note that the formula for the variance of the estimator a_{π} is modified with respect to the classical case where $\sigma_a^2 = \alpha^2 \sigma_g^2$.

Proposition 5 *Under the simple expectations-based reaction function (2.3), if $\gamma_{\pi} \in S_1$, then the weaker the response to expected inflation (the smaller γ_{π}), the greater the asymptotic variance of the limiting distribution, σ_a^2.*

Proof. See Appendix 2.E. ■

Looking at the formula for the asymptotic variance (2.16) it is possible to understand the role of policy decisions in determining the speed of convergence to the REE: for a weaker response to expected inflation, the slope of the $T(.)$ mapping is steeper and the convergence is slower in the sense that the asymptotic variance of the limiting distribution is greater.

Now, let us define $S_2 = \left\{ \gamma_{\pi} : \frac{\alpha\varphi+\beta-1}{\alpha\varphi} < \gamma_{\pi} < \frac{\alpha\varphi+\beta-1/2}{\alpha\varphi} \right\}$. If $\gamma_{\pi} \in S_2$, the estimates $a_{\pi,t}$ converge to the REE $\overline{a}_{\pi}$, but at a speed different from root-t. In this case, as Marcet and Sargent (1995) suggest, the importance of initial conditions fails to die out at an exponential rate (as is needed for *root-t convergence*) and the learning equilibrium converges to the REE at a rate slower than root-t. In particular, even when $\gamma_{\pi} \in S_2$ it is possible to show by means of simulations that as the slope of the $T(.)$ mapping increases, the speed of convergence decreases[13]. Figure 2 shows examples for the two cases where $\gamma_{\pi} \in S_1$ and $\gamma_{\pi} \in S_2$.

Since the least squares algorithm adjusts each parameter towards the truth when new information is received, the new belief $a_{\pi,t+1}$ will be an average of the previous beliefs $a_{\pi,t}$ and the actual value $T(a_{\pi,t})$ plus an error. When the reaction of the policy maker to expected inflation is *strong* ($\gamma_{\pi} \in S_1$), the derivative of $T(.)$ is smaller than (or equal to) $1/2$ and $T(a_{\pi,t})$ is close to $\overline{a}_{\pi}$; when the reaction is *weak* ($\gamma_{\pi} \in S_2$), the derivative of $T(.)$ is larger than $1/2$ and $T(a_{\pi,t})$ is close to $a_{\pi,t}$ instead of being close to $\overline{a}_{\pi}$, so the average can stay far from the REE for a long time.

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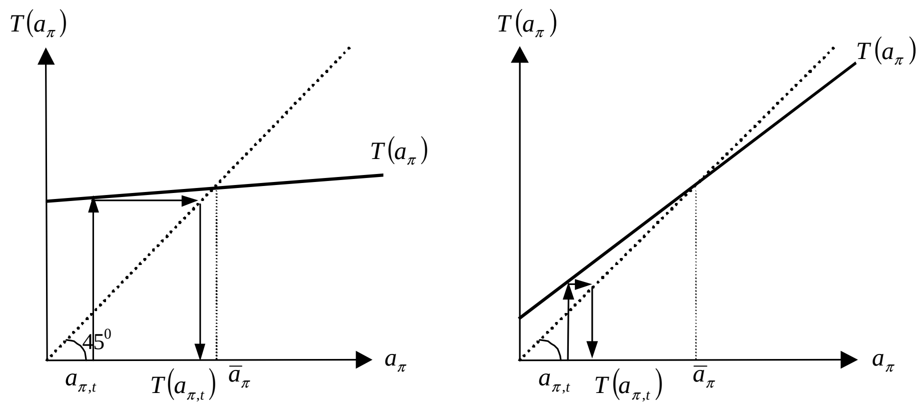
[13] See section 4 for simulations that relate speed of convergence and the slope of the $T()$ mapping.

Fig. 2

Mapping from PLM to ALM and the speed of convergence

a) Low γ_π b) High γ_π



It is worth noting that even though the transition is quite different in the two cases analyzed here, the learning equilibrium could end up converging to the same REE and, according to policy-maker preferences, the speed of convergence could become a relevant variable in the policy decision problem.

2.4 *Optimal monetary policy under discretion*

The reason the analysis starts with the *simple expectations-based reaction function* (2.3) is that it simplifies the dynamics under learning. I now consider the optimal monetary policy problem without commitment (discretionary policies), where any promises made in the past by the policy-maker do not constrain current decisions. In deriving the optimal discretionary policy, I follow Evans and Honkapohja (2002), assuming that the policy-maker cannot manipulate private agent's beliefs. This assumption implies that the optimality conditions derived under learning are equivalent to the ones obtained under RE.

The policy problem consists in choosing the time path for the instrument i_t to engineer a contingent plan for target variables π_t and $(x_t - \overline{x})$ that maximizes the objective function

(2.17) $$\underset{x_t,\pi_t}{Max} - E_0 \sum_{t=0}^{\infty} \beta^t L(\pi_t, x_t)$$

where

(2.18) $$L(\pi_t, x_t) = \frac{1}{2}\left[\pi_t^2 + \lambda (x_t - \overline{x})^2\right]$$

subject to the constraints (2.1) and (2.2) and $E_t^* \pi_{t+1}$, $E_t^* x_{t+1}$ given.

The solution of this problem[14], as derived in Evans and Honkapohja (2002), yields the following optimality conditions[15]

(2.19) $$\pi_t = \frac{\lambda\alpha}{(\lambda+\alpha^2)}\overline{x} + \frac{\lambda\beta}{(\lambda+\alpha^2)}E_t^*\pi_{t+1}$$

(2.20) $$x_t = \frac{\lambda}{(\lambda+\alpha^2)}\overline{x} - \frac{\alpha\beta}{(\lambda+\alpha^2)}E_t^*\pi_{t+1}.$$

The optimal outcome could be written as a reaction function that relates the policy instrument i_t to the current state of the economy and the expectations of private agents:

(2.21) $$i_t = \gamma^\star + \gamma_x^\star E_t^* x_{t+1} + \gamma_\pi^\star E_t^* \pi_{t+1} + \gamma_g^\star g_t$$

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[14] I consider λ as an exogenous policy parameter, as is often done in monetary policy literature. An alternative approach is to obtain λ as the result of the general equilibrium problem. In this case λ would depend on representative consumer preferences and firms' price setting rules.

[15] To obtain this result note that, after substituting the constraints (2.1) and (2.2) into the loss function, the problem becomes

$$\underset{i_t}{Max} - \frac{1}{2}\left[(\alpha E_t^* x_{t+1} - \alpha\varphi i_t + \alpha g_t + (\beta+\alpha\varphi) E_t^*\pi_{t+1})^2 + \lambda (E_t^* x_{t+1} - \varphi (i_t - E_t^*\pi_{t+1}) + g_t - \overline{x})^2 \right]$$

s.t $E_t^* x_{t+1}, E_t^*\pi_{t+1}$ given

and the FOC is:

$$i_t = -\frac{\lambda}{(\lambda+\alpha^2)\varphi}\overline{x} + \frac{1}{\varphi}E_t^* x_{t+1} + \left(1 + \frac{\alpha\beta}{(\lambda+\alpha^2)\varphi}\right) E_t^*\pi_{t+1} + \frac{g_t}{\varphi}$$

and (2.19), (2.20) are obtained by substituting this expression into (2.1) and (2.2).

where

$$\begin{aligned} \gamma^\star &= -\frac{\lambda}{(\lambda+\alpha^2)\,\varphi}\overline{x} \\ \gamma_x^\star &= \frac{1}{\varphi} \\ \gamma_\pi^\star &= 1+\frac{\alpha\beta}{(\lambda+\alpha^2)\,\varphi} \\ \gamma_g^\star &= \frac{1}{\varphi} \end{aligned} \tag{2.22}$$

Since interest rate rule (2.21) states that the policy maker should react to the expected inflation and output gap, it is sometimes called the *optimal expectations-based reaction function (*Evans and Honkapohja, 2002). However, to stress the fact that this policy is optimal under rational expectations but is not necessarily optimal under learning, I call it the *RE-optimal expectations-based reaction function*[16].

Under rational expectations (i.e. $E_t^* x_{t+1} = E_t x_{t+1}$ and $E_t^* \pi_{t+1} = E_t \pi_{t+1}$) the equilibrium of the dynamic system defined by (2.19) and (2.20) is:

$$\pi_t = E_t \pi_{t+1} = \overline{a}_\pi \quad \text{and} \quad x_t = E_t x_{t+1} = \overline{a}_x,$$

where $\overline{a}_\pi, \overline{a}_x$ are

$$\overline{a}_\pi = \frac{\lambda\alpha}{(\lambda+\alpha^2)-\lambda\beta}\overline{x} \quad \text{and} \quad \overline{a}_x = \frac{\lambda\,(1-\beta)}{(\lambda+\alpha^2)-\lambda\beta}\overline{x}.$$

Assuming, instead, that private agents do not know $\overline{a}_\pi$ and $\overline{a}_x$ but estimate them recursively, the expected inflation and output gap would be given by

$$E_t^* \pi_{t+1} = a_{\pi,t} \quad \text{and} \quad E_t^* x_{t+1} = a_{x,t},$$

where $a_{\pi,t}$ and $a_{x,t}$ are inferred recursively from past data according to

$$a_{\pi,t} = a_{\pi,t-1} + t^{-1}\left(\pi_{t-1} - a_{\pi,t-1}\right)$$

16 In sections 4 and 5 I call it *Evans and Honkapohja (EH) policy*, to avoid notational flutter.

$$a_{x,t} = a_{x,t-1} + t^{-1}\left(x_{t-1} - a_{x,t-1}\right)$$

and the ALM of the economy would be

$$\left(\pi_t, x_t\right) = T\left(a_{\pi,t}, a_{x,t}\right),$$

where

$$T\left(a_{\pi,t}, a_{x,t}\right) = \left(\frac{\lambda\alpha}{(\lambda+\alpha^2)}\overline{x} + \frac{\lambda\beta}{(\lambda+\alpha^2)}a_{\pi,t}, \frac{\lambda}{(\lambda+\alpha^2)}\overline{x} - \frac{\alpha\beta}{(\lambda+\alpha^2)}a_{\pi,t}\right) \tag{2.25}$$

is the mapping from PLM to ALM.

Now, since the right-hand side of (2.25) does not depend on $a_{x,t}$, as in the previous section, properties of the learning equilibrium can be described simply by focusing on the mapping from perceived inflation to actual inflation

$$\pi_t = \frac{\lambda\alpha}{(\lambda+\alpha^2)}\overline{x} + \frac{\lambda\beta}{(\lambda+\alpha^2)}a_{\pi,t}. \tag{2.26}$$

For reasonable values of the parameters[17], the following lemma could be derived (Evans and Honkapohja, 2002):

Lemma 6 *Under the RE-optimal expectations-based reaction function the rational expectations equilibrium is unique and E-stable.*

Proof. See Evans and Honkapohja (2002). ■

Lemma 6 states that the *RE-optimal expectations-based policy rule* derived as the optimal solution of the problem under discretion and rational expectations is a "good" policy, not only under RE but also under the hypothesis that private agents update their forecasts through recursive least squares estimates. Here with "good" policy I refer to the criterion used

---

[17] Clarida, Galí and Gertler (2000) and Woodford (1999) derive from regressions on US data respectively, the following values for the economy parameters

$$\varphi = 1,\ \alpha = 0.3,\ \beta = 0.99,\ \rho_u = 0.35$$
$$\varphi = 0.17,\ \alpha = 0.024,\ \beta = 0.99,\ \rho_u = 0.35$$

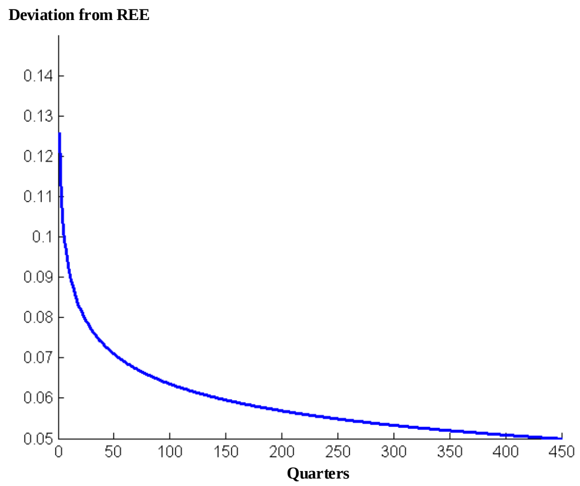
by Bullard and Mitra (2001) to evaluate policy rules, based on determinacy and E-stability of the REE.

However, simulating the model under the *RE-optimal expectations-based reaction function*, with $\lambda = 0.5$ and Clarida, Galì and Gertler (2000) calibration, the distance between the learning equilibrium and the REE would be significatively different from zero for many periods.

Figure 3 shows the evolution of perceived inflation under learning for this policy. Assuming an initial perceived inflation 15 per cent higher than the REE, more than 450 periods (quarters) are needed to reduce the distance between the perceived inflation and the REE from 15 to 5 per cent[18]!

Fig 3

Deviation of actual inflation from the REE



Applying a similar argument to that used in propositions 4 and 5 it is possible to state the following proposition about the speed of convergence and the role of the weight to output gap in the welfare function, λ.

Proposition 7 *Under the RE-optimal expectations-based reaction function, the speed of convergence of the learning process depends negatively on the weight that the policy-maker*

[18] With Woodford (1999) calibration the convergence is even slower.

gives to output gap relative to inflation. In particular, under flexible inflation targeting policies ($\lambda > 0$), the greater the weight to output gap, the slower the learning process.

Proof. See Appendix 2.F. ■

Under learning, with the *RE-optimal expectations-based reaction function*, proposition 7, by looking at the slope of the mapping from perceived inflation to actual inflation, relates the speed of convergence of the learning equilibrium to the relevance of output gap in the objective function (2.17).

Fig 4

Slope of the mapping from PLM to ALM

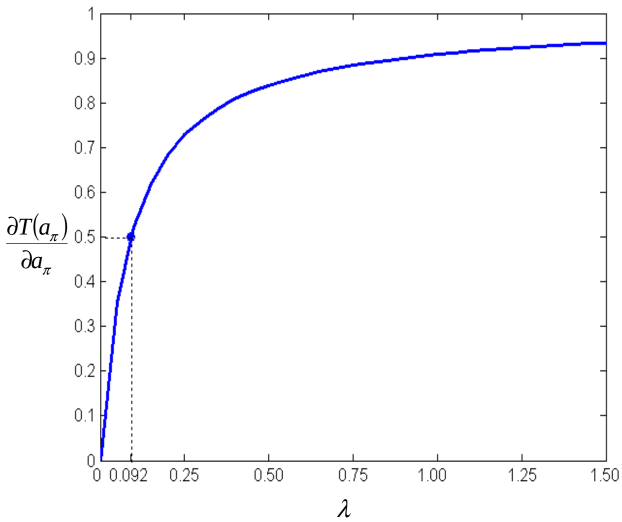


Figure 4 shows how the slope of the mapping from the PLM to ALM of inflation changes as the weight that the policy-maker gives to the output gap relatively to inflation increases[19].

Table 1 shows that when the policy-maker cares equally about output gap and inflation ($\lambda = 1$), the slope of the mapping of PLM to ALM is around 0.9; when he cares less about output gap than inflation the slope is smaller (for example if $\lambda = 0.5$, then the slope is 0.84) but, unless λ is smaller than 0.1, *root-t convergence* is never reached.

---

[19] I use the Clarida, Gali and Gertler (CGG) calibration for US. Similar results obtain with the Woodford (W) calibration.

Tab 1

Slope of the mapping from PLM to ALM

λ	$\frac{dT(a_\pi)}{da_\pi}$
0	0
0.09	0.49
0.1	0.52
0.5	0.84
1	0.91
2	0.95
100	0.99

The fact that the learning speed could be very slow (or very fast) depending on policy decisions[20], suggests that when they consider the monetary policy design problem under learning, policy-makers should take into account the transition to the REE. E-stability, once transition is taken into account, is not itself sufficient to characterize a policy in a context of bounded rationality, and policy (2.21), which is optimal under rational expectations, may not be optimal under learning.

In the following sections I show that, in general, the analysis of the speed of convergence is helpful in evaluating policy rules.

3. Speed of convergence and policy design

Let us consider a third and more generic set of *expectations-based reaction functions*

(3.1) $$i_t = \gamma + \gamma_x E_t^* x_{t+1} + \gamma_\pi E_t^* \pi_{t+1} + \gamma_g g_t$$

and show how to discriminate between the elements of this set using a measure of the speed of convergence.

---

[20] This result could be applied to the problem of "optimal delegation", justifing a conservative central bank when fast convergence is required.

I define

$$Y_t = \begin{bmatrix} \pi_t \\ x_t \end{bmatrix}$$

and write the ALM of inflation and output gap, obtained by substituting (3.1) into (2.1) and (2.2):

(3.2) $$Y_t = Q + F \times E_t^* Y_{t+1} + S g_t,$$

where Q and S are vectors that depend respectively on policy parameters γ and γ_g and F is a matrix that depends on the policy parameters γ_x, γ_π

$$Q = \begin{bmatrix} \alpha\varphi\gamma \\ \varphi\gamma \end{bmatrix}, \quad S = \begin{bmatrix} \alpha\left(1 - \varphi\gamma_g\right) \\ \left(1 - \varphi\gamma_g\right) \end{bmatrix} \tag{3.3}$$

$$F = \begin{bmatrix} \left(\beta + \alpha\varphi\left(1 - \gamma_\pi\right)\right) & \alpha\left(1 - \varphi\gamma_x\right) \\ \varphi\left(1 - \gamma_\pi\right) & \left(1 - \varphi\gamma_x\right) \end{bmatrix}. \tag{3.4}$$

The REE is

(3.5) $$Y_t = \overline{A} + S g_t,$$

where

(3.6) $$\overline{A} = \begin{bmatrix} \frac{-\alpha\gamma}{\gamma_x(1-\beta)+(\gamma_\pi-1)\alpha} \\ \frac{-\gamma(1-\beta)}{\gamma_x(1-\beta)+(\gamma_\pi-1)\alpha} \end{bmatrix}.$$

The following lemma provides a characterization of the REE obtained under the *generic expectations-based reaction functions* (3.1), showing how the choice of the reaction to the expected inflation and output gap (i.e., the value of γ_π and γ_x) determines the equilibrium level of inflation and output gap under the rational expectations hypothesis.

Lemma 8 *In the REE, under expectations-based reaction functions (3.1), for* $\gamma \neq 0$*, for* $\gamma_\pi > 1$ *the higher the value of* γ_π *the higher inflation and output gap (in absolute values), for* $\gamma_\pi < 1$ *the higher the value of* γ_π *the lower the inflation and the output gap level; moreover, the higher the value of* γ_x *the lower the inflation and output gap level (in absolute values).*

Proof. See Appendix 2.G. ■

3.1 *The learning mechanism*

Under the *generic expectations-based reaction functions* (3.1), if private agents update their forecasts through recursive least squares algorithms, expected inflation and output gap evolve in a more complex way then described in section 2.3. As both the IS and the AS relations also depend on the expected output gap[21], the learning equilibrium cannot be described only by focusing on beliefs regarding expected inflation (see Appendix 2.H for a complete description of the learning mechanism under the generic reaction function of the type (3.1)).

In this case, if agents do not know $\overline{A}$ but estimate it recursively, expectations are given by:

$$E_t^* Y_{t+1} = A_t, \tag{3.7}$$

where A_t are statistics inferred from past data according to

$$A_t = A_{t-1} + t^{-1} \left(Y_{t-1} - A_{t-1} \right). \tag{3.8}$$

By substituting (3.7) into (3.2) I obtain the ALM of inflation and output gap under adaptive learning

$$Y_t = T\left(A_t\right) + S g_t,$$

where $T\left(.\right)$ is the mapping from PLM to ALM

$$T\left(A_t\right) = Q + F \times A_t$$

and

$$\frac{dA}{d\tau} = T\left(A\right) - A \tag{3.9}$$

---

[21] In equation (3.2) I do not impose that the elements in the second column of the R matrix are not necessarily zero.

is the associated *ordinary differential equation*. Note that the unique fixed point of the T-map is the vector $\overline{A}$ derived in equation (3.6). As in section 2.3, the REE is said to be E-stable if it is locally asymptotically stable under equation (3.9) and under some regularity conditions. Underlying the role of policy decisions, the following lemma describes the necessary and sufficient conditions under which the REE (3.5) is E-stable.

Lemma 9 *Under a generic expectations-based reaction function (3.1), the necessary and sufficient conditions for a rational expectations equilibrium to be E-stable are*

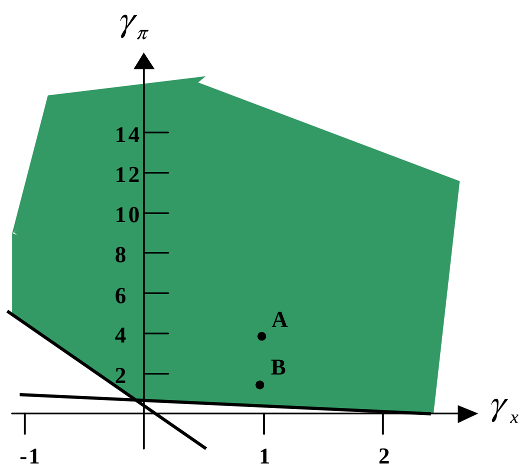
$$\begin{aligned} \gamma_\pi &> 1 - \frac{1-\beta}{\alpha\varphi} - \frac{\gamma_x}{\alpha} \quad \text{for} \quad \gamma_x \leq \frac{-1+\beta}{\varphi\beta} \\ \gamma_\pi &> 1 - \frac{(1-\beta)}{\alpha}\gamma_x \quad \text{for} \quad \gamma_x \geq \frac{-1+\beta}{\varphi\beta} \end{aligned}$$

Proof. See Appendix 2.H. ■

Figure 5 shows, with CGG calibration, all the combinations (γ_π, γ_x) under which the REE is E-stable[22].

Fig 5

E-stable region under the expectations-based policy rule



Note that, since the *RE-optimal expectations-based policy rule* (2.21) is an element of the set of *generic expectations-based policy rules* (3.1), points A and B represent the combination

22 A similar figure was obtained under Woodford (1999) calibration.

$\gamma_{\pi}^{\star}, \gamma_{x}^{\star}$ in the two extreme cases where policy-makers do not care about the output gap, $\lambda = 0$ (point A), and where they give equal weight to both inflation and the output gap, $\lambda = 1$ (point B). Figure 5 shows that in both cases the REE is E-Stable[23]. However, for $\lambda = 1$ this is very close to the bounds of the E-stability region; in this case, if the policy-maker chooses the *RE-optimal expectations-based policy rule* but improperly calibrates the model it can easily end up outside the stability region, enforcing a non-stationary policy.

Finally, the fact that the origin is not in the stable region is consistent with the non-convergence result of Evans and Honkapohja (2002): policies that react only to shocks, ignoring expectations, are unstable under learning.

3.2 *The transition to the REE*

In the previous sections, I have shown that policy-makers settle the coefficients of matrix F, by means of reaction functions. This means that the evolution of estimated coefficients in private agents' forecasts (i.e., the speed at which private agents learn) strictly depends on policy decisions.

Proposition 10 *Let us define*

$$Q_1 = \left\{ \gamma_{\pi}, \gamma_x : \ \gamma_x \leq \frac{4\beta - 1}{4\varphi\beta} \ \textit{and} \ \gamma_{\pi} > 1 + \frac{\beta}{\alpha\varphi} - \frac{\gamma_x}{\alpha} \right\}$$

$$Q_2 = \left\{ \gamma_{\pi}, \gamma_x : \ \gamma_x \geq \frac{4\beta - 1}{4\varphi\beta} \ \textit{and} \ \gamma_{\pi} > 1 + \frac{1 - 2\beta}{2\alpha\varphi} - \frac{1 - 2\beta}{\alpha}\gamma_x \right\}.$$

Under expectations-based reaction functions (3.1), if $(\gamma_{\pi}, \gamma_x) \in Q_1$ or Q_2, then the speed of convergence of the learning equilibrium is ***Root-t****, that is*

$$\sqrt{t}\,(A_t - A) \xrightarrow{D} N\,(0, \Omega)$$

---

[23] It is possible, moreover, to show that for any positive and finite value of λ, i.e., for all *flexible inflation targeting* policies under the *optimal expectation-based reaction function* (2.21) the rational expectation equilibrium is E-Stable (Evans and Honkapohja, 2002).

where the matrix Ω satisfies

(3.10) $$\left[\frac{I}{2}(F-I)\right]\Omega+\Omega\left[\frac{I}{2}(F-I)\right]'+SS'\sigma_g^2=0$$

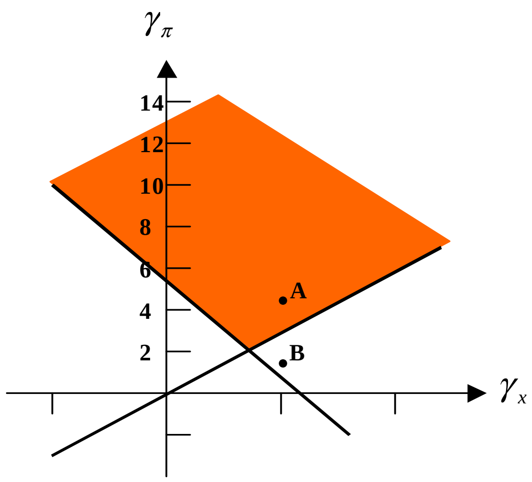
Proof. See Appendix 2.I. ■

Under the generic expectations-based reaction function (3.1), if the REE is E-stable but conditions in proposition 10 are not satisfied, then not all the eigenvalues of the matrix F are smaller than one half**.** In this case, as suggested in section 2.3, the learning equilibrium converges to the REE at a slower rate than root-t.

Figure 6 shows all combinations of γ_π and γ_x for which there is root-*t* convergence.

Fig 6

Root-t convergence under the expectations-based policy rule



By comparing Figure 5 and Figure 6, it is apparent that the set of combinations (γ_x, γ_π) resulting in root-*t* convergence is much smaller than the one under which E-stability holds. Points A and B in Figure 6 show the combination $\gamma_\pi^\star, \gamma_x^\star$, i.e., the reaction to expected inflation and output gap under the *RE-optimal expectations-based policy rule*, in the two extreme cases where policy-makers do not care about the output gap, $\lambda = 0$ (point A), and where they give equal weight to both inflation and the output gap, $\lambda = 1$ (point B). As derived in section 2.4, Figure 6 shows that when the policy-maker gives weight $\lambda = 1$ there is no *root-t* convergence.

In the previous sections, in order to characterize how policies determine the speed of converge to REE, I focused only on one policy parameter at a time (γ_π in section 2.3 and λ in section 2.4). Here, on the contrary, since the speed of convergence, as suggested by Marcet and Sargent (1995), is determined by the eigenvalues of F and this matrix depends on both γ_π and γ_x, I have to focus on two policy parameters at a time. For this reason I define the speed of convergence isoquants that map elements of the set of *expectations-based reaction functions* into a speed of convergence measure.

Definition 1 *A speed of convergence isoquant is a curve in R^2 along which all points (i.e., combinations (γ_π, γ_x) of an expectations-based reaction function (3.1)) result in the same real part of the largest eigenvalue $z_1$of the matrix F*[24].

For simplicity I restrict the analysis to the set

$$\Gamma = \{\gamma_\pi, \gamma_x : \gamma_\pi > 0, \gamma_x > 0 \text{ and } 0 \leq z_1 < 1\}.$$

The following definition and proposition describe the main properties of the speed of convergence isoquants:

Definition 2 *The speed of convergence, represented by the speed of convergence isoquants, is monotonically increasing in the reaction to expected inflation (γ_π) if, given the reaction to the expected output gap (γ_x), the real part of the largest eigenvalue $z_1$of the matrix F is decreasing in γ_π.*

A similar definition for monotonicity with respect to the expected output gap could be settled.

Proposition 11 *The speed of convergence relation, represented by the speed of convergence isoquants and defined over Γ is: (i) monotonically increasing in γ_π, (ii) not monotonic with respect to γ_x.*

Proof. See Appendix 2.L. ■

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[24] In the definition I relate speed of convergence to the eigenvalues of the matrix F. In general, as shown in previous sections, the speed of convergence is related to the eigenvalues of the derivatives of the mapping from PLM to ALM, $T(A)$. In this case, the derivative is equal to F.

Proposition 13 states that, for a given reaction to output gap expectations, the policy-maker, by increasing the reaction to expected inflation increases monotonically the speed at which private agents learn. On the contrary, for a given reaction to expected inflation, by increasing the reaction to the expected output gap, private agents could learn both faster or slower, depending on the value of γ_π.

Figure 7 shows the speed of convergence isoquants: the lower the isoquant, the slower the convergence. In fact, the larger the real part of z_1, the lower the isoquant and, from Marcet and Sargent (1995), the larger the real part of z_1, the slower the convergence.

Fig 7

The speed of learning isoquants

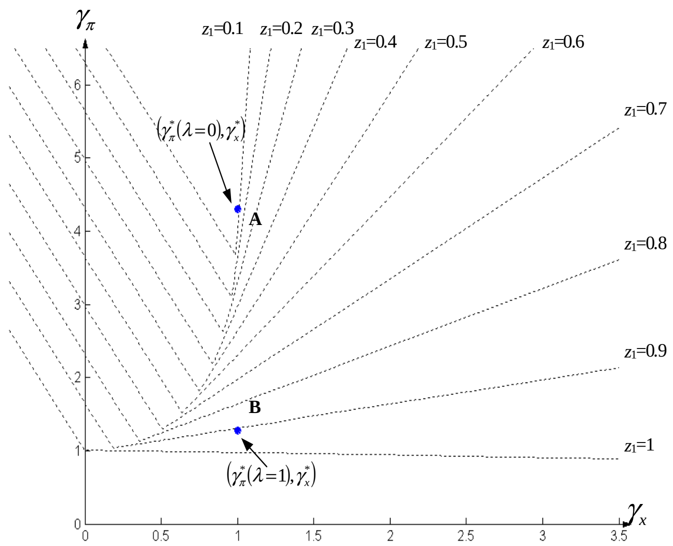

Figure 7 illustrates a practical way of using the speed of convergence of the learning equilibrium to characterize monetary policies. For example, a combination of γ_π and γ_x just above the isoquant $z_1 = 1$ (for example, point B) determines an E-stable REE, but would imply very slow convergence. The combinations of γ_π and γ_x that stay above the isoquant $z_1 = 0.1$ imply a very fast learning process. The combinations of γ_π and γ_x that stay above the isoquant $z_1 = 0.5$ imply a learning process that converges to the REE at a *root-t* speed.

Let us now see how to make active use of the speed of convergence in the study of optimal policies under discretion.

4. Discretionary policy and learning

In section 2.4, it was shown that under rational expectations the optimal monetary policy under discretion is given by the reaction function (2.21). Now, considering the generic set of *expectations-based reaction functions* (3.1) we have that

Proposition 12 *Under rational expectations, there are infinitely many expectations-based reaction functions, i.e., combinations of* $\gamma, \gamma_x, \gamma_\pi, \gamma_g$ *that result in the optimal REE for* $\{\pi_t, x_t\}$ *defined in (2.23) and (2.24).*

Proof. See Appendix 2.M. ■

Evans and Honkapohja (2002) say that the *expectations-based reaction function* (2.21) is not only a "good" policy because it determines an E-stable REE, but it also "*implements optimal discretionary policy in every period and for all values of private expectations*" in a context where "*private agents behave in a boundedly rational way*". In order to identify (2.21) as the optimal policy rule under discretion and learning, however, the crucial assumption is that "*the policy-maker does not make active use of learning behaviour on the part of agents*" (Evans and Honkapojha, 2002).

If, under rational expectations, the problem of optimal "discretionary policy" implies, by definition, that policy-makers cannot affect private agents' expectations, under the hypothesis of bounded rational private agents, since policy decisions affect the learning process, a rational policy-maker with full information should take properties of the learning equilibrium into account in solving the monetary policy design problem. In fact, if private agents' expectations are the result of the estimates of the learning parameters that depend on past values of the monetary policy instrument, the policy-maker's decisions, will affect future estimates and, consequently, the private agents' learning process[25]. The *expectations-based reaction function* (2.21) is not necessarily optimal under learning but could be defined as *asymptotically-optimal*. However, if private agents' perceived law of motion is well specified, once the learning process has converged to rational expectations, not only the policy rule (2.21) will be optimal, but proposition 14 says there is a continuum of *expectations-based policy rules* that result in the same REE. Since these policies could determine different learning equilibria along the

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[25] See Dedola and Ferrero (2003) for a derivation of the truely optimal policy under learning.

transition to the REE, a device for discriminating between them is required[26] and the speed of convergence isoquants derived in the previous section could be a useful starting point.

Let us consider a restricted set of *asymptotically-optimal expectations-based reaction functions* and describe the speed of convergence that these determine along the transition to the REE.

Proposition 13 *The maximum speed of convergence of the learning process that could be reached under the restricted set of asymptotically-optimal expectations-based reaction functions,*

$$i_t = \gamma' + \gamma'_x E_t x_{t+1} + \gamma'_\pi E_t \pi_{t+1} + \gamma'_g g_t, \tag{4.1}$$

with

$$\begin{aligned} \gamma'_g &= \gamma^*_g = \frac{1}{\varphi} \\ \gamma' &= \gamma^* = -\frac{\lambda}{(\lambda+\alpha^2)\varphi}\overline{x} \\ \gamma'_\pi &= \frac{\gamma^R - \gamma^*}{a_\pi} - \frac{a_x}{a_\pi}\gamma'_x = \frac{\lambda\left((1+\alpha\varphi)(\lambda+\alpha^2) - \lambda\beta\right)}{(\lambda+\alpha^2)-\lambda\beta}\overline{x} - \frac{1-\beta}{\alpha}\gamma'_x, \end{aligned} \tag{4.2}$$

depends negatively on the weight that the policy-maker gives to output gap relative to inflation.

Proof. See Appendix 2.N. ■

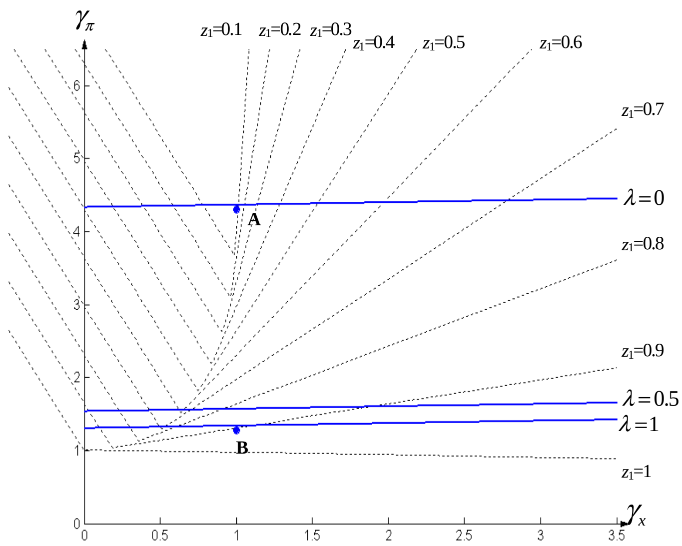
Proposition 15 states that taking the set of reaction functions with coefficients γ' and γ'_g equal to the ones in the *optimal expectations-based reaction function* (2.21) and $\gamma'_\pi = \frac{\lambda\left((1+\alpha\varphi)\left(\lambda+\alpha^2\right)-\lambda\beta\right)}{(\lambda+\alpha^2)-\lambda\beta}\overline{x} - \frac{1-\beta}{\alpha}\gamma'_x$, the economy converges asymptotically to the optimal REE under discretion, but for a given λ the policy-maker can bring about a different speed of convergence. Note, instead, that under policy rule (2.21) for each λ there was a given speed of convergence. In particular, under *asymptotically optimal expectations-based reaction functions* (4.1), the larger the relative weight on output gap (λ), the larger will be the real part of the biggest eigenvalue of the F matrix and the slower the fastest speed of convergence that a policy-maker could induce.

[26] *"There is no single policy rule that is uniquely consistent with the optimal equilibrium. Many rules may be consistent with the same equilibrium, even though they are not equivalent insofar as they imply a commitment to different sorts of out-of equilibrium behaviour" (Svensson and Woodford, 1999).*

Figure 8 shows, in the same picture, the speed of learning isoquants and, for given λ, combinations of γ_x and γ_x under which the economy will converge asymptotically to the optimal REE under discretion.

Fig 8

Asymptotically-optimal expectations-based reaction functions



The line $\lambda = 0$ shows that if the policy-maker does not care about the output gap, by imposing $\gamma_\pi = \gamma'_\pi$, he can choose combinations of γ_π and γ_x such that the speed of convergence ranges from very slow to very fast: the line $\lambda = 0$, in fact, intersects isoquants $z_1 = 0.1$ (very fast speed), $z_1 = 0.7$ (speed of convergence slower than root-t) and many others. If, instead, the relative weight to output gap is one half, i.e., the line $\lambda = 0.5$, the policy-maker could choose only combinations of γ_π and γ_x such that the speed of convergence is slower than root-t: the line $\lambda = 0.5$ does not intersect any isoquant with $z_1 \leq 0.5$; if the policy-maker cares equally about inflation and output gap, i.e., the line $\lambda = 1$, he can choose only combinations of γ_π and γ_x such that the speed of convergence is very slow, slower than root-t, since the line $\lambda = 1$ does not intersect any isoquant with $z_1 \leq 0.7$.

Points A and B in Figure 8 also show another important result that will be analyzed further in the next section: for a given value of λ there are infinitely many expectations-based policies that determine asymptotically the same REE, but induce a faster (or slower) speed of convergence than the one determined by policy (2.21).

4.1 *The mapping from PLM to ALM*

In order to show how the central bank can make active use of private agents' learning behaviour in the monetary policy design problem under discretion, I now consider more in detail the mapping from perceived to actual variables.

To avoid notational flutter, let us define the parameters

$$\Gamma^* = \frac{\lambda\beta}{(\lambda + \alpha^2)}, \Phi^* = \frac{\lambda\alpha}{(\lambda + \alpha^2)}\overline{x}$$

and rewrite the coefficients in the *RE-optimal expectations-based reaction function* (2.21),

$$\gamma^* = -\frac{\Phi^*}{\varphi\alpha}, \gamma_x^* = \gamma_g^* = \frac{1}{\varphi}, \gamma_\pi^* = \left(1 + \frac{\beta - \Gamma^*}{\alpha\varphi}\right) \tag{4.3}$$

and the optimality conditions (2.19) and (2.20)

$$\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \begin{bmatrix} \Phi^* \\ \frac{\Phi^*}{\alpha} \end{bmatrix} + \begin{bmatrix} \Gamma^* & 0 \\ -\frac{(\beta-\Gamma^*)}{\alpha} & 0 \end{bmatrix} \begin{bmatrix} E_t\pi_{t+1} \\ E_t x_{t+1} \end{bmatrix}. \tag{4.4}$$

Under least square learning the mapping from PLM to ALM (2.26) can be rewritten as

$$T\left(a_{\pi,t}, a_{x,t}\right) = \left(\Phi^* + \Gamma^* a_{\pi,t}, \frac{\Phi^*}{\alpha} - \frac{(\beta - \Gamma^*)}{\alpha} a_{\pi,t}\right). \tag{4.5}$$

In order to study the convergence of the learning equilibrium to the REE, section 2.3 showed that the analysis could be concentrated on the mapping from perceived inflation to actual inflation

$$T\left(a_{\pi,t}\right) = \Phi^* + \Gamma^* a_{\pi,t} \tag{4.6}$$

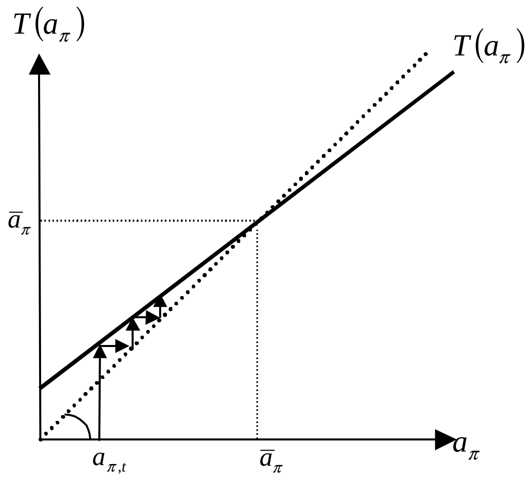
The necessary and sufficient condition for E-stability reduces to $\Gamma^* < 1$.

To give an example, since I consider λ to be an exogenous policy parameter, let us assume that the policy-maker gives a positive weight $\lambda = 0.5$ (note that with this weight I assume that the policy-maker cares twice as strongly about inflation than about output gap). In this case the mapping $T\left(a_{\pi,t}\right)$ has a slope equal to 0.84 under CGG parametrization. Figure 9 shows the mapping from PLM to ALM.

Figure 9 shows that, even if initial perceived inflation is not too far from the REE, since the slope of the $T(.)$ mapping is close to 1, the transition from the learning to the RE equilibrium is very slow.

Fig 9

The mapping T $(a_{\pi,t})$ from PLM to ALM ($\boldsymbol{\lambda} = \mathbf{0.5}$)



4.2 *Adjusting the learning speed*

In the previous section, under the *RE-optimal expectations-based reaction function* (2.21) suggested by Evans and Honkapohja (2002) (*EH policy* from now on), if the policy-maker follows a flexible inflation targeting policy rule with $\lambda = 0.5$, the private agents' learning process will converge very slowly to the RE (slower than root-*t* convergence). The question now is how a policy-maker who wants in the long run (i.e., once the private agents have learned the REE) to reach the same REE determined by the reaction function (2.21) can speed up or slow down the private agents' learning process. To answer to this question let us introduce a new expectations-based policy rule[27]:

[27] At the beginning of this section I showed an asymptotically-optimal policy (4.1) that allowed a choice to be made among different speeds of convergence. However, under that policy, the analysis of the learning dynamics involved a mapping from PLM to ALM with both perceived inflation and output gap. Here, instead, I consider a policy that allows a choice between diffent speeds of convergence just by looking at a mapping from PLM to ALM involving only expected inflation, as under the *RE-optimal expectations-based reaction function* (2.21).

Definition 3 *The Adjusted Learning Speed-*Γ' *(ALS-*Γ'*) policy rule, is an expectations-based reaction function*

(4.7) $$i_t^{ALS}(\Gamma') = \gamma^{ALS} + \gamma_x^{ALS} E_t x_{t+1} + \gamma_\pi^{ALS} E_t \pi_{t+1} + \gamma_g^{ALS} g_t$$

with coefficients

$$\begin{aligned} \gamma^{ALS} &= -\frac{\Phi^*(1-\Gamma')}{(1-\Gamma^*)\,\alpha\varphi} = -\frac{\lambda\overline{x}\,(1-\Gamma')}{\varphi\,(\lambda+\alpha^2)-\varphi\lambda\beta} \\ \gamma_x^{ALS}, \gamma_g^{ALS} &= \frac{1}{\varphi} \\ \gamma_\pi^{ALS} &= \left(1+\frac{\beta-\Gamma'}{\alpha\varphi}\right) \end{aligned} \qquad (4.8)$$

where $1 < \Gamma' < 0$ *is the slope of the new mapping from perceived inflation to actual inflation obtained under the ALS-*Γ' *policy:*

(4.9) $$T'(a_{\pi,t}) = \frac{(1-\Gamma')}{(1-\Gamma^*)}\Phi^* + \Gamma' a_{\pi,t}.$$

Note that, under least square learning, the ALS-Γ' policy leads to a mapping from PLM to ALM

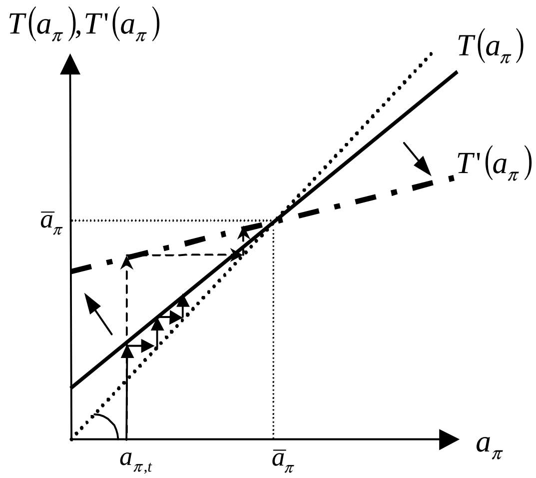
(4.10) $$T'(a_{\pi,t}, a_{x,t}) = \left(\frac{(1-\Gamma')}{(1-\Gamma^*)}\Phi^* + \Gamma' a_\pi, \frac{\Phi^*(1-\Gamma')}{(1-\Gamma^*)\,\alpha} - \frac{(\beta-\Gamma')}{\alpha}a_{\pi,t}\right)$$

that does not depend on the perceived output gap. Therefore, in order to study convergence of the learning equilibrium to the REE, as under EH policy, the analysis can concentrate on the mapping from perceived inflation to actual inflation (4.9).

Figure 10 shows the new mapping $T'(a_{\pi,t})$ under the *ALS-*Γ'*policy*. In particular, it can be observed that $T'(a_{\pi,t})$ has the same fixed point, $\overline{a}_\pi$, as under the *EH policy*, but the intercept and the slope are different. The policy-maker, in order to speed up (slow down) the transition to the REE should follow an expectations-based reaction function that induces a rotation of the mapping from PLM to ALM around the fixed-point (i.e., the REE), with a slope Γ' lower (higher) than under the EH policy.

Fig 10

The mapping $\mathbf{T}'_{a_\pi}(a_{\pi,t})$ under the ALS-Γ' policy ($\Gamma' < \Gamma^*$)



The following proposition compares the REE and asymptotic properties of the learning equilibrium under ALS-Γ' and EH policies.

Proposition 14 *Under rational expectations, the ALS-*Γ' *policy results in the same REE for* $\{\pi_t, x_t\}$ *derived under the EH policy. Under least squares learning, the ALS-*Γ' *policy results asymptotically in the same REE for* $\{\pi_t, x_t\}$ *derived under the EH policy.*

Proof. See Appendix 2.O. ■

To stress the differences in the role of the policy-maker in determining the properties of the learning equilibrium along the transition to the REE, under ALS-Γ' and under the EH policies, consider again equations (4.3) and (4.8).

Taking parameters α, φ, β as given, under the EH policy, the speed of convergence of the learning equilibrium to the REE relies entirely on λ. By choosing a λ the policy-maker is also choosing the slope of the $T(.)$ mapping (in the previous example, with $\lambda = 0.5$, the slope was equal to 0.84) and, as shown in section 2, he determines the speed of convergence.

On the contrary, under *ALS-*Γ' *policy*, the policy-maker could choose separately the relative weight on output gap and the speed at which agents learn, i.e., the slope of the $T(.)$ mapping, without affecting the REE. To see this, note that expression (4.8) can give

(4.11) $$\Gamma' = \left(1 - \gamma_{\pi}^{ALS}\right) \alpha\varphi + \beta.$$

Equation (4.11) shows that, given the parameters α, φ, β, each value of the policy reaction parameter γ_{π}^{ALS} has a corresponding slope of the $T(.)$ mapping, Γ', independently from λ. In particular,

Lemma 15 *The response of interest rate to a rise in expected inflation is higher under the ALS-*Γ' *than under the EH policy if* $\Gamma' < \Gamma^*$*, is lower if* $\Gamma' > \Gamma^*$*.*

Proof. See Appendix 2.P. ■

The following proposition formally compares the transition under ALS-Γ' and under the EH policies.

Proposition 16 *Assume that private agents form expectations through recursive least squares learning; define* $a_{\pi,t}\left(i^{ALS}\left(\Gamma'\right)\right)$ *and* $a_{\pi,t}\left(i^{EH}\right)$ *the perceived inflation under ALS-*Γ' *and EH policies,* $\pi_t\left(i^{ALS}\left(\Gamma'\right)\right)$ *and* $\pi_t\left(i^{EH}\right)$ *actual inflation under ALS-*Γ' *and EH policies. Finally, assume that the economy starts from a point where the learning equilibrium and the REE do not coincide, in particular*

$$a_{\pi,0}\left(i^{ALS}\left(\Gamma'\right)\right) = a_{\pi,0}\left(i^{EH}\right) \neq \overline{a}_{\pi}$$

– if $\gamma_{\pi}^{ALS} > \gamma_{\pi}^{*}$, than for every $0 < t < \infty$

$$\left|a_{\pi,t}\left(i^{ALS}\left(\Gamma'\right)\right) - \overline{a}_{\pi}\right| < \left|a_{\pi,t}\left(i^{EH}\right) - \overline{a}_{\pi}\right|$$

and

$$\left|\pi_t\left(i^{ALS}\left(\Gamma'\right)\right) - \overline{a}_{\pi}\right| < \left|\pi_t\left(i^{EH}\right) - \overline{a}_{\pi}\right|$$

– if $\gamma_{\pi}^{ALS} < \gamma_{\pi}^{*}$, than for every $0 < t < \infty$

$$\left|a_{\pi,t}\left(i^{ALS}\left(\Gamma'\right)\right) - \overline{a}_{\pi}\right| > \left|a_{\pi,t}\left(i^{EH}\right) - \overline{a}_{\pi}\right|$$

and

$$\left|\pi_t\left(i^{ALS}\left(\Gamma'\right)\right)-\overline{a}_\pi\right|>\left|\pi_t\left(i^{EH}\right)-\overline{a}_\pi\right|$$

Proof. See Appendix 2.Q. ■

Proposition 19 says that if initial perceived inflation is the same under both policies but different from the REE, under an *ALS*-Γ' *policy* that induces a flat slope in the $T()$ mapping, perceived and actual inflation will always be closer to the REE than under the *EH policy*. The opposite is true when the mapping under *ALS*-Γ' *policy* is steeper than under *EH policy*.

In general, comparing two *ALS*-Γ' *policies* gives the following corollary to Proposition 19.

Corollary 17 *Consider two ALS policies* $i^{ALS}\left(\Gamma'_1\right)$ *and* $i^{ALS}\left(\Gamma'_2\right)$ *with* $0<\Gamma'_1<\Gamma'_2<1$ *and*

$$a_{\pi,0}\left(i^{ALS}\left(\Gamma'_1\right)\right)=a_{\pi,0}\left(i^{ALS}\left(\Gamma'_2\right)\right)\neq\overline{a}_\pi$$

than

$$\left|a_{\pi,t}\left(i^{ALS}\left(\Gamma'_1\right)\right)-\overline{a}_\pi\right|<\left|a_{\pi,t}\left(i^{ALS}\left(\Gamma'_2\right)\right)-\overline{a}_\pi\right|$$

$$\left|\pi_t\left(i^{ALS}\left(\Gamma'_1\right)\right)-\overline{a}_\pi\right|<\left|\pi_t\left(i^{ALS}\left(\Gamma'_2\right)\right)-\overline{a}_\pi\right|$$

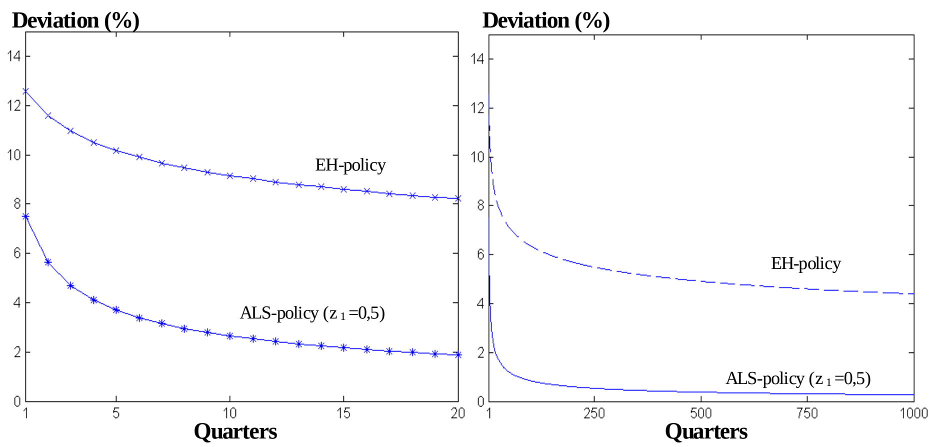
The intuition is the following: if the policy-maker reacts strongly to a change in expected inflation, the difference between private agents' expectations and actual inflation will be greater; since for $0<t<\infty$ if private agents make larger errors they will adapt their estimates faster, the transition to the REE will be shorter. In other words, the stronger the policy-maker's response to a change in private agents' expectations, the faster private agents learn and the shorter the transition to the REE.

The fact that under the ALS policy for every $0<t<\infty$ the distance from the REE could be smaller (greater) than under the EH policy could be used to address the following question: how long does it take under the two policies to get ε-close to the REE, i.e., starting from the same distance from the REE, $|a_{\pi,0} - \overline{a}_\pi| > \varepsilon$, how many periods are needed under the two policies to have $|\pi_t - \overline{a}_\pi| < \varepsilon$?

Figure 11 compares the results of a simulation under the EH policy rule (2.21) and under an ALS policy with $\Gamma' = 0.5$ (i.e., root-t convergence is imposed) assuming an initial perceived inflation 15 per cent higher than the REE.

Fig 11

Deviation of actual inflation from the REE



Under the ASL policy, after 1 quarter the distance to the REE is already halved, after 3 quarters the distance is below 5 per cent and after 20 quarters it is below 2 per cent. On the contrary, under EH policy, more than 100 quarters are necessary to halve the distance and more than 500 to reduce it to 5 percent.

Table 2 compares the transition of the learning equilibrium to the REE for different ALS-Γ' policies. Let us consider, for example, the ALS-Γ' policy with $\gamma_\pi^{ALS} = 2.6$. Assuming an initial perceived inflation 15 per cent higher than the REE, the distance from the REE can be reduced to less than 5 per cent, in:

- $1/2$ of the time needed under the ALS-Γ' with $\gamma_\pi^{ALS} = 2.3$

- approximately $1/5$ of the time needed under the ALS-Γ' with $\gamma_\pi^{ALS} = 2$

- approximately $1/40$ of the time needed under the ALS-Γ' with $\gamma_\pi^{ALS} = 1.6$

- approximately $1/150$ of the time needed under the EH policy!

Tab 2

Transition under the ALS-Γ' policy

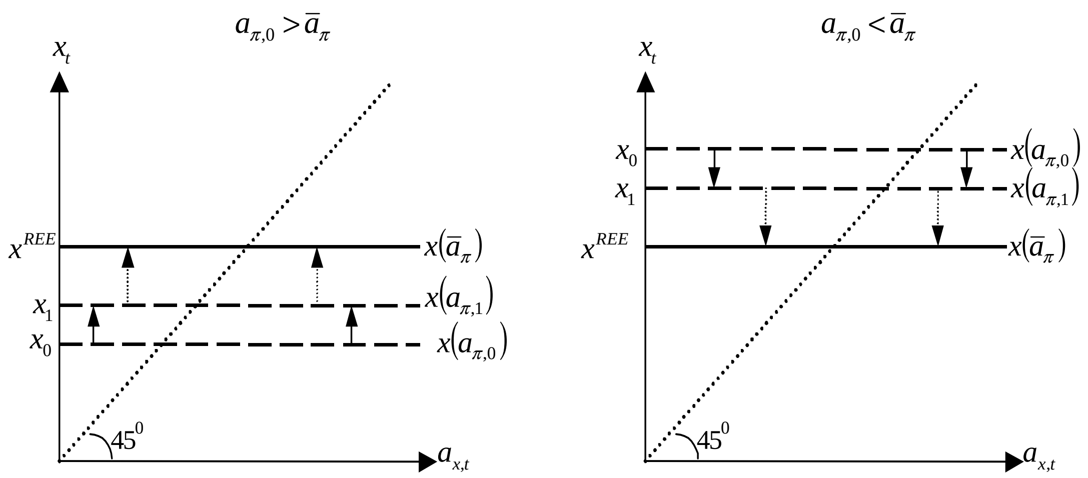
γ_π^{ALS}	Γ'	T(<10%)	T(<5%)	T(<3%)	T(<1%)
4.0	0.1	1	1	1	2
3.6	0.2	1	1	2	5
3.3	0.3	1	1	2	10
3	0.4	1	2	4	24
2.6	0.5	1	3	8	72
2.3	0.6	1	6	21	322
2	0.7	2	16	90	3489
1.6	0.8	4	114	1461	> 10000
1.5[1]	0.84	6	445	> 10000	> 10000
1.3	0.9	30	> 10000	> 10000	> 10000

[1]For $\gamma_\pi^{ALS} = 1.5$, the ALS and EH policies coincide.

This section looked at the role of policy decisions in determining the speed of convergence under learning, focusing on the mapping from perceived inflation to actual inflation. Now, before asking how the policy-maker can make use of this role to increase social welfare, a brief analysis is made of the behaviour of the output gap along the transition.

Fig. 12

Output gap under EH policy and ALS-Γ' policy



Mappings (4.5) and (4.10) show that the actual output gap, under both EH and ALS policies, depends only on perceived inflation (Figure 12). In particular,

Lemma 18 *Under EH and ALS policies, when initial perceived inflation is higher (lower) than the REE, the output gap converges to the REE from below (above).*

Proof. See Appendix 2.R. ■

Now it is possible to return to the question addressed at the beginning of the paper: is the *RE-optimal expectations-based reaction function* still optimal under learning? Are policies that speed up the learning process always better than policies that involve a slow transition to the REE?

5. Welfare analysis

In January 1999, with the start of stage 3 of the Economic and Monetary Union, monetary competencies were transferred from each country of the European Union to the European Central Bank. Before that date people were accustomed to take into account the monetary policy of their own country when making economic decisions. After the start of stage 3, they faced a new policy-maker (and a new monetary policy) and inflation and output gap equilibria determined under the new policy regime were, in some cases, different from the ones implied by the previous policies. Let us consider, for example, countries like Italy or Spain, whose rates of inflation are historically higher than in other member states, and assume that in those two countries expected inflation at the start of the EMU was higher than the REE determined by the new monetary regime. Under the assumption that private agents need time to learn the new equilibrium, it is clear that the dynamics of the learning equilibrium along the transition to the REE play an important role in the analysis of monetary policy decisions based on welfare measures. Questions like the ones raised at the end of the previous section show up spontaneously.

To answer to those questions I consider separately the two cases where initial expected inflation is higher than the REE and where it is lower. The reason why I proceed in this way is that in the literature it is well known that under the loss function (2.18) the first best plan would be, for all t, to have inflation and output gap at their target levels, i.e., $\pi_t^{FB} = 0$ and $x_t^{FB} = \overline{x}$.

As many works have shown, under no commitment, the first best solution is not feasible if $\overline{x} \neq 0$. The optimal (time-consistent) policy in this case leads to a REE with inflation higher than the first best and output gap lower[28]:

$$\pi_t^{REE} = \frac{\lambda\alpha}{(\lambda+\alpha^2)-\lambda\beta}\overline{x} \; > \pi_t^{FB} \text{ for all } t$$

$$x_t^{REE} = \frac{\lambda(1-\beta)}{(\lambda+\alpha^2)-\lambda\beta}\overline{x} < x_t^{FB} \quad \text{for all } t$$

Under learning, however, monetary policies could result in a learning equilibrium that remains far from the REE for a long (or short) time. Therefore, if initial perceived inflation is higher than the REE, as in previous section, actual inflation will be higher and output gap lower than the REE along the transition. In this case, a policy-maker who bases decisions on the loss function (2.18) would prefer policies that make inflation fall and output gap rise quickly to the REE. On the contrary, if initial perceived inflation is lower than the REE, the policy-maker would prefer policies that make inflation climbing and output gap landing slowly to the REE. Since *EH policy* is not taking into account the transition, I claim that there are *ALS*-Γ' *policies* that will make our economy better off.

In order to verify this claim, let us start by assuming that the *EH policy* (which is optimal under RE) is also optimal when private agents form expectations through adaptive learning. The aim is to compute the welfare cost of alternative monetary policies, i.e., ALS-Γ', that asymptotically result in the same REE as the *RE-optimal expectations-based reaction function*, but along the transition result in different learning equilibria.

The social loss associated with *EH policy* is defined as:

$$L_0^{EH} = E_0 \sum_{t=0}^{\infty} \beta^t L\left(\pi_t\left(i^{EH}\right), x_t\left(i^{EH}\right)\right),$$

[28] Under CGG parametrization, assuming $\lambda = 0.5$, the REE would be

$$\pi_t = 1.57 * \overline{x} \quad \text{and} \quad x_t = 0.05 * \overline{x}$$

where $L\left(\pi_t\left(i^{EH}\right), x_t\left(i^{EH}\right)\right)$ is the period t loss function defined in equation (2.18) and $\pi_t\left(i^{EH}\right), x_t\left(i^{EH}\right)$ denote the contingent plans for inflation and output gap under *EH policy*. Similarly, the social loss associated with *ALS*-Γ' *policies* is defined as

$$L_0^{ALS}\left(\Gamma'\right) = E_0 \sum_{t=0}^{\infty} \beta^t L\left(\pi_t\left(i^{ALS}\left(\Gamma'\right)\right), x_t\left(i^{ALS}\left(\Gamma'\right)\right)\right).$$

Three measures of the welfare cost (or gain) of adopting policy *ALS*-Γ' instead of the reference *EH policy* are considered.

5.1 *Percentage loss in total welfare*

This measure of the welfare loss (gain) is merely the percentage increase (decrease) in the social loss of moving from *EH* to *ALS*-Γ'*policy*:

$$\omega\left(L_0^{ALS}\left(\Gamma'\right)\right) = \left(\frac{L_0^{ALS}\left(\Gamma'\right) - L_0^{EH}}{L_0^{EH}}\right) * 100.$$

Note that for values of $\omega\left(L_0^{ALS}\left(\Gamma'\right)\right) < 0$ there is a welfare gain in adopting ALS-Γ' policy instead of EH, while for $\omega\left(L_0^{ALS}\left(\Gamma'\right)\right) > 0$, there is a welfare loss.

I run simulations of the model for 10000 periods, assuming that the policy-maker follows a flexible inflation targeting policy rule with $\lambda = 0.5$, the output gap target is $\overline{x} = 0.02$ and using CGG calibration. I start with the assumption of an initial expected inflation 15 per cent higher than the REE and I compute social losses under the EH and ALS-Γ' policies for different values of γ_π^{ALS} (i.e., different Γ').

Figure 13 shows that ALS-Γ' policies with $\Gamma' < \Gamma^*$, by inducing a fast convergence, reduce the social loss up to 11 per cent relative to EH policy. Policies with $\Gamma' > \Gamma^*$, on the contrary, increase the social loss by up to 10 per cent. In particular, a central bank that follows an ALS-Γ' policy with $\gamma_\pi^{ALS} = 2.6$ can, by increasing the speed of convergence to root-t, lower the value of the loss function by 8.6 per cent relative to the EH policy. In order to analyze how the percentage increase (decrease) in the social loss evolves along the transition simulations are also run for $T < 10000$ periods.

Fig 13

Percentage loss in total welfare ($\pi_0 > \pi^{RE}$)

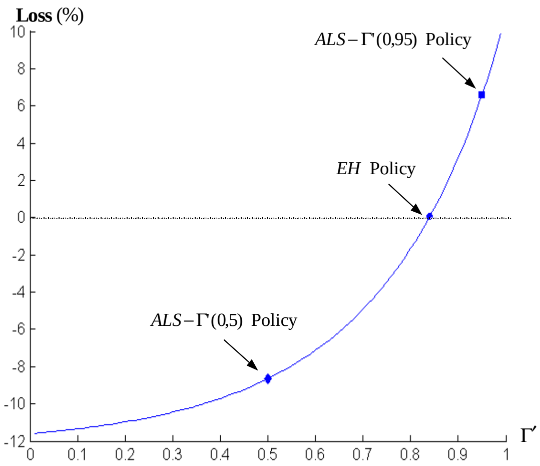


Table 3 shows the results for different T, pointing out that most of the gain from using an ALS-Γ' policy with fast transition is concentrated in the first 20 periods.

Tab 3

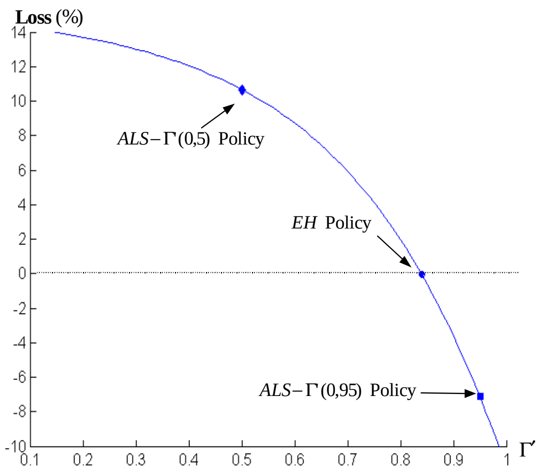
Percentage loss in total welfare after T periods ($\pi_0 > \pi^{RE}$)

γ_π^{ALS}	Γ'	T=10	T=20	T=50	T=100	T=10000
4.0	0.1	-9.7	-11.3	-12.0	-11.9	-11.3
3.6	0.2	-8.9	-10.6	-11.4	-11.4	-10.9
3.3	0.3	-8.0	-9.6	-10.6	-10.7	-10.5
3	0.4	-7.0	-8.5	-9.6	-9.8	-9.7
2.6	0.5	-5.8	-7.2	-8.3	-8.6	-8.6
2.3	0.6	-4.4	-5.6	-6.6	-7.0	-7.1
2	0.7	-2.8	-3.6	-4.3	-4.7	-4.9
1.6	0.8	-0.9	-1.1	-1.4	-1.5	-1.6
1.5[1]	0.84	0	0	0	0	0
1.3	0.9	1.5	2.0	2.5	2.9	3.2
1.1	0.95	2.8	3.8	5.0	5.8	6.6

Figure 14 and Table 4 show that under the assumption of an initial expected inflation 15 per cent lower than the REE, by inducing a slower convergence, the policy-maker could greatly reduce the welfare loss.

Fig 14

Percentage loss in total welfare $(\pi_0 < \pi^{RE})$



A central bank that follows an ALS-Γ′ policy with $\gamma_\pi^{ALS} = 1.1$ can, by increasing the slope of the mapping from perceived inflation to actual inflation to $\Gamma' = 0.95$, slow down the transition and lower the value of the loss function by approximately 7 per cent relative to the EH policy. On the contrary, a policy-maker who speeds up the transition to root-t convergence, following an ALS-Γ′ policy with $\gamma_\pi^{ALS} = 2.6$, would increase the value of the loss function by approximately 11 per cent relative to the EH policy (Figure 14).

Tab 4

Percentage loss in total welfare after T periods $(\pi_0 < \pi^{RE})$

γ_π^{ALS}	Γ'	T=10	T=20	T=50	T=100	T=10000
4.0	0.1	15.7	16.3	16.0	15.3	14.2
3.6	0.2	14.2	15.1	15.1	14.6	13.7
3.3	0.3	12.6	13.7	14.0	13.7	13.0
3	0.4	10.7	12.0	12.6	12.5	12.0
2.6	0.5	8.6	9.9	10.7	10.9	10.7
2.3	0.6	6.3	7.5	8.4	8.7	8.7
2	0.7	3.8	4.7	5.4	5.7	5.9
1.6	0.8	1.1	1.4	1.7	1.8	1.9
1.5	0.84	0	0	0	0	0
1.3	0.9	-1.7	-2.3	-2.9	-3.3	-3.6
1.1	0.95	-3.1	-4.3	-5.6	-6.3	-7.1

Table 4 shows that most of the loss from using an *ALS*-Γ' *policy* with fast transition is concentrated in the first 20 periods.

While the percentage loss (gain) in total welfare from using alternative ALS policies shows the importance of the speed of convergence when private agents form expectations through recursive leats squares estimate, this measure of the welfare loss has no simple economic interpretation. As an alternative measure, the "inflation equivalent" can be calculated.

5.2 *The inflation equivalent*

This measure of the welfare loss, denoted by $\omega_\pi^{ALS}(\Gamma')$, is computed as the fraction of inflation under *EH policy* that a central bank is willing to accept above $\pi_t\left(i^{EH}\right)$ to be as well off under policy *ALS*-Γ' as under policy *EH*. Formally the "inflation equivalent" is implicitly defined by:

$$L_0^{ALS}(\Gamma') = E_0 \sum_{t=0}^{\infty} \beta^t L\left(\left(1+\omega_\pi^{ALS}(\Gamma')\right)\pi_t\left(i^{EH}\right), x_t\left(i^{EH}\right)\right).$$

Note that for values of $\omega_{\pi}^{ALS}(\Gamma') > 0$ there is a welfare gain in adopting *ALS*-Γ' *policy* instead of *EH*, while for $\omega_{\pi}^{ALS}(\Gamma') < 0$ there is a welfare loss.

Solving for $\omega_{\pi}^{ALS}(\Gamma')$, for different values of γ_{π}^{ALS}, I obtain similar qualitative results as before (Table 5). However, by computing the "inflation equivalent" I can say, for example, that when initial expected inflation is 15 per cent higher than the REE, the inefficiency of EH policy is equivalent to an inflation up to 7 per cent higher than $\pi_t\left(i^{EH}\right)$. In other words, under an *ALS*-Γ' *policy* with $\gamma_{\pi}^{ALS} = 3.6$, welfare can be increased by an amount equivalent to a reduction of inflation of 7 per cent below the $\pi_t\left(i^{EH}\right)$ level. Again, most of the welfare differences under ALS and EH policies are concentrated in the first 20 periods.

Tab 5

Inflation equivalent after T periods

$(\pi_0 > \pi^{RE})$

γ_{π}^{ALS}	Γ'	T=10	T=20	T=50	T=100	T=10000
3.6	0.2	5.4	6.4	7.0	6.9	6.7
2.6	0.5	3.5	4.3	5.0	5.2	5.2
2	0.7	1.6	2.1	2.6	2.8	2.9
1.5	0.84	0	0	0	0	0
1.1	0.95	-1.9	-2.2	-2.9	-3.4	-3.8

$(\pi_0 < \pi^{RE})$

γ_{π}^{ALS}	Γ'	T=10	T=20	T=50	T=100	T=10000
3.6	0.2	-8.1	-8.5	-8.6	-8.3	-7.8
2.6	0.5	-5.0	-5.7	-6.1	-6.2	-6.1
2	0.7	-2.2	-2.7	-3.2	-3.3	-3.4
1.5	0.84	0	0	0	0	0
1.1	0.95	1.8	2.5	3.3	3.8	4.3

Before concluding I wish to emphasize two aspects concerning the robustness of welfare results.

5.3 *Robustness*

In the previous section the speed of convergence and welfare were studied by running simulations with $\lambda = 0.5$ and $\overline{x} = 0.02$. Changing these parameters would not change the finding that *EH policy*, that is optimal under rational expectations, is not optimal under learning and that, when initial perceived inflation is higher than the REE, the central bank could increase welfare by inducing a faster transition. However, in the extreme case where $\overline{x} = 0$, if initial inflation is lower than the REE, the finding that a slower convergence to the REE increases welfare does not hold anymore. In fact, when $\overline{x} = 0$, in our model, the optimal policy under discretion results in a REE with inflation and output gap equal to the first best, and the faster the transition the higher will be welfare under learning.

The new-Keynesian model analyzed in this paper is derived assuming that only one shock affects the economy. Under this assumption the policy-maker neutralizes the real effects of the shock whether it follows the *EH policy* or an *ALS*-Γ' *policy,* i.e., $\gamma_g^* = \gamma_g^{ALS} = \frac{1}{\varphi}$. However, when an additional shock hits the economy (for example, a "cost-push shock", u_t) the policy-maker cannot, in general, neutralize both shocks at the same time. In this case, since the two policies along the transition to the REE would react differently to u_t, welfare analysis could be affected. Simulations show that the introduction of a cost-push shock affects the results only in the amount of the welfare gain (or loss).

Tab 6

Percentage loss in total welfare with cost-push shocks (T $= 10000$)

$(\pi_0 > \pi^{RE})$

γ_π^{ALS}	Γ'	$100 * \omega_\pi^{ALS}(\Gamma')$	$\omega\left(L_0^{ALS}(\Gamma')\right)$
4.0	0.1	5.6	-11.1
3.6	0.2	5.4	-10.8
2.6	0.5	4.5	-8.9
1.6	0.7	2.9	-5.2
1.5	0.84	0	0
1.1	0.95	-2.3	7.3

$(\pi_0 < \pi^{RE})$

γ_π^{ALS}	Γ'	$100 * \omega_\pi^{ALS}(\Gamma')$	$\omega\left(L_0^{ALS}(\Gamma')\right)$
4.0	0.1	-15.0	24.2
3.6	0.2	-14.5	23.3
2.6	0.5	-11.1	17.7
1.6	0.7	-5.9	9.8
1.5	0.84	0	0
1.1	0.95	9.0	-11.2

Table 6 shows that adding an AR(1) shock u_t in the aggregate supply expression[29], when initial private agents' perceived inflation is higher than the REE, a central bank that follows an ALS-Γ' policy with $\gamma_\pi^{ALS} = 2.6$ can lower the value of the loss function by approximately 9 per cent relative to the EH policy (8.6 per cent without cost-push shocks); when initial private agents' perceived inflation is lower than the REE, an ALS-Γ' policy with $\gamma_\pi^{ALS} = 1.1$ can lower the value of the loss function by approximately 11.2 per cent (7.1 per cent without cost-push shocks). In terms of inflation equivalent, in the first case, under an *ALS*-Γ' *policy* with $\gamma_\pi^{ALS} = 2.6$, welfare could be increased by an amount equivalent to a reduction of inflation of 4.5 per cent below the $\pi_t\left(i^{EH}\right)$ level (5.2 per cent without cost-push shocks); in the second case, an *ALS*-Γ' *policy* with $\gamma_\pi^{ALS} = 1.1$, the reduction of inflation would be 9 per cent below the $\pi_t\left(i^{EH}\right)$ level (4.3 per cent without cost-push shocks).

The results obtained in this section show that optimal policies derived under RE are not optimal under learning. Using results for the speed of convergence could help to increase social welfare by taking into account the transition from learning equilibrium to the REE. Solving for the true optimal policy under discretion and learning would envolve taking into account that the policy-maker could make active use of private agents' learning behaviour. However, since the optimal monetary policy has to be derived by substituting the private agents' PLM into the objective function, it would be time-dependent. Further analysis in this direction is required and will be left for future research.

6. Conclusions

In this paper I have shown that considering learning in a model of monetary policy design is particularly important in order to describe not only the asymptotic properties of rational expectations equilibrium to which the economy could converge, but even to describe the dynamics that characterize the transition to this equilibrium.

The central message of the paper is that policy-makers should not only look at monetary policies that determine a stable equilibrium under learning, but also take into account how policy decisions affect the speed at which learning converges to rational expectations. In particular, under certain policies, the REE is E-stable, but the period needed to converge to

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[29] I assume $\lambda = 0.5$, $\overline{x} = 0.02$, $u_t = \rho_u u_{t-1} + \varepsilon_{u,t}$ with $\rho_u = 0.5$ and $\varepsilon_{u,t} \sim N\left(0, 0.05\right)$

this equilibrium could be incredibly long. Reacting strongly to expected inflation, a central bank would shorten the transition and increase the speed of convergence from the learning equilibrium to the REE.

A policy-maker who considers his role in determining the dynamics of the private agents' learning process could choose a policy rule that induces agents to learn at a given speed, affecting the welfare of society. In particular, if the policy-maker knows that after a regime change private agents' perceived inflation would be higher than the REE, by choosing a policy that reacts strongly to expected inflation he would determine a fast convergence and could increase social welfare. If, instead, perceived inflation is initially lower than the REE, a slow transition is preferred when the output gap target is greater than zero, a fast transition when the target is equal to zero.

Appendix 1

To derive the REE, consider the mapping from the PLM to the ALM

$$T\left(a_{\pi},a_{x}\right)=\left(\alpha\varphi\gamma+\left(\beta+\alpha\varphi\left(1-\gamma_{\pi}\right)\right)a_{\pi},\varphi\gamma+\varphi\left(1-\gamma_{\pi}\right)a_{\pi}\right)$$

and analyze the ordinary differential equation (ODE):

$$\frac{d}{d\tau}\left(a_{\pi},a_{x}\right)=T\left(a_{\pi},a_{x}\right)-\left(a_{\pi},a_{x}\right)$$

where τ denotes notional time. The REE can be derived by looking at the fixed point of the differential equation, by imposing $\frac{d}{d\tau}\left(a_{\pi},a_{x}\right)=0$:

$$a_{\pi}=\alpha\varphi\gamma\left(1-\beta-\alpha\varphi\left(1-\gamma_{\pi}\right)\right)^{-1}$$

$$a_{x}=\varphi\gamma+\varphi\left(1-\gamma_{\pi}\right)a_{\pi}$$

The REE is:

$$\begin{bmatrix}\pi_t\\ x_t\end{bmatrix}=\begin{bmatrix}\frac{\alpha\varphi\gamma}{(1-\beta-\alpha\varphi(1-\gamma_{\pi}))}\\ \frac{\varphi\gamma(1-\beta)}{(1-\beta-\alpha\varphi(1-\gamma_{\pi}))}\end{bmatrix}+\begin{bmatrix}\alpha\\ 1\end{bmatrix}g_t.$$

Appendix 2

A. PROOF OF LEMMA 1

The necessary and sufficient condition for determinancy of the REE is given by all the eigenvalues of the matrix

$$\begin{bmatrix} \beta + \alpha\varphi\left(1-\gamma_\pi\right) & 0 \\ \varphi\left(1-\gamma_\pi\right) & 0 \end{bmatrix}$$

being inside the unit circle; since the eigenvalues are equal to 0 and $\beta + \alpha\varphi\left(1-\gamma_\pi\right)$ it is only necessary to check for $-1 < \beta + \alpha\varphi\left(1-\gamma_\pi\right) < 1$.

B. PROOF OF LEMMA 2

Consider (2.6) and take derivatives

$$\frac{\partial\alpha\varphi\gamma\left[1-\beta-\alpha\varphi\left(1-\gamma_\pi\right)\right]^{-1}}{\partial\gamma_\pi} = \frac{\alpha^2\varphi^2\gamma\left(1-\gamma_\pi\right)}{\left[1-\beta-\alpha\varphi\left(1-\gamma_\pi\right)\right]^2}$$

and

$$\frac{\partial\varphi\gamma\left(1-\beta\right)\left[1-\beta-\alpha\varphi\left(1-\gamma_\pi\right)\right]^{-1}}{\partial\gamma_\pi} = \frac{\alpha\varphi^2\gamma\left(1-\beta\right)\left(1-\gamma_\pi\right)}{\left[1-\beta-\alpha\varphi\left(1-\gamma_\pi\right)\right]^2}$$

that is

$$\frac{\partial\overline{a}_\pi}{\partial\gamma_\pi} > 0 \quad iff \quad \gamma_\pi > 1$$

$$\frac{\partial\overline{a}_x}{\partial\gamma_\pi} > 0 \quad iff \quad \gamma_\pi > 1$$

C. PROOF OF LEMMA 3

Consider the ordinary differential equation (ODE)

$$\begin{aligned} \frac{da_\pi}{d\tau} &= T\left(a_\pi\right) - a_\pi \\ &= \alpha\varphi\gamma + \left[\beta + \alpha\varphi\left(1-\gamma_\pi\right)\right]a_\pi - a_\pi \end{aligned}$$

The solution of $\frac{da_\pi}{d\tau} = 0$ is

$$\overline{a}_\pi = \alpha\varphi\gamma \left[1 - \beta - \alpha\varphi\left(1 - \gamma_\pi\right)\right]^{-1}$$

is E-stable under the dynamics of the ODE (Marcet and Sargent (1989) and Evans and Honkapohja, 2001) if and only if

$$\left[\beta + \alpha\varphi\left(1 - \gamma_\pi\right)\right] < 1$$

that is

$$\gamma_\pi > 1 - \frac{1 - \beta}{\alpha\varphi}$$

D. Proof of Proposition 4

Given the recursive stochastic algorithm

$$a_{\pi,t} = a_{\pi,t-1} + t^{-1}\left(\alpha\varphi\gamma + \left[\beta + \alpha\varphi\left(1 - \gamma_\pi\right)\right] a_{\pi,t-1} + \alpha g_{t-1} - a_{\pi,t-1}\right)$$

let

$$h\left(a_\pi\right) = \left[\alpha\varphi\gamma + \left[\beta + \alpha\varphi\left(1 - \gamma_\pi\right)\right] a_\pi - a_\pi\right]$$

and let a_π be such that $h\left(a_\pi\right) = 0$. By the theorem of Benveniste et. al.(Theorem 3, page 110), if the derivative of $h\left(a_\pi\right)$ is smaller than $-1/2$, then

$$\sqrt{t}\left(a_{\pi,t} - \overline{a}_\pi\right) \xrightarrow{D} N\left(0, \sigma_a^2\right)$$

where σ_a^2 satisfies

$$\left[h'\left(\overline{a}_\pi\right)\right]\sigma_a^2 + E\left[\alpha\varphi\gamma + \left[\beta + \alpha\varphi\left(1 - \gamma_\pi\right)\right]\overline{a}_\pi - \overline{a}_\pi + \alpha g_t\right]^2 = 0$$

Note that the derivative of $E\left[\alpha\varphi\gamma+\left[\beta+\alpha\varphi\left(1-\gamma_{\pi}\right)\right]a_{\pi}-a_{\pi}\right]$ being smaller than $-1/2$ coincides with $\left[\beta+\alpha\varphi\left(1-\gamma_{\pi}\right)\right]$ being smaller than $1/2$, i.e., γ_{π} being larger than $1-\frac{1/2-\beta}{\alpha\varphi}$

E. PROOF OF PROPOSITION 5

The formula for the asymptotic variance of the limiting distribution is

$$\sigma_a^2=\frac{\alpha^2}{\left[1-\beta-\alpha\varphi\left(1-\gamma_{\pi}\right)\right]}\sigma_g^2$$

and for $\gamma_{\pi}\in S$,

$$\frac{\partial\sigma_a^2}{\partial\gamma_{\pi}}=-\frac{\alpha\varphi\left(1-\beta-\alpha\varphi\left(1-\gamma_{\pi}\right)\right)}{\left[1-\beta-\alpha\varphi\left(1-\gamma_{\pi}\right)\right]^2}\alpha^2\sigma_g^2<0$$

F. PROOF OF PROPOSITION 7

The argument is similar to the one used in the proof of Propositions 4 and 5.

In order to have root-t convergence there must be

$$\frac{\lambda\beta}{\left(\lambda+\alpha^2\right)}<1/2$$

that is

$$\lambda<\frac{\alpha^2}{2\beta-1}$$

For values of $\lambda>\frac{\alpha^2}{2\beta-1}$ there is no root-t convergence and, as in lemma 3, convergence will be slower.

G. PROOF OF LEMMA 8

Consider (3.5) and (3.6) and take derivatives

$$\frac{\partial\alpha\gamma\left[\left(1-\beta\right)\gamma_x-\left(1-\gamma_{\pi}\right)\alpha\right]^{-1}}{\partial\gamma_{\pi}}=\frac{\alpha^2\gamma\left(1-\gamma_{\pi}\right)}{\left[\left(1-\beta\right)\gamma_x-\left(1-\gamma_{\pi}\right)\alpha\right]^2}$$

and

$$\frac{\partial\gamma\left(1-\beta\right)\left[\left(1-\beta\right)\gamma_x-\left(1-\gamma_{\pi}\right)\alpha\right]^{-1}}{\partial\gamma_{\pi}}=\frac{\alpha\gamma\left(1-\beta\right)\left(1-\gamma_{\pi}\right)}{\left[\left(1-\beta\right)\gamma_x-\left(1-\gamma_{\pi}\right)\alpha\right]^2}$$

that is

$$\frac{\partial a_\pi}{\partial \gamma_\pi} > 0 \quad iff \quad \gamma_\pi > 1$$

$$\frac{\partial a_x}{\partial \gamma_\pi} > 0 \quad iff \quad \gamma_\pi > 1$$

$$\frac{\partial \alpha\gamma \left[(1-\beta)\gamma_x - (1-\gamma_\pi)\alpha\right]^{-1}}{\partial \gamma_x} = -\frac{\alpha\gamma(1-\beta)}{\left[(1-\beta)\gamma_x - (1-\gamma_\pi)\alpha\right]^2} < 0$$

and

$$\frac{\partial \gamma(1-\beta)\left[(1-\beta)\gamma_x - (1-\gamma_\pi)\alpha\right]^{-1}}{\partial \gamma_\pi} = -\frac{\gamma(1-\beta)^2}{\left[(1-\beta)\gamma_x - (1-\gamma_\pi)\alpha\right]^2} < 0$$

H. Proof of Lemma 9

Let us define

$$Y_t = \begin{bmatrix} \pi_t \\ x_t \end{bmatrix}$$

the (3.2) can be rewritten as

$$Y_t = g\left(E_t Y_{t+1}, \varepsilon_{\pi t}, \varepsilon_{xt}, \eta\right)$$

where η is a vector of parameters in the economy that includes parameters of monetary policy.

Under least squares learning hypothesis, it is assumed that the private agents do not know the effective value of the a_π, a_x coefficients, but estimate them through recursive least square regressions. In this case, agents' expectations are given by:

$$E_t Y_{t+1} = h\left(u_t, a_{\pi,t}(\mu), a_{x,t}(\mu)\right)$$

where $a_{\pi,t}(\mu)$ and $a_{x,t}(\mu)$ are certain statistics inferred from past data and h is the forecast function that depends on today's state and the statistics. These statistics are generated by learning mechanisms f_x and f_π

$$a_{\pi,t}(\mu) = f_\pi\left(a_{\pi,t-1}(\mu), \varepsilon_{\pi t}, \mu\right)$$

$$a_{x,t}(\mu) = f_x\left(a_{x,t-1}(\mu), \varepsilon_{xt}, \mu\right)$$

where μ are certain learning parameters that govern how past data are used to form the statistics.

In the context of the present model,the function h will be

$$E_t Y_{t+1} = \begin{pmatrix} a_{\pi,t} \\ a_{x,t} \end{pmatrix}$$

I assume learning mechanisms f_x and f_π

$$a_{\pi,t} = a_{\pi,t-1} + t^{-1}\left(\pi_{t-1} - a_{\pi,t-1}\right)$$

$$a_{x,t} = a_{x,t-1} + t^{-1}\left(x_{t-1} - a_{x,t-1}\right)$$

The PLM of the boundedly rational agents is assumed to be well specified[30]. Under least square learning, agents at time t estimate the model

$$\begin{aligned} \pi_t &= a_\pi + \kappa_{\pi t} \\ x_t &= a_x + \kappa_{xt} \end{aligned}$$

by running a least squares regression of π_t and x_t on an intercept using available data. Let ($a_{\pi,t}$, $a_{x,t}$) denote the least squares estimate using data on π_i, x_i , $i = 1, ..., t-1$. Expectations

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30 A well-specified PLM is one that considers all the state variables that we have unde RE:

$$\pi_t = a_\pi$$
$$x_t = a_x$$

are then given by

$$\begin{bmatrix} E_t\pi_{t+1} \\ E_t x_{t+1} \end{bmatrix} = A_t$$

where

$$A_t = \begin{bmatrix} a_{\pi,t} \\ a_{x,t} \end{bmatrix}$$

Then, the ALM of inflation and output gap is

$$\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = Q + FA_t$$

Thus the mapping from PLM to ALM takes the form

$$T\left(A_t'\right) = Q + FA_t$$

Consider the stability under learning (E-stability) of the rational expectation solution $\overline{A}$ as the situation where the estimated parameters A_t converge to $\overline{A}$ over time.

From Evans and Honkapohja (2001), the E-stability is determined by the following matrix differential equation

$$\frac{d}{d\tau}\left(A'\right) = T\left(A'\right) - A'$$

For this framework E-stability conditions are readily obtained by computing the derivative of $T\left(A'\right) - A'$ and imposing that the determinant of the matrix with the derivatives of the previous differential equation with respect to A is greater than zero and the trace of the matrix with the derivative is greater than zero. In particular, the eigenvalues of F, z_1 and z_2, must have real parts less than one (let us define the biggest eigenvalue of the F matrix as z_1).

Then, let us distinguish between the two cases:

A. The "real" case.

In this case two conditions must be satisfied in order to have convergence to the REE:

I. For reality

$$\left(\alpha\varphi\left(1-\gamma_\pi\right)+\beta+\left(1-\varphi\gamma_x\right)\right)^2 - 4\beta\left(1-\varphi\gamma_x\right) > 0$$

II. $\boxed{z_1 < 1}$

That is

$\frac{(\alpha\varphi(1-\gamma_\pi)+\beta+(1-\varphi\gamma_x))}{2} + \frac{\sqrt{(\alpha\varphi(1-\gamma_\pi)+\beta+(1-\varphi\gamma_x))^2-4\beta(1-\varphi\gamma_x)}}{2} < 1$

Under equality

$$\gamma_\pi = 1 - \frac{(1-\beta)}{\alpha}\gamma_x$$

Since by hypothesis $z_1 \geq z_2$, if $z_1 < 1$ then also $z_2 < 1$.

Assuming the Clarida, Gali and Gertler calibration, the relation

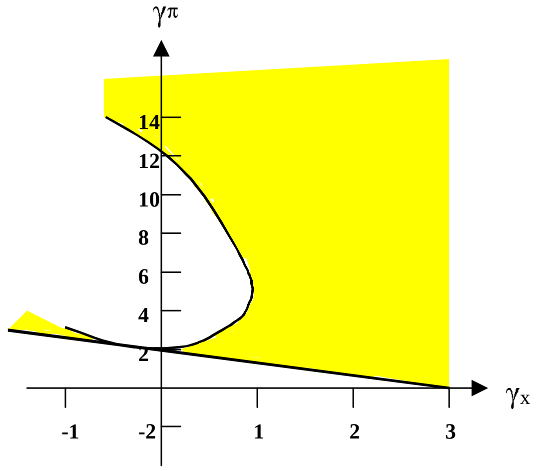
$$1.24 - 1.32\gamma_\pi + 0.09\gamma_\pi^2 - 0.8\gamma_x + \gamma_x^2 + 0.6\gamma_\pi\gamma_x > 0$$

shows the combinations of γ_π, γ_x for which the eigenvalues z_1 and z_2 are real.

In order to have $z_1 < 1$

$$\gamma_\pi > 1 - \frac{1}{3}\gamma_x$$

Graphically, it is necessary to be **inside** the shadowed area.



B. The "complex" case.

In this case two conditions must be satisfied in order to have convergence to the REE:

I. For the solution to be imaginary,

$$(\alpha\varphi(1-\gamma_\pi) + \beta + (1-\varphi\gamma_x))^2 - 4\beta(1-\varphi\gamma_x) < 0$$

II. Real part of $z_1 < 1$

$$\frac{(\alpha\varphi(1-\gamma_\pi)+\beta+(1-\varphi\gamma_x))}{2}<1$$

That is

$$\gamma_\pi > 1-\frac{1-\beta}{\alpha\varphi}-\frac{\gamma_x}{\alpha}$$

Since by hypothesis $z_1 \geq z_2$, if $z_1 < 1$ then also $z_2 < 1$.

Assuming the Clarida, Galí and Gertler's calibration, the relation

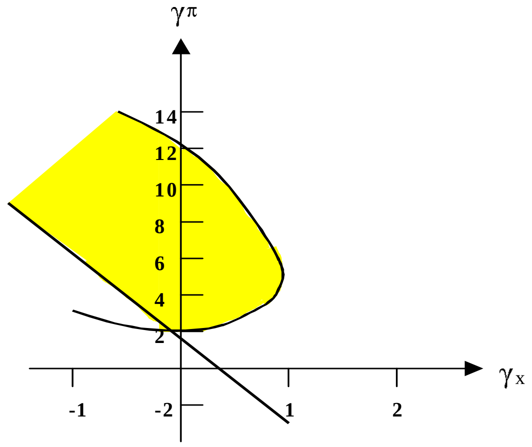
$$(0.3(1-\gamma_\pi)+0.9+(1-\gamma_x))^2-4*0.9(1-\gamma_x)<0$$

shows the combinations of γ_π, γ_x for which the eigenvalues z_1 and z_2 are complex.

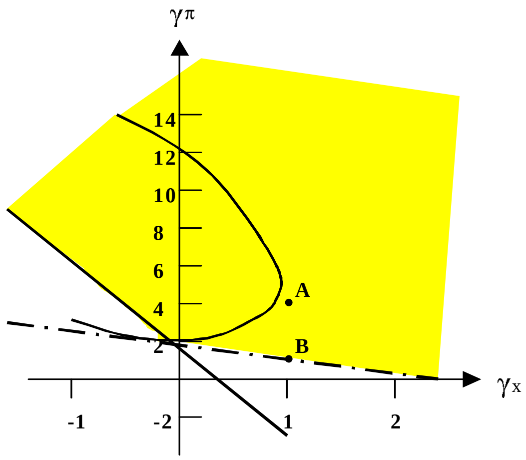
In order to have the eigenvalue z_1 inside the unit circle,

$$\gamma_\pi > \frac{2}{3}-\frac{10}{3}\gamma_x$$

Graphically, it is necessary to be **inside** the shadowed area.



In the following pictures, the shadowed areas represent the combinations of γ_π and γ_x for which least squares learning converges to rational expectations, under the alternative values of the parameters given by Clarida, Gali and Gertler (1999).



Note that the "optimal" combinations of γ_π,γ_x with the CGG (2000) parameters are:

For $\lambda = 0$

$$\begin{aligned} \gamma_\pi^\star &= 4 \\ \gamma_x^\star &= 1 \end{aligned}$$

For $\lambda = 1$

$$\begin{aligned} \gamma_\pi^\star &= 1.25 \\ \gamma_x^\star &= 1 \end{aligned}$$

I. Proof of Proposition 10

Consider again the mapping from PLM to ACL under the least square learning hypothesis:

$$T\left(A_t'\right) = Q + FA_t$$

From Marcet and Sargent (1992) it follows that in order to have root-t convergence the eigenvalues of F must have the real part smaller than $\frac{1}{2}$.

Then, let us distinguish between the two cases:

A. The "real" case.

In this case two conditions must be satisfied in order to have convergence to the REE:

I. For reality

$$\left(\alpha\varphi\left(1-\gamma_{\pi}\right)+\beta+\left(1-\varphi\gamma_{x}\right)\right)^{2}-4\beta\left(1-\varphi\gamma_{x}\right)>0$$

II. $\boxed{z_1 < 0.5}$

That is

$\frac{(\alpha\varphi(1-\gamma_{\pi})+\beta+(1-\varphi\gamma_{x}))}{2}+\frac{\sqrt{(\alpha\varphi(1-\gamma_{\pi})+\beta+(1-\varphi\gamma_{x}))^{2}-4\beta(1-\varphi\gamma_{x})}}{2}<\frac{1}{2}$

Consider what happens under equality:

$$\gamma_{\pi}=1+\frac{1-2\beta}{2\alpha\varphi}-\frac{1-2\beta}{\alpha}\gamma_{x}$$

Note that if $z_1$is smaller than $\frac{1}{2}$ then even z_2 is smaller than $\frac{1}{2}$.

Assuming the values of the parameters of Clarida, Gali and Gertler (1999), the relation

$$1.24-1.32\gamma_{\pi}+0.09\gamma_{\pi}^{2}-0.8\gamma_{x}+\gamma_{x}^{2}+0.6\gamma_{\pi}\gamma_{x}>0$$

shows the combinations of γ_{π},γ_{x} for which the eigenvalues z_1 and z_2 are real.

In order to have $z_1<\frac{1}{2}$,

$$\gamma_{\pi}>-0.33+2.67\gamma_{x}$$

B. The "complex" case.

In this case two conditions to be satisfied in order to have root-t convergence:

I. For the solution to be imaginary,

$$\left(\alpha\varphi\left(1-\gamma_{\pi}\right)+\beta+\left(1-\varphi\gamma_{x}\right)\right)^{2}-4\beta\left(1-\varphi\gamma_{x}\right)<0$$

II. Real part of $z_1 < \frac{1}{2}$

$$\frac{\left(\alpha\varphi\left(1-\gamma_{\pi}\right)+\beta+\left(1-\varphi\gamma_{x}\right)\right)}{2}<\frac{1}{2}$$

That is

$$\gamma_{\pi}>1+\frac{\beta}{\alpha\varphi}-\frac{\gamma_{x}}{\alpha}$$

Note that if $z_1$is smaller than $\frac{1}{2}$ then even z_2 is smaller than $\frac{1}{2}$.
Assuming the values of the parameters of Clarida, Gali and Gertler (1999), the relation

$$1.24-1.32\gamma_{\pi}+0.09\gamma_{\pi}^{2}-0.8\gamma_{x}+\gamma_{x}^{2}+0.6\gamma_{\pi}\gamma_{x}<0$$

shows the combinations of γ_{π},γ_{x} for which the eigenvalues z_1 and z_2 have an imaginary part.
In order to have the real part of $z_1 < \frac{1}{2}$,

$$\gamma_{\pi}>4.0-3.33\gamma_{x}$$

L. Proof of Proposition 13

Consider the set $\Gamma = \{\gamma_{\pi},\gamma_{x} : \gamma_{\pi}>0, \gamma_{x}>0 \text{ and } 0\leq z_1<1\}$.

Monotonically increasing with respect to γ_{π}: for every $h=(\gamma_{\pi}^{1},\gamma_{x}^{1})\in\Gamma$ and $w=(\gamma_{\pi}^{2},\gamma_{x}^{1})\in\Gamma$, with $\gamma_{\pi}^{2}\geq\gamma_{\pi}^{1}$, w implies a value for the real part of z_1 smaller or equal to the one with h.

Proof. z_1 is the biggest eigenvalue of F:

$$\begin{aligned} z_1 = & \frac{\left(\alpha\varphi\left(1-\gamma_{\pi}\right)+\beta+\left(1-\varphi\gamma_{x}\right)\right)}{2}+ \\ & +\frac{\sqrt{\left(\alpha\varphi\left(1-\gamma_{\pi}\right)+\beta+\left(1-\varphi\gamma_{x}\right)\right)^{2}-4\beta\left(1-\varphi\gamma_{x}\right)}}{2} \end{aligned}$$

Consider a $h=(\gamma_{\pi}^{1},\gamma_{x}^{1})\in\Gamma$ such that $z_1=z_1^1$ is real. In this case

$$\left(\alpha\varphi\left(1-\gamma_{\pi}\right)+\beta+\left(1-\varphi\gamma_{x}\right)\right)^{2}-4\beta\left(1-\varphi\gamma_{x}\right)>0$$

For every $\varepsilon \geq 0$ there is a $w = (\gamma_\pi^2, \gamma_x^2) = (\gamma_\pi^1 + \varepsilon, \gamma_x^1) \in \Gamma$ with $\gamma_\pi^2 \geq \gamma_\pi^1$.

For the combination $(\gamma_\pi, \gamma_x) = w$, the biggest eigenvalue of F, z_1^2 is equal to

$$z_1^2 = \frac{(\alpha\varphi(1-(\gamma_\pi^1+\varepsilon)) + \beta + (1-\varphi(\gamma_x^1)))}{2} + \\ + \frac{\sqrt{(\alpha\varphi(1-(\gamma_\pi^1+\varepsilon)) + \beta + (1-\varphi(\gamma_x^1)))^2 - 4\beta(\gamma_x^1)}}{2}$$

There could be two cases:

A. w is such that z_1^2 is real. In this case

$$\left(\alpha\varphi\left(1-\left(\gamma_\pi^1+\varepsilon\right)\right) + \beta + \left(1-\varphi\gamma_x^1\right)\right)^2 - 4\beta\gamma_x^1 > 0$$

Now, it is obvious that $z_1^2 - z_1^1 < 0$ and monotonicity with respect to γ_π is satisfied.

B. w is such that z_1^2 is complex. In this case z_1^1 should be compared with the real part of z_1^2: $\frac{\left(\alpha\varphi\left(1-\left(\gamma_\pi^1+\varepsilon\right)\right)+\beta+\left(1-\varphi\gamma_x^1\right)\right)}{2}$

Since

$$\frac{(\alpha\varphi(1-(\gamma_\pi^1+\varepsilon)) + \beta + (1-\varphi\gamma_x^1))}{2} - \frac{(\alpha\varphi(1-\gamma_\pi) + \beta + (1-\varphi\gamma_x))}{2} < 0$$

monotonicity with respect to γ_π is satisfied.

Consider an $h = (\gamma_\pi^1, \gamma_x^1)$ such that z_1^1 is complex. In this case only the real part of z_1^1: $\frac{(\alpha\varphi(1-\gamma_\pi)+\beta+(1-\varphi\gamma_x))}{2}$ is of interest.

Take a $w = (\gamma_\pi^1 + \varepsilon, \gamma_x^1)$, in this case $\|w - h\| = \left[\left(\gamma_\pi^1 + \varepsilon - \gamma_\pi^1\right)^2\right]^{\frac{1}{2}} = \varepsilon$. In the point $(\gamma_\pi, \gamma_x) = w$, the biggest eigenvalue of F, z_1^2 is equal to

$$z_1^2 = \frac{(\alpha\varphi(1-(\gamma_\pi^1+\varepsilon)) + \beta + (1-\varphi\gamma_x^1))}{2} + \\ + \frac{\sqrt{(\alpha\varphi(1-(\gamma_\pi^1+\varepsilon)) + \beta + (1-\varphi\gamma_x^1))^2 - 4\beta\gamma_x^1}}{2}$$

Note now, that if z_1^1 is complex, z_1^2 cannot be real: if z_1^1 is complex $4\beta\gamma_x^1 > (\alpha\varphi(1-\gamma_\pi^1) + \beta + (1-\varphi\gamma_x^1))^2$.

Now, since

$$\left(\alpha\varphi\left(1-\gamma_{\pi}^{1}\right)+\beta+\left(1-\varphi\gamma_{x}^{1}\right)\right)^{2}>\left(\alpha\varphi\left(1-\left(\gamma_{\pi}^{1}+\varepsilon\right)\right)+\beta+\left(1-\varphi\gamma_{x}^{1}\right)\right)^{2}$$

then $4\beta\gamma_x^1 > (\alpha\varphi(1-(\gamma_\pi^1+\varepsilon))+\beta+(1-\varphi\gamma_x^1))^2$, i.e., z_1^2 is complex. In this case it is obvious that monotonicity with respect to γ_π is satisfied. ■

No Monotonicity with respect to γ_x: Consider an $h = (\gamma_\pi^1, \gamma_x^1) \in \Gamma$ and a $w = (\gamma_\pi^1, \gamma_x^2) = (\gamma_\pi^1, \gamma_x^1 + \varepsilon) \in \Gamma$ such that z_1^1 and z_1^2 are complex. In this case it is easy to see (using a similar argument to the previous proof) that $z_1^1 \le z_1^2$; take now $h = (\gamma_\pi^1, \gamma_x^1) \in \Gamma$ and a $w = (\gamma_\pi^1, \gamma_x^2) = (\gamma_\pi^1, \gamma_x^1 + \varepsilon) \in \Gamma$ such that z_1^1 and z_1^2 are real and it is easy to see that $z_1^2 \le z_1^1$.

M. Proof of Proposition 14

Substituting the value of the conditional expectations into (2.21), the optimal policy rule could be written as:

$$i_t = \gamma^R + \gamma_g^R g_t$$

$$\gamma^R = \frac{\lambda\alpha}{(\lambda+\alpha^2)-\lambda\beta}\overline{x}$$
$$\gamma_g^R = \frac{1}{\varphi}$$

This expression says that the policy-maker should offset demand shocks (g_t) by adjusting the nominal interest rate in order to neutralize any shock to the IS curve. Since this optimal policy rule involves only the fundamentals of the economy (demand and supply shocks), it could be defined as the *optimal fundamentals-based reaction function* under rational expectations (Evans and Honkapohja (2002))[31].

Now, consider a generic *expectations-based* policy rule of the form:

$$i_t = \gamma + \gamma_x E_t x_{t+1} + \gamma_\pi E_t \pi_{t+1} + \gamma_g g_t$$

31 Many autors (see for example Woodford (1999)) have shown that this interest rate rule leads to indeterminacy, i.e., a multiplicity of rational expectations equilibria.

Assuming rational expectations, expected values could be substituted in the previous expression to obtain the following policy rule:

$$i_t = (\gamma + \gamma_x a_x + \gamma_\pi a_\pi) + \gamma_g g_t$$

By comparing this equation with the *optimal fundamentals-based policy rule,* a system of two equations on four unknowns ($\gamma, \gamma_x, \gamma_\pi, \gamma_g$) is obtained:

$$\begin{aligned} \gamma^R &= (\gamma + \gamma_x a_x + \gamma_\pi a_\pi) \\ \gamma_g^R &= \gamma_g \end{aligned}$$

Obviously, this system has multiple solutions.

N. PROOF OF PROPOSITION 15

By considering the values of the coefficients of the reaction function γ_g^*, γ^*,γ^R and the rational expectations values a_x, a_π given, the combinations of γ_x and γ_π are obtained that determine asymptotically the same equilibrium derived under the *optimal expectations-based reaction function* (2.21):

$$\gamma_\pi = \frac{(\lambda + \alpha^2)(1 + \alpha\varphi) - \lambda\beta}{\alpha(\lambda + \alpha^2)\varphi} - \frac{(1-\beta)}{\alpha}\gamma_x$$

Consider the isoquants of Figure 8:

$$\gamma_\pi = 1 - \frac{(1 - z_1)(\beta - z_1)}{z_1 \alpha \varphi} + \frac{(\beta - z_1)}{z_1 \alpha}\gamma_x \qquad \text{for } \gamma_x < \widehat{\gamma}_x$$

$$\gamma_\pi = 1 + \frac{\beta + 1 - 2z_1}{\alpha\varphi} - \frac{1}{\alpha}\gamma_x \qquad \text{for } \gamma_x \geq \widehat{\gamma}_x$$

with a kink on

$$\left(\widehat{\gamma}_x = \frac{-z_1^2 + \beta}{\varphi\beta}, \widehat{\gamma}_\pi = 1 + \frac{(\beta - z_1)^2}{\alpha\varphi\beta}\right)$$

Restricting the analysis to the set $\Gamma_\beta = \{\gamma_\pi, \gamma_x : 0 < z_1 < \beta, \gamma_\pi > 0, \gamma_x > 0\}$, now the maximum speed of convergence problem defined for $0 < z_1 < \beta$:

$$\begin{aligned} &\min_{\gamma_\pi, \gamma_x} Z(\gamma_\pi, \gamma_x) \\ s.t.\quad \gamma_\pi \quad &= \quad \frac{(\lambda + \alpha^2)(1 + \alpha\varphi) - \lambda\beta}{\alpha(\lambda + \alpha^2)\varphi} - \frac{(1-\beta)}{\alpha}\gamma_x \end{aligned}$$

has a solution (use proposition 3.D.1 in Mas-Colell et al., 1995), and there is also an indirect speed of convergence function $v(\lambda)$ that is strictly decreasing on λ (use proposition 3.D.3 in Mas-Colell et al., 1995). The maximum speed of convergence that could be induced by a combination (γ_π, γ_x) for a given λ will always coincide with the kink. Note that

$$\frac{\partial \widehat{\gamma}_x}{\partial z_1} = \frac{-2z_1}{\varphi\beta} < 0$$

$$\frac{\partial \widehat{\gamma}_\pi}{\partial z_1} = \frac{-2(\beta - z_1)}{\alpha\varphi\beta} < 0 \quad \text{for} \quad z_1 < \beta$$

Now, since the higher the level curve, the faster the convergence, it must be shown that as λ increases, the line

$$\gamma_\pi = \frac{(\lambda + \alpha^2)(1 + \alpha\varphi) - \lambda\beta}{\alpha(\lambda + \alpha^2)\varphi} - \frac{(1-\beta)}{\alpha}\gamma_x$$

moves downward and the fastest speed of convergence that is feasible is lower, or in other words the smallest z_1 that can be reached is larger.

O. Proof of Proposition 17

Under the EH policy, the economy evolves according to the following dynamic system:

$$\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \begin{bmatrix} \Phi^* \\ \frac{\Phi^*}{\alpha} \end{bmatrix} + \begin{bmatrix} \Gamma^* & 0 \\ -\frac{(\beta - \Gamma^*)}{\alpha} & 0 \end{bmatrix} \begin{bmatrix} E_t\pi_{t+1} \\ E_t x_{t+1} \end{bmatrix}$$

The REE under EH policy is

$$\pi_t = \frac{\Phi^*}{(1 - \Gamma^*)} \text{ and } x_t = \frac{\Phi^*(1-\beta)}{(1-\Gamma^*)\alpha}$$

Under the ALS-Γ' policy, the economy evolves according to the following dynamic system:

$$\begin{bmatrix} \pi_t \\ x_t \end{bmatrix} = \begin{bmatrix} \frac{\Phi^*(1-\Gamma')}{(1-\Gamma^*)} \\ \frac{\Phi^*(1-\Gamma')}{(1-\Gamma^*)\alpha} \end{bmatrix} + \begin{bmatrix} \Gamma' & 0 \\ -\frac{(\beta-\Gamma')}{\alpha} & 0 \end{bmatrix} \begin{bmatrix} E_t\pi_{t+1} \\ E_t x_{t+1} \end{bmatrix}$$

It is evident that the REE under ALS-Γ' policy is the same as under the EH policy.

Under learning, when both Γ' and Γ^* are smaller than one the REE is E-stable.

P. Proof of Lemma 18

If

$$1 > \Gamma^* > \Gamma'$$

then

$$\gamma'_\pi = \left(1 + \frac{(\beta - \Gamma')}{\alpha\varphi}\right) > \left(1 + \frac{\beta - \Gamma^*}{\alpha\varphi}\right) = \gamma^*_\pi$$

Similarly if

$$1 > \Gamma' > \Gamma^*$$

Q. Proof of Proposition 19

$$\overline{a}_\pi = \frac{\Phi^*}{(1 - \Gamma^*)}$$

$$\pi_t\left(i^{EH}\right) = \Phi^* + \Gamma^* a_{\pi,t}\left(i^{EH}\right)$$

$$\pi_t\left(i^{ALS}\left(\Gamma'\right)\right) = \Phi^*\left(1 - \Gamma^*\right)^{-1}\left(1 - \Gamma'\right) + \Gamma' a_{\pi,t}\left(i^{ALS}\left(\Gamma'\right)\right)$$

$$a_{\pi,t}\left(i^{EH}\right)=a_{\pi,t-1}\left(i^{EH}\right)+t^{-1}\left(\pi_{t-1}\left(i^{EH}\right)-a_{\pi,t-1}\left(i^{EH}\right)\right)$$

$$a_{\pi,t}\left(i^{ALS}\left(\Gamma'\right)\right)=a_{\pi,t-1}\left(i^{ALS}\left(\Gamma'\right)\right)+t^{-1}\left(\pi_{t-1}\left(i^{ALS}\left(\Gamma'\right)\right)-a_{\pi,t-1}\left(i^{ALS}\left(\Gamma'\right)\right)\right)$$

$$a_{\pi,0}\left(i^{ALS}\left(\Gamma'\right)\right)=a_{\pi,0}\left(i^{EH}\right)=a_{\pi,0}\neq\overline{a}_{\pi}$$

I will proove the proposition for $\gamma_{\pi}^{ALS}>\gamma_{\pi}^{*}$. A similar procedure could be used for the case $\gamma_{\pi}^{ALS}<\gamma_{\pi}^{*}$.

Let $\gamma_{\pi}^{ALS}>\gamma_{\pi}^{*}$, then

$$\Gamma'<\Gamma^{*}$$

Now, for $t=0$, since

$$\pi_{0}\left(i^{EH}\right)-\overline{a}_{\pi}=\Gamma^{*}\left(a_{\pi,0}-\frac{\Phi^{*}}{(1-\Gamma^{*})}\right)$$

and

$$\pi_{0}\left(i^{ALS}\left(\Gamma'\right)\right)-\overline{a}_{\pi}=\Gamma'\left(a_{\pi,0}-\frac{\Phi^{*}}{(1-\Gamma^{*})}\right)$$

then

$$\left|\pi_{0}\left(i^{ALS}\left(\Gamma'\right)\right)-\overline{a}_{\pi}\right|<\left|\pi_{0}\left(i^{EH}\right)-\overline{a}_{\pi}\right|$$

For $t=1$, since

$$a_{\pi,1}\left(i^{EH}\right)-\overline{a}_{\pi}=\pi_{\mathbf{0}}\left(i^{EH}\right)-\overline{a}_{\pi}$$

$$a_{\pi,t}\left(i^{ALS}\left(\Gamma'\right)\right)-\overline{a}_{\pi}=\pi_{\mathbf{0}}\left(i^{ALS}\left(\Gamma'\right)\right)-\overline{a}_{\pi}$$

then

$$\left|a_{\pi,t}\left(i^{ALS}\left(\Gamma'\right)\right)-\overline{a}_{\pi}\right|<\left|a_{\pi,1}\left(i^{EH}\right)-\overline{a}_{\pi}\right|$$

Moreover, since

$$\pi_1\left(i^{EH}\right)-\overline{a}_{\pi}=\Phi^*+\Gamma^*\left(\Phi^*+\Gamma^*a_{\pi,\mathbf{0}}\right)-\overline{a}_{\pi}$$

$$\pi_1\left(i^{ALS}\left(\Gamma'\right)\right)-\overline{a}_{\pi}=\Phi^*\left(1-\Gamma'\right)\left(1-\Gamma^*\right)^{-1}+\Gamma'\left(\Phi^*\left(1-\Gamma^*\right)^{-1}\left(1-\Gamma'\right)+\Gamma'a_{\pi,\mathbf{0}}\right)-\overline{a}_{\pi}$$

$$x_1\left(i^{ALS}\left(\Gamma'\right)\right)-\overline{a}_{\pi}=\frac{\Phi^*\left(1-\Gamma'\right)}{\left(1-\Gamma^*\right)\alpha}-\frac{\left(\beta-\Gamma'\right)}{\alpha}\left(\Phi^*\frac{\left(1-\Gamma'\right)}{\left(1-\Gamma^*\right)}+\Gamma'a_{\pi,\mathbf{0}}\right)-\overline{x}$$

then

$$\pi_1\left(i^{EH}\right)-\overline{a}_{\pi}=\Gamma^{*2}\left(a_{\pi,\mathbf{0}}-\frac{\Phi^*}{\left(1-\Gamma^*\right)}\right)$$

$$\pi_1\left(i^{ALS}\left(\Gamma'\right)\right)-\overline{a}_{\pi}=\Gamma'^2\left(a_{\pi,\mathbf{0}}-\frac{\Phi^*}{\left(1-\Gamma^*\right)}\right)$$

and since

$$\Gamma^{*2}>\Gamma'^2$$

then

$$\left|\pi_1\left(i^{ALS}\left(\Gamma'\right)\right)-\overline{a}_\pi\right|<\left|\pi_1\left(i^{EH}\right)-\overline{a}_\pi\right|$$

Similarly for $t>1$.

R. PROOF OF LEMMA 21

Given that

$$x^{REE}=\frac{\Phi^*\left(1-\beta\right)}{\left(1-\Gamma^*\right)\alpha}$$

it must be shown that if $a_{\pi,0}\left(i^{ALS}\left(\Gamma'\right)\right)=a_{\pi,0}\left(i^{EH}\right)>\overline{a}_\pi$, then for every $0\leq t<\infty$,

$$x_t\left(i^{ALS}\left(\Gamma'\right)\right),x_t\left(i^{EH}\right)<x^{REE}$$

and for all $0\leq t',t<\infty$, with $t'>t$

$$x_t\left(i^{ALS}\left(\Gamma'\right)\right)<x_{t'}\left(i^{ALS}\left(\Gamma'\right)\right)<x^{REE}\text{ and }x_t\left(i^{EH}\right)<x_{t'}\left(i^{EH}\right)<x^{REE}$$

If $a_{\pi,0}\left(i^{ALS}\left(\Gamma'\right)\right)=a_{\pi,0}\left(i^{EH}\right)<\overline{a}_\pi$, then for every $0\leq t<\infty$,

$$x_t\left(i^{ALS}\left(\Gamma'\right)\right),x_t\left(i^{EH}\right)>x^{REE}$$

and for all $0\leq t',t<\infty$, with $t'>t$

$$x_t\left(i^{ALS}\left(\Gamma'\right)\right)>x_{t'}\left(i^{ALS}\left(\Gamma'\right)\right)>x^{REE}\text{ and }x_t\left(i^{EH}\right)>x_{t'}\left(i^{EH}\right)>x^{REE}$$

$$x_t\left(i^{EH}\right)=\frac{\Phi^*}{\alpha}-\frac{\left(\beta-\Gamma^*\right)}{\alpha}a_{\pi,t}\left(i^{EH}\right)$$

$$x_t\left(i^{ALS}\left(\Gamma'\right)\right)=\frac{\Phi^*\left(1-\Gamma'\right)}{\left(1-\Gamma^*\right)\alpha}-\frac{\left(\beta-\Gamma'\right)}{\alpha}a_{\pi,t}\left(i^{ALS}\left(\Gamma'\right)\right)$$

let $a_{\pi,0} < \overline{a}$. Since

$$\frac{(\Gamma' - \beta)}{\alpha}, \frac{(\Gamma^* - \beta)}{\alpha} < 0$$

in

$$x_0\left(i^{EH}\right) - x^{REE} = \frac{(\Gamma^* - \beta)}{\alpha}\left(a_{\pi,0} - \frac{\Phi^*}{(1-\Gamma^*)}\right)$$

$$x_0\left(i^{ALS}(\Gamma')\right) - x^{REE} = \frac{(\Gamma' - \beta)}{\alpha}\left(a_{\pi,0} - \frac{\Phi^*}{(1-\Gamma^*)}\right)$$

then $x_0\left(i^{EH}\right), x_0\left(i^{ALS}(\Gamma')\right) < 0$.

For $t = 1$, we have

$$x_1\left(i^{EH}\right) - x^{REE} = \Gamma^* \frac{(\Gamma^* - \beta)}{\alpha}\left(a_{\pi,0} - \frac{\Phi^*}{(1-\Gamma^*)}\right)$$

$$x_1\left(i^{ALS}(\Gamma')\right) - \overline{a}_\pi = \Gamma' \frac{(\Gamma' - \beta)}{\alpha}\left(a_{\pi,0} - \frac{\Phi^*}{(1-\Gamma^*)}\right)$$

and again it is obvious that $x_1\left(i^{EH}\right), x_1\left(i^{ALS}(\Gamma')\right) < 0$ and

$$x_1\left(i^{ALS}(\Gamma')\right) > x_0\left(i^{ALS}(\Gamma')\right) \text{ and } x_1\left(i^{EH}\right) > x_0\left(i^{EH}\right)$$

similarly for t=2,3,... and for the case $a_{\pi,0}\left(i^{ALS}(\Gamma')\right) = a_{\pi,0}\left(i^{EH}\right) > \overline{a}_\pi$.

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