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Gaussian inference on certain long-range dependent volatility models

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GAUSSIAN INFERENCE ON CERTAIN LONG-RANGE DEPENDENT VOLATILITY MODELS

Paolo Zaffaroni *

Abstract

For a class of long memory volatility models, we establish the asymptotic distribution theory of the Gaussian estimator and the Lagrange multiplier test. Both the case of estimation of martingale difference and ARMA levels are considered. A Montecarlo exercise is presented to assess the small sample properties of the Gaussian estimator and the Lagrange multiplier test. An empirical application, using foreign exchange rates and stock index returns, suggests the potential of these models to capture the dynamic features of the data.

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Keywords: volatility model; nonlinear moving average model; long memory; Whittle estimation; asymptotic distribution theory.

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1 Introduction*

The importance of modelling volatility lies in the dependence of any financial investment decision on the forecasts of asset risks and returns, formalized in asset pricing theory following the Sharpe (1964) and Lintner (1965) capital asset pricing model. Moreover, a deep understanding of the data generating process underlying a time series is a compulsory step towards forecasting it and, when dealing with financial assets, is crucial for a successful implementation of any option-pricing model. Indeed, since the introduction of the autoregressive conditional heteroskedasticity (ARCH) model of Engle (1982), a great deal of research has focused on modelling the volatility of financial asset returns as they feature very complicated dynamics, synthesized in a number of well-known stylized facts, dynamic conditional heteroskedasticity in particular. Estimation of these models is usually based on a maximum likelihood estimator (MLE) approach, mainly using the assumption of Gaussignify of rescaled innovations. This is easily implementable but derivation of the statistical properties of MLE, pseudo MLE (PMLE) when the distributional assumptions are violated, appears a very difficult task because of the nonlinear feature of the model. Indeed, consistency and asymptotic normality have been established for the Gaussian PMLE of the basic ARCH(p)model (see Weiss (1986)) and for the generalized ARCH(1, 1) (GARCH(1, 1)) only (see Lee and Hansen (1994) and Lumsdaine (1996)), despite the innumerable developments of ARCH-type volatility modelling. (See e.g. Bollerlsev, Chou, and Kroner (1992) or Bollerslev, Engle, and Nelson (1995) for complete surveys.)

In this paper we consider an alternative way of modelling changing volatility by means of a family of nonlinear moving average (nonlinear MA) models, introduced by Robinson and Zaffaroni (1997),

$$x_t = \mu + u_t, \qquad t \in \mathbb{Z}, \tag{1}$$

with

$$u_t = \epsilon_t h_{t-1}, \quad h_{t-1} = \rho + \sum_{i=1}^{\infty} \alpha_i \epsilon_{t-i}, \quad \sum_{i=1}^{\infty} \alpha_i^2 < \infty, \quad (2)$$

where the x_t represent the observed time series and the ϵ_t the unobservable independent identically distributed (*i.i.d.*) zero mean innovations. The constant μ defines the (conditionally constant) first moment of the x_t whereas ρ and the α_i characterize the memory of squares and higher-order statistical properties, as discussed below. Robinson (2001) shows that for Gaussian ϵ_t the nonlinear MA model (2) is nested within a large class of volatility models, which includes the standard stochastic volatility (SV) model of Taylor

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(1986) and a simple case of the exponential GARCH (EGARCH) model of Nelson (1991).

The main contribution of this paper is to establish the asymptotic statistical properties of the frequency domain Gaussian estimator in the sense of Whittle (1962), for the nonlinear MA (1)-(2), employing the squares as the observable. We focus on parameterizations for the model that allow long memory autocorrelated squares. This choice is motivated on both theoretical and empirical grounds. In fact, although not formally established, it is likely that generalizations of previous work on Gaussian estimation and methodof-moment estimation, along the lines of Robinson (1977, 1978), would yield asymptotically correct statistical inference for short memory parameterizations of the squares. On the other hand, relatively recent empirical research on asset returns volatility suggests that the effect of shocks on the conditional variance is very persistent though eventually absorbed with time (see Gallant, Rossi, and Tauchen (1993)), ruling out integrated GARCH behaviour. That is, sample autocorrelations in squared returns tend to decline very slowly in contrast with the fast exponential decay implied by standard ARCH-type models. This is, in turn, consistent with the notion of long memory when theoretical autocovariances are not summable, or when, alternatively, the power spectrum is unbounded at zero frequency.

With respect to ARCH-type models various alternatives have been proposed to account for this aspect of asset returns dynamics. In order to develop a test for no-ARCH Robinson (1991) introduced the ARCH(∞), a possibly long memory generalized ARCH model. Baillie, Bollerslev, and Mikkelsen (1996) considered a particular case of the ARCH(∞) denominated as fractionally integrated GARCH (FIGARCH), and proposed a time domain Gaussian PMLE estimation approach. Note that $ARCH(\infty)$ is a nonlinear autoregressive model and cannot be nested within the Robinson (2001) class of nonlinear moving average models. Nelson (1991), introducing the EGARCH, mentioned the possibility of long memory parameterizations explicitly developed in Bollerlsev and Mikkelsen (1996, the fractionally integrated EGARCH). These long memory volatility models are characterized by the lack of asymptotic distribution theory for the PMLE, as well as for any other estimator, and the available asymptotic results for estimation of short memory ARCH-type models and linear long memory processes are not readily extendable.

By contrast, in this paper we provide a formal framework to assess whether long memory volatility models represent a valid alternative to short memory ones.

Making distributional assumptions on the innovations, the nonlinear MA could be estimated by (exact) MLE. However, asymptotic properties of the MLE would depend on invertibility of the model and the invertibility the-

ory of linear processes is not immediately extendable to nonlinear cases, as noted in Granger and Andersen (1978). Furthermore, for the nonlinear MA, likelihood and scores would only be computable recursively and expensively, which would render asymptotic theory of exact MLE extremely cumbersome to establish. Moreover, given the dynamic structure of the model, the likelihood necessarily depends on the initial conditions which will probably have non-negligible effects in finite samples, especially within a long memory parameterization. More importantly, the asymptotic properties of the MLE may not be robust with respect to deviations from the postulated distributional assumption, as often happens for nonlinear econometric models.

These arguments do not apply to the Gaussian estimator considered here, partly owing to its frequency domain specification. We establish that the Gaussian estimator is strongly \sqrt{T} -consistent with a normal asymptotic distribution, T denoting the sample size. This result does not follow directly from the asymptotic results on ARCH estimation nor from the Gaussian estimation theory for linear long memory models (see e.g. Giraitis and Surgailis (1990), Hosoya (1997)), but depends upon a central limit theorem (CLT) for quadratic forms in nonlinear long memory variates established in this paper. Secondly, we develop a pseudo Gaussian Lagrange multiplier (LM) testing procedure designed to detect dynamic conditional heteroskedasticity that uses the nonlinear MA as a class of alternatives. Despite the long memory feature of these alternative hypotheses, the usual asymptotic chi-square distribution is obtained under the null hypothesis of homoskedasticity.

The paper is organized as follows. In section 2 we briefly recall the loworder statistical properties of nonlinear MA: memory, kurtosis and 'leverage' effect. Estimation of the model is discussed in section 3, where we introduce the Gaussian estimator and the LM multiplier test and establish their asymptotic statistical properties. The martingale difference assumption for (demeaned) levels x_t implied by (1) and (2) makes it possible to focus exclusively on the volatility dynamics of the nonlinear MA model. However, it represents at best an approximation for practical use of the model. A finite-order autoregressive moving average (ARMA) model represents a simple yet appealing parameterization of the conditional mean of the observable x_t . In section 4, we present the asymptotic properties of the Whittle estimator of the parameters of ARMA models with innovations u_t described by the nonlinear MA (2). Preliminary estimation of the nonlinear MA model is not required. Next, we present the asymptotic properties of the Gaussian estimator of the nonlinear MA based on the estimated ARMA residuals. Section 5 reports the result of Montecarlo exercises to assess the small-sample performance of the Gaussian estimator and of the LM test. In section 6 we present an empirical application based on observed time series of stock index and foreign exchange rate returns. The nonlinear MA and its likely

main competitor, the ARCH(∞), are compared in terms of goodness-of-fit using the Gaussian estimator. Concluding remarks are in section 7. The results are formally stated in theorems the proofs of which are reported in the mathematical appendixes, together with a number of technical lemmas.

2 Low-order statistical properties of the nonlinear moving average model

Let the innovation ϵ_t in (2) satisfy the following assumption. Assumption A. The process $\{\epsilon_t\}$ is i.i.d. with $E(\epsilon_t^4) < \infty$ and

$$\begin{split} E(\epsilon_t) &= 0, \\ E(\epsilon_t^2) &= \sigma^2, \ 0 < \sigma^2 < \infty, \\ E(\epsilon_t^3) &= 0. \end{split}$$

Let κ denote the fourth-order cumulant of the ϵ_t , yielding $E(\epsilon_t^4) = \kappa + 3\sigma^4$.

We do not normalize the innovation variance σ^2 to be equal to 1. More importantly, we do not constrain the magnitude of the fourth moment and of the fourth-order cumulant. Assumption A is satisfied, for instance, for Student's $t \epsilon_t$ with n degrees of freedom whenever n > 4. In this case $\kappa = n^2 B(\frac{5}{2}, \frac{n-4}{2})/B(\frac{1}{2}, \frac{n}{2}) - 3(\frac{n}{n-2})^2$ with $\kappa > 0$ where $B(\cdot, \cdot)$ indicates the Beta function. When the ϵ_t are Gaussian then $\kappa = 0$ whereas $\kappa < 0$ for ϵ_t uniformly distributed over [-a, a] for some finite a > 0. Note that, in general, bounded fourth moment and non degeneracy of the ϵ_t ($0 < \operatorname{var}(\epsilon_t^2) < \infty$) implies $\kappa > -2\sigma^4$. When Assumption A holds, the demeaned x_t are martingale difference but not independent as the squares $y_t = x_t^2$ are autocorrelated (cf. Robinson and Zaffaroni (1997)).

A frequency domain characterization of the memory properties of the x_t and the y_t is central to the paper, in particular with respect to the estimation part (cf. section 3). For this purpose, let $\alpha(\lambda)$ ($-\pi \leq \lambda < \pi$) be the transfer function of the sequence { α_i }, implying

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \alpha(\omega) e^{i\omega u} d\omega = \alpha_u, \ u \ge 1.$$
(3)

Set

$$\beta(\lambda) = 2\mathcal{R}e(\alpha(\lambda)), \quad -\pi \le \lambda < \pi, \tag{4}$$

 $\mathcal{R}e(\cdot)$ denoting the real part of its argument and let $f_{zz}(\lambda)$ $(-\pi \leq \lambda < \pi)$ be the power spectrum (when it exists) of a weakly stationary process $\{z_t\}$. Hereafter, set $\delta_{ab}(u) = \sum_{i=1}^{\infty} a_i b_{i+u} (u = 0, \pm 1, ...)$ for any square summable sequences $\{a_i\}, \{b_i\}$. **Theorem 1** Under Assumption A, for $-\pi \leq \omega < \pi$,

$$f_{xx}(\omega) = \frac{\sigma^2}{2\pi} (\rho^2 + \sigma^2 \delta_{\alpha\alpha}(0)),$$

$$f_{yy}(\omega) = \frac{(2\sigma^4 + \kappa)(\rho^2 + \sigma^2 \delta_{\alpha\alpha}(0))}{(2\pi)^2} \int_{-\pi}^{\pi} \beta(\lambda)\beta(\omega - \lambda)d\lambda + \frac{4\sigma^4 \rho^2 \mu}{2\pi}\beta(\omega)$$

$$+ \frac{2\sigma^8}{(2\pi)^2} \int_{-\pi}^{\pi} |\alpha(\lambda)|^2 |\alpha(\omega - \lambda)|^2 d\lambda + \frac{4\sigma^6 \mu}{(2\pi)^2} \int_{-\pi}^{\pi} \beta(\lambda) |\alpha(\omega - \lambda)|^2 d\lambda$$

$$+ \frac{4\sigma^6 \rho^2}{2\pi} |\alpha(\omega)|^2 + \frac{\sigma^4 \kappa}{(2\pi)^3} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \alpha(\lambda)\alpha(\theta)\alpha(\omega - \theta)\alpha(-\omega - \lambda)d\lambda d\theta$$

$$+ \frac{\nu_y}{2\pi},$$

where ν_y is a strictly positive constant.

Proof: see appendix A.

The power spectrum of the y_t is a linear combination of convolutions of the $|\alpha(\omega)|^2$, $\beta(\omega)$, given the nonlinear transformation involved $y_t = x_t^2$ (see Hannan (1970, p.82)).

Alternatively, under Assumption A the y_t have autocorrelation function (ACF)

$$\gamma_{yy}(l) = 2\sigma^{8}\alpha_{|l|}^{2}\delta_{\alpha\alpha}(0) + 2\sigma^{8}\delta_{\alpha\alpha}^{2}(l) + 4\rho^{2}\sigma^{6}\delta_{\alpha\alpha}(l) + \sigma^{4}\kappa(\delta_{\alpha\alpha}(0)\alpha_{|l|}^{2} + \delta_{\alpha^{2}\alpha^{2}}(l)) + 4\sigma^{4}\mu\alpha_{|l|}\left[\rho^{2} + \sigma^{2}\delta_{\alpha\alpha}(l)\right] + 2\rho^{2}\sigma^{6}\alpha_{|l|}^{2} + \nu_{y}\delta(l,0), \quad l = 0, \pm 1, \dots$$
(5)

where, henceforth we set $\gamma_{zu}(l) = \operatorname{cov}(z_t, u_{t+l}), l = 0, \pm 1, \ldots$, for stationary processes $\{z_t, u_t\}$ and by $\delta(u, v)$ the Kronecker delta. Robinson and Zaffaroni (1997), hereafter RZ, establish (5) for the case $\rho = \mu = 0$. We skip details for simplicity's sake but, in the proof of Theorem 1, sketch the derivation of (5) in the more general case here considered.

By (5) and square summability of the α_i (see RZ), as $l \to \infty$,

$$\gamma_{yy}(l) \sim \begin{cases} 2\sigma^8 \delta_{\alpha\alpha}^2(l), & \rho = 0, \\ 4\rho^2 \sigma^6 \delta_{\alpha\alpha}(l), & \rho \neq 0, \end{cases}$$
(6)

where \sim denotes asymptotic equivalence: $a(x) \sim b(x)$ as $x \to x_0$ when $a(x)/b(x) \to 1$ as $x \to x_0$. When

$$\alpha_j \sim k \, j^{d-1} \quad \text{as} \quad j \to \infty$$

with 0 < d < 1/2, $0 < |k| < \infty$, the y_t display an hyperbolically decaying ACF, exhibiting long memory when $\rho \neq 0$ or for d > 1/4 when $\rho = 0$ (cf. RZ, section 4). In particular, when $\rho \neq 0$

$$\gamma_{yy}(l) \sim k \, l^{2d-1} \quad \text{as} \quad l \to \infty,$$
(7)

and both long memory and covariance stationarity are achieved for 0 < d < 1/2.

As a by-product of the nonlinearity, the nonlinear MA exhibits fatter tails than the Gaussian case, thus matching another well-known stylized fact of asset return dynamics, since Fama (1965). Under Assumption A, the model kurtosis is

$$\frac{E(x_t - \mu)^4}{\gamma_{xx}^2(0)} = 3 + \frac{6}{(\bar{\rho}^2 + \delta_{\alpha\alpha}(0))^2} (\delta_{\alpha\alpha}^2(0) + 2\bar{\rho}^2 \delta_{\alpha\alpha}(0)) \\
+ \frac{\bar{\kappa}}{(\bar{\rho}^2 + \delta_{\alpha\alpha}(0))^2} (\bar{\rho}^4 + 6\bar{\rho}^2 \delta_{\alpha\alpha}(0) + 3\delta_{\alpha\alpha}^2(0) + (3 + \bar{\kappa})\delta_{\alpha^2\alpha^2}(0))$$

setting $\bar{\rho} = \rho/\sigma$, $\bar{\kappa} = \kappa/\sigma^4$. The second term on the right-hand side expresses the deviation from the Gaussian kurtosis case due to the nonlinearity of the model whereas the third term depends on the fourth-order cumulant. It will be zero only for $\kappa = 0$ but it can be arbitrarily large, depending on the tail behaviour of the marginal distribution of the ϵ_t .

Since Black (1976), another feature of asset return empirical distributions is the so-called 'leverage' effect, i.e. an asymmetric response of future (*l*period ahead) volatility to past and present shocks implied by a negative value of $cov(x_t, y_{t+l})$. Insights into this dynamic asymmetry can be obtained generalizing h_t in (2) by

$$\tilde{h}_t = \rho + \sum_{i=1}^{\infty} \alpha_i g(\epsilon_{t-i}), \qquad (8)$$

where $g(\epsilon_t) = \vartheta \epsilon_t + \gamma(|\epsilon_t| - E |\epsilon_t|)$ for some constant real ϑ, γ . The specification of the $g(\cdot)$ function, known as the 'news impact curve' in volatility literature, was introduced by Nelson (1991). Under Assumption A, setting (for simplicity) $\mu = 0$ and $E(\epsilon_t |\epsilon_t|) = 0$, e.g. implied by symmetric ϵ_t , one obtains for l > 0

$$\operatorname{cov}(x_t, y_{t+l}) = 2\sigma^4 \vartheta \alpha_l \left((\sigma^2(\vartheta^2 + \gamma^2) - \gamma^2 E^2 |\epsilon_t|) \delta_{\alpha\alpha}(l) + \rho^2 \right).$$

It follows that the crucial (necessary) condition to account for the 'leverage' effect is $\vartheta \neq 0$. The interplay of ϑ and γ would impart great flexibility in modelling asymmetry expressed by shifts and rotations in the 'news impact curve' (see e.g. Hentschel (1995)). We decided to focus on the particular yet asymmetric case h_t (cf. (2)), setting $\vartheta = 1, \gamma = 0$ in \tilde{h}_t . In fact, both the non-differentiability of the absolute-value function in $g(\cdot)$ and the possibility of leaving ϑ , γ unrestricted are likely to induce a great deal of technical difficulties when establishing the asymptotics for the class of estimators considered in this paper (cf. section 3). The 'leverage' effect, at lag L, arises whenever $\alpha_L < 0$ independently of the value of ρ . The nonlinear MA is also

apt to display negative 'simultaneous' correlation between asset return and asset volatility, as observed in Campbell and Hentschel (1990).

In contrast to nonlinear autoregressive models (e.g. ARCH-type models), derivation of the (low-order) statistical properties of the nonlinear MA appears straightforward. Note, however, that this is not achieved by introducing a second latent sequence of innovations, as for example for SV models. Thus, the usual difficulties arising with the estimation and filtering of SV models are avoided in our nonlinear MA framework. We now discuss estimation and testing of the model.

3 Estimation and testing

3.1 Parameterization

Within this section, let us set $\mu = 0$ for the sake of simplicity. The more complicated case of observable x_t with time-varying conditional mean will be described in section 4. Let us introduce a sequence of functions $\alpha_i(\theta)$ $(i \ge 1)$ of a $p \times 1$ vector $\theta \in \Theta \subset \mathbb{R}^p$. Within the set-up of the previous sections, this implies introducing new functions $\alpha(\lambda; \theta)$ and $\beta(\lambda; \theta)$ of the frequency $\lambda(-\pi \le \lambda < \pi)$ and θ . Replace $\alpha(\lambda)$ and $\beta(\lambda)$ in $f_{yy}(\lambda)$ (cf. Theorem 1) with $\alpha(\lambda; \theta)$ and $\beta(\lambda; \theta)$. Recalling $\bar{\kappa} = \kappa/\sigma^4$, $\bar{\rho}^2 = \rho^2/\sigma^2$, set $\psi = (\bar{\kappa}, \theta')'$ and $\bar{\rho}^2 = 1$ for model identification. This yields $f_{yy}(\lambda; \psi, \sigma^2)$ for $\psi \in \Psi \subset \mathbb{R}^{p+1}$, Ψ denoting the parameter space. It is convenient to express $f_{yy}(\lambda; \psi, \sigma^2)$ as $\sigma^8 h_y(\lambda; \psi)$ with $h_y(\cdot; \psi) = f_{yy}(\cdot; \psi, 1)$ because, as shown below, rescaling κ and ρ^2 by suitable powers of σ^2 allows factorizing and hence concentrating σ^2 out. In the same way, replacing the α_i with the $\alpha_i(\theta)$ in (5), we define the sequence $\gamma_{yy}(l; \psi, \sigma^2)$, $l = 0, \pm 1, \ldots$, which can be re-written as $\gamma_{yy}(l; \psi, \sigma^2) = \sigma^8 \bar{\gamma}_{yy}(l; \psi)$. Denoting by ψ_0 the true value for ψ , with $\psi_0 \in \Psi$, it follows that, for any $-\pi \le \lambda < \pi$ and $l = 0, \pm 1, \ldots$,

$$f_{yy}(\lambda) = \sigma^8 h_y(\lambda; \psi_0) = \sigma^8 h_y(\lambda), \quad \gamma_{yy}(l) = \sigma^8 \bar{\gamma}_{yy}(l; \psi_0), \quad \alpha_l = \alpha_l(\theta_0).$$

An example of a parsimonious parameterization, adopted in our empirical application below (cf. section 6), is obtained by choosing the $\alpha_i(\theta)$ as the $MA(\infty)$ coefficients of the autoregressive fractional moving average model of order p, d, q (ARFIMA(p, d, q)) of Granger and Joyeux (1980), for integers $p, q \geq 0$ and real |d| < 1/2:

$$1 + \alpha(L;\theta) = (1-L)^{-d} \frac{\chi(L)}{\phi(L)} = (1-L)^{-d} \frac{(1+\chi_1 L + \dots + \chi_q L^q)}{(1-\phi_1 L - \dots - \phi_p L^p)}, \qquad (9)$$

setting $\theta = (\chi_1, \ldots, \chi_q, \phi_1, \ldots, \phi_p, d)'$ and assuming that the polynomials in the lag operator L (where $x_{t-1} = Lx_t$), $\chi(L)$ and $\phi(L)$, have roots strictly

outside the unit circle in the complex plane and never common. Remembering that non-negativity constraints on the $\alpha_i(\theta)$ need not be imposed, one can allow for cyclical behaviour in the $\alpha_i(\theta)$; e.g. by the standard reparameterization of the AR(p) coefficients, with p = 2, as $\phi_1 = 2\tilde{\phi}_1 \cos(\tilde{\phi}_2)$, $\phi_2 = -\tilde{\phi}_1^2$ with $0 \leq \tilde{\phi}_1 < 1$ and $0 \leq \tilde{\phi}_2 \leq \pi$.

3.2 Limiting distribution theory of the Gaussian estimator

The Gaussian estimator, originally proposed in RZ, is based on Whittle's function (see Whittle (1962)), i.e. the frequency domain approximation of the Gaussian log likelihood, used as a measure of distance between the periodogram of squared observations and the model spectral density $f_{yy}(\lambda; \psi, \sigma^2)$. In particular, setting $\lambda_j = 2\pi j/T$, we consider the following discrete version of Whittle's function

$$Q_T(\sigma^2, \psi) = \frac{1}{T} \sum_{j=1}^{T-1} \left[\ln(\sigma^8 h_y(\lambda_j; \psi)) + \frac{I_{yy}(\lambda_j)}{\sigma^8 h_y(\lambda_j; \psi)} \right],$$
(10)

where $I_{yy}(\omega)$ defines the periodogram based on consecutive T observations of the y_t :

$$I_{yy}(\omega) = \frac{1}{2\pi T} \left| \sum_{t=1}^{T} (y_t - \bar{y}) e^{i t \omega} \right|^2, \quad -\pi \le \omega < \pi$$
(11)

with $\bar{y} = 1/T \sum_{t=1}^{T} y_t$. We skip the zero frequency in (10) due to mean correction. Concentrating σ^8 out yields

$$Q_T(\psi) = \ln\left(\hat{\sigma}_T^{8}(\psi)\right) + \frac{1}{T} \sum_{j=1}^{T-1} \ln h_y(\lambda_j; \psi) + 1, \qquad (12)$$

setting

$$\hat{\sigma}_T^8(\psi) = \frac{1}{T} \sum_{j=1}^{T-1} \frac{I_{yy}(\lambda_j)}{h_y(\lambda_j;\psi)}.$$

Set $\overline{\Psi}$ equal to the compact closure of Ψ . The Gaussian estimator $\hat{\psi}_T$ is defined by

$$\hat{\psi}_T = argmin_{\psi \in \bar{\Psi}} Q_T(\psi)$$
 .

The main motivation for employing Whittle's function is that it naturally takes into account the asymptotic behaviour of the ACF of the y_t , as the sample size goes to infinity. Therefore we expect it to be very sensitive to the degree of dependence of the process in a second-order sense. Indeed, in a semiparametric framework Robinson (1995a) showed how the most efficient (semiparametric) estimator of the memory parameter is obtained via a local Gaussian log likelihood. Moreover, Whittle's function does not depend on initial conditions and its computation and asymptotic behaviour is independent from the invertibility of the model. On the other hand, the Gaussian estimator requires knowledge of the precise functional form of the model spectral density (cf. Theorem 1). Moreover, independently of the distributional assumption made on the ϵ_t , the y_t can never be Gaussian by construction and thus the Gaussian estimator can never achieve asymptotic efficiency in contrast to MLE.

One cannot implement the Gaussian estimator by employing the levels x_t as the observable. In this case, parameter non-identification arises because the x_t display a perfectly flat power spectrum (cf. Theorem 1). In principle, different instantaneous transformations may be considered but, using the squares $y_t = x_t^2$ as the observable represents the simplest feasible approach.

This estimation approach can be extended considering a multivariate version. Let us take the bivariate process $z_t = (x_t, y_t)$ as the observable. Under the presumption that the z_t are jointly Gaussian, one obtains the 'bivariate' Gaussian estimator by minimizing the 'bivariate' Whittle's function. Using this 'bivariate' Gaussian estimator one achieves a certain efficiency gain with respect to the (univariate) Gaussian estimator (see Zaffaroni (1997)). Within an ARCH-type setting, Meddahi and Renault (1997) show that using a time domain 'bivariate' Gaussian PMLE, based on the correct specification of the first and second conditional moment of the observable, one achieves the generalized method of moment efficiency bound in the sense of Chamberlain (1987). This suggests that a similar result might hold for the 'bivariate' Gaussian estimator. However, in this paper we will focus only on the asymptotics of the (univariate) Gaussian estimator and leave the analysis of the multivariate Gaussian estimator, and the related efficiency issue, to future research.

In order to establish the asymptotic properties of the Gaussian estimator $\hat{\psi}_T$ we introduce the following assumptions. Let k, K denote arbitrary constants, possibly varying in each context. They might depend on ψ , so that e.g. $k = k(\psi)$, but for the sake of simplicity we will not make this explicit.

Assumptions B. Let ψ_0 be an interior point of $\overline{\Psi}$. For any $\theta, \theta^* \in \Theta$:

$$\begin{split} & \mathbf{B}_1 \qquad \alpha_j(\theta) \sim k j^{d(\theta)-1}, \ 0 < \mid k \mid < \infty, \ \text{ as } j \to \infty, \\ & \text{ where the function } d : \Theta \to (0, \frac{1}{2}) \text{ is continuous.} \\ & \mathbf{B}_2 \qquad \mid \alpha_j(\theta) - \alpha_{j+1}(\theta) \mid \leq k \frac{\mid \alpha_j(\theta) \mid}{j}, \ \forall j > J, \ \text{ some } J < \infty, 0 < k < \infty. \end{split}$$

$$\begin{split} & \mathrm{B}_{3} \qquad \frac{\partial \alpha_{j}(\theta)}{\partial \theta_{i}} \sim k_{i}' L\left(j\right) j^{d(\theta)-1}, 0 < |k_{i}'| < \infty, \ as \ j \to \infty, \ for \ i = 1, 2, ..., p. \\ & \mathrm{B}_{4} \qquad \left| \frac{\partial \alpha_{j}(\theta)}{\partial \theta_{i}} - \frac{\partial \alpha_{j+1}(\theta)}{\partial \theta_{i}} \right| \leq \frac{k'}{j} \left| \frac{\partial \alpha_{j}(\theta)}{\partial \theta_{i}} \right|, \\ & \forall j > J', \ some \ J' < \infty, 0 < k' < \infty, \ for \ i = 1, 2, ..., p. \\ & \mathrm{B}_{5} \qquad \frac{\partial^{2} \alpha_{j}(\theta)}{\partial \theta_{i_{1}} \partial \theta_{i_{2}}} \sim k_{i_{1,i_{2}}}'' L_{i_{1}}(j) \ L_{i_{2}}(j) \ j^{d(\theta)-1}, \ 0 < |k_{i_{1,i_{2}}}'| < \infty, \ as \ j \to \infty, \\ & for \ i_{1}, i_{2} = 1, 2, ..., p. \\ & \mathrm{B}_{6} \qquad \left| \frac{\partial^{2} \alpha_{j}(\theta)}{\partial \theta_{i_{1}} \partial \theta_{i_{2}}} - \frac{\partial^{2} \alpha_{j+1}(\theta)}{\partial \theta_{i_{1}} \partial \theta_{i_{2}}} \right| \leq \frac{k''}{j} \left| \frac{\partial^{2} \alpha_{j}(\theta)}{\partial \theta_{i_{1}} \partial \theta_{i_{2}}} \right|, \ \forall j > J'' \\ & , \ some \ J'' < \infty, \ 0 < k'' < \infty, \ for \ i_{1}, i_{2} = 1, 2, ..., p. \\ & \mathrm{B}_{7} \qquad \alpha_{j}(\theta) \neq \alpha_{j}(\theta^{*}), \forall j > J''', \ some \ J''' < \infty \ and \ \theta \neq \theta^{*}. \\ & \mathrm{B}_{8} \qquad There \ are \ 2(p+1) \ coefficients \ \alpha_{u_{i}}(\theta) \ and \\ & \partial \alpha_{u_{i}}(\theta) / \partial \theta \ (i = 1, \ldots, p+1) \ such \ that \ \partial \bar{\gamma}_{yy}(u_{j}; \psi) / \partial \psi \ (j = 1, \ldots, q), \end{split}$$

with $q \ge p+1$, are linearly independent vectors,

where L(j), $L_{i_1}(j)$, $L_{i_2}(j)$, j = 1, 2, ... are either constant or equal to $\ln(j)$ and J, J', J'', J''' are equal to some finite integers.

Remarks B. (a) Assumption B_1 defines the $\alpha_i(\theta)$ to behave asymptotically as the MA coefficients of stationary ARFIMA(p, d, q), with real 0 < d < 1/2and integers p, q. The imposing of an exact rate in B_1 , rather than an upper bound, plays a crucial role in the asymptotic results, e.g. together with B_7 it guarantees identification of the model. Assumption B_2 implies that the sequence $\{\alpha_i(\theta)\}$ converges to zero smoothly enough or, more formally, that it is quasi monotonically convergent to zero (see Yong (1974, p. 2)). Thus, besides determining the behaviour near the zero frequency of the $f_{yy}(\lambda;\psi,\sigma^2), B_2$ imposes a Lipschitz condition (see Zygmund (1977, p. 42)) on the model spectrum in the interval $(0, \pi]$. Assumptions B_3 , B_4 and B_5 , B_6 impart the same degree of smoothness as B_1 and B_2 to $\partial f_{yy}(\lambda; \psi, \sigma^2)/\partial \psi$ and to $\partial^2 f_{yy}(\lambda; \psi, \sigma^2) / \partial \psi \partial \psi'$ respectively. Assumptions B_5 , B_6 are needed in order to establish the CLT of the Gaussian estimator, imposing the required degree of smoothness to the Hessian. B_7 represents an identification condition. Assumption B_8 guarantees that the Hessian is nonsingular and it is a rather mild identification condition, ruling out over-parameterizations. B_8 is based on the short run behaviour of the $\alpha_i(\theta)$ and derivative of, in contrast to B_3 which implies that the $\partial \alpha_u(\theta)/\partial \theta$ are, as $u \to \infty$, asymptotically linearly dependent. We are not assuming, for the sake of simplicity, the possibility of slowly varying factors in the definition of the $\alpha_i(\theta)$ but they can nevertheless arise as a by-product (expressed by the logarithm function $\ln(\cdot)$ when differentiating. Finally, note that nowhere are non-negativity

constraints imposed. This implies a great deal of flexibility when considering finite-parameterization the $\alpha_i(\theta)$ compatible with Assumptions B.

(b) Time domain regularity assumptions, such as Assumptions B, are not common in the long memory parametric (see Fox and Taqqu (1986), Dahlhaus (1989), Giraitis and Surgailis (1990), Hosoya (1997) among others) and seminonparametric literature, with the exception of Robinson (1994a). Indeed, regularity conditions are usually expressed in terms of a certain degree of smoothness in the model power spectrum. (Fox and Taqqu (1986, remark at p.529) acknowledge that, in certain cases, Tauberian theorems can be applied to obtain the asymptotic behaviour of the Fourier coefficients of the model power spectrum and its derivatives.) However, whereas in a seminonparametric framework those assumptions represent a natural choice, they may not be as well motivated in a parametric framework. In fact, it might be very difficult in general, within a specific nonlinear parameterization, to check that the model power spectra driven by this parameterization satisfy such frequency domain assumptions. In our nonlinear parametric framework the statistical properties of the model are completely determined by the behaviour of the unobservables ϵ_t and of the coefficients $\alpha_i(\theta)$. Therefore it seems natural to impose regularity conditions as stated by Assumptions B above. These conditions appear very easy to check. We show that the MA coefficients of ARFIMA(p, d, q) do satisfy Assumptions B, representing a simple yet feasible parameterization of the model (cf. appendix E).

The asymptotic properties, consistency and asymptotic normality, of the Gaussian estimators $\hat{\psi}$ and $\hat{\sigma}_T^2(\hat{\psi}_T)$ follow.

Theorem 2 Under Assumptions $A, B_1, B_2, B_3, B_4, B_7$, as $T \to \infty$,

$$\hat{\psi}_T \to_{a.s.} \psi_0, \hat{\sigma}_T^2(\hat{\psi}_T) \to_{a.s.} \sigma^2,$$

where $\rightarrow_{a.s.}$ denotes convergence almost surely (a.s.).

Proof: see appendix B.2.

We need to reinforce Assumption A as follows.

Assumption A'. The ϵ_t satisfy Assumption A with $E\epsilon_t^8 < \infty$ and zero cumulants of order s = 5, ..., 8.

Theorem 3 Under Assumptions $A', B, as T \to \infty$,

$$T^{\frac{1}{2}}(\psi_T - \psi_0) \to_d \mathcal{N}_{p+1}(0, M^{-1}VM^{-1}),$$

where

$$M(\psi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} N(\lambda; \psi) N(\lambda; \psi)' d\lambda,$$

with

$$N(\lambda;\psi) = \frac{\partial \ln h_y(\lambda;\psi)}{\partial \psi} - \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial \ln h_y(\omega;\psi)}{\partial \psi} d\omega\right],$$

and

$$V(\psi) = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\omega; \psi) g(\omega; \psi)' h_y^2(\omega) d\omega + \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} g(\omega_1; \psi) g(\omega_2; \psi)' \overline{Q}_{yyy}(-\omega_1, \omega_2, -\omega_2) d\omega_1 d\omega_2,$$

with $g(\lambda; \psi) = \frac{\partial}{\partial \psi} h_y^{-1}(\lambda; \psi)$, $M = M(\psi_0)$, $V = V(\psi_0)$ and $\overline{Q}_{yyy}(.,.,.) = Q_{yyy}(.,.,.)/\sigma^{16}$, $Q_{yyy}(.,.,.)$ denoting the trispectrum of the y_t (the Fourier transform of the fourth-order cumulants of the y_t), \rightarrow_d denotes convergence in distribution and $\mathcal{N}_q(\cdot, \cdot)$ a q-dimensional normal r.v.

Proof; see appendix B.3.

Remarks 2 and 3.(a) The important aspect of Assumption A' is that it imposes the strong *i.i.d.* property on the ϵ_t , imparting strict stationarity and ergodicity to the y_t . Zero restrictions on cumulants are made for simplicity's sake but play no substantial role in the proofs and the results of Theorem 2 and Theorem 3 are unchanged with constant cumulants and a bounded eighth moment.

(b) A strong law of large number (LLN) for $\hat{\psi}_T$ and $\hat{\sigma}_T^2(\hat{\psi}_T)$ is obtained adapting the Hannan (1973) approach although, unlike the scalar-valued Gaussian and linear long memory case (see e.g. Fox and Taqqu (1986), Dahlhaus (1989), Giraitis and Surgailis (1990)), we cannot employ Hannan (1973) directly. In fact, as a by-product of the nonlinearity, the model features a non-standard parameterization in the sense that $f_{yy}(\lambda; \psi, \sigma^2)$ does not seem to be factored, unlike the linearly regular process case (see Hannan (1970, Chapter II-5)). In fact, although the Wold representation theorem implies that $f_{yy}(\lambda)$ can be written, under Assumption A, as

$$f_{yy}(\lambda) = \frac{\tau}{2\pi} \mid 1 + \sum_{j=1}^{\infty} \beta_j e^{ij\lambda} \mid^2, \quad -\pi \le \lambda < \pi, \tag{13}$$

the determination of the β_j and τ as closed-form, functionally independent, functions of the α_i and σ^2 , κ , ρ^2 , μ , appears very difficult, practically unfeasible. Robinson (1978) discusses several cases where the spectral density of the observable is not easily factored and provides the asymptotics of the Gaussian estimator for those cases without assuming linearity. However, only short memory specifications of the nonlinear MA might be covered in that framework as strong smoothness assumptions on the model spectrum are imposed, ruling out long memory.

(c) The theoretical result, on which Theorem 3 is based, is a CLT for quadratic forms in the y_t , a non-Gaussian and nonlinear sequence with long memory, which the statistical literature on Gaussian estimation for linear long memory processes does not cover. Limit laws for quadratic form in scalar-valued long memory Gaussian and linear processes are in Fox and Taqqu (1987) and Giraitis and Surgailis (1990), respectively. The former uses Gaussianity in an essential way, employing the exact expression for the cumulants of a quadratic form in Gaussian variates (see Grenander and Szegö (1958)) whereas the latter, though relaxing the Gaussianity assumption, exploits the simple structure of the spectrum and trispectrum of a linear process (cf. (13)). More recently Heyde and Gay (1993) and Hosoya (1997) have developed CLT for vector-valued linear non-Gaussian processes under weaker conditions on the innovations, the latter also allowing for non-standard conditions and non-factorization in the model spectral density. However, those results are ultimately based on some martingale approximation, making a basic use of the maintained structure of linearity and imposing some form of mixing condition on the linear innovations, stronger than the white-noise property implied by the Wold representation. Our method of proof, instead, takes explicitly into account the nonlinear structure of the model and consists in approximating the quadratic form in the y_t (embedded in Whittle's function) with another quadratic form, asymptotically equivalent to the former, for which a CLT follows using standard results. In order to establish this approximation we use certain results of the asymptotic behaviour of the trace of Toeplitz matrices (see Fox and Taggu (1987)).

(d) The relevance of Theorem 3 stems from two reasons. First, the long memory feature of the squares y_t might have generated 'per se' non-standard asymptotic distribution such as nonlinear functionals of fractional Brownian motion (see Taqqu (1975)), or a rate of convergence depending on the degree of memory of the process, as for the case of the sample mean of T consecutive observations of a long memory process. Instead, using the Gaussian estimator standard asymptotic inference can be made on the nonlinear MA such as Wald-type tests. This is due to the typical compensation effect of Whittle's function which smoothes the possible lack of square integrability of $f_{yy}(\lambda)$. Second, and more importantly, only for a limited number of volatility models the asymptotic properties of certain estimators (usually the Gaussian PMLE) have been established (see Weiss (1986) for ARCH(p) and Lee and Hansen (1994), Lumsdaine (1996) GARCH(1, 1)). Long memory parameterizations are always excluded. This lack of results in fact characterizes all long memory volatility models, including SV models. The only exception is Breidt, Crato, and deLima (1998) who establish consistency of the Gaussian estimator for the long memory SV model of Harvey (1998), exploiting the linearity of $\ln y_t$ for long memory SV x_t .

(e) As a simple case of Robinson (2001), (7) generalizes to

$$\operatorname{cov}\left(g(\epsilon_t \, h_{t-1}), g(\epsilon_{t+u} \, h_{t+u-1})\right) \sim k \, u^{2d-1} \quad \text{as } u \to \infty,$$

for Gaussian ϵ_t and any arbitrary function $g(\cdot)$ satisfying $Eg^2(\epsilon_t h_{t-1}) < \infty$ and $E h_{t-1} g(\epsilon_t h_{t-1}) \neq 0$, with g(xy) either equal to g(x)g(y) or g(x) + g(y). Particular cases are $g(t) = |t|^{\alpha}$, $\alpha > 0$ and $g(t) = \ln t^2$. Therefore, for nonlinear MA the y_t , the $|x_t|$ and the $\ln y_t$ have precisely the same degree of long memory. As $\ln y_t$ is defined over the entire real line and thus closer to a Gaussian variate than y_t , it would probably be more efficient to estimate the memory parameter d by fitting the spectral density of the $\ln y_t$ rather than $f_{yy}(\lambda)$. Moreover, Wright (2000) indicates that, when the levels x_t are conditionally leptokurtic, a non-negligible small sample negative bias arises systematically when estimating semiparametrically the memory parameter dusing the y_t as the observables, compared with the case when the $|x_t|$ or the $\ln y_t$ are used. Although in principle other instantaneous transformations of nonlinear MA x_t could be used besides $y_t = x_t^2$, the Gaussian estimator requires knowledge of the precise functional form of the model spectral density for the transformed data. This task appears cumbersome, if not practically unfeasible, to be derived for the $|x_t|$ and the $\ln y_t$ of nonlinear MA.

3.3 Testing long memory dynamic heteroskedasticity

When limiting the attention to short memory specification of the model, any LM test for ARCH effects that is asymptotically equivalent to Engle (1982) test provides an efficient way to detect short memory conditional heteroskedasticity. In this section we introduce a family of LM tests specifically designed to have good power against long memory dynamic conditional heteroskedastic alternatives. The test statistics introduced here are developed assuming that under the alternative hypothesis the observable is described by the nonlinear MA. Nonparametric (see e.g. Lo (1991)) and semiparametric approaches (see e.g. Robinson (1994b, 1995a, 1995b)) can still be applied but with the usual low performance in terms of power. In a parametric framework the LM tests of Robinson (1991) can be considered but they are derived under ARCH-type alternatives. The test we propose follows the approach of Robinson (1991) but, by construction, it has more local asymptotic power when the alternative is represented by the nonlinear MA. This is, obviously, particularly important when fitting the nonlinear MA to the data. The following assumptions concerning both the $\alpha_i(\theta)$ and the ϵ_t are needed.

Assumptions C.

 $C_1 \text{ The coefficients } \{\alpha_i(\theta)\} \text{ are invertible functions of a } p \times 1 \text{ vector } \theta$ such that $\alpha_i(\theta) = 0 \ (i \ge 1)$ if and only if $\theta = 0$. The partial derivatives of $\alpha_i(\theta)$ are square summable for $\theta = 0$ where we set $\tau_i(\theta) = \frac{\partial}{\partial \theta} \alpha_i(\theta)$ with $\tau_i = \tau_i(0)$. $C_2 \text{ The matrix } \Gamma = \sum_{i=1}^{\infty} \tau_i \tau'_i \text{ is invertible.}$

 $C_3 \rho^2 = 1.$

Assumption D. The process $\{\epsilon_t\}$ behaves as a Gaussian noise sequence up to the eighth moment.

We define the null hypothesis as

$$H_0: \ \alpha_i(0) = 0, \ i \ge 1.$$

so that under H_0 both the x_t and the y_t are not autocorrelated.

Assumption C_1 guarantees that the null hypothesis can be equivalently characterized in terms of the $\alpha_i(\theta)$ or, alternatively, in terms of θ . Hence we can write H_0 : $\theta = 0$. Assumptions C_2 and C_3 are minimal identification conditions. In fact, $x_t = \mu + \rho \epsilon_t$ under H_0 implying $\operatorname{var}(x_t) = \rho^2 \sigma^2$ and ρ is not identified under H_0 . Finally, note that nowhere in Assumption D did we assume the ϵ_t to be an *i.i.d.* sequence.

Departures from H_0 do not necessarily display long memory, but express behaviours for the $\alpha_i(\theta)$ with at least a finite number of them non-zero, including any short memory parameterization. Note that any sequence $\alpha_i(\theta)$ satisfying Assumption C_1 , C_2 is compatible with both H_0 and Assumptions B.

Setting $w = \mu^2/\sigma^2$, the parameterized model is described by the parameter vector $(w, \sigma^2, \theta')'$. The LM test statistic for testing H_0 is (cf. appendix D):

$$\Phi = \phi'\phi, \tag{14}$$

with

$$\phi = \phi_{T-1}(\hat{\sigma}^2, \hat{w}) = (T \, e(\hat{w}) \, \hat{\sigma}^2 \, \Gamma_{T-1})^{-\frac{1}{2}} \sum_{i=1}^{T-1} \tau_i \, \eta_i(\hat{\sigma}^2, \hat{w}), \tag{15}$$

setting

$$\eta_i(\sigma^2, w) = \sum_{t=1+i}^T (x_{t-i} - \bar{x}) X_t(\sigma^2, w),$$

$$X_t(\sigma^2, w) = \left(\left(\frac{y_t}{\sigma^2} - w - 1\right) \left(\left(\frac{y_t}{\sigma^2} - w - 1\right) \frac{(1+w)}{(1+2w)} + 1 \right) - 2(1+w) \right) + 1 \right)$$

and $e(w) = 9 + (65 + 150w^3 + 338w^2 + 280w)/(1 + 2w)^2$, $\Gamma_T = \sum_{i=1}^T \tau_i \tau'_i$ where $\hat{\sigma}^2 = \frac{1}{T} \sum_{t=1}^T (x_t - \bar{x})^2$, $\bar{x} = \frac{1}{T} \sum_{i=1}^T x_i$, $\hat{w} = \bar{x}^2/\hat{\sigma}^2$. One can use Γ instead of Γ_T if the former has a closed-form expression. As a discrete convolution is involved, we can make use of the Fast Fourier Transform (FFT) with the usual computational gain. Indeed, a frequency domain version of the LM test can be computed:

$$\tilde{\Phi} = \tilde{\phi}' \tilde{\phi} \,, \tag{16}$$

with

$$\tilde{\phi} = (T e(\hat{w}) \hat{\sigma}^2 \Gamma_T)^{-\frac{1}{2}} \frac{1}{T} \sum_{a=0}^{T-1,*} \tilde{\tau}(\omega_a) \tilde{x}(\omega_a) \tilde{X}(-\omega_a) ,$$

setting $\tilde{X}(\omega) = \sum_{a=0}^{T-1} X_{a+1}(\hat{\sigma^2}, \hat{w}) e^{ia\omega}$, $\tilde{x}(\omega) = \sum_{a=0}^{T-1} (x_{a+1} - \bar{x}) e^{ia\omega}$, $\tilde{\tau}(\omega) = \sum_{a=0}^{\infty} \tau_{a+1} e^{ia\omega}$, with \sum^* meaning that the summation excludes values of ω_a for which $\tilde{\tau}(\omega_a)$ is unbounded. The matrix Γ can now be expressed as $\Gamma = \frac{1}{4\pi} \int_{-\pi}^{\pi} \tilde{\tau}(\omega) \tilde{\tau}'(-\omega) d\omega$, which, in turn, can be approximated by its Riemann sum. Thus

Theorem 4 Under the null hypothesis H_0 and Assumptions C and D,

$$\Phi \to_d \chi_p^2 \quad as \quad T \to \infty,$$

where χ_p^2 denotes a central chi-square r.v. with p degrees of freedom.

Proof: see appendix D.

Remarks 4. (a) A standard asymptotic distribution is obtained though the test has been designed for possibly long memory alternatives.

(b) The two test statistics Φ and Φ are asymptotically equivalent, given that $\Phi - \tilde{\Phi} = o_p(1)$, so Theorem 4 applies to $\tilde{\Phi}$ as well.

(c) From a computational point of view, calculating Φ involves no more than a few summations and the inversion of a matrix of dimension $p \times p$ (θ is of dimension $p \times 1$) and thus not a function of T. The LM ARCH test of Engle (1982), instead, requires inverting a matrix of dimension $r \times r$, with r being the number of lags considered. It follows that taking into account the behaviour at long lags, for example in order to capture long memory behaviour, makes the LM ARCH test unfeasible.

(d) The statistic Φ is based on the Gaussian log likelihood, so it is in fact a Gaussian pseudo LM test. Unlike the estimation part, the time domain Gaussian log likelihood is used as the objective function. In fact, under H_0 the model is trivially invertible given that $x_t = \mu + \epsilon_t$.

(e) Theorem 4 effectively establishes that ϕ is converging in distribution to a multivariate standard normal so that, when p = 1, a one-sided test of H_0 against the alternative (say) $\theta > 0$ can be performed with the usual gain in power (cf. Robinson (1991)). (f) The obvious, mild condition for consistency of Φ is $\sum_{i=1}^{\infty} \tau_i E(\eta_i) \neq 0$ when H_0 is false.

In terms of the assumptions made, D imposes a relatively stringent structure on the moments of the unobservable ϵ_t . For this reason we propose a robustified LM test statistic, which has the same asymptotic distribution under H_0 as the Φ statistic, but allowing a great deal of heterogeneity in the ϵ_t . Details are reported in appendix D.1.

4 ARMA with nonlinear MA innovations

To relax the hypothesis of conditionally constant first moment of the levels x_t we extend (1) to

$$x_t = \sum_{j=0}^{\infty} \beta_j u_{t-j}, \quad \beta_0 = 1, \quad \sum_{j=0}^{\infty} \beta_j^2 < \infty, \quad t \in \mathbb{Z},$$

$$(17)$$

with nonlinear MA innovations u_t described by (2). With no loss of generality, $E(x_t) = 0$ under (17) and Assumption A.

For parametric estimation of the β_j , a flexible and computationally convenient parameterization is provided by the ARMA(r, q) model, whereby

$$x_t = \omega_{1,0}x_{t-1} + \omega_{2,0}x_{t-2} + \dots + \omega_{r,0}x_{t-r} + u_t + \xi_{1,0}u_{t-1} + \dots + \xi_{q,0}u_{t-q}, \quad (18)$$

yielding $\beta_j = \beta_j(\zeta_0)$ with $\zeta_0 = (\omega'_0, \xi'_0)'$ for constant coefficients

 $\omega_0 = (\omega_{1,0}, ..., \omega_{r,0})', \ \xi_0 = (\xi_{1,0}, ..., \xi_{q,0})', \ \text{such that the polynomials } \Omega_0(z) = 1 - \omega_{1,0} z - ... \omega_{r,0} z^r \ \text{and } \Xi_0(z) = 1 + \xi_{1,0} z + ... \xi_{q,0} z^q \ \text{satisfy}$

$$\Xi_0(z)\Omega_0(z) \neq 0, \ |z| \le 1,$$
(19)

implying (17). Clearly $\beta(L;\zeta_0) = \sum_{i=0}^{\infty} \beta_i(\zeta_0) L^i = \Xi_0(L) / \Omega_0(L).$

Model (18) with (2) could be estimated by a two-stage approach. In the first stage, one estimates the conditional mean ARMA parameters ζ_0 . Given the martingale difference property of the innovations u_t and $\gamma_{uu}(0) < \infty$, implied by Assumption A, the observable x_t have spectral density now equal to

$$f_{xx}(\lambda) = \frac{\gamma_{uu}(0)}{2\pi} \frac{|\Xi_0(e^{i\lambda})|^2}{|\Omega_0(e^{i\lambda})|^2} = \frac{\gamma_{uu}(0)}{2\pi} p_x(\lambda;\zeta_0), \quad -\pi \le \lambda < \pi.$$
(20)

Given (20) the mean parameters could be estimated by the Whittle estimator $\hat{\zeta}$, defined by

$$\hat{\zeta}_T = \operatorname{argmin}_{\zeta \in \bar{\mathcal{Z}}} P_T(\zeta), \tag{21}$$

setting

$$P_T(\zeta) = \frac{2\pi}{T} \sum_{j=1}^{T-1} \frac{I_{xx}(\lambda_j)}{p_x(\lambda_j;\zeta)}$$

where $p_x(\lambda;\zeta) = |\beta(e^{i\lambda};\zeta)|^2$ defines the parametrized model spectrum for the x_t (up to a constant of proportionality), $\mathcal{Z} = \{\zeta \in \mathbb{R}^{r+q} \mid \Omega(z)\Xi(z) \neq 0$ for $|z| \leq 1$ and

 $\Omega(z), \Xi(z)$ have no common roots }, $\overline{\mathcal{Z}}$ defines the compact closure of \mathcal{Z} , and

$$I_{ab}(\lambda) = \frac{1}{2\pi T} \sum_{t,s=1}^{T} (a_t - \bar{a})(b_s - \bar{b})e^{-i(t-s)\lambda}, \quad -\pi \le \lambda < \pi$$

for any sequences $\{a_t\}$, $\{b_t\}$. Likewise, we define the cross-spectrum (when it exists)

$$f_{ab}(\lambda) = \frac{1}{2\pi} \sum_{u=-\infty}^{\infty} \operatorname{cov}(a_t, b_{t+u}) e^{i\lambda u}, \quad -\pi \le \lambda < \pi.$$

The asymptotic properties of $\hat{\zeta}_T$ follows.

Theorem 5 Under Assumption A, for ζ_0 being an interior point of \overline{Z} , as $T \to \infty$,

$$\hat{\zeta}_T \to_{a.s.} \zeta_0,$$

and

$$T^{\frac{1}{2}}(\hat{\zeta}_T - \zeta_0) \to_d \mathcal{N}_{r+q}\left(0, F^{-1}GF^{-1}\right),$$

where

$$F(\zeta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} b(\lambda;\zeta) \, b(\lambda;\zeta)' d\lambda, \qquad (22)$$

$$G(\zeta) = 2F(\zeta) + \frac{1}{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} b(\omega_1; \zeta) b(\omega_2; \zeta)' \frac{f_{u^2 u^2}(\omega_1 + \omega_2)}{\gamma_{uu}^2(0)} d\omega_1 d\omega_2$$
(23)

and

$$b(\lambda;\zeta) = \frac{\partial}{\partial\zeta} \ln p_x(\lambda;\zeta) \tag{24}$$

with $F = F(\zeta_0)$, $G = G(\zeta_0)$, $b(\lambda) = b(\lambda; \zeta_0)$ and $f_{u^2u^2}(\lambda)$ defines the power spectrum of the u_t^2 .

Proof: see appendix C.

Remarks 5. (a) Expression (23) gives the general form of the matrix $G(\zeta)$ whenever one considers Whittle estimation of ARMA(r, q) allowing for conditionally heteroskedastic innovations u_t with bounded fourth moment. This includes not only nonlinear MA but also, for example GARCH (imposing bounded fourth moment). (23) simplifies to $G(\zeta) = 2F(\zeta)$ when the u_t are

independent or, more generally, conditionally homoskedastic.

(b) The expression for the power spectrum of the u_t^2 , $f_{u^2u^2}(\lambda)$ enters into the definition of G. The exact expression for $f_{u^2u^2}(\lambda)$ is given in Theorem 1 (setting $\mu = 0$ in $f_{yy}(\lambda)$). However, note that a consistent estimate of G does not require the knowledge of this expression and can be obtained replacing $f_{u^2u^2}(\lambda)$ with $I_{u^2u^2}(\lambda)$ and adapting Taniguchi (1982).

(c) $P_T(\tilde{\zeta}_T)$ represents a strongly consistent estimator of $\gamma_{uu}(0)$ but, although $\gamma_{uu}(0) = \sigma^4(\bar{\rho}^2 + \delta_{\alpha\alpha}(0))$, this cannot be used to identify and thus estimate the nonlinear MA parameters, as previously noted.

(d) When assuming q = 0, for AR(p) x_t , the Whittle estimator of $\zeta_0 = \omega_0$ becomes the Yule-Walker estimator, which has a closed-form expression and is asymptotically equivalent to the least squares estimator (see Brockwell and Davis (1987, section 8.1)).

Given (19) the $AR(\infty)$ representation of the x_t follows:

$$x_t - \sum_{j=1}^{\infty} \pi_j(\zeta_0) x_{t-j} = u_t,$$

with $1 - \sum_{j=1}^{\infty} \pi_j(\zeta_0) L^j = \Omega_0(L) / \Xi_0(L)$, from which one gets the estimated residuals as

$$\hat{u}_t = x_t - \sum_{j=1}^{t-1} \pi(\hat{\zeta}_T) x_{t-j}, \ t = 1, ..., T,$$

where $\sum_{i=a}^{b} c_i = 0$ for a > b for any sequence $\{c_i\}$. Other methods of estimation of the residuals can be used such as the Kalman filter (see Brockwell and Davis (1987, section 12.2)).

Given the \hat{u}_t one then estimates the nonlinear MA parameters $\psi = (\bar{\kappa}, \theta')'$ with the Gaussian estimator, replacing the y_t with the \hat{u}_t^2 in section 3. Let

$$\tilde{Q}_T(\psi) = \ln\left(\tilde{\sigma}_T^8(\psi)\right) + \frac{1}{T} \sum_{j=1}^{T-1} \ln h_y(\lambda_j;\psi) + 1, \qquad (25)$$

setting

$$\tilde{\sigma}_T^8(\psi) = \frac{1}{T} \sum_{j=1}^{T-1} \frac{I_{\hat{u}^2 \hat{u}^2}(\lambda_j)}{h_y(\lambda_j; \psi)}$$

The Gaussian estimator $\tilde{\psi}_T$, based on the estimated residuals \hat{u}_t , is defined by

$$\tilde{\psi}_T = argmin_{\psi \in \bar{\Psi}} \tilde{Q}_T(\psi)$$

It turns out that $\tilde{\psi}_T$ is still (strongly) \sqrt{T} -consistent and asymptotically normally distributed, although with a different, larger asymptotic variancecovariance matrix from the one of $\hat{\psi}_T$ (stated in Theorem 3). **Theorem 6** Under the same assumptions made in Theorem 3, as $T \to \infty$,

$$\psi_T \rightarrow_{a.s.} \psi_0$$

and

$$T^{\frac{1}{2}}(\tilde{\psi}_T - \psi_0) \to_d \mathcal{N}_{p+1}(0, M^{-1}UM^{-1}),$$

where

$$U(\psi) = V(\psi) + H(\psi)F^{-1}GF^{-1}H(\psi)', \quad U = U(\psi_0),$$

setting

$$H(\psi) = \frac{2}{\pi\sigma^8} \int_{-\pi}^{\pi} g(\lambda;\psi) \,\mathcal{R}e(f_{u^2 \, uv'}(\lambda)) d\lambda$$

with the (r+q)-vector valued random variables $v_t = \sum_{j=1}^{\infty} \vartheta_j x_{t-j}$, $\vartheta_j(\zeta) = \partial \pi_j(\zeta) / \partial \zeta$, $\vartheta_j = \vartheta_j(\zeta_0) \ (j \ge 1)$, and with M, V and $g(\lambda; \psi)$ defined in Theorem 3 and F, G defined in Theorem 5.

Proof: see appendix C.

Remarks 6. (a) The shape of the asymptotic variance-covariance matrix of $\tilde{\psi}_T$ shows that there is an efficiency loss when estimating the residuals compared with the case when the levels x_t are martingale difference themselves. (b) A more efficient estimation method is based on joint estimation of ζ_0 and ψ_0 , such as by using the bivariate Gaussian estimation approach mentioned in section 3.2. The two-stage procedure here proposed, though, appears very appealing computationally compared with practical use of the bivariate Gaussian estimator. Moreover, as discussed above, full efficiency can never be attained by the bivariate Gaussian estimator either. Finally, having established the asymptotics for the univariate Gaussian estimator, the incremental mathematical achievement for the bivariate Gaussian estimator will not be significant.

(c) The result can be easily extended to the case of exogenous regressors such as $x_t = \beta' z_t + \beta(L, \zeta_0) u_t$ under suitable regularity conditions on the z_t (see Hannan (1973, section 4)).

(d) Using $\hat{\zeta}_T$ and $\tilde{\psi}_T$ a consistent estimate of U can be obtained, replacing $f_{u^2 uv'}(\lambda)$ with $I_{u^2 u\hat{v}'}(\lambda)$, setting $\hat{v}_t = \sum_{j=1}^{t-1} \vartheta_j(\hat{\zeta}_T) x_{t-j}$, and adapting Taniguchi (1982).

5 Montecarlo experiments

We have performed a relatively extensive Montecarlo exercise to assess the small sample performance of the Gaussian estimator and of the LM test. We have considered both the case of martingale difference and ARMA observables. In the latter case, we have also reported the Montecarlo results of the first-stage estimation of ARMA parameters. We consider the following exact, up to the second moment, simulation approach. First, we simulate T+1 innovations ϵ_t ($0 \le t \le T$) as *i.i.d.* draw from a Student's *t* distribution with *n* degrees of freedom, variance 1 and fourth-order cumulant $\bar{\kappa}$. Normally distributed ϵ_t are obtained when $n = \infty$ ($\bar{\kappa} = 0$). Recall that $\theta = (\phi', \chi', d)'$ and $\zeta = (\omega', \xi')'$. *T* defines sample size. Then we construct the sequence u_t ($1 \le t \le T$) as

$$u_t = \epsilon_t \left(\sigma \bar{\rho} + \sqrt{\delta_t(\theta)} \epsilon_0 + \sum_{j=1}^{t-1} \alpha_j(\theta) \epsilon_{t-j} \right), \quad 1 \le t \le T,$$

setting

$$\delta_t(\theta) = \begin{cases} \sum_{j=1}^{\infty} \alpha_j^2(\theta), & t = 1, \\ \delta_{t-1}(\theta) - \alpha_{t-1}^2(\theta), & t \ge 2. \end{cases}$$

When considering the simple case of martingale difference levels ($\zeta = 0$) then set $x_t = u_t (1 \le t \le T)$ otherwise

$$x_t = \sqrt{\eta_t(\zeta)}u_1 + \sum_{j=0}^{t-2} \beta_j(\zeta)u_{t-j}, \quad 1 \le t \le T,$$

setting

$$\eta_t(\zeta) = \begin{cases} \sum_{j=0}^{\infty} \beta_j^2(\zeta), & t = 1, \\ \eta_{t-1}(\zeta) - \beta_{t-2}^2(\zeta), & t \ge 2. \end{cases}$$

Throughout the exercise we always consider 1000 replications and sample size T equal to 1024, 2048, 4096.

At first we assess the performance of the Gaussian estimator assuming martingale difference levels $x_t = u_t$. We focus on the following parameterization:

$$\alpha(L;\theta) = \frac{(1-L)^{-d}}{(1-\phi_1 L)}, \ 0 < d < 1/2, \ |\phi_1| < 1,$$
(26)

yielding $\psi = (\bar{\kappa}, \phi_1, d)$, with $\bar{\kappa} \in \{0, 1.5, 3.0\}$ (*n* equal to infinity, 8 and 6 respectively), $\phi_1 \in \{-0.8, -0.4, 0.4, 0.8\}$ and $d \in \{0.15, 0.25, 0.45\}$. We set $\rho^2 = 1, \sigma^2 = 1, \mu = 0$. Tables 1.1, 1.2 and 1.3 reports the sample mean and its standard deviation of the estimates across the 1000 replications.

The results show that for all parameters increasing sample size improves remarkably the results in terms of bias reduction and accuracy. There is, though, a marked difference in the accuracy of the estimates of $\bar{\kappa}$ compared with the other parameters ϕ_1 , d. This is probably a consequence of the fact that the Gaussian estimator is not a likelihood-method and, therefore, information on the conditional distribution of the x_t (unconditional distribution of the ϵ_t) enters only indirectly into Whittle's function, through $\bar{\kappa}$. This contrasts with likelihood-methods, such as PMLE, which by construction take directly into account the conditional distribution of the x_t . Estimates of d frequently exhibit a negative bias increasing in absolute value with $\bar{\kappa}$. This agrees with the finding of Wright (2000) in a semiparametric setting (cf. remark (e) to Theorem 3). Although estimation of $\bar{\kappa}$ should potentially reduce this problem, the simulations show often that a significant bias arises. Moreover, estimates of d exhibit a greater accuracy compared with estimates of ϕ_1 . However, the Gaussian estimator shows a good performance with respect to the well-known difficulty of disentangling the effect of the long memory parameter from pure autoregressive parameters in estimation of parametric long memory models. Tables 3.1 and 3.2 report the empirical frequencies for confidence intervals at 95%, based on estimates of the parameter d and their asymptotic standard errors, for parameterization (26). The results show a sizeble improvement, as the sample size increases, for medium and large values of d. In contrast, no substancial improvement is obtained for small values of d. The results tend to worsen for large values of $\bar{\kappa}$, especially for d greater than 0.15.

Next, we consider the case of AR(1) levels, $x_t = \omega_1 x_{t-1} + u_t$ with $|\omega_1| < 1$, and the nonlinear MA coefficients still given by (26), following the approach indicated in section 4. Thus $\zeta = \omega_1$ and $\psi = (\bar{\kappa}, \phi_1, d)$, where $\omega_1 = 0.5$, $\psi_1 \in \{-0.4, 0.4\}$ and $d \in \{0.15, 0.25, 0.45\}$. The results are presented in Tables 2.1, 2.1 and 2.3. The conditional mean parameter ω_1 appears much more precisely estimated compared with the conditional variance parameters ψ . The efficiency loss emerging when estimating ψ in the second stage, based on estimated residuals, does not appear too serious in most cases. Again, estimates of $\bar{\kappa}$ are more imprecise compared with other parameter' estimates. For all parameters bias reduction and accuracy improve as T increases.

Finally, Table 4 presents evidence of the small-sample performance, both in terms of size and power, of the LM test Φ of section 3.3. The data are simulated as described above, with Gaussian ϵ_t , and

$$\alpha(L;\theta) = (1-L)^{-d}, \ 0 < d < 1/2.$$

This yields $\tau_i = 1/i$ (cf. Assumption C) and $\Gamma = \pi^2/6$ (cf. Robinson (1991)). Good power arises for moderate and large values of d but it is poor when d = 0.1. Moreover, the power decreases when d is very close to 1/2, in the proximity of the non-stationary region, possibly the effect of imposing the boundary condition. Overall, the results improve markedly as T increases.

6 Empirical application

In this section we assess the performance of the nonlinear MA to fit time series of observed asset returns using the Gaussian estimator. The results are compared with the performance of its most natural competitor, the $ARCH(\infty)$ of Robinson (1991):

$$x_t = \nu + \epsilon_t \sigma_t, \ t \in \mathbb{Z}, \tag{27}$$

$$\sigma_t^2 = \tau + \sum_{j=1}^{\infty} \psi_j y_{t-j}, \ a.s., \ \sum_{j=1}^{\infty} \psi_j < \infty,$$
 (28)

where $\tau, \psi_j \ge 0 \ (j \ge 1)$ and the ϵ_t satisfy Assumption A.

Various stationarity conditions and memory properties of $\operatorname{ARCH}(\infty)$ are discussed in RZ, Giraitis, Kokoska, and Leipus (1998) and Zaffaroni (2000). In particular, long memory is ruled out when $Ex_t^2 < \infty$. Imposing $Ey_t^2 < \infty$, the spectral density for $\operatorname{ARCH}(\infty) y_t$ can be factored exhibiting a representation (13), in contrast to nonlinear MA. The (necessary and sufficient) condition for covariance stationarity of $\operatorname{ARCH}(\infty) y_t$ is (using the notation of section 2):

$$0 \le \Sigma = \frac{2\sigma^4 \tau^2}{\Psi^2(1)} \left(1 - 2\sigma^4 \sum_{u=-\infty}^{\infty} \delta_{\psi\psi}(u) (\delta_{\gamma\gamma}(u) + \gamma_u) \right)^{-1} < \infty,$$
(29)

setting $\Psi(L) = 1 - \sigma^2 \sum_{i=1}^{\infty} \psi_i L^i$, $\Gamma(L) = 1 + \sum_{i=1}^{\infty} \gamma_i L^i = \Psi(L)^{-1}$ and $\kappa = 0$ (see Zaffaroni (2000, Theorem 4)). When (29) holds, ARCH(∞) y_t have a well-defined power spectrum:

$$f_{yy}(\lambda) = \frac{\Sigma}{2\pi} |\Psi(e^{i\lambda})|^{-2}, \quad -\pi \le \lambda < \pi.$$
(30)

By finite-parameterizing the ψ_i , inference on ARCH(∞) can be performed using the Gaussian estimator of section 3.2. Giraitis and Robinson (1998) establish consistency and asymptotic normality of the Gaussian estimator of ARCH(∞). Given our emphasis on long memory parameterizations, we consider the following parsimonious specification for the ψ_i :

$$1 - \sum_{j=1}^{\infty} \psi_j(\theta) L^j = (1 - L)^d \frac{\phi(L)}{\chi(L)} = (1 - L)^d \frac{(1 - \phi_1 L - \dots - \phi_p L^p)}{(1 + \chi_1 L + \dots + \chi_q L^q)}, \quad (31)$$

where $\theta = (\chi_1, .., \chi_q, \phi_1, .., \phi_p, d)'$, with d > 0 and, hereafter, we assume that $\phi(L), \chi(L)$ have all zeros outside the unit circle on the complex plane. (These stationarity and invertibility conditions can be easily imposed using the mapping between the ARMA coefficients and the partial autocorrelation function (see Monahan (1984)).) Subsequently, one gets $\Sigma(\theta)$ from (29) replacing the ψ_i with the $\psi_i(\theta)$ and

$$\Psi(L;\theta) = 1 - \sigma^2 \sum_{j=1}^{\infty} \psi_j(\theta) L^j.$$
(32)

Condition (29) and parameterization (31) require $\sigma^2 < 1/2$ (cf. Zaffaroni (2000, Remark 4.4)). We set $\sigma^2 = 0.4$.

In analogy to (31) the following parameterization is considered for the nonlinear MA (cf. (9)):

$$1 + \alpha(L;\theta) = (1-L)^{-d} \frac{\chi(L)}{\phi(L)} = (1-L)^{-d} \frac{(1+\chi_1L+.+\chi_qL^q)}{(1-\phi_1L-.-\phi_pL^p)}.$$
 (33)

Estimation of the nonlinear MA requires |d| < 1/2 and the invertibility conditions on $\phi(L), \chi(L)$. In addition to these conditions, estimation of ARCH(∞) with parameterization (31) requires also $\psi_i(\theta) \ge 0$ ($i \ge 1$) and $\Sigma(\theta) < \infty$. These conditions impose severe nonlinear restrictions on θ and are computationally very demanding.

We consider seven time series, three stock indexes returns (S&P 500, FTSE All, FTSE 100) and four foreign exchange rates returns (Yen/US Dollar spot and 1 month forward, US Dollar/Pound spot and 1 month forward). The data are daily, from 1 January 1985 to 1 June 1998, with 3,500 observations (source: Datastream). Returns are calculated as $x_t = \ln(P_t/P_{t-1})$ where P_t denotes the speculative price of the asset. The ACF for the data in levels and in squares are plotted in Figure 1 for lags 1-300. For all the time series the graphs clearly indicate a marked difference between the ACF in the levels (first and third row) and the ACF in the squares (second and fourth row), the latter exhibiting an approximate hyperbolic behaviour as the lag increases. Table 5.1 reports the results for the LM test Φ , based on $\tau_i = 1/i$ and $\Gamma = \pi^2/6$ (cf. Robinson (1991)). As suggested by the graphs, Φ is always highly significant for all but the Yen/US Dollar spot series, which is not significant at 1%, providing strong indications of dynamic heteroskedasticity in the data.

As a preliminary exercise, we estimate semiparametrically the parameter of long memory using the local Gaussian estimator of Robinson (1995b). The results, in Table 5.2, clearly support the hypothesis of long memory in squared returns for all the series. Next, the nonlinear MA and ARCH(∞) are estimated by the Gaussian estimator. The second and third row of Table 5.2 report the estimates of d for ARCH(∞) and nonlinear MA respectively. (Asymptotic standard errors use the estimates of the trispectrum integral proposed by Taniguchi (1982) with a Fejer window.) For the nonlinear MA we also report, in square brackets, the estimates of the parameter $\bar{\kappa}$, the fourth-order cumulant of the standartized residuals. For easy reference, the estimates of the other parameters are not reported. The reported estimates correspond to the parameterization (values of p and q) that yields the best performance (with significant parameters estimates), expressed by the degree of white-noisiness of the estimated residuals in levels and squares based on Milhøj (1981) approach. For all the observed time series of returns there is clear evidence that shocks to conditional volatility do decay with time, but very slowly, in agreement with previous empirical studies (see Ding, Granger, and Engle (1993) and Baillie, Bollerslev, and Mikkelsen (1996) among others). Estimates of $\bar{\kappa}$ show a substancial degree of leptokurtosis, especially for the stock indexes return data. This could explain the smaller values obtained for the semiparametricestimates of d (see Wright (2000)). Estimating $\bar{\kappa}$ freely rather than equating it a priori to zero should instead attenuate the effect of the negative bias for the nonlinear MA estimates of d.

A formal analysis of the goodness-of-fit performance of the two competing models is carried out based on the estimated residuals. The results are reported in Table 5.3, within columns two and five. Milhøj (1981) test statistic is calculated under the hypothesis that the power spectrum of the residuals, in levels and squares, is constant at all frequencies, viz. that the residuals and the residuals squared are white-noise. Under the hypothesis of correct model specification Milhøj (1981) test statistic converges in probability to $1/\pi = 0.318...$ Moreover, the last two columns of Table 5.3 report the values of the test statistic using the raw data, respectively in the levels and the squares, providing a benchmark for the results of the fitted models reported in the first four columns. The results of Table 5.3 indicate a better relative performance of the nonlinear MA in terms of goodness-of-fit with respect to ARCH(∞) for all but the stock index S&P 500 and the foreign exchange rate US Dollar/Pound spot data, where the latter model appears superior.

The different performance of the two models is probably caused by the severe parameter restrictions on $\operatorname{ARCH}(\infty)$, in particular $\Sigma(\theta) < \infty$. For instance, when d = q = 0, p = 1 and $\sigma^2 = 1$ (the $\operatorname{ARCH}(1)$), the latter restriction delivers the well-known condition $0 \leq \phi_1 < 1/\sqrt{3}$. Hence, (29) rules out a lot of possible behaviours for the model ACF. This contrasts with the nonlinear MA case, where this condition is not required and, indeed, one can freely allow $-1 < \phi_1 < 1$ when d = q = 0, p = 1.

All the computations and the optimizations are based on Gauss and Fortran codes using an IBM RS 6000 and the NAG library optimization routine E04UCF.(All the codes are available on request from the author.) In terms of computational burden, the nonlinear MA involves calculations of the convolutions in the model power spectrum (cf. Theorem 1) whereas for ARCH(∞) the burden comes from imposing non-negativity of the $\psi_j(\theta)$ and boundedness of $\Sigma(\theta)$. Therefore, despite the apparently simpler expression for the power spectrum of ARCH(∞) (cf. (30)), estimation of the nonlinear MA appears computationally straightforward.

7 Conclusions

In this paper we consider the family of nonlinear MA models introduced by Robinson and Zaffaroni (1997) and show that, for a long memory specification, inference of the model can be performed by a frequency domain Gaussian estimator and by a Gaussian pseudo LM test for dynamic conditional heteroskedasticity, establishing the related asymptotic distribution theory. Finally, we have analyzed the effect of allowing for a time-varying conditional mean for the observables when parameterized by stationary ARMA. An interesting development would be a multivariate extension of the nonlinear MA and related inference by a multivariate Gaussian estimator. Further research is needed to accomplish these goals.

A Second-order properties

Proof of Theorem 1.(i) The $x_t - \mu$ are square integrable martingale difference, by Assumption A, thus displaying a constant power spectrum. The result follows by simple calculations.

(ii) The ACF of the y_t , under Assumption A and $\mu = \rho = 0$ was originally given in RZ, section 4. Let us relax Assumption A setting $\mu_3 = E(\epsilon_t^3)$. For integers a, b, c such that $a > b \ge c$, set $h_{a,(b,c)} = \rho + \sum_{k=a-b}^{a-c} \alpha_k \epsilon_{t+a-k}$. Then, for any l > 0 (without loss of generality),

$$\gamma_{yy}(l) = (i) + (ii) + (iii) + (iv),$$

where

$$\begin{aligned} (i) &= \operatorname{cov}(\epsilon_t^2 h_{0,(-1,-\infty)}^2, \epsilon_{t+l}^2 h_{l,(l-1,-\infty)}^2), \\ (ii) &= \operatorname{cov}(2\mu\epsilon_t h_{0,(-1,-\infty)}, 2\mu\epsilon_{t+l} h_{l,(l-1,-\infty)}), \\ (iii) &= \operatorname{cov}(\epsilon_t^2 h_{0,(-1,-\infty)}^2, 2\mu\epsilon_{t+l} h_{l,(l-1,-\infty)}), \\ (iv) &= \operatorname{cov}(2\mu\epsilon_t h_{0,(-1,-\infty)}, \epsilon_{t+l}^2 h_{l,(l-1,-\infty)}^2). \end{aligned}$$

Applying the cumulant' theorem (see Leonov and Shiryaev (1959)) to each term and with simple but tedious calculations, for any $l = 0, \pm 1, \ldots$,

$$\begin{split} \gamma_{yy}(l) &= \delta_{\alpha\alpha}(0)(\kappa\sigma^{4} + 2\sigma^{8})\alpha_{|l|}^{2} + 4\rho^{2}\sigma^{6}\delta_{\alpha\alpha}(l) + \sigma^{4}\kappa\delta_{\alpha^{2}\alpha^{2}}(l) + 2\sigma^{8}\delta_{\alpha\alpha}^{2}(l) \\ &+ 2\sigma^{2}\mu\alpha_{|l|}\left[\rho\alpha_{|l|}\mu_{3} + 2\rho^{2}\sigma^{2} + 2\sigma^{4}\delta_{\alpha\alpha}(l)\right] + \rho^{2}(\kappa\sigma^{2} + 2\sigma^{6})\alpha_{|l|}^{2} \\ &+ 2\mu_{3}^{2}\alpha_{|l|}\delta_{\alpha^{2}\alpha}(l) + \mu_{3}\left[2\rho\sigma^{4}\delta_{\alpha\alpha^{2}}(l) + 4\rho\sigma^{4}\alpha_{|l|}\delta_{\alpha\alpha}(l) + 2\rho^{3}\sigma^{2}\alpha_{|l|} \\ &+ 2\rho\sigma^{4}\alpha_{|l|}\delta_{\alpha\alpha}(0) + 2\rho\sigma^{4}\delta_{\alpha^{2}\alpha}(l) + \delta(l,0)\nu_{y} \end{split}$$

(35)

where

$$\nu_{y} = (\kappa + 2\sigma^{4}) \left((\rho^{2} + \sigma^{2}\delta_{\alpha\alpha}(0))^{2} + 2\sigma^{2}\delta_{\alpha\alpha}(0)(2\rho^{2} + \sigma^{2}\delta_{\alpha\alpha}(0)) \right) + \kappa \left(4\mu_{3}\rho\delta_{\alpha^{2}\alpha}(0) + (\kappa + 2\sigma^{4})\delta_{\alpha^{2}\alpha}(0) \right) + 4\mu_{3}\mu \left[\rho^{3} + 3\rho\sigma^{2}\delta_{\alpha\alpha}(0) + \mu_{3}\delta_{\alpha^{2}\alpha}(0) \right] + 4\rho\sigma^{4}\mu_{3}\delta_{\alpha^{2}\alpha}(0).$$

Turning to the power spectrum evaluation, under Assumption A, using (3) and (4) and setting $k_1 = \sigma^2(\kappa + 2\sigma^4)(\rho^2 + \sigma^2\delta_{\alpha\alpha}(0)) + \sigma^4\kappa\delta_{\alpha\alpha}(0), k_2 = 2\sigma^8, k_3 = 4\sigma^6\mu, k_4 = 4\sigma^6\rho^2, k_5 = 4\sigma^4\rho^2\mu, k_6 = \sigma^4\kappa$, yields

$$\begin{split} \gamma_{yy}(l) &= \frac{k_1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \beta(\omega)\beta(\lambda)e^{i\omega l}e^{i\lambda l}d\omega d\lambda \\ &+ \frac{k_2}{(2\pi)^2} \int_{-\pi}^{\pi} |\alpha(\omega)|^2 |\alpha(\lambda)|^2 e^{i\omega l}e^{i\lambda l}d\omega d\lambda \\ &+ \frac{k_3}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \beta(\lambda) |\alpha(\omega)|^2 e^{i\omega l}e^{i\lambda l}d\omega d\lambda \\ &+ \frac{k_4}{2\pi} \int_{-\pi}^{\pi} |\alpha(\lambda)|^2 e^{i\lambda l}d\lambda + \frac{k_5}{2\pi} \int_{-\pi}^{\pi} \beta(\lambda)e^{i\lambda l}d\lambda \\ &+ \frac{k_6}{(2\pi)^3} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \alpha(-\omega - \lambda - \mu)\alpha(\lambda)\alpha(\mu)\alpha(\omega)e^{il(\omega + \lambda)}d\omega d\lambda d\mu. \end{split}$$

Then make the change of variable from ω to $\nu = \omega + \lambda$ and equate the integrand with respect to ν to $f_{yy}(\nu)e^{il\nu}$. \Box

B Asymptotic properties of the Gaussian estimator: martingale difference levels

B.1 Lemmata

In this section we will establish a number of properties for the nonlinear MA model (needed for the LLN and the CLT results) mainly in terms of the behaviour of the model spectral densities $f_{yy}(\lambda; \psi, \sigma^2)$ and its derivatives. In turn, these are implied by establishing certain asymptotic features of the $\alpha_i(\theta)$ and their convolution. We recall that \sim denotes asymptotic equivalence, $1_{\mathcal{A}}$ denotes the indicator function, $\delta(u, v)$ the Kronecker delta so that $\delta(u, v) = 1_{u=v}, tr(.)$ the trace operator and $\| \cdot \|$ the Euclidean norm. Constants will be denoted k (not always the same) although these will, in general, be a function of the parameter $(k = k(\theta))$ but we will not write this explicitly.

Lemma 1 Under Assumption B_1 , for some integer m, n with $m, n \ge 0$ and $m + n \ge 2$, as $l \to \infty$,

$$\sum_{j=1}^{\infty} \alpha_j^m(\theta) \, \alpha_{j+l}^n(\theta) \, \sim \, k (l^{n(d(\theta)-1)} + l^{(n+m)(d(\theta)-1)+1}) \,,$$

for some constant k.

Proof. As $l \to \infty$, $\sum_{j=1}^{\infty} \alpha_j^m(\theta) \alpha_{j+l}^n(\theta) = \sum_{j=1}^l \alpha_j^m(\theta) \alpha_{j+l}^n(\theta) + \sum_{j=l+1}^{\infty} \alpha_j^m(\theta) \alpha_{j+l}^n(\theta)$ $\sim k \left(l^{n(d(\theta)-1)} \sum_{j=1}^l \alpha_j^m(\theta) + \sum_{j=l+1}^{\infty} \alpha_j^{(m+n)}(\theta) \right) \sim k \left(l^{n(d(\theta)-1)} + l^{(m+n)(d(\theta)-1)+1} \right). \square$

Lemma 2 Let $\{p_j, j = 0, \pm 1...\}$ and $\{q_j, j = 0, \pm 1...\}$ be two square summable sequences satisfying, for $r_j \in \{p_j, q_j\}$,

(a)
$$|r_j - r_{j+1}| \le |r_j| k/j$$
, all $j > J$, some $0 < k, J < \infty$,
(b) $r_j \to 0$, as $j \to \infty$.

Then, the product sequence and the convolution sequence, defined by $m_u = p_u q_u$, $n_u = \sum_{i=-\infty}^{\infty} p_i q_{i+u}$, satisfy (a) and (b). Note that condition (a) is stronger than quasi monotonic convergence to zero (QMC), implying bounded variation as well (Yong 1974, section I.1).

Proof. Let us start from the product sequence m_j . From $m_{j+1} - m_j = p_{j+1}(q_{j+1} - q_j) + q_j(p_{j+1} - p_j) = (i) + (ii)$. From the assumptions made one obtains, for some $0 < k, k', J < \infty$ and all j > J, $|(i)| \le k/j |p_{j+1}q_j|, |(ii)| \le k' 1/j |p_j q_{j+1}|$, so that as $j \to \infty$, $|m_j - m_{j+1}| = O(|m_j| / j)$. For the convolution, as $j \to \infty$, $n_{j+1} - n_j = \sum_{i=-\infty}^{\infty} p_i(q_{i+j+1} - q_{i+j}) = \sum_{|i| \le j} p_i(q_{i+j+1} - q_{i+j}) + \sum_{|i| > j} p_i(q_{i+j+1} - q_{i+j}) = (iii) + (iv)$. Then, as $j \to \infty$, $|(iii)| = O(|\sum_{|i| \le j} (p_i q_{i+j})/(i+j) |) = O(|q_j| / j \sum_{|i| \le j} |p_i|)$,

 $\begin{aligned} |(iii)| &= O(|\sum_{|i| \le j} (p_i q_{i+j})/(i+j)|) = O(|q_j| / j \sum_{|i| \le j} |p_i|), \\ |(iv)| &= O(|\sum_{|i| > j} p_i (q_{i+1} - q_i)|) = O(1/j \sum_{|i| > j} |p_i q_i|). \ \Box \end{aligned}$

Lemma 3 Let the $\delta_{\alpha\alpha}(l;\theta)$ be given by replacing the α_i with the $\alpha_i(\theta)$ in $\delta_{\alpha\alpha}(l)$. By Assumptions B_1 , B_2 , the following sequences are QMC: $\delta_{\alpha\alpha}(l;\theta)$, $\delta^2_{\alpha\alpha}(l;\theta)$, $\alpha^2_{|l|}(\theta)$, $\alpha_{|l|}(\theta)\delta_{\alpha\alpha}(l;\theta)$, $l = \pm 1, \ldots$ Assuming further B_3 , B_4 the following are QMC: $\frac{\partial}{\partial \theta_i}\delta^2_{\alpha\alpha}(l;\theta)$, i = 1, 2, ..., p, $l = \pm 1, \ldots$ Assuming further B_5 , B_6 , the following are QMC: $\frac{\partial^2}{\partial \theta_{i_1}\partial \theta_{i_2}}\delta^2_{\alpha\alpha}(l;\theta)$, $i_1, i_2 = 1, 2, ..., p$, $l = \pm 1, \ldots$

Proof. The result follows by the direct use of Lemma 2 in each of the sequences obtained as products or convolutions of the $\alpha_j(\theta)$ and their derivatives. \Box

Lemma 4 Assume A, B_1 , B_2 . For any $\psi \in \overline{\Psi}$, for constant $0 < k < \infty$,

$$f_{yy}(\lambda;\psi,\sigma^2) \sim k \mid \lambda \mid^{-2d(\theta)} as \lambda \to 0^+.$$

Let k be a constant $0 < |k| < \infty$, varying in each sentence below and let $L(\cdot)$ be either constant or equal to $\ln(\cdot)$. Assume further B_3 , B_4 . For $j = 1, \ldots, p$, as $\lambda \to 0^+$,

$$\begin{array}{ll} \frac{\partial}{\partial \theta_j} f_{yy}(\lambda; \psi, \sigma^2) & \sim k \, L(\frac{1}{|\lambda|}) \mid \lambda \mid^{-2d(\theta)}, \\ \frac{\partial}{\partial \bar{\kappa}} f_{yy}(\lambda; \psi, \sigma^2) & \sim k. \end{array}$$

Assume further B_5 , B_6 and let $L_i(\cdot)$ be either constant or equal to $\ln(\cdot)$. For $i_1, i_2 = 1, \ldots, p$, as $\lambda \to 0^+$,

$$\frac{\partial^2}{\partial \theta_{i_1} \partial \theta_{i_2}} f_{yy}(\lambda; \psi, \sigma^2) \sim k L_{i_1}(\frac{1}{|\lambda|}) L_{i_2}(\frac{1}{|\lambda|}) |\lambda|^{-2d(\theta)},$$

$$\frac{\partial^2}{\partial \bar{\kappa} \partial \theta_i} f_{yy}(\lambda; \psi, \sigma^2) \sim k.$$

Proof. Let us focus on the model spectrum, which we write as

$$f_{yy}(\nu;\psi,\sigma^2) = f_1(\nu) + f_2(\nu) + f_3(\nu) + f_4(\nu) + \frac{\nu_y(\psi,\sigma^2)}{2\pi},$$

with

$$f_1(\nu) = \frac{k_1(\psi,\sigma^2)}{2\pi} \sum_{l=-\infty}^{\infty} \alpha_{|l|}^2(\theta) e^{i\nu l}, \qquad f_2(\nu) = \frac{k_2(\psi,\sigma^2)}{2\pi} \sum_{l=-\infty}^{\infty} \delta_{\alpha\alpha}^2(l;\theta) e^{i\nu l},$$
$$f_3(\nu) = \frac{k_3(\psi,\sigma^2)}{2\pi} \sum_{l=-\infty}^{\infty} \delta_{\alpha\alpha}(l;\theta) e^{i\nu l}, \quad f_4(\nu) = \frac{k_4(\psi,\sigma^2)}{2\pi} \sum_{l=-\infty}^{\infty} \delta_{\alpha^2\alpha^2}(l)(\theta) e^{i\nu l}$$

where the $k_1(.,.), k_2(.,.), k_3(.,.), k_4(.,.)$ and $\nu_y(.,.)$ can be easily derived from the proof of Theorem 1 with the α_i replaced by the $\alpha_i(\theta)$. By Lemma 1 and Assumption B_1 it follows that the dominating term is $f_4(.)$. Finally, apply Yong (1974, Theorem III-14). The proofs for the first and second derivative of the model spectrum follow along the same lines, where $L(\cdot), L_{i_1}(\cdot), L_{i_2}(\cdot)$ indicate either a constant or $\ln(\cdot)$. \Box

Lemma 5 Under Assumptions $A, B_1, B_2, f_{yy}(\lambda; \psi, \sigma^2)$ is continuous for all ψ and λ such that $\lambda \neq 0$. Further assuming B_3, B_4 , the same applies to $\frac{\partial}{\partial \psi_i} f_{yy}(\lambda; \psi, \sigma^2)$ (i = 1, 2, ..., p+1). Further assuming B_5, B_6 , the same applies to $\frac{\partial^2}{\partial \psi_{i_1} \psi_{i_2}} f_{yy}(\lambda; \psi, \sigma^2)$ $(i_1, i_2 = 1, 2, ..., p+1)$.

Proof. Application of Robinson (1994a, Lemma 8) yields the results. Note that for the derivatives of $f_{yy}(\lambda; \psi, \sigma^2)$ a slowly varying function appears, so that in fact an approximate Lipschitz continuity condition holds. \Box

Lemma 6 Under Assumptions A, B_1 , for any $0 \le \lambda \le \pi$,

$$\inf_{\psi\in\bar{\Psi}}f_{yy}(\lambda;\psi,\sigma^2)\geq\delta>0,$$

for some δ independent from ψ .

Proof. For $\lambda \to 0^+$ the result comes directly from Lemma 4. We will then consider the case $0 < \lambda \leq \pi$. The result follows decomposing $f_{yy}(\lambda; \psi, \sigma^2)$ as the sum of a non-negative function of λ and of a strictly positive quantity, constant in λ . In fact, from the nonlinear structure of the process y_t , the model ACF exhibits a discontinuity (a positive jump) at lag u = 0. Given that $\sigma^2 > 0$, we need to show that $h_y(\lambda; \psi)$ is strictly positive. From Theorem 1, for any l > 0 (with no loss of generality),

$$\bar{\gamma}_{yy}(l;\psi) = 2(\delta^2_{\alpha\alpha}(l;\theta) - \delta_{\alpha^2\alpha^2}(l;\theta)) + 4\delta_{\alpha\alpha}(l;\theta) + (\bar{\kappa}+2)\delta_{\alpha^2\alpha^2}(l;\theta) + (2+\bar{\kappa})\alpha_l^2(\theta)(\delta_{\alpha\alpha}(0;\theta)+1),$$
(36)

and, by simple manipulation,

$$\bar{\gamma}_{yy}(0;\psi) = \tilde{\gamma}_{yy}(0;\psi) + \delta(\psi),$$

setting

$$\delta(\psi) = (2+\bar{\kappa}) \left(4\delta_{\alpha\alpha}(0;\theta) + 2(\delta_{\alpha\alpha}^2(0;\theta) - \delta_{\alpha^2\alpha^2}(0;\theta)) \right) + (2+\bar{\kappa})^2 \delta_{\alpha^2\alpha^2}(0;\theta),$$

and

$$\tilde{\gamma}_{yy}(0;\psi) = 2(\delta^2_{\alpha\alpha}(0;\theta) - \delta_{\alpha^2\alpha^2}(0;\theta)) + 4\delta_{\alpha\alpha}(0;\theta) + (\bar{\kappa}+2)\delta_{\alpha^2\alpha^2}(0;\theta) + (2+\bar{\kappa})(1+\delta_{\alpha\alpha}(0;\theta))^2.$$
(37)

Define another sequence $\tilde{\gamma}_{yy}(l;\psi)$ such that $\tilde{\gamma}_{yy}(l;\psi) = \bar{\gamma}_{yy}(l;\psi)$ $(l \neq 0)$ with $\tilde{\gamma}_{yy}(0;\psi)$ defined as above. We need to show only that $\tilde{\gamma}_{yy}(l;\psi)$ $(l = 0, \pm 1, \ldots)$ behaves as an autocovariance function by its positive (semi) definiteness and so its Fourier transform is non-negative (see Rozanov (1963, Theorem 5.1)). This property of the $\tilde{\gamma}_{yy}(l;\psi)$ is inherited from the $\bar{\gamma}_{yy}(l;\psi)$ once we show $|\tilde{\gamma}_{yy}(l;\psi)| \leq \tilde{\gamma}_{yy}(0;\psi)$ $(l = 0, \pm 1, \ldots)$. However, this easily follows comparing (36) and (37) term by term. Hence, the Fourier transform of the sequence $\tilde{\gamma}_{yy}(l;\psi)$ differs from $h_y(\lambda;\psi)$ by $\delta(\psi)$, which is strictly positive for any $\bar{\kappa} > -2$, $d(\theta) > 0$. Finally, set $\delta = \inf_{\psi \in \bar{\Psi}} \delta(\psi)$. \Box

B.2 Consistency

In the following three lemmas we will establish the conditions required in order to be able to apply Hannan (1973, Theorem 1). The other two lemmas which follow, used in the proof of Theorem 2, represent the usual steps to establish consistency for M-estimators.

Lemma 7 Under Assumption A the process y_t is bounded almost surely, ergodic and strictly stationary.

Proof. We adapt the proof of Nelson (1991, Theorem 2.1). Using Billingsley (1986, Theorem 22.6), the h_t , and thus the y_t , are finite *a.s.* given the square summability of the α_i and the independence and finite variance of the ϵ_t , stated in Assumption A. Ergodicity and strict stationarity follows by Stout (1974, Theorem 3.5.8). \Box

Lemma 8 Under Assumptions A the process y_t is purely non-deterministic. **Proof.** We need to show that (see Hannan (1970))

$$\int_{-\pi}^{\pi} \ln f_{yy}(\omega;\psi,\sigma^2) d\omega > -\infty.$$

From $|\ln f_{yy}(\omega; \psi, \sigma^2)| \leq max \left(|f_{yy}(\omega; \psi, \sigma^2) - 1|, |1 - f_{yy}^{-1}(\omega; \psi, \sigma^2)| \right)$, the result follows by stationarity of the y_t and continuity of $f_{yy}^{-1}(\lambda; \psi, \sigma^2)$ by Lemma 6. \Box

Lemma 9 Under Assumptions $A, B_1, B_2, B_3, B_4, B_7$, for any $\psi, \psi^* \in \overline{\Psi}$ we obtain $h_y(\lambda; \psi) \neq h_y(\lambda; \psi^*)$, for any $-\pi \leq \lambda < \pi$, whenever $\psi \neq \psi^*$.

Proof. Let us choose two vectors of parameters ψ, ψ^* such that for some $(p+1) \times 1$ vector of constants $\eta \neq 0$ we can write $\psi = \psi^* + \eta$. Let us start, contradicting the conclusion of the Lemma, assuming that for some $-\pi \leq \lambda < \pi, h_y(\lambda; \psi) = h_y(\lambda; \psi^*)$. We will see that the condition $\psi \neq \psi^*$ will contradict the hypothesis of the Lemma. By using the fact that the ACF uniquely identifies the power spectrum, it follows that $\bar{\gamma}_{yy}(u; \psi) = \bar{\gamma}_{yy}(u; \psi^*)$, for any $u = \pm 1, \ldots$ In particular, by the mean value theorem, considering the u_1, \ldots, u_{p+1} of Assumption B_8 , this implies

$$\begin{pmatrix} \bar{\gamma}_{yy}(u_1;\psi) \\ \vdots \\ \bar{\gamma}_{yy}(u_{p+1};\psi) \end{pmatrix} = \begin{pmatrix} \bar{\gamma}_{yy}(u_1;\psi) \\ \vdots \\ \bar{\gamma}_{yy}(u_{p+1};\psi) \end{pmatrix} + \begin{pmatrix} \frac{\partial \bar{\gamma}_{yy}(u_1;\tilde{\psi})}{\partial \psi'} \\ \vdots \\ \frac{\partial \bar{\gamma}_{yy}(u_{p+1};\tilde{\psi})}{\partial \psi'} \end{pmatrix} (\psi - \psi^*)$$

where $\tilde{\psi}$ is such that $\tilde{\psi} = \psi + R(\psi^* - \psi)$ with R being $(p+1) \times (p+1)$ matrix satisfying $\| \tilde{\psi} - \psi \| \leq \| \psi^* - \psi \|$. However, by Assumption B_8 , this holds only for $\psi = \psi^*$. \Box

Lemma 10

(i) Under Assumptions A, B_1, B_2 , uniformly in $\psi \in \Psi$,

$$\lim_{T \to \infty} Q_T(\psi) = Q(\psi) \ a.s.,$$

with

$$Q(\psi) = \ln\left(\frac{\sigma^8}{2\pi} \int_{-\pi}^{\pi} \frac{h_y(\lambda)}{h_y(\lambda;\psi)} d\lambda\right) + \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln h_y(\lambda;\psi) + 1$$

(ii) Under Assumptions B_1 , B_2 , B_7 , for any $\psi \in \overline{\Psi}$,

$$Q(\psi) \ge Q(\psi_0) = \ln(\sigma^8) + \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln h_y(\lambda) + 1,$$

so that the minimum is attained for $\psi = \psi_0$.

Proof. All the convergences below hold as $T \to \infty$. (i) For the first stochastic term on the right hand side of $Q_T(\psi)$ (see 12)), the result follows directly from Hannan (1973, Lemma 1) and continuity of the logarithmic function. Let us consider the second non-stochastic term. Let j_{ϵ} be an integer such that $\frac{1}{T}2\pi j_{\epsilon} \to \epsilon > 0$ for an arbitrary $0 < \epsilon < 1/2$. Also let m be an integer such that $\frac{1}{m} + \frac{m}{T} \to 0$ so that we can always take $2\pi m/T < \epsilon$ for T large enough. For $h_y(\lambda; \psi)$ being an even function of λ , one need only consider

$$\frac{1}{T}\sum_{j=1}^{[T/2]} \ln h_y(\lambda_j; \psi) = (a) + (b) + (c),$$

with

$$(a) = \frac{1}{T} \sum_{j=j_{\epsilon}+1}^{[T/2]} \ln h_y(\lambda_j; \psi), (b) = \frac{1}{T} \sum_{j=m}^{j_{\epsilon}} \ln h_y(\lambda_j; \psi), (c) = \frac{1}{T} \sum_{j=1}^{m-1} \ln h_y(\lambda_j; \psi).$$

By the continuity of the spectrum away from the zero frequency

$$(a) \to \frac{\int_{\epsilon}^{\pi} \ln h_y(\lambda;\psi) d\lambda}{2\pi}$$

Secondly, from $1 - 1/X \le \ln X \le X - 1$ for X > 0, one obtains (c) = o(1).

In fact
$$(c)' = \frac{m}{T} - \frac{1}{T} \sum_{j=1}^{m-1} \frac{1}{h_y(\lambda_j; \psi)} \le (c) \le -\frac{m}{T} + \frac{1}{T} \sum_{j=1}^{m-1} h_y(\lambda_j; \psi) = (c)''.$$

 $(c)' = O(\frac{m}{T}) = o(1)$ by the continuity of the inverse of the spectrum, and $(c)'' = O((\lambda_m)^{2\delta}) = o(1)$ by Robinson (1994b, Lemma 4) where because of B_1 we can always choose $\delta > 0$ such that $\sup_{\psi \in \bar{\Psi}} d(\theta) < 1/2 - \delta < 1/2$. For (b) one obtains

$$|(b) - \frac{1}{2\pi} \int_0^{\epsilon} \ln h_y(\omega; \psi) d\omega | \le |(b)'| + |(b)''|,$$

with

$$(b)' = \left(\frac{1}{T}\sum_{j=m}^{j\epsilon} \ln h_y(\lambda_j;\psi) - \frac{1}{2\pi}\int_{\lambda_m}^{\epsilon} \ln h_y(\lambda;\psi)d\lambda\right),$$
$$(b)'' = \frac{1}{2\pi}\int_0^{\lambda_m} \ln h_y(\lambda;\psi)d\lambda = O(\lambda_m\ln(\lambda_m)).$$

In turn

$$(b)' \mid \leq | \frac{\ln(h_y(\lambda_m; \psi))}{T} | + \frac{1}{2\pi} | \sum_{j=m}^{(j_\epsilon)-1} \int_{\lambda_j}^{\lambda_{j+1}} \ln(\frac{h_y(\lambda_{j+1}; \psi)}{h_y(\lambda; \psi)}) d\lambda |$$

= | (b.1)' | + | (b.2)' |,

and $|(b.1)'| = O(\ln T/T)$. Finally, from

$$\frac{2\pi}{T} - \frac{\lambda_{j+1}^2 - \lambda_j^2}{2\lambda_{j+1}} \le \int_{\lambda_j}^{\lambda_{j+1}} \ln(\frac{\lambda_{j+1}}{\lambda}) d\lambda \le \lambda_{j+1} \ln(\frac{\lambda_{j+1}}{\lambda_j}) - \frac{2\pi}{T}$$

one gets $\frac{\pi}{T(j+1)} \leq \int_{\lambda_j}^{\lambda_{j+1}} \ln(\frac{\lambda_{j+1}}{\lambda}) d\lambda \leq \frac{2\pi}{T_j}$ yielding $| (b.2)'' | = O(\ln j_{\epsilon}/T)$, where all the bounds do not depend on ψ , by the compactness assumption made on the parameter space.

(ii) ψ_0 is the unique minimizer of $Q(\psi)$, given the identification condition, by which $h_y(\lambda; \psi) \neq h_y(\lambda; \psi_0) (-\pi < \lambda \leq \pi)$ whenever $\psi \neq \psi_0$ (cf. Lemma 9). Therefore

$$\begin{split} Q(\psi) &= \log(\sigma^8) + \log\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{h_y(\lambda)}{h_y(\lambda;\psi)} d\lambda\right) + \frac{1}{2\pi} \int_{-\pi}^{\pi} \log h_y(\lambda;\psi) d\lambda + 1 \\ &= \log(\sigma^8) + \frac{1}{2\pi} \int_{-\pi}^{\pi} \log h_y(\lambda) d\lambda + 1 \\ &+ \log\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{h_y(\lambda)}{h_y(\lambda;\psi)} d\lambda\right) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \log\left(\frac{h_y(\lambda)}{h_y(\lambda;\psi)}\right) d\lambda \\ &= Q(\psi_0) \\ &+ \log\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{h_y(\lambda)}{h_y(\lambda;\psi)} d\lambda\right) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \log\left(\frac{h_y(\lambda)}{h_y(\lambda;\psi)}\right) d\lambda \geq Q(\psi_0), \end{split}$$

using Jensen's inequality for integrals (see Hardy, Littlewood, and Polya (1964)). \Box

Proof of Theorem 2. By Lemmas 7-10 and Hannan (1973, Lemma 1), strong consistency follows by Hannan (1973, Theorem 1). \Box

We exploited the ergodicity property for the *a.s.* convergence of the sample autocovariance. Alternatively, one could use the fourth-order cumulant expression for the y_t in order to derive the convergence in probability of the sample covariances of y_t . In fact, one obtains that as $T \to \infty$ $\operatorname{var}(c(u)) = O(T^{2d-1})$, where c(u), $u = 0, \pm 1, \ldots$ denote the sample autocovariances made of $(y_1, \ldots, y_T)'$ (cf. Zaffaroni (1997)).
B.3 Asymptotic distribution

A number of technical lemmas follow, which allow us to establish the CLT for the score and the convergence of the Hessian. In particular, we establish that the quadratic form $\frac{1}{2\pi} \int_{-\pi}^{\pi} g(\lambda; \psi_0) I_{yy}(\lambda) d\lambda$, opportunely normalized, is asymptotically normal. This will be derived by studying the asymptotic behaviour of its cumulants of order two and four. The approach here considered is similar to Giraitis and Surgailis (1990, Theorem 1) although there the analysis is based on looking at the power spectrum and trispectrum (the Fourier transform of the fourth-order cumulant) given their simple structure. In our context we need to develop the analysis in terms of the cumulants because, unlike the linear process case of Giraitis and Surgailis (1990), the trispectrum of the nonlinear MA has a very complicated expression (cf. Zaffaroni (1997)).

Lemma 11 Under Assumptions $A, B_1, B_2, B_3, B_4, B_5, B_6, \int_{-\pi}^{\pi} \ln h_y(\omega, \psi) d\omega$ can be differentiated twice under the integral operator for any $\psi \in \overline{\Psi}$.

Proof. We follow Fox and Taqqu (1986) and skip the second differentiation case for simplicity. Denoting the *j*-th unit vector in \mathbb{R}^{p+1} by e_j , write $\frac{1}{2} \left(\int_{-\infty}^{\pi} \ln h_j \left(\frac{1}{2} \ln h_j \right) \right) \right)} \right)\right)$

$$\frac{1}{\epsilon} \left(\int_{-\pi}^{\pi} \ln h_y(\omega; \psi + e_j \epsilon) d\omega - \int_{-\pi}^{\pi} \ln h_y(\omega; \psi) d\omega \right) =$$

 $\frac{1}{\epsilon} \int_{-\pi}^{\pi} (\ln h_y(\omega; \psi + e_j \epsilon) d\omega - \ln h_y(\omega; \psi)) d\omega \text{ for some arbitrary } \epsilon > 0. \text{ Thus} by the mean value theorem the last integral, ignoring constant terms, is bounded by}$

$$\int_{-\pi}^{\pi} |\omega|^{2(d_l - d_u) - \delta} d\omega < \infty,$$

where $d_l = inf_{\theta \in \Theta}d(\theta)$, $d_u = sup_{\theta \in \Theta}d(\theta)$, given $(d_l - d_u) > -1/2$, where δ can be taken as arbitrarily small. Use of the dominated convergence theorem concludes. \Box

Lemma 12

(i) Under Assumptions A, B₁, B₂, B₇, the Fourier coefficients of $h_y^{-1}(\lambda; \psi)$ ($\pi \leq \lambda < \pi$) are QMC.

(ii) Let $\tilde{g}_{\psi}(u;\psi)$ $(u=0,\pm 1,...)$ be the Fourier coefficients of

$$g(\lambda;\psi) = \frac{\partial}{\partial\psi} h_y^{-1}(\lambda;\psi).$$

Under Assumptions $A, B_1, B_2, B_3, B_4, B_7$, for any $\psi \in \overline{\Psi}$ and constant $0 < |k| < \infty$ (varying in each sentence below) the $\tilde{g}_{\psi}(u; \psi)$ form a vector QMC sequence and, as $u \to \infty$,

$$\begin{split} \tilde{g}_{\theta_i}(u;\psi) &\sim k \, L(|u|) \, |u|^{-1-2d(\theta)}, \\ \tilde{g}_{\bar{\kappa}}(u;\psi) &\sim k \, |u|^{-2-6d(\theta)}, \end{split}$$

for i = 1, ..., p, where the $L(\cdot)$ are either equal to 1 or $\ln(\cdot)$.

Proof. (i) The continuity and thus the integrability of the inverse of the spectrum yields the convergence to zero of its Fourier coefficients by the Riemann-Lebesgue Theorem. Then the Lipschitz condition (Lemma 5) allows us to use Zygmund (1977, Theorem 4.7-(i) p.46) and so to bound the Fourier coefficients of $h_y^{-1}(\lambda; \psi)$ as $\int_{-\pi}^{\pi} h_y^{-1}(\lambda; \psi) e^{i\lambda u} d\lambda = O(|u|^{-k})$ for some 0 < k < 1. By using Yong (1974, Lemma I-1, p.4) the result follows. (ii) The function $g(\lambda; \psi)$ is given by the product of $h_y^{-2}(\lambda; \psi)$ times $-\frac{\partial}{\partial \psi}h_y(\lambda; \psi)$. Thus, each $\tilde{g}_{\psi_i}(u;\psi)$ is equivalent to the convolution of two series of QMC coefficients by Lemma 3 and part (i) of this Lemma, and therefore it is QMC itself by Lemma 2. By Yong (1974, Theorem III-33-(ii)) and Lemma 4 the asymptotic relations for the Fourier coefficients follow. \Box

Lemma 13 Under Assumptions $A', B, as T \to \infty$,

$$T^{\frac{1}{2}}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}g(\lambda;\psi_0)\left(I_{yy}(\lambda)-EI_{yy}(\lambda)\right)d\lambda\right)\to_d\mathcal{N}_{p+1}(0,\bar{V}),$$

for some positive semi definite matrix \overline{V} .

Proof. All the convergences below hold as $T \to \infty$. We establish the result in two steps. In the first we approximate $\frac{1}{2\pi} \int_{-\pi}^{\pi} g(\lambda; \psi_0) I_{yy}(\lambda) d\lambda$ by another quadratic form such that they share the same asymptotic distribution. Then, we establish the asymptotic distribution of this second quadratic form, adapting standard limit laws. By Fox and Taqqu (1987, Lemma 8.1), substitution of \bar{y} with $E(y_t)$ in the periodogram is (asymptotically) equivalent. Setting $\tilde{g}_{\psi}(u) = \tilde{g}_{\psi}(u; \psi_0) (u = 0, \pm 1, ...)$ and choosing an integer $0 < M < \infty$, define

Rather than establishing $\operatorname{var}(Q_T - S_T) = o(T)$ we will obtain the slightly weaker condition: $\operatorname{var}(Q_T - S_T) = O(\eta(M)T)$ where $0 < \eta(M) < \infty$ and $\eta(M) \downarrow 0$ as $M \to \infty$. Then

$$\begin{aligned} \hat{y}_t \hat{y}_s - \hat{y}_t^{(M)} \hat{y}_s^{(M)} &= \left(y_t - E(y_t) - y_t^{(M)} + E(y_t^{(M)}) \right) \left(y_s - E(y_s) \right) \\ &+ \left(y_t^{(M)} - E(y_t^{(M)}) \right) \left(y_s - E(y_t) - y_s^{(M)} + E(y_t^{(M)}) \right), \\ y_t - y_t^{(M)} &= \epsilon_t^2 \Big((\rho + w_{t-1})^2 - (\rho + w_{t-1}^{(M)})^2 \Big) \\ &= \epsilon_t^2 \left(w_{t-1}^2 - (w_{t-1}^{(M)})^2 + 2\rho(w_{t-1} - w_{t-1}^{(M)}) \right) \\ &= \epsilon_t^2 \left(2\rho \sum_{i=M}^{\infty} \alpha_i \epsilon_{t-i} + \left(\sum_{i=M}^{\infty} \alpha_i \epsilon_{t-i} \right) \left(\sum_{i=1}^{\infty} \bar{\alpha}_i \epsilon_{t-i} \right) \right), \end{aligned}$$

where

$$\bar{\alpha}_i = \begin{cases} \alpha_i, & i \ge M, \\ 2\alpha_i, & i < M. \end{cases}$$

Thus setting $z_t = \left(y_t - y_t^{(M)} - E(y_t) + E(y_t^{(M)})\right)$ we get $\hat{y}_t \hat{y}_s - \hat{y}_t^{(M)} \hat{y}_s^{(M)} = z_t \hat{y}_s + \hat{y}_t^{(M)} z_s$, yielding

$$\operatorname{var}(Q_{T} - S_{T}) = (38.1) + (38.2) + (38.3), \quad (38)$$

with
$$(38.1) = \operatorname{var}(\sum_{t,s=1}^{T} \tilde{g}_{\psi}(t-s)\hat{y}_{t}z_{s}), \quad (38.2) = \operatorname{var}(\sum_{t,s=1}^{T} \tilde{g}_{\psi}(t-s)\hat{y}_{t}^{(M)}z_{s}), \quad (38.3) = 2\operatorname{cov}\left(\sum_{t_{1},s_{1}=1}^{T} \tilde{g}_{\psi}(t_{1} - s_{1})\hat{y}_{t_{1}}z_{s_{1}}, \sum_{t_{2},s_{2}=1}^{T} \tilde{g}_{\psi}(t_{2} - s_{2})\hat{y}_{t_{2}}^{(M)}z_{s_{2}}\right).$$

By the cumulant' theorem (see Brillinger (1975)), denoting by $c_k(a_1, \dots, a_k)$ the k-th order cumulant in possibly k different arguments a_1, \dots, a_k , the first term of (38) is equivalent to

$$(38.1) = (a) + (b) + (c),$$

$$(a) = \sum_{t_1, s_1, t_2, s_2 = 1}^{T} \tilde{g}_{\psi_i}(t_1 - s_1) \, \tilde{g}_{\psi_j}(t_2 - s_2) c_2(\hat{y}_{t_1}, \hat{y}_{t_2}) c_2(z_{s_1}, z_{s_2}),$$

$$(b) = \sum_{t_1, s_1, t_2, s_2 = 1}^{T} \tilde{g}_{\psi_i}(t_1 - s_1) \, \tilde{g}_{\psi_j}(t_2 - s_2) c_2(\hat{y}_{t_1}, z_{s_2}) c_2(\hat{y}_{t_2}, z_{s_1}),$$

$$(c) = \sum_{t_1, s_1, t_2, s_2 = 1}^{T} \tilde{g}_{\psi_i}(t_1 - s_1) \, \tilde{g}_{\psi_j}(t_2 - s_2) c_4(\hat{y}_{t_1}, \hat{y}_{t_2}, z_{s_1}, z_{s_2}),$$

for i, j = 1, ..., p+1. It can be shown that term (38.2) has the same asymptotic behaviour as term (38.1) and both, by Schwarz inequality, determine the behaviour of the covariance term (38.3). Therefore we will develop case (38.1) only. Let us evaluate each of these components. Starting with term (a) above write $z_t = A_t + B_t$ with

$$A_t = 2\rho\epsilon_t^2 (\sum_{i=M}^{\infty} \alpha_i \epsilon_{t-i}), \quad B_t = \epsilon_t^2 (\sum_{i=M}^{\infty} \alpha_i \epsilon_{t-i}) (\sum_{i=1}^{\infty} \bar{\alpha}_i \epsilon_{t-i}),$$

and $\hat{y}_t = C_t + D_t + E_t$ with

$$C_t = \rho^2 \epsilon_t^2, \quad D_t = 2\rho \epsilon_t^2 \sum_{i=1}^{\infty} \alpha_i \epsilon_{t-i}, \quad E_t = \epsilon_t^2 \sum_{i,j=1}^{\infty} \alpha_j \alpha_i \epsilon_{t-j} \epsilon_{t-i}.$$

Note that each of the A_t, B_t, D_t, E_t is made by products of terms with at least one having mean zero, e.g. $A_t = 2\rho\epsilon_t^2(\sum_{i=M}^{\infty} \alpha_i\epsilon_{t-i})$ and $E(\sum_{i=M}^{\infty} \alpha_i\epsilon_{t-i}) = 0$. Thus we get

$$c_{2}(z_{s_{1}}, z_{s_{2}}) = c_{2}(A_{s_{1}}, A_{s_{2}}) + c_{2}(B_{s_{1}}, B_{s_{2}}) + c_{2}(A_{s_{1}}, B_{s_{2}}) + c_{2}(B_{s_{1}}, A_{s_{2}}), (39)$$

$$c_{2}(\hat{y}_{t_{1}}, \hat{y}_{t_{2}}) = c_{2}(C_{t_{1}}, C_{t_{2}}) + c_{2}(C_{t_{1}}, D_{t_{2}}) + c_{2}(C_{t_{1}}, E_{t_{2}})$$

$$+ c_{2}(D_{t_{1}}, C_{t_{2}}) + c_{2}(D_{t_{1}}, D_{t_{2}}) + c_{2}(D_{t_{1}}, E_{t_{2}})$$

$$+ c_{2}(E_{t_{1}}, C_{t_{2}}) + c_{2}(E_{t_{1}}, D_{t_{2}}) + c_{2}(E_{t_{1}}, E_{t_{2}}).$$

$$(40)$$

Obviously there are fewer terms to be evaluated, given that indexes can always be interchanged. Let us set for integers j, u

$$\gamma_{j,(u)} = \sum_{i=j}^{\infty} \alpha_i \alpha_{i+u}, \quad \gamma_{(u)} = \gamma_{1,(u)}, \quad \gamma_j = \gamma_{j,(0)}, \quad \gamma = \gamma_{1,(0)} = \gamma_{(0)},$$

$$\delta_{j,(u)} = \sum_{i=j}^{\infty} \alpha_i^2 \alpha_{i+u}^2, \quad \delta_{(u)} = \delta_{1,(u)}, \quad \delta_j = \delta_{j,(0)}, \quad \delta = \delta_{1,(0)} = \delta_{(0)}.$$

Note that in terms of the previous definition (cf. section 2) $\gamma_{(u)} = \delta_{\alpha\alpha}(u)$ but we use the newer notation for simplicity. Starting from $c_2(z_{s_1}, z_{s_2})$, using the cumulant' theorem and setting $s = s_2 - s_1$ with $s \ge 0$

$$\begin{split} c_2(A_{s_1}, A_{s_2}) &= \begin{cases} (3\sigma^4 + \kappa)\gamma_{M,(s)}, \ s > 0, \\ 0, \qquad s = 0, \end{cases} \\ c_2(A_{s_1}, B_{s_2}) &= c_2(B_{s_1}, A_{s_2}) = 0, \\ c_2(B_{s_1}, B_{s_2}) &= \begin{cases} \sigma^4 \kappa (\alpha_s^2 \mathbf{1}_{s \ge M} \gamma_M + \delta_{M,(s)}) + \\ \sigma^8 (\gamma_{(s)} \gamma_{M,(s)} + \gamma_{M,(s)} (\mathbf{1}_{s < M} \gamma_{M - s,(s)} + \mathbf{1}_{s \ge M} \gamma_{M,(s)}))), \\ (\kappa + 2\sigma^4) \sigma^4 \gamma_M^2 + (\kappa + 3\sigma^4) (\sigma^4 \gamma_M^2 + \kappa \delta_M), \end{cases} \quad s > 0, \\ c_2(C_{s_1}, C_{s_2}) &= \begin{cases} \rho^4 (\kappa + 2\sigma^4), \ s = 0, \\ 0, \qquad s > 0, \end{cases} \\ c_2(C_{s_1}, D_{s_2}) &= c_2(D_{s_1}, C_{s_2}) = c_2(D_{s_1}, E_{s_2}) = c_2(E_{s_1}, D_{s_2}) = 0, \\ c_2(C_{s_1}, E_{s_2}) &= \begin{cases} \sigma^2 (\kappa + 3\sigma^4), \quad s = 0, \\ \sigma^2 (\kappa + 2\sigma^4) \alpha_s^2, \quad s > 0, \end{cases} \\ c_2(D_{s_1}, D_{s_2}) &= \begin{cases} \sigma^2 (\kappa + 3\sigma^4) \gamma, \quad s = 0, \\ \sigma^6 \gamma_{(s)}, \qquad s > 0, \end{cases} \\ c_2(E_{s_1}, C_{s_2}) &= \begin{cases} \sigma^2 (\kappa + 3\sigma^4) \gamma, \quad s = 0, \\ 0, \qquad s > 0, \end{cases} \\ c_2(E_{s_1}, C_{s_2}) &= \begin{cases} \sigma^4 [(\kappa + 2\sigma^4) \gamma \alpha_s^2 + \kappa \delta_{(s)} + 2\sigma^4 \gamma_{(s)}^2], \quad s > 0, \\ \sigma^4 [(\kappa + 2\sigma^4) \gamma + \kappa \delta + 2\sigma^4 \gamma^2)], \qquad s = 0, \end{cases} \end{split}$$

For case (b) we no longer have symmetry so we need to take t > s, t = s, t < s separately, obtaining

$$c_2(A_t, C_s) = 0$$
 for any t, s ,

$$c_{2}(A_{t}, D_{s}) = \begin{cases} 0, & t \neq s, \\ \sigma^{2}(\kappa + 3\sigma^{4})\gamma_{M}, & t = s, \end{cases}$$

$$c_{2}(A_{t}, E_{s}) = 0 \text{ for any } t, s, \\c_{2}(B_{t}, C_{s}) = \begin{cases} 0, & t \neq s, \\ \sigma^{2}(\kappa + 3\sigma^{4})\gamma_{M}, & t = s, \end{cases}$$

$$c_{2}(B_{t}, D_{s}) = 0 \text{ for any } t, s, \end{cases}$$

$$c_{2}(B_{t}, E_{s}) = \begin{cases} \sigma^{4}(\gamma(\kappa + 2\sigma^{4})\alpha_{t-s}^{2}\mathbf{1}_{t-s\geq M} + \kappa\delta_{M-(t-s)+1,(t-s)}\gamma_{(t-s)}), & t > s, \end{cases}$$

$$c_{2}(B_{t}, E_{s}) = \begin{cases} \sigma^{4}(\gamma(\kappa + 2\sigma^{4})(\gamma + \gamma_{M}) + \sigma^{2}(\kappa\delta_{M} + 2\sigma^{4}\gamma\gamma_{M}), & t = s, \\ \sigma^{2}((\kappa + 2\sigma^{4})(\gamma + \gamma_{M}) + \sigma^{2}(\kappa\delta_{M} + 2\sigma^{4}\gamma\gamma_{M}), & t = s, \end{cases}$$

As $s \to \infty$, assuming M < s, writing $\sum_{i=M}^{\infty} = \sum_{i=M}^{s} + \sum_{i=s+1}^{\infty}$ yields

$$|\gamma_{M,(s)}| = O(s^{d-1}(s^d + M^d) + s^{2d-1}) = O(s^{2d-1+\delta}M^{-\delta}),$$
(41)

for some arbitrary constant $\delta > 0$. Likewise, when s < M

$$|\gamma_{M,(s)}| = O(\sum_{i=M}^{\infty} \alpha_i^2) = O(M^{-\delta} s^{2d-1+\delta}),$$

where $0 < \eta(M) = M^{-\delta} < \infty$ can be made arbitrarily small when M gets arbitrarily large. The same bound applies for $\delta_M(s)$. Finally, apply Fox and Taqqu (1987, Theorem 1-(a)), with (using their notation) p = 2, $\alpha = 2d$, $\beta = -2d = -\alpha$, yielding $2(\alpha + \beta) = 0 < 1$. Thus $(a) = O(\eta(M)T)$, $(b) = O(\eta(M)T)$.

To evaluate the asymptotic behaviour of the cumulant expression (c) in (38.1) we need to focus on the following possibilities, setting $t = t_1 = t_2$ when $t_1 = t_2$ and $s = s_1 = s_2$ when $s_1 = s_2$ where, by symmetry, we can always take $t_1 \leq t_2$ and $s_1 \leq s_2$:

We consider the first few cases, given that the same argument will carry through in all cases. We need to evaluate the following fourth-order cumulants:

$$c_4(x, x', y, y')$$
, with $x, x' \in \{A_t, B_s\}, y, y' \in \{C_r, D_u, E_v\}, t, s, r, u, v = 1, \dots, T$

Considering the product expression for A_t, B_t, C_t, D_t, E_t , note that when the number of factors involved is odd the cumulant vanishes by $E(\epsilon_t) = 0$. For

example, the cumulant $c_4(A_t, A_s, D_r, E_v)$ involves 2 + 2 + 3 + 4 = 11 factors and thus is equal to zero. For the non-zero terms we need to evaluate the cumulant directly.

Case s = t:

case s = t. $c_4(A_t, A_t, C_t, C_t) = (c_4(\epsilon_t^2, \epsilon_t^2, \epsilon_t^2, \epsilon_t^2) + 2(c_2(\epsilon_t^2, \epsilon_t^2))^2 + 3\sigma^2 c_3(\epsilon_t^2, \epsilon_t^2, \epsilon_t^2)) \sigma^2 \gamma_M$, so that $|c_4(A_t, A_t, C_t, C_t)| \leq K |\gamma_M|$, where the constant K involves terms such as σ^2 , κ and their power. Let us denote by K_j , $j = 1, 2, \ldots$ some constants, (different from time to time) that, like K, might involve terms such as σ^2 , κ , γ , δ and their powers. Then

$$\begin{aligned} c_4(A_t, A_t, C_t, E_t) &= (K_1 \gamma_M^2 + K_2 \delta_M), \ c_4(A_t, A_t, D_t, D_t) = (K_1 \gamma_M^2 + K_2 \delta_M), \\ c_4(A_t, A_t, E_t, E_t) &= (K_1 \gamma_M^2 + K_2 \delta_M + K_3 \gamma_M + K_4 \gamma_M \delta_M), \\ c_4(A_t, B_t, C_t, D_t) &= (K_1 \gamma_M^2 + K_2 \delta_M + K_3 \gamma_M), \\ c_4(A_t, B_t, D_t, E_t) &= (K_1 \gamma_M^2 + K_2 \delta_M + K_3 \gamma_M), \\ c_4(B_t, B_t, C_t, C_t) &= (K_1 \gamma_M^2 + K_2 \delta_M + K_3 \gamma_M), \\ c_4(B_t, B_t, D_t, D_t) &= (K_1 \gamma_M^2 + K_2 \delta_M + K_3 \gamma_M), \\ c_4(B_t, B_t, D_t, D_t) &= (K_1 \gamma_M^2 + K_2 \delta_M + K_3 \gamma_M), \\ c_4(B_t, B_t, E_t, E_t) &= (K_1 \gamma_M^2 + K_2 \delta_M + K_3 \gamma_M), \end{aligned}$$

and for large M

$$| c_4(A_t, A_t, C_t, E_t) | \leq K | \delta_M |, | c_4(A_t, A_t, D_t, D_t) | \leq K | \delta_M |, | c_4(A_t, A_t, E_t, E_t) | \leq K | \gamma_M |, | c_4(A_t, B_t, C_t, D_t) | \leq K | \gamma_M |, | c_4(A_t, B_t, D_t, E_t) | \leq K | \gamma_M |, | c_4(B_t, B_t, C_t, C_t) | \leq K | \delta_M |, | c_4(B_t, B_t, C_t, E_t) | \leq K | \gamma_M |, | c_4(B_t, B_t, D_t, D_t) | \leq K | \gamma_M |, | c_4(B_t, B_t, E_t, E_t) | \leq K | \gamma_M |.$$

Case s > t:

$$\begin{split} c_4(A_t, A_t, C_s, E_s) &= \begin{pmatrix} K_1 \, \alpha_{s-t}^2 \gamma_M + K_2 \alpha_{s-t}^2 \gamma_{M,(s-t)} + K_3 \delta_{M,(s-t)} \end{pmatrix}, \\ c_4(A_t, A_t, E_s, E_s) &= \begin{pmatrix} K_1 \, \alpha_{s-t}^4 \gamma_M + K_2 \alpha_{s-t}^2 \gamma_{M,(s-t)}^2 + K_3 \alpha_{s-t}^2 \delta_{M,(s-t)} \\ &+ K_4 \gamma_{M,(s-t)} \delta_{M,(s-t)} + K_5 \gamma_M + K_6 \gamma_{M,(s-t)}^2 + K_7 \delta_{M,(s-t)} \end{pmatrix}, \\ c_4(A_t, B_t, C_s, D_s) &= \begin{pmatrix} K_1 \, \delta_{M,(s-t)} + K_2 \gamma_M \gamma_{M,(s-t)} + K_3 \gamma_{M,(s-t)} \end{pmatrix}, \\ c_4(A_t, B_t, D_s, E_s) &= \begin{pmatrix} K_1 \, \alpha_{s-t}^2 \delta_{M,(s-t)} + K_2 \alpha_{s-t}^2 \gamma_{M,(s-t)} \gamma_{(s-t)} + K_3 \alpha_{s-t}^2 \gamma_M \gamma_{(s-t)} \\ &+ K_4 \gamma_{M,(s-t)} \delta_{M,(s-t)} + K_5 \gamma_{(s-t)} \delta_{M,(s-t)} + K_6 \delta_{M,(s-t)} \\ &+ K_7 \delta_{(s-t)} \gamma_M + K_8 \gamma_M \gamma_{M,(s-t)} + K_9 \gamma_{M,(s-t)}^2 \rangle, \\ c_4(B_t, B_t, C_s, E_s) &= \begin{pmatrix} K_1 \, \delta_M + K_2 \gamma_M^2 + K_3 \delta_{M,(s-t)} + K_4 \gamma_{M,(s-t)}^2 \end{pmatrix}, \\ c_4(B_t, B_t, D_s, D_s) &= \begin{pmatrix} K_1 \, \delta_M + K_2 \gamma_M^2 + K_3 \delta_{M,(s-t)} + K_4 \gamma_{M,(s-t)}^2 \end{pmatrix}, \\ c_4(B_t, B_t, E_s, E_s) &= \begin{pmatrix} K_1 \, \delta_M + K_2 \gamma_M^2 + K_3 \delta_{M,(s-t)} + K_4 \gamma_{M,(s-t)}^2 \end{pmatrix}, \\ c_4(B_t, B_t, E_s, E_s) &= \begin{pmatrix} K_1 \, \delta_M + K_2 \gamma_M^2 + K_3 \delta_{M,(s-t)} + K_4 \gamma_{M,(s-t)}^2 \rangle, \\ + K_5 \delta_{M,(s-t)} \gamma_M + K_6 \delta_{M,(s-t)} \gamma_{M,(s-t)} + K_7 \delta_{M,(s-t)} \gamma_{M,(s-t)} \end{pmatrix}, \\ c_4(B_t, B_t, E_s, E_s) &= \begin{pmatrix} K_1 \, \delta_M + K_2 \gamma_M^2 + K_3 \delta_{M,(s-t)} + K_4 \gamma_{M,(s-t)}^2 \rangle, \\ + K_1 \delta_{M,(s-t)} + K_1 4 \gamma_M \gamma_{(s-t)} \gamma_{M,(s-t)} + K_1 2 \gamma_{M,(s-t)}^2 \end{pmatrix}, \\ c_4(B_t, B_t, E_s, E_s) &= \begin{pmatrix} K_1 \, \delta_M + K_2 \gamma_M^2 + K_1 \delta_M + K_{11} \gamma_M^2 + K_{12} \gamma_{M,(s-t)}^2 + K_{13} \delta_{M,(s-t)} + K_{13} \delta_{M,(s-t)} + K_{16} \gamma_M \delta_{M,(s-t)} \gamma_{M,(s-t)} + K_{17} \gamma_M \delta_{(s-t)} + K_{18} \gamma_M \gamma_{(s-t)}^2 \rangle. \end{split}$$

So, given that as $u \to \infty$, $\alpha_u^m = o(\gamma_{(u)})$, $m \ge 1$ and $\delta_{(u)} = o(\gamma_{(u)})$, one obtains as $(s-t) \to \infty$ and for large M

$$\begin{aligned} | c_4(A_t, A_t, C_s, E_s) | &\leq K \alpha_{s-t}^2 | \gamma_{M,(s-t)} |, & | c_4(A_t, A_t, E_s, E_s) | \leq K | \gamma_M |, \\ | c_4(A_t, B_t, C_s, D_s) | &\leq K | \gamma_{M,(s-t)} |, & | c_4(A_t, B_t, D_s, E_s) | \leq K | \delta_{M,(s-t)} |, \\ | c_4(B_t, B_t, C_s, E_s) | &\leq K | \delta_M |, & | c_4(B_t, B_t, D_s, D_s) | \leq K | \delta_M |, \\ | c_4(B_t, B_t, E_s, E_s) | &\leq K | \gamma_M |. \end{aligned}$$

The other cases follow along the same lines, so that by (41) and applying Fox and Taqqu (1987, Theorem 1 a)) $(c) = O(\eta(M)T)$, concluding the main part of the proof. Finally, given that S_T is a quadratic form in *M*-dependent variates and thus ϕ -mixing with arbitrarily fast decreasing mixing coefficient, Ibragimov and Linnik (1971, Theorem 18.5.2) applies. \Box

Lemma 14 Under Assumptions $A, B_1, B_2, B_3, B_4, B_5, B_6$, as $T \to \infty$,

$$M_T(\psi) = \frac{\partial^2}{\partial \psi \partial \psi'} Q_T(\psi) \to_{a.s.} M(\psi),$$

uniformly in $\psi \in \overline{\Psi}$.

Proof. All the convergences below hold as $T \to \infty$. $M_T(\psi) = (a) + (b) + (c) + (d)$,

with

$$(a) = 1/\left(\frac{1}{T}\sum_{j}\frac{I_{yy}(\lambda_{j})}{h_{y}(\lambda_{j};\psi)}\right)\frac{1}{T}\sum_{j}\frac{\partial^{2}}{\partial\psi\partial\psi'}h_{y}^{-1}(\lambda_{j};\psi)I_{yy}(\lambda_{j}),$$

$$(b) = -1/\left(\frac{1}{T}\sum_{j}\frac{I_{yy}(\lambda_{j})}{h_{y}(\lambda_{j};\psi)}\right)^{2}\left(\frac{1}{T}\sum_{j}\frac{\partial}{\partial\psi}h_{y}^{-1}(\lambda_{j};\psi)I_{yy}(\lambda_{j})\right)$$

$$\times\left(\frac{1}{T}\sum_{j}\frac{\partial}{\partial\psi'}h_{y}^{-1}(\lambda_{j};\psi)I_{yy}(\lambda_{j})\right),$$

$$(c) = \frac{1}{T}\sum_{j}\frac{\partial}{\partial\psi}h_{y}^{-1}(\lambda_{j};\psi)\frac{\partial}{\partial\psi'}h_{y}(\lambda_{j};\psi),$$

$$(d) = \frac{1}{T}\sum_{j}h_{y}^{-1}(\lambda_{j};\psi)\frac{\partial^{2}}{\partial\psi\partial\psi'}h_{y}(\lambda_{j};\psi),$$

where the index j in the summations ranges over $1 \le j \le T-1$. Now, for (a) and (b) by Lemma 5 and following the argument of Hannan (1973, Lemma 1)

$$(a) \to_{a.s.} \frac{1}{\left(\int_{-\pi}^{\pi} \frac{h_y(\lambda)}{h_y(\lambda;\psi)} d\lambda\right)} \left(\int_{-\pi}^{\pi} \frac{\partial^2}{\partial\psi\partial\psi'} h_y^{-1}(\lambda;\psi) h_y(\lambda) d\lambda\right),$$

$$(b) \to_{a.s} -\frac{1}{\left(\int_{-\pi}^{\pi} \frac{h_y(\lambda)}{h_y(\lambda;\psi)}\right)^2} \left(\int_{-\pi}^{\pi} g(\lambda;\psi) h_y(\lambda) d\lambda\right) \left(\int_{-\pi}^{\pi} g'(\lambda;\psi) h_y(\lambda) d\lambda\right).$$

The last two terms, (c) and (d), though not stochastic, need further attention owing to the behaviour near zero frequency of the model power spectrum. We consider the proof for (c) only, the result being valid for (d) as well given that they display the same behaviour in the neighborhood of the zero frequency (cf. Lemma 4). Following the proof of Lemma 10, let us consider

$$\frac{1}{T} \sum_{j=1}^{|T/2|} \frac{\partial h_y^{-1}(\lambda_j;\psi)}{\partial \psi} \frac{\partial h_y(\lambda_j;\psi)}{\partial \psi'} = \frac{1}{T} \sum_{j=1}^{m-1} \frac{\partial h_y^{-1}(\lambda_j;\psi)}{\partial \psi} \frac{\partial h_y(\lambda_j;\psi)}{\partial \psi'} + \frac{1}{T} \sum_{j=m}^{|T/2|} \frac{\partial h_y^{-1}(\lambda_j;\psi)}{\partial \psi} \frac{\partial h_y(\lambda_j;\psi)}{\partial \psi'} + \frac{1}{T} \sum_{j=j_{\epsilon}+1}^{|T/2|} \frac{\partial h_y(\lambda_j;\psi)^{-1}}{\partial \psi} \frac{\partial h_y(\lambda_j;\psi)}{\partial \psi'} = (c.1) + (c.2) + (c.3).$$

By Lemma 5

$$(c.3) \to \frac{1}{2\pi} \int_{\epsilon}^{\pi} \frac{\partial h_y(\lambda;\psi)^{-1}}{\partial \psi} \frac{\partial h_y(\lambda;\psi)}{\partial \psi'} d\lambda.$$

For any $i, j = 1, 2, \dots, (p+1)$, by Lemma 4, as $\lambda \to 0^+$,

$$\left|\frac{\partial h_y^{-1}(\lambda;\psi)}{\partial \psi_i}\frac{\partial h_y(\lambda;\psi)}{\partial \psi_j}\right| = O(\ln^2(\frac{1}{\lambda})),$$

yielding $|(c.1)| = O(\sum_{j=1}^{m-1} \ln^2(\lambda_j)/T) = O(\ln^2(\lambda_m)\lambda_m) = o(1)$. Finally consider $|(c.2)| \le |(c.2)'| + |(c.2)''|,$

$$|(c.2)| \leq |(c.2)|$$

setting

$$(c.2)' = \frac{\sum_{j=m}^{j_{\epsilon}} \ln^2(\lambda_j)}{T} - \frac{\int_{\lambda_m}^{\epsilon} \ln^2(\lambda) d\lambda}{2\pi}, \ (c.2)'' = \int_0^{\lambda_m} \ln^2(\lambda) d\lambda.$$

By solving the integral $|(c.2)''| = O(\lambda_m \ln^2(\lambda_m)) = o(1)$ and |(c.2)'| = o(1) given

$$|(c.2)'| \leq |\frac{\ln^2(\lambda_m)}{T}| + \frac{1}{2\pi} \sum_{j=m}^{j_{\epsilon}-1} \int_{\lambda_j}^{\lambda_{j+1}} |(\ln^2(\lambda_{j+1}) - \ln^2(\lambda))| d\lambda,$$

and bounding the second term on the right hand side above as

$$|\sum_{j=m}^{j_{\epsilon}-1} \int_{\lambda_{j}}^{\lambda_{j+1}} \left(\ln^{2}(\lambda_{j+1}) - \ln^{2}(\lambda) \right) d\lambda | \leq |\frac{2\pi}{T} \sum_{j=m}^{j_{\epsilon}-1} \left(\ln^{2}(\lambda_{j+1}) - \ln^{2}(\lambda_{j}) \right) |$$

$$= \frac{2\pi}{T} |\left(\ln^{2}(\lambda_{j_{\epsilon}-1}) - \ln^{2}(\lambda_{m}) \right) | = \frac{2\pi}{T} |\left(\ln(j_{\epsilon}/m) \ln(\lambda_{j_{\epsilon}+m-1}) \right) | = O(\frac{\ln(j_{\epsilon})}{T}),$$

where all the bounds are independent of ψ . \Box

Lemma 15 Assume A', B_1, B_2, B_3, B_4 . (i) Setting $g(\lambda) = g(\lambda; \psi_0)$,

$$\lim_{T \to \infty} T \operatorname{var} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} g(\lambda) I_{yy}(\lambda) d\lambda \right)$$
$$= \frac{1}{\pi} \int_{-\pi}^{\pi} g(\omega) g'(\omega) f_{yy}^{2}(\omega) d\omega$$
$$+ \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} g(\omega_{1}) g'(\omega_{2}) Q_{yyy}(-\omega_{1}, \omega_{2}, -\omega_{2}) d\omega_{1} d\omega_{2}$$

(ii) As $T \to \infty$

$$E\left(\frac{1}{\sigma^{8}2\pi}\int_{-\pi}^{\pi}g(\lambda)I_{yy}(\lambda)d\lambda + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial}{\partial\psi}\ln h_{y}(\lambda)d\lambda\right) = o(T^{-\frac{1}{2}}).$$
(42)

Proof. (i) Given the non-Gaussianity of the process, one obtains the two covariance terms and the fourth-order cumulant term:

$$\operatorname{var}(\frac{1}{2\pi}\int_{-\pi}^{\pi}g(\lambda)I_{yy}(\lambda)d\lambda) = (a) + (b),$$

$$(a) = \frac{1}{T^2(2\pi)^2} \sum_{\substack{t_1, s_1, t_2, s_2 \\ =1}}^T \tilde{g}_{\psi}(t_1 - s_1) \tilde{g}'_{\psi}(t_2 - s_2) \times (\gamma_{yy}(t_1 - s_2) \gamma_{yy}(t_2 - s_1) + \gamma_{yy}(t_1 - t_2) \gamma_{yy}(s_2 - s_1)) (b) = \frac{1}{T^2(2\pi)^2} \sum_{\substack{t_1, s_1, t_2, s_2 \\ =1}}^T \tilde{g}_{\psi}(t_1 - s_1) \tilde{g}'_{\psi}(t_2 - s_2) cum_{yyy}(s_1 - t_1, t_2 - t_1, s_2 - t_1),$$

where $cum_{yyy}(.,.,.)$ denotes the fourth order cumulant of y_t . With respect to (a) the result follows by Fox and Taqqu (1987, Theorem 1-(i)), setting $\alpha = -\beta$ and with respect to (b) by adapting Hosoya (1993, Lemma A4.2) with simple yet tedious calculations.

(ii) From $\int_{-\pi}^{\pi} w(\lambda; \psi) d\lambda = -\int_{-\pi}^{\pi} g(\lambda; \psi) h_y(\lambda; \psi) d\lambda$, by Parseval relation and re-arranging terms, the left hand side of (42) is equal to

$$-2\left(\sum_{u=T-1}^{\infty}\tilde{g}_{\psi}(u)\,\gamma_{yy}(u)\,+\sum_{u=1}^{T-1}\,\frac{u}{T}\tilde{g}_{\psi}(u)\gamma_{yy}(u)\right)\sim k\,L(T)\,T^{-1}\text{ as }T\to\infty,$$

using (7) and Lemma 12-(ii) where $|L(T)| = O(\ln(T))$.

Proof of Theorem 3. Assuming that the true value ψ_0 lies in the interior of Ψ and given the consistency of $\hat{\psi}_T$, by Lemma 11 and using the Delta method

$$\frac{\partial Q_T(\psi_T)}{\partial \psi} = 0 = \frac{\partial Q_T(\psi_0)}{\partial \psi} + \frac{\partial^2 Q_T(\psi)}{\partial \psi \partial \psi'} (\hat{\psi}_T - \psi_0),$$

where $\tilde{\psi}$ is such that $\tilde{\psi} = \psi_0 + R(\hat{\psi}_T - \psi_0)$ with R being $(p+1) \times (p+1)$ matrix such that $\| \tilde{\psi} - \psi_0 \| \leq \| \hat{\psi}_T - \psi_0 \|$. The result follows combining Lemmas 11-15, yielding for $T \to \infty$

$$M_T(\psi) \to_{a.s.} M(\psi)$$
, uniformly for $\psi \in \overline{\Psi}$,
 $T^{\frac{1}{2}} \frac{\partial Q_T(\psi_0)}{\partial \psi} \to_d \mathcal{N}_{p+1}(0, V)$. \Box

C Asymptotic properties of the Gaussian estimator: ARMA levels

In this section we show how to adapt the results in appendixes B.2 and B.3 for estimation of the nonlinear MA parameters ψ , using the squared estimated residuals obtained from preliminary estimation of the ARMA parameters ζ .

At first, we need to establish the asymptotic properties of the Whittle estimator of the ARMA parameters.

Proof of Theorem 5. This follows adapting Hannan (1973) (see also Brockwell and Davis (1987, section 10.8) for more details). For $1 + \delta_{\alpha\alpha}(0) > 0$ and

under Assumption A, the u_t are white noise not degenerate, and the strong consistency proof of Hannan (1973, Theorem 1) applies. For the asymptotic distribution one gets:

$$\left[-\frac{\partial^2 P_T(\tilde{\zeta})}{\partial \zeta \partial \zeta'}\right] T^{\frac{1}{2}}(\hat{\zeta}_T - \zeta_0) = T^{\frac{1}{2}} \frac{\partial P_T(\zeta_0)}{\partial \zeta},$$

for T sufficiently large such that the matrix in square brackets is nonsingular, where $\tilde{\zeta}$ satisfies $\|\tilde{\zeta} - \zeta_0\| \leq \|\hat{\zeta}_T - \zeta_0\|$.

All the convergences below hold as $T \to \infty$. By the same arguments of Hannan (1973, Lemma 1) and simple manipulations

$$\frac{\partial^2 P_T(\tilde{\zeta})}{\partial \zeta \partial \zeta'} \to_{a.s} \gamma_{uu}(0) F,$$

with F defined in (22). Next, we need to show that

$$T^{\frac{1}{2}} \frac{\partial P_T(\zeta_0)}{\partial \zeta} \to_d N_{r+q}(0, \gamma_{uu}^2(0) G),$$

with G defined in (23). Exploiting the martingale difference property of the u_t (see Hannan (1973, Lemma 5)) we just need to establish the CLT for

$$\frac{2\pi}{T^{\frac{1}{2}}}\sum_{j=1}^{T-1}I_{uu}(\lambda_j)b(\lambda_j),$$

where $b(\lambda)$ is defined in (24). The u_t are not conditionally homoskedastic, violating (6) of Hannan (1973)). However, by a truncation argument precisely along the lines of Lemma 13, we approximate the u_t with $u_t^{(N)} = \epsilon_t(\rho + \sum_{j=1}^N \alpha_j \epsilon_{t-j})$ for some integer $N < \infty$. By the N-dependence of the $u_t^{(N)}$

$$\frac{2\pi}{T^{\frac{1}{2}}} \sum_{j=1}^{T-1} I_{u^{(N)}u^{(N)}}(\lambda_j) b(\lambda_j) \to_d N_{r+q}(0,\bar{V}),$$

for some positive semi definite matrix $\bar{V} = \bar{V}(N)$ and, with simple yet tedious calculations,

$$\operatorname{var}\left(\frac{2\pi}{T^{\frac{1}{2}}}\sum_{j=1}^{T-1} (I_{u^{(N)}u^{(N)}}(\lambda_j) - I_{uu}(\lambda_j))b(\lambda_j)\right) = O(\eta(N)),$$

for some function $0 < \eta(N) < \infty$ with $\eta(N) \downarrow 0$ as $N \to \infty$. Note, though, that the special structure (2) of the u_t implies a form of the asymptotic variance-covariance matrix of the $\hat{\zeta}_T$ that is different from the conditionally

homoskedastic case of Hannan (1973). In fact, setting $\tilde{b}_{\zeta}(u) = 1/2\pi \int_{-\pi}^{\pi} b(\lambda) e^{-iu\lambda} d\lambda \ (u = 0, \pm 1, ...),$

$$\operatorname{var}\left(\frac{2\pi}{T^{\frac{1}{2}}}\sum_{j=1}^{T-1}I_{uu}(\lambda_{j})b(\lambda_{j})\right)$$

$$=\frac{1}{T}\sum_{h_{1},h_{2}=-\infty}^{\infty}\tilde{b}_{\zeta}(h_{1})\tilde{b}_{\zeta}(h_{2})'\sum_{s_{1}=1}^{T-|h_{1}|}\sum_{s_{2}=1}^{T-|h_{2}|}E(u_{s_{1}}u_{s_{1}+h_{1}}u_{s_{2}}u_{s_{2}+h_{2}})+o(1)$$

$$=2\sum_{h=-T+1}^{T-1}\tilde{b}_{\zeta}(h)\tilde{b}_{\zeta}(h)'(1-\frac{|h|}{T})E(u_{s}^{2}u_{s+h}^{2})+o(1)$$

$$=2\gamma_{uu}^{2}(0)\sum_{h=-T+1}^{T-1}(1-\frac{|h|}{T})\tilde{b}_{\zeta}(h)\tilde{b}_{\zeta}(h)'$$

$$+2\sum_{h=-T+1}^{T-1}(1-\frac{|h|}{T})\tilde{b}_{\zeta}(h)\tilde{b}_{\zeta}(h)'\gamma_{u^{2}u^{2}}(h)+o(1),$$

given that the contribution of $E(u_{s_1}u_{s_1+h_1}u_{s_2}u_{s_2+h_2})$ is non zero only for cases $s_1 = s_2, h_1 = h_2$ and $s_1 = s_2 + h_2, s_2 = s_1 + h_1$ using $1/2\pi \int_{-\pi}^{\pi} b(\lambda)d\lambda = \tilde{b}_{\zeta}(0) = 0$ (vector of zeros) and $\tilde{b}_{\zeta}(-h) = \tilde{b}_{\zeta}(h)$. The last equality above is obtained writing $E(u_s^2u_{s+h}^2) = \gamma_{uu}^2(0) + \gamma_{u^2u^2}(h)$. The standard case of independent u_t , or more generally with conditional homoskedasticity is obtained when $\gamma_{u^2u^2}(u) = 0$ for $u \neq 0$.

Thus,

$$\operatorname{var}\left(\frac{2\pi}{T^{\frac{1}{2}}}\sum_{j=1}^{T-1}I_{uu}(\lambda_{j})b(\lambda_{j})\right) \to 2\gamma_{uu}^{2}(0)\sum_{h=-\infty}^{\infty}\tilde{b}_{\zeta}(h)\tilde{b}_{\zeta}(h)'$$
$$+2\sum_{h=-\infty}^{\infty}\tilde{b}_{\zeta}(h)\tilde{b}_{\zeta}(h)'\gamma_{u^{2}u^{2}}(h)$$
$$=\frac{\gamma_{uu}^{2}(0)}{\pi}\int_{-\pi}^{\pi}b(\lambda)b(\lambda)'d\lambda + \frac{1}{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f_{u^{2}u^{2}}(\lambda_{1})b(\lambda_{2})b(\lambda_{1}-\lambda_{2})'d\lambda_{1}d\lambda_{2}. \ \Box$$

Proof of Theorem 6. All the limits below are taken for $T \to \infty$. For consistency, we just need to show that

$$\sup_{\psi \in \bar{\Psi}} | Q_T(\psi) - \tilde{Q}_T(\psi) |_{\to_{a.s}} 0, \tag{43}$$

so that Theorem 2 applies. For this purpose, given the smoothness of the $\pi_i(\zeta)$ and using the mean value theorem

$$\hat{u}_t = u_t + \sum_{j=1}^{t-1} (\pi_j(\zeta_0) - \pi_j(\hat{\zeta}_T)) x_{t-j} + \sum_{j=t}^{\infty} \pi_j(\zeta_0) x_{t-j}$$

$$= u_t + (\zeta_0 - \hat{\zeta}_T)'\tilde{v}_t + B_t,$$

setting $\tilde{v}_t = \sum_{j=1}^{t-1} \vartheta_j(\tilde{\zeta}) x_{t-j}$, for some $\tilde{\zeta}$ satisfying $\|\tilde{\zeta} - \zeta_0\| \leq \|\hat{\zeta}_T - \zeta_0\|$, and $B_t = \sum_{j=0}^{\infty} \pi_{j+t}(\zeta_0) x_{-j}$. Thus

$$\hat{u}_t^2 = u_t^2 + ((\zeta_0 - \hat{\zeta}_T)'\tilde{v}_t)^2 + B_t^2 + 2u_t(\zeta_0 - \hat{\zeta}_T)'\tilde{v}_t + 2B_t(\zeta_0 - \hat{\zeta}_T)'\tilde{v}_t + 2u_tB_t.$$
(44)

By Theorem 5, (44) and $|B_t| = O(a^t)$, $t \to \infty$ a.s. for some 0 < a < 1 one gets

$$I_{\hat{u}^{2}\hat{u}^{2}}(\lambda) = I_{u^{2}u^{2}}(\lambda) + (\zeta_{0} - \hat{\zeta}_{T})' 4\mathcal{R}e(I_{u^{2}u\tilde{v}}(\lambda)) + o(T^{-\frac{1}{2}})R_{T}(\lambda) \quad a.s., \ -\pi \leq \lambda < \pi_{2}$$
(45)

for some real random variable $R_T(\lambda)$ $(-\pi \leq \lambda < \pi)$, function of the u_t , B_t and \tilde{v}_t $(1 \leq t \leq T)$ satisfying, by the same arguments of Hannan (1973, Lemma 1),

$$\frac{2\pi}{T} \sum_{j=1}^{T-1} R_T(\lambda) \to_{a.s.} \int_{-\pi}^{\pi} R(\lambda) d\lambda,$$

for some real integrable (by Assumption A') function $R(\lambda)$. Likewise,

$$\frac{4}{T}\sum_{j=1}^{T-1} \mathcal{R}e(I_{u^2\,u\tilde{v}}(\lambda_j)) \to_{a.s.} \frac{2}{\pi} \int_{-\pi}^{\pi} \mathcal{R}e(f_{u^2\,uv}(\lambda))d\lambda,$$

with a finite limit (by Assumption A'), where we recall that, given $\lambda_j = 2\pi/T$,

$$I_{u^2 u \tilde{v}}(\lambda_j) = \frac{1}{2\pi T} \sum_{t,s=1}^{T-1} u_t^2 u_s \tilde{v}_s e^{i(t-s)\lambda_j}, \ 1 \le j \le T-1.$$

Using $\ln(1 + a/b) \leq a/b$ for any positive a, b, and (45), Lemma 6 and the consistency part of Theorem 5 then

$$|Q_{T}(\psi) - Q_{T}(\psi)|$$

$$\leq \frac{|\zeta_{0} - \hat{\zeta}_{T}|'}{\hat{\sigma}_{T}^{8}(\psi)} \frac{4}{T} \sum_{j=1}^{T-1} \frac{|\mathcal{R}e(I_{u^{2}u\tilde{v}}(\lambda_{j}))|}{h_{y}(\lambda_{j};\psi)} + \frac{o(|\zeta_{0} - \hat{\zeta}_{T}|')}{\hat{\sigma}_{T}^{8}(\psi)} \frac{\frac{1}{T} \sum_{j=1}^{T-1} |R_{T}(\lambda_{j})|}{h_{y}(\lambda_{j};\psi)}$$

$$\leq \frac{1}{\delta \hat{\sigma}_{T}^{8}(\psi)} o\left(\frac{4}{T} \sum_{j=1}^{T-1} |\mathcal{R}e(I_{u^{2}u\tilde{v}}(\lambda_{j}))| + \frac{1}{T} \sum_{j=1}^{T-1} |R_{T}(\lambda_{j})|\right) \quad a.s.,$$

where the term in square brackets does not depend on ψ and $\inf_{\psi \in \bar{\Psi}} \hat{\sigma}_T^8(\psi) > 0$, given $\sup_{\psi \in \bar{\Psi}} h_y(\lambda; \Psi) < \infty$ for $0 < \lambda \leq \pi$ by Lemma 5.

For the asymptotic distribution result we just need to look at the asymptotic distribution of the normalized score evaluated at ψ_0 , which satisfies

$$T^{\frac{1}{2}} \frac{\partial \tilde{Q}_{T}(\psi_{0})}{\partial \psi} = \frac{\hat{\sigma}_{T}^{8}(\psi_{0})}{\tilde{\sigma}_{T}^{8}(\psi_{0})} T^{\frac{1}{2}} \frac{\partial Q_{T}(\psi_{0})}{\partial \psi} + \frac{1}{\tilde{\sigma}_{T}^{8}(\psi_{0})} \frac{4}{T} \sum_{j=1}^{T-1} g(\lambda_{j};\psi_{0}) \mathcal{R}e(I_{u^{2}}_{u\tilde{v}'}(\lambda_{j})) T^{\frac{1}{2}}(\zeta_{0} - \hat{\zeta}_{T}) + o(1) \quad a.s.$$

by (45). Given that

$$\frac{4}{T}\sum_{j=1}^{T-1}g(\lambda_j;\psi_0)\mathcal{R}e(I_{u^2\,u\tilde{v}'}(\lambda_j)) \to_{a.s.} H(\psi_0), \quad \frac{\hat{\sigma}_T^8(\psi_0)}{\tilde{\sigma}_T^8(\psi_0)} \to_{a.s.} 1,$$

the normalized score is asymptotically equivalent to

$$T^{\frac{1}{2}} \frac{\partial Q_T(\psi_0)}{\partial \psi} + H(\psi_0) T^{\frac{1}{2}}(\zeta_0 - \hat{\zeta}_T) + o(1), \ a.s.$$

By Theorem 3 and 5 then

$$T^{\frac{1}{2}} \frac{\partial Q_T(\psi_0)}{\partial \psi} \to_d \mathcal{N}_{p+1}(0, V + H(\psi_0) F^{-1} G F^{-1} H(\psi_0)' + H(\psi_0) Z + Z' H(\psi_0)'),$$

setting

$$Z = \lim_{T \to \infty} T \operatorname{cov} \left((\zeta_0 - \hat{\zeta}_T), \frac{\partial Q_T(\psi_0)}{\partial \psi'} \right).$$

It turns out that Z = 0 under Assumption A'. In fact, we just need to look at

$$T \operatorname{cov}\left(\frac{1}{T} \sum_{j_1=1}^{T-1} I_{uu}(\lambda_{j_1}) b(\lambda_{j_1}), \frac{1}{T} \sum_{j_2=1}^{T-1} I_{u^2 u^2}(\lambda_{j_2}) g(\lambda_{j_2})'\right)$$
$$= \frac{1}{T} \sum_{t_1, t_2, s_1, s_2=1}^{T} \tilde{b}_{\zeta}(t_1 - t_2) \tilde{g}_{\psi}'(s_1 - s_2) \operatorname{cov}(u_{t_1} u_{t_2}, u_{s_1}^2 u_{s_2}^2) + o(1).$$

Given that $\tilde{b}_{\zeta}(0) = 0$ the only non zero terms would be $\tilde{b}_{\zeta}(t-s) \tilde{g}'_{\psi}(t-s)E(u_t^3 u_s^3)$ $(1 \le t \ne s \le T)$ and $\tilde{b}_{\zeta}(t-s) \tilde{g}'_{\psi}(0)E(u_t^5 u_s)$ $(1 \le s < t \le T)$. However, $E(\epsilon_t^3) = E(\epsilon_t^5) = 0$ by Assumption A' yielding $E(u_t^3 u_s^3) = E(u_t^5 u_s) = 0$ for $t \ne s$. \Box

D Asymptotic distribution of the LM test under H_0

In this section we derive the asymptotic distribution of the LM test statistics Φ (cf. (14)) under H_0 . Finally, we will introduce a robustified LM test statistic and show that it shares the same asymptotic distribution of Φ under H_0 . By Assumption A

$$E(y_t|\mathcal{F}_{t-1}) = \mu^2 + \sigma^2 h_{t-1}^2, \quad \operatorname{var}(y_t|\mathcal{F}_{t-1}) = 2\left(\sigma^4 h_{t-1}^4 + 2\,\sigma^2 \mu^2 \,h_{t-1}^2\right),$$

where \mathcal{F}_{t-1} expresses the sigma-algebra generated by the $\{\epsilon_s, s \leq t-1\}$. The (time domain) log likelihood, under the pretence that the y_t are Gaussian, is

$$\begin{aligned} Q_T^{time}(\sigma^2, w, \theta) &= -\frac{T}{2} \ln(4\pi) - T \ln(\sigma^2) \\ &- \frac{1}{2} \ln\left(\prod_{t=1}^T 2(h_{t-1}^4(\theta) + 2w h_{t-1}^2(\theta))\right) \\ &- \frac{1}{4} \sum_{t=1}^T (\frac{y_t}{\sigma^2} - w - h_{t-1}^2(\theta))^2 / (h_{t-1}^4(\theta) + 2w h_{t-1}^2(\theta)), \end{aligned}$$

where $w = \mu^2 / \sigma^2$. Expression (14) follows readily by applying the LM principle to $Q_T^{time}(\cdot, \cdot, \cdot)$. For an integer *m* with m < T set

$$\phi_m = \frac{1}{(T \, e(w) \, \sigma^2)^{\frac{1}{2}}} \Gamma_m^{\frac{1}{2}} \sum_{i=1}^m \tau_i \, \eta_i$$

obtained by substituting T - 1 with m and $\hat{\sigma}^2$, \hat{w} with σ^2 , w in $\phi_{T-1}(\hat{\sigma}^2, \hat{w})$ (cf. (15)) and writing $\eta_i = \eta_i(\sigma^2, w)$.

Proof of Theorem 4. This follows adapting the proof of Robinson (1991, Theorem 2.1). Setting $\phi_m = (T e(w) \sigma^2)^{-\frac{1}{2}} \Gamma^{-\frac{1}{2}} \sum_{i=1}^m \tau_i \bar{\eta}_i$, $\bar{\eta}_i = \sum_{t=1}^T \epsilon_{t-i} X_t$, from $E(X_t \epsilon_{t-i} | \mathcal{F}_{t-1}) = 0$, for any $i \geq 1$ it follows that $E \parallel \sum_{i=1}^m \tau_i \tilde{\eta}_i \parallel^2 = O(m^2)$ with $\tilde{\eta}_i = \bar{\eta}_i - \eta_i = \sum_{t=1}^i \epsilon_{t-i} X_t$, yielding $\phi_m - \phi_m = o_p(1)$. Under H_0 the covariance matrix of the score is block-diagonal with respect to (σ^2, w) on one hand and θ on the other, yielding the usual expression for the LM test statistic (14). Note that by square summability of the τ_i , $\lim_{t\to\infty} \sum_{i=t}^\infty \tau_i \tau_i' = 0$, the objective function $Q_T^{time}(\cdot, \cdot, \cdot)$ and the conditional objective function, obtained substituting the ϵ_t with the $\bar{\epsilon}_t = \epsilon_t \mathbf{1}_{t>0}$, yield asymptotically equivalent scores. Further, from simple but tedious calculations, under H_0

$$E\left(\frac{\partial Q_T^{time}(\sigma^2, w, 0)}{\partial \theta} \frac{\partial Q_T^{time}(\sigma^2, w, 0)}{\partial \theta'}\right) = \frac{e(w)}{(1+2w)^2} \sigma^2 \Gamma_{T-1},$$

and $E(X_t^2 | \mathcal{F}_{t-1}) = e(w)$. Under Assumptions *C*, in particular C_2 and *D*, a martingale CLT (see Brown (1971)) applies, yielding, as $T \to \infty$ and finite m,

$$\bar{\phi_m} \to_d \mathcal{N}_p(0, I_p).$$

Under H_0 , $y_t^2 - E(y_t^2)$ and $y_t - E(y_t)$ are stationary square integrable martingale differences and thus $\hat{\sigma^2} \rightarrow_p \sigma^2$ and $\hat{w} \rightarrow_p w$, as $T \rightarrow \infty$ by Slutzky lemma (\rightarrow_p denoting convergence in probability), yielding

$$\eta_i(\hat{\sigma}^2, \hat{w}) - \eta_i = \sum_{t=i+1}^T \left((x_{t-i} - \hat{\mu}) X_t(\hat{\sigma}^2, \hat{w}) - (x_{t-i} - \mu) X_t \right) = o_p(T^{\frac{1}{2}}).$$

Finally from $E(\bar{\eta}_i \, \bar{\eta}_{i+j}) = e(w) \, T \, \sigma^2 \delta(0, j), \, i \neq 0$ we get as $m \to \infty$

$$E \parallel \sum_{i=m}^{T-1} \tau_i \bar{\eta_i} \parallel^2 \le Te(w) \, \sigma^2 \, \sum_{i=m}^{\infty} \parallel \tau_i \parallel^2 = o(T)$$

and Bernstein's Lemma (Hannan 1970, p.242) applies. \Box

D.1 The robustified LM test

We relax Assumption D as follows: Assumptions D'.

$$\begin{split} D_1' \, E(\epsilon_t | \mathcal{F}_{t-1}) &= 0, \ E(\epsilon_t^2 | \mathcal{F}_{t-1}) = \sigma^2, \ E[(\epsilon_t^2 - \sigma^2)\epsilon_t^2 | \mathcal{F}_{t-1}] = 2\sigma^4 \\ D_2' \, E(\epsilon_1^{16}) &< \infty \,. \end{split}$$

$$\begin{split} D_3' \ For \ k, r &\geq 1 \\ E(X_1 X_{1+i} \epsilon_{1-k} \epsilon_{1-r}^a) &= \begin{cases} 0, & a = 0, \\ 0, & i \geq 1, \\ f_k, & i = 0, \ k = r, \end{cases} \quad a = 1. \end{split}$$

$$\begin{split} D_4' \ \inf_{i \geq 1} \ E(X_1^2 \epsilon_{1-i}^2) &> 0 \,. \end{aligned}$$

$$\begin{split} D_5' \ \frac{1}{T} \sum_{t=1}^T \epsilon_{t-i} \epsilon_{t-j} E(X_t^2 | \mathcal{F}_{t-1}) \to_{a.s.} \ f_i \delta(i, j), \ as \ T \to \infty. \end{split}$$

Assumptions D' allow a great deal of heterogeneity in the ϵ_t . In fact we only need the ϵ_t to have constant conditional moment up to the fourth order, the trade off being the strict unconditional moment condition D'_2 . Assumption D'_3 expresses the minimal degree of stationarity required in the ϵ_t and Assumption D'_4 guarantees that the distribution of the score is non-singular asymptotically. Assumption D'_5 is a mild ergodicity condition. The robustified test statistic is

$$\Phi_R = \phi'_R \phi_R,\tag{46}$$

with $\phi_R = (T\hat{\Gamma}_R)^{-\frac{1}{2}} \sum_{i=1}^{T-1} \tau_i \eta_i(\hat{\sigma^2}, \hat{w})$ setting $\hat{\Gamma}_R = \sum_{i=1}^{T-1} \tau_i \tau'_i \bar{f}_i(\sigma^2, w)$, $\bar{f}_i(\sigma^2, w) = \frac{1}{T} \sum_{t=i+1}^T X_t^2(\sigma^2, w)(x_{t-i} - \bar{x})^2$. A frequency domain expression can be readily obtained. The considerations made for Φ concerning the computational gains are even more important for the Φ_R statistic, owing to the $\bar{f}_i(\sigma^2, w)$. We now establish the asymptotic distribution of Φ_R under H_0 .

Theorem 7 Under the null hypothesis H_0 and Assumptions C and D', as $T \to \infty$,

$$\Phi_R \to_d \chi_p^2$$
.

Proof. The proof closely follows that of Theorem 4 and therefore it will be sketched only. The block-diagonality of the variance-covariance matrix of the score and the substitution of the $\bar{\eta}_i$ with η_i follow from Assumptions D'_1, D'_2 . Then by Assumptions D'_1, D'_3, D'_5 we obtain that, as $T \to \infty$,

$$\frac{1}{T^{\frac{1}{2}}} \sum_{i=1}^m \tau_i \bar{\eta_i} \to_d \mathcal{N}_p(0, \sum_{i=1}^m \tau_i \tau_i f_i) \,.$$

Assumption D'_4 guarantees that the matrix Γ_R is invertible. Assumption D'_1 yields $X_t - X_t(\hat{\sigma}^2, \hat{w}) = o_p(1)$ and thus, for fixed m and $T \to \infty$, $\sum_{i=1}^m \tau_i \tau'_i(\bar{f}_i - \bar{f}_i(\hat{\sigma}^2, \hat{w})) \to_p 0$, and $\sum_{i=1}^m \tau_i \tau'_i(\bar{f}_i - f_i) \to_p 0$ by D'_5 and $E|\bar{f}_i| = O(1), E|\bar{f}_i(\hat{\sigma}^2, \hat{w})| = O(1)$ by D'_2 . Thus, as m (and then T) goes to infinity,

$$\sum_{i=m}^{T-1} \tau_i \tau'_i(E|\bar{f}_i| + E|\bar{f}_i(\hat{\sigma^2}, \hat{w})| + f_i) \to_p 0.\Box$$

E A feasible parameterization: the ARFIMA(p, d, q) coefficients

In this section we show that the parameterization (9) employed in the empirical application of section 6 satisfies Assumptions *B*. Let us start by considering the case p = q = 0, $d(\theta) = \theta$, setting

$$c_i(\theta) = \begin{cases} \prod_{k=1}^{i} \frac{k-1+\theta}{k} & , i \ge 1, \\ 1 & , i = 0, \end{cases}$$
(47)

and $\alpha_i(\theta) = c_i(\theta)$, with $\alpha(\lambda) = (1 - e^{i\lambda})^{-\theta} (-\pi \le \lambda < \pi)$. Using Stirling's formula (Brockwell and Davis 1987, p.522), $\alpha_i(\theta) \sim k i^{\theta-1}$, as $i \to \infty$, and

 B_1 holds. From (47) one gets $\alpha_i(\theta) - \alpha_{i+1}(\theta) = \alpha_i(\theta) \left(\frac{1-\theta}{i+1}\right)$ and thus B_2 is trivially satisfied. Concerning B_3 , $\ln \alpha_i(\theta) = \sum_{k=1}^i \ln \left((k-1+\theta)/k\right)$. Differentiating yields $\frac{\partial}{\partial \theta} \ln \alpha_i(\theta) = \sum_{k=1}^i \ln \left(\frac{1}{k-1+\theta}\right)$. As $n \to \infty$, $\sum_{k=1}^n \frac{1}{k} \sim \ln(n)$, yielding $\frac{\partial}{\partial \theta} \alpha_i(\theta) \sim k \, i^{\theta-1} \ln(i)$ as $i \to \infty$. Likewise from $\frac{\partial}{\partial \theta} \ln \alpha_i(\theta) - \frac{\partial}{\partial \theta} \ln \alpha_{i+1}(\theta) = \ln(i+\theta)$, as $i \to \infty$,

$$\frac{\partial}{\partial \theta} \alpha_i(\theta) - \frac{\partial}{\partial \theta} \alpha_{i+1}(\theta) = \frac{1-\theta}{i+\theta} \frac{\partial}{\partial \theta} \alpha_{i+1}(\theta) + \alpha_i(\theta) \ln (i+\theta) \sim k \alpha_i(\theta) \ln (i),$$

and thus B_4 holds. By an identical argument it can be shown that Assumptions B_5 and B_6 are satisfied. Finally, regarding B_7 , imposing the condition $\alpha_i(\theta) = \alpha_i(\theta^*)$, for any integer $i \ge 1$ yields

$$(\theta^{i} - (\theta^{*})^{i})c_{i} + (\theta^{i-1} - (\theta^{*})^{i-1})c_{i-1} + \ldots + (\theta - \theta^{*})c_{1} = 0,$$

where the coefficients c_j (j = 1, ..., i) are positive functions of the integers 1, 2, ..., i only. Hence the above condition is true if and only if $\theta = \theta^*$. B_8 follows stacking $\partial \bar{\gamma}_{yy}(u; \psi) / \partial \psi$, for u = 0, 1, 2, in a 3×3 matrix and showing, by algebraic calculations, that it is nonsingular. These arguments can be easily extended to the case p, q > 0, setting:

$$\alpha_i(\theta) = \phi_1 \alpha_{i-1}(\theta) + ... + \phi_p \alpha_{i-p}(\theta) + c_i(d) + \chi_1 c_{i-1}(d) + ... + \chi_q c_{i-q}(d),$$

where $\theta = (\chi_1, ..., \chi_q, \phi_1, ..., \phi_p, d)'$.



Figure 1: Correlograms for stock index returns series, levels and squares (first and second row), and foreign exchange rate returns, levels and squares (third and fourth row).



						1.1					
					Sample	e size: 1024					
Parameter	Value	Mean	Std.Dev.	Parameter	Value	Mean	Std.Dev.	Parameter	Value	Mean	Std.Dev.
$\bar{\kappa}$	0	-0.0888	0.0216	$ar{\kappa}$	0	-0.2370	0.0211	$ar{\kappa}$	0	-0.0828	0.0242
ϕ_1	-0.8	-0.7609	0.0030	ϕ_1	-0.8	-0.7530	0.0030	ϕ_1	-0.8	-0.7621	0.0039
d	0.15	0.1430	0.0043	d	0.25	0.1767	0.0045	d	0.45	0.3411	0.0045
$\bar{\kappa}$	0	-0.1695	0.0221	$\bar{\kappa}$	0	-0.1176	0.0247	$\bar{\kappa}$	0	0.0466	0.0258
ϕ_1	-0.4	-0.3323	0.0065	ϕ_1	-0.4	-0.3531	0.0064	ϕ_1	-0.4	-0 2604	0.0097
d^{φ_1}	0.15	0.0850	0.0023	d^{φ_1}	0.25	0.2017	0.0031	d^{φ_1}	0.45	0 4041	0.0030
u	0.10	0.0000	0.0020	u	0.20	0.2011	0.0001	u	0.40	0.4041	0.0000
Ţ.	0	0.1759	0.0257	-	0	0 1651	0.0947	-	0	0 9909	0.0927
κ	0	-0.1752	0.0237	κ	0	-0.1001	0.0247	κ	0	0.2292	0.0237
ϕ_1	0.4	0.3475	0.0100	ϕ_1	0.4	0.3640	0.0106	ϕ_1	0.4	0.3782	0.0123
d	0.15	0.0867	0.0027	d	0.25	0.1556	0.0045	d	0.45	0.3066	0.0050
$ar{\kappa}$	0	0.0975	0.0242	$ar{\kappa}$	0	0.0764	0.0232	$ar\kappa$	0	0.1831	0.0212
ϕ_1	0.8	0.7427	0.0054	ϕ_1	0.8	0.7606	0.0060	ϕ_1	0.8	0.7742	0.0073
d	0.15	0.1203	0.0038	d	0.25	0.1711	0.0045	d	0.45	0.33906	0.0047
	0.20	000			0.20	0.2	0.00-0		0.10	0.00000	0.00
\overline{k}	15	1 5800	0.0586	\overline{k}	15	1.0074	0.0568	\overline{k}	15	1 5100	0.0648
n t	1.0	0.7611	0.0000	n L	1.5	0.7650	0.0000	n t	1.0	0.7425	0.0040
φ_1	-0.8	-0.7011	0.0042	φ_1	-0.8	-0.7030	0.0041	φ_1	-0.8	-0.7455	0.0005
d	0.15	0.1257	0.0046	d	0.25	0.1536	0.0048	d	0.45	0.3048	0.0054
$ar{\kappa}$	1.5	1.3448	0.0603	$ar{\kappa}$	1.5	0.8106	0.0596	$ar{\kappa}$	1.5	1.3842	0.0638
ϕ_1	-0.4	-0.3376	0.0067	ϕ_1	-0.4	-0.3440	0.0070	ϕ_1	-0.4	-0.2270	0.00110
d	0.15	0.0769	0.0025	d	0.25	0.1826	0.0033	d	0.45	0.3862	0.0037
			'								
$\bar{\kappa}$	1.5	1.0881	0.0679	$\bar{\kappa}$	1.5	0.9772	0.0635	$\bar{\kappa}$	1.5	1.5241	0.0638
φ1	0.4	0.3505	0.0111	φ1	0.4	0 4013	0.0091	φ1	0.4	0 4078	0.0120
φ_1	0.15	0.0761	0.0111	φ_1	0.1	0.1010	0.0001	φ_1	0.1	0.2050	0.0120
u	0.15	0.0701	0.0021	u	0.20	0.1431	0.0041	u	0.40	0.2950	0.0000
_	1 5	1 40 40	0.0004	_	1 5	1 45 45	0.0505	_	1 5	1.0754	0.0505
κ	1.5	1.4940	0.0634	κ	1.5	1.4547	0.0585	κ	1.5	1.6754	0.0525
ϕ_1	0.8	0.7155	0.0068	ϕ_1	0.8	0.7353	0.0074	ϕ_1	0.8	0.7576	0.0085
d	0.15	0.1232	0.0036	d	0.25	0.1723	0.0042	d	0.45	0.3470	0.0046
$ar{\kappa}$	3.0	2.8591	0.0951	$ar{\kappa}$	3.0	2.4178	0.0975	$ar\kappa$	3.0	2.7228	0.1042
ϕ_1	-0.8	-0.7579	0.0046	ϕ_1	-0.8	-0.7514	0.0047	ϕ_1	-0.8	-0.7341	0.0067
d	0.15	0.1166	0.0044	d	0.25	0.1489	0.0050	d	0.45	0.2951	0.0056
			1								
Ē	3.0	2 8950	0 0927	Ē	3.0	2 3159	0.0927	Ē	3.0	2.2586	0.0987
d.	0.0	0.3571	0.0021	d.	0.4	0.3671	0.0021	d-	0.0	0.2525	0.0307
φ_1	-0.4	-0.3371	0.0009	φ_1	-0.4	-0.3071	0.0001	φ_1	-0.4	-0.2323	0.0108
a	0.15	0.0708	0.0026	a	0.25	0.1819	0.0039	a	0.45	0.3704	0.0040
			o 40-0 I	1							
$\bar{\kappa}$	3.0	1.9126	0.1070	$ar{\kappa}$	3.0	2.0682	0.1054	$ar{\kappa}$	3.0	2.4803	0.1031
ϕ_1	0.4	0.3755	0.0091	ϕ_1	0.4	0.4036	0.0095	ϕ_1	0.4	0.4052	0.0123
d	0.15	0.0797	0.0032	d	0.25	0.1406	0.0043	d	0.45	0.2830	0.0053
$ar{\kappa}$	3.0	2.4853	0.0975	$\bar{\kappa}$	3.0	2.5792	0.0942	$\bar{\kappa}$	3.0	2.4803	0.1031
ϕ_1	0.8	0.7212	0.0067	ϕ_1	0.8	0.7471	0.0067	ϕ_1	0.8	0.7642	0.0075
d	0.15	0.1191	0.0033	d	0.25	0.1748	0.0043	d	0.45	0.3592	0.0044
Parama	torizot	$an a (I \cdot A)$	(1 I)	-d(1 + 1)	<u>)-1</u>	Design: a^2	$-1 \sigma^2 - 1$	u = 0	0.10	0.000-	
_ i arame	uerizati	$\alpha(L; \sigma)$	j = (1 - L)	$(1 - \psi_1)$., .L	p = sign: p^{-1} :	_ 1, 0	$\mu = 0.$			

Table 1.1

					Samala	<u>1.4</u> 					
D	37.1	M	CULD.	D	Sample	size: 2040	CLID.	D	37.1	14.	CLID.
Parameter		0.0621	0.0107	Parameter		0 1092	0.0106	Parameter	Value	0.0544	0.0214
n d	0 8	-0.0031	0.0197	n d	0	-0.1920 0.7710	0.0190	n d	0	-0.0344	0.0214 0.0041
φ_1	-0.8	-0.1113	0.0021	φ_1	-0.8	-0.1056	0.0019	φ_1	-0.8	-0.7854	0.0041
a	0.15	0.1480	0.0038	a	0.25	0.1850	0.0041	a	0.45	0.3814	0.0041
_	0	0 1 4 0 -	0.000	_	0	0 1055	0.0051	_	0	0.0005	0.0050
κ	0	-0.1467	0.0207	κ	0	-0.1257	0.0251	κ	0	-0.0665	0.0253
ϕ_1	-0.4	-0.3445	0.0056	ϕ_1	-0.4	-0.3481	0.0051	ϕ_1	-0.4	-0.3109	0.0077
d	0.15	0.0943	0.0037	d	0.25	0.2186	0.0021	d	0.45	0.4161	0.0025
$\bar{\kappa}$	0	-0.2168	0.0246	$\bar{\kappa}$	0	-0.1452	0.0227	$ar{\kappa}$	0	0.1079	0.0225
ϕ_1	0.4	0.3695	0.0078	ϕ_1	0.4	0.3797	0.0073	ϕ_1	0.4	0.3706	0.0110
d	0.15	0.1086	0.0023	d	0.25	0.1933	0.0029	d	0.45	0.3475	0.0041
			'				I				
$ar{\kappa}$	0	0.0902	0.0218	$\bar{\kappa}$	0	0.0425	0.0201	$\bar{\kappa}$	0	0.1061	0.0192
ϕ_1	0.8	0.7663	0.0043	ϕ_1	0.8	0.7737	0.0053	ϕ_1	0.8	0.7814	0.0062
d	0.15	0.1227	0.0031	d	0.25	0.1774	0.0038	d	0.45	0.3591	0.0040
a	0.10	0.1221	0.0001	a	0.20	0.1111	0.0000	cu -	0.10	0.0001	0.0010
$\bar{\kappa}$	15	1 3265	0.0552	$\bar{\kappa}$	15	0.9625	0.0531	\overline{k}	15	1 3/01	0.0574
n d	1.0	0.8060	0.0002 0.0017	n d	1.0	0.9020	0.0001	n d	1.5	0.79191	0.0014
φ_1	-0.8	-0.0009	0.0017	φ_1	-0.8	-0.0004	0.0012	φ_1	-0.8	-0.7612	0.0040
a	0.15	0.1227	0.0042	a	0.25	0.1834	0.0048	a	0.45	0.3304	0.0052
_		0 5 1 5 0	0.0550	_		0.0051	0.0500	_		0.0045	0.000
$\bar{\kappa}$	1.5	0.5476	0.0572	$\bar{\kappa}$	1.5	0.9651	0.0529	$\bar{\kappa}$	1.5	0.9345	0.0605
ϕ_1	-0.4	-0.3693	0.0058	ϕ_1	-0.4	-0.4258	0.0064	ϕ_1	-0.4	-0.2744	0.0091
d	0.15	0.0931	0.0013	d	0.25	0.2421	0.0028	d	0.45	0.3959	0.0032
$\bar{\kappa}$	1.5	1.1763	0.0585	$\bar{\kappa}$	1.5	1.2395	0.0609	$\bar{\kappa}$	1.5	1.5391	0.0562
ϕ_1	0.4	0.4382	0.0033	ϕ_1	0.4	0.3733	0.0038	ϕ_1	0.4	0.4323	0.0109
d	0.15	0.0889	0.0019	d	0.25	0.1929	0.0037	d	0.45	0.3321	0.0046
			'				I				
$\bar{\kappa}$	1.5	1.8294	0.0634	$\bar{\kappa}$	1.5	1.8599	0.0543	$\bar{\kappa}$	1.5	1.5616	0.0467
ϕ_1	0.8	0.7208	0.0044	ϕ_1	0.8	0.7485	0.0052	ϕ_1	0.8	0.7514	0.0079
d^{φ_1}	0.15	0.1442	0.0023	d^{φ_1}	0.25	0 2197	0.0035	d^{φ_1}	0.45	0.3547	0.0044
u	0.10	0.1112	0.0020	u	0.20	0.2101	0.0000	a	0.10	0.0011	0.0011
\overline{k}	3.0	9 4789	0.0012	\overline{k}	3.0	2 2024	0.0810	\overline{k}	3.0	1 1638	0.0801
h	0.8	0 7718	0.0312	n du	0.0	2.0004 0.7062	0.0013	h	0.0	0.8681	0.0031 0.0017
φ_1	-0.8	-0.1110	0.0030	φ_1	-0.8	-0.7902	0.0034	φ_1	-0.8	-0.0001	0.0017
a	0.15	0.0779	0.0052	a	0.25	0.2165	0.0041	a	0.45	0.5928	0.0054
_	2.0	0.0001	0.0054	_	2.0	0.9094	0.0010	_	2.0	2 7020	0.0054
κ	3.0	2.8291	0.0854	κ	3.U	2.3934	0.0819	κ	3.U	3.7839	0.0854
ϕ_1	-0.4	-0.3908	0.0058	ϕ_1	-0.4	-0.3735	0.0071	ϕ_1	-0.4	-0.1072	0.0085
d	0.15	0.0955	0.0024	d	0.25	0.2288	0.0032	d	0.45	0.4651	0.0018
$\bar{\kappa}$	3.0	1.9903	0.1037	$ar{\kappa}$	3.0	2.3729	0.0888	$\bar{\kappa}$	3.0	3.5272	0.0789
ϕ_1	0.4	0.3965	0.0079	ϕ_1	0.4	0.2303	0.0060	ϕ_1	0.4	0.6246	0.0043
d	0.15	0.0903	0.0028	d	0.25	0.2844	0.0051	d	0.45	0.4095	0.0037
$\bar{\kappa}$	3.0	2.3209	0.0819	$\bar{\kappa}$	3.0	3.3355	0.0839	$\bar{\kappa}$	3.0	2.7774	0.0686
ϕ_1	0.8	0.7189	0.0061	ϕ_1	0.8	0.6375	0.0176	ϕ_1	0.8	0.8411	0.0078
\overline{d}	0.15	0.1236	0.0029	\overline{d}	0.25	0.2702	0.0041	\overline{d}	0.45	0.4423	0.0024
Parame	terizati	ion: $\alpha(L;\theta)$) = (1 - L)	$-d(1-\phi_1 I)$	$(L)^{-1}$. L	esign: ρ^2	$=1, \sigma^2 = 1$	$\mu = 0.$			

Table 1.2

					Samula	1.3 .:					
	37.1		a i b b	D (Sample	size: 4090	au D	D .	37.1		() I D
Parameter	Value	Mean	Std.Dev.	Parameter	Value	Mean	Std.Dev.	Parameter	Value	Mean	Std.Dev.
$\kappa_{_{\perp}}$	0	-0.0119	0.0170	$\kappa_{_{\perp}}$	0	-0.1240	0.0170	$\kappa_{_{\perp}}$	0	0.0033	0.0105
φ_1	-0.8	-0.1879	0.0013	φ_1	-0.8	-0.7841	0.0013	φ_1	-0.8	-0.7942	0.0017
d	0.15	0.1382	0.0032	d	0.25	0.1900	0.0037	d	0.45	0.4113	0.0031
=	0	0 1719	0.0105	-	0	0.0400	0.0949	=	0	0.0547	0.0025
κ	0	-0.1712	0.0185	κ	0	-0.0409	0.0242	κ_{\prime}	0	-0.0547	0.0235
ϕ_1	-0.4	-0.3747	0.0036	ϕ_1	-0.4	-0.3783	0.0036	ϕ_1	-0.4	-0.3017	0.0074
d	0.15	0.1081	0.0017	d	0.25	0.2376	0.0016	d	0.45	0.4252	0.0023
-	0	0 1000	0.0001	-	0	0.9074	0.0000	=	0	0.0564	0.0166
κ	0	-0.1000	0.0251	κ	0	-0.2074	0.0202	κ	0	0.2004	0.0100
ϕ_1	0.4	0.3925	0.0046	ϕ_1	0.4	0.4152	0.0052	ϕ_1	0.4	0.4298	0.0039
d	0.15	0.1223	0.0017	d	0.25	0.2061	0.0023	d	0.45	0.4320	0.0024
Ē	0	0.0463	0.0187	Ē	0	0 1781	0.0155	Ē	0	0.2454	0.0163
n d	0	0.0403 0.7771	0.0107	n d	0	0.1701	0.0100	h d	0.8	0.2404	0.0103
φ_1	0.0	0.1111	0.0057	φ_1	0.0	0.0027	0.0029	φ_1	0.8	0.8333	0.0059
a	0.15	0.1267	0.0027	a	0.25	0.2192	0.0039	a	0.45	0.4292	0.0023
\overline{k}	15	1 6228	0.0484	\overline{k}	15	1 3077	0.0472	\bar{k}	15	1 5222	0.0512
n d	1.0	1.0220 0.7050	0.0404	n d	1.5	1.3077	0.0472	h d	1.5	1.0000	0.0012
ψ_1	-0.0	-0.7950	0.0022	φ_1	-0.0	-0.7930	0.0021	φ_1	-0.8	-0.7652	0.0022
a	0.15	0.1073	0.0035	a	0.25	0.1540	0.0042	a	0.45	0.3712	0.0044
Ē	15	1 3475	0.0484	Ē	15	1 9439	0.0501	Ē	15	1 5295	0.0531
h h	-0.4	-0.3080	0.0404	h h	-0.4	-0 3830	0.0001	h h	-0.4	-0.3300	0.0001
ψ_1	0.15	0.000	0.0044	φ_1	0.4	0.0000	0.0000	ψ_1	0.45	0.3074	0.0000
u	0.15	0.1089	0.0021	u	0.20	0.2433	0.0029	u	0.40	0.3974	0.0029
$\bar{\kappa}$	1.5	0.8611	0.0587	$\bar{\kappa}$	1.5	1.9787	0.0564	$\bar{\kappa}$	1.5	1.0910	0.0559
ф1	0.4	0.4062	0.0059	ф1	0.4	0 4176	0.0062	φ1	0.4	0.3914	0.0103
φ_1	0.15	0.1002 0.1174	0.0000	φ_1	0.1	0.1110	0.0036	φ_1	0.1	0.3624	0.0105
u	0.10	0.11111	0.0021	u	0.20	0.1010	0.0000	u	0.10	0.0021	0.0000
$\bar{\kappa}$	1.5	1.3712	0.0487	$\bar{\kappa}$	1.5	1.4489	0.0454	$\bar{\kappa}$	1.5	1.7064	0.0413
ϕ_1	0.8	0.7643	0.0044	ϕ_1	0.8	0.7748	0.0051	ϕ_1	0.8	0.7706	0.0067
d	0.15	0.1173	0.0026	d	0.25	0.1916	0.0036	d	0.45	0.3998	0.0034
	0.20		0.00-0		0.20	0.2020			0.100		0.000-
$\bar{\kappa}$	3.0	2.9487	0.0810	$ar{\kappa}$	3.0	4.4445	0.0761	$\bar{\kappa}$	3.0	4.7166	0.0843
ϕ_1	-0.8	-0.7924	0.0032	ϕ_1	-0.8	-0.8650	0.0010	ϕ_1	-0.8	-0.8729	0.0014
d	0.15	0.0927	0.0034	d	0.25	0.1898	0.0055	d	0.45	0.4008	0.0054
			1				I				
$\bar{\kappa}$	3.0	2.8843	0.0755	$\bar{\kappa}$	3.0	1.9387	0.0886	$\bar{\kappa}$	3.0	2.5579	0.0751
ϕ_1	-0.4	-0.4077	0.0046	ϕ_1	-0.4	-0.3822	0.0065	ϕ_1	-0.4	-0.3088	0.0074
d	0.15	0.1102	0.0025	d	0.25	0.2420	0.0027	d	0.45	0.3933	0.0034
			I				I				
$\bar{\kappa}$	3.0	1.5057	0.0924	$\bar{\kappa}$	3.0	1.9387	0.0886	$\bar{\kappa}$	3.0	2.5579	0.0852
ϕ_1	0.4	0.4087	0.0062	ϕ_1	0.4	0.4131	0.0069	ϕ_1	0.4	0.3872	0.0099
d	0.15	0.0935	0.0024	d	0.25	0.1765	0.0032	d	0.45	0.3522	0.0042
			'				I				
$\bar{\kappa}$	3.0	2.5540	0.0774	$\bar{\kappa}$	3.0	2.4468	0.0751	$ar{\kappa}$	3.0	2.9603	0.0722
ϕ_1	0.8	0.7910	0.0027	ϕ_1	0.8	0.7650	0.0057	ϕ_1	0.8	0.7877	0.0061
d	0.15	0.1358	0.0041	d	0.25	0.1903	0.0035	d	0.45	0.4008	0.0033
Parame	terizati	ion: $\alpha(L;\theta)$) = (1 - L)	$-d(1-\phi_1 I)$	$(L)^{-1}$. L	Design: ρ^2	$= 1, \sigma^2 = 1$	$\mu = 0.$			

Table 1.3

Table 2.1											
					\mathbf{Sample}	size: 1024					
Parameter	Value	Mean	Std.Dev.	Parameter	Value	Mean	Std.Dev.	Parameter	Value	Mean	Std.Dev.
$\bar{\kappa}$	0	-0.2033	0.0226	$\bar{\kappa}$	0	-0.1382	0.0253	$\bar{\kappa}$	0	0.0178	0.0265
ω_1	0.5	0.4982	0.0008	ω_1	0.5	0.4967	0.0009	ω_1	0.5	0.4933	0.0011
ϕ_1	-0.4	-0.3303	0.0062	ϕ_1	-0.4	-0.3367	0.0066	ϕ_1	-0.4	-0.2526	0.0108
d	0.15	0.0876	0.0022	d	0.25	0.1978	0.0030	d	0.45	0.4013	0.0031
Ē	0	-0 1562	0.0252	$\bar{\kappa}$	0	-0 1577	0 0244	Ē	0	0 1490	0.0238
(1)1	0.5	0.4921	0.0013	(1)1	0.5	0 4934	0.0014	(1)1	0.5	0 4901	0.0015
ϕ_1	$0.0 \\ 0.4$	0.3383	0.0101	ω_1	0.0	0.3756	0.0104	ω_1 ϕ_1	0.0	0.3659	0.0010 0.0127
d^{φ_1}	0.15	0.0942	0.0028	$\overset{\varphi_1}{d}$	0.25	0.1587	0.0037	$\overset{\varphi_1}{d}$	0.45	0.3132	0.0049
-	1 5	1 9059	0.0005	-	1 5	1.0500	0.0679	-	1 5	1 1 6 7 0	0.0640
κ	1.5	1.2053	0.0685	κ	1.5	1.0502	0.0073	κ	1.5	1.1079	0.0649
ω_1	0.5	0.4978	0.0009	ω_1	0.5	0.4919	0.0014	ω_1	0.5	0.4966	0.0009
ϕ_1	-0.4	-0.3429	0.0065	ϕ_1	-0.4	-0.3546	0.0076	ϕ_1	-0.4	-0.2573	0.0111
d	0.15	0.0810	0.0025	d	0.25	0.1846	0.0076	d	0.45	0.3804	0.0038
$\bar{\kappa}$	1.5	1.0502	0.0673	$ar{\kappa}$	1.5	1.1679	0.0649	$ar{\kappa}$	1.5	1.7450	0.0612
ω_1	0.5	0.4919	0.0014	ω_1	0.5	0.4921	0.0015	ω_1	0.5	0.4911	0.0015
ϕ_1	0.4	0.3428	0.0109	ϕ_1	0.4	0.4102	0.0095	ϕ_1	0.4	0.4011	0.0128
d	0.15	0.0812	0.0029	d	0.25	0.1484	0.0043	d	0.45	0.3076	0.0052
Ē	3.0	2 9649	0 0923	Ē	3.0	2.2467	0 0943	Ē	3.0	2 2600	0 0000
(1)1	0.5	0 4974	0.0020	(1)1	0.5	0.4973	0.0010	(1)1	0.5	0.4933	0.0000
ϕ_1	-0.4	-0.3479	0.0006	ω_1	-0.4	-0.3573	0.0003	ω_1	-0.4	-0.2527	0.0012
ψ_1	0.4	0.0787	0.0000	φ_1	0.4 0.25	0.0010 0.1785	0.0010	φ_1	0.45	0.2021 0.3715	0.0110
u	0.15	0.0101	0.0020	u	0.20	0.1765	0.0058	u	0.40	0.3715	0.0042
$\bar{\kappa}$	3.0	2.068	0.1059	$ar\kappa$	3.0	2.3136	0.1038	$ar{\kappa}$	3.0	2.7811	0.1029
ω_1	0.5	0.4924	0.0014	ω_1	0.5	0.4909	0.0015	ω_1	0.5	0.4863	0.0015
ϕ_1	0.4	0.3717	0.0103	ϕ_1	0.4	0.4178	0.0093	ϕ_1	0.4	0.4084	0.0165
d	0.15	0.0835	0.0031	d	0.25	0.1507	0.0044	d	0.45	0.32814	0.0054
Parame	terizati	on: $\beta(L;\zeta)$	$) = (1 - \omega_1)$	$L)^{-1}, \alpha(L)$	θ) = (1	$-L)^{-d}(1 -$	$(-\phi_1 L)^{-1}$.	Design: ρ	$^{2} = 1, \sigma$	$\mu^2 = 1, \ \mu =$	0.

Table 2.2											
					Sample s	size: 2048					
Parameter	Value	Mean	Std.Dev.	Parameter	Value	Mean	Std.Dev.	Parameter	Value	Mean	Std.Dev
$\bar{\kappa}$	0	-0.1756	0.0209	$\bar{\kappa}$	0	-0.0888	0.0249	$\bar{\kappa}$	0	0.0532	0.0250
ω_1	0.5	0.4992	0.0006	ω_1	0.5	0.4984	0.0006	ω_1	0.5	0.4966	0.0078
ϕ_1	-0.4	-0.3514	0.0051	ϕ_1	-0.4	-0.3602	0.0050	ϕ_1	-0.4	-0.2756	0.0088
d	0.15	0.0981	0.0021	d	0.25	0.2238	0.0021	d	0.45	0.4172	0.0026
	0	0 1069	0 0998		0	0 1799	0 0921	T.	0	0 1020	0.0221
ĸ	0	-0.1902	0.0238	ĸ	0	-0.1755	0.0231	κ	0	0.1029 0.4027	0.0221
ω_1	0.5	0.4973	0.0009	ω_1	0.5	0.4904	0.0010	ω_1	0.5	0.4937	0.0011
ϕ_1	0.4	0.3857	0.0069	ϕ_1	0.4	0.3916	0.0075	ϕ_1	0.4	0.3410	0.0173
d	0.15	0.1056	0.0023	d	0.25	0.1883	0.0029	d	0.45	0.3485	0.0042
$\bar{\kappa}$	1.5	1.3419	0.0534	$\bar{\kappa}$	1.5	1.1364	0.0626	$\bar{\kappa}$	1.5	0.9927	0.0594
ω_1	0.5	0.4979	0.0006	ω_1	0.5	0.4935	0.0005	ω_1	0.5	0.4976	0.0008
ϕ_1	-0.4	-0.3649	0.0058	ϕ_1	-0.4	-0 4146	0.0057	ϕ_1	-0.4	-0 4967	0.0008
d^{φ_1}	0.15	0.0929	0.0023	d^{φ_1}	0.25	0.2296	0.0028	$\overset{\varphi_1}{d}$	0.45	0.3940	0.0032
			l				1				
$\bar{\kappa}$	1.5	1.8963	0.0627	$ar{\kappa}$	1.5	1.2007	0.0601	$\bar{\kappa}$	1.5	0.4973	0.0603
ω_1	0.5	0.4986	0.0010	ω_1	0.5	0.4968	0.0011	ω_1	0.5	0.4917	0.0011
ϕ_1	0.4	0.3922	0.0079	ϕ_1	0.4	0.4229	0.0078	ϕ_1	0.4	0.3787	0.0112
d	0.15	0.0895	0.0026	d	0.25	0.1773	0.0036	d	0.45	0.3195	0.0046
_	0.0	0.007	0.0079	_	0.0	0.0000	0.001 7	_	0.0	0 1 400	0.0000
κ	3.0	2.907	0.0873	κ	3.0	2.3920	0.0817	κ	3.0	2.1439	0.0930
ω_1	0.5	0.4993	0.0007	ω_1	0.5	0.4983	0.0006	ω_1	0.5	0.4968	0.0009
ϕ_1	-0.4	-0.3842	0.0059	ϕ_1	-0.4	-0.3773	0.0069	ϕ_1	-0.4	-0.2871	0.0092
d	0.15	0.0874	0.0024	d	0.25	0.2217	0.0032	d	0.45	0.3908	0.0036
Ē	3.0	1 0337	0 0002	\bar{k}	3.0	2 1064	0.0082	Ē	3.0	2 5204	0.0021
n	0.5	1.9997	0.0992	n.	0.5	2.1904	0.0902	n.	0.5	2.0204	0.0921
ω_1	0.5	0.4970	0.0010	ω_1	0.5	0.4909	0.0011	ω_1	0.5	0.4921	0.0012
ψ_1	0.4	0.3942	0.0077	φ_1	0.4	0.4501	0.0078	φ_1	0.4	0.4009	0.0112
<i>d</i>	0.15	0.0911	0.0027		0.25	0.1650	0.0040	<u>a</u>	0.45	0.3269	0.0048
Parame	terizati	ion: $eta(L;\zeta$	$) = (1 - \omega_1)$	$(L)^{-1}, \alpha(L)$	$(\theta) = (1)$	$(-L)^{-a}(1 -$	$(-\phi_1 L)^{-1}$.	Design: ρ^2	$^{2} = 1, \sigma$	$^{2} = 1, \ \mu =$	0.

	Table 2.3											
					Sample s	size: 4096						
Parameter	Value	Mean	Std.Dev.	Parameter	Value	Mean	Std.Dev.	Parameter	Value	Mean	Std.Dev.	
$\bar{\kappa}$	0	-0.1369	0.0201	$ar{\kappa}$	0	-0.0866	0.0240	$\bar{\kappa}$	0	0.0110	0.0230	
ω_1	0.5	0.4998	0.0004	ω_1	0.5	0.4998	0.0004	ω_1	0.5	0.4983	0.0006	
ϕ_1	-0.4	-0.3729	0.0034	ϕ_1	-0.4	-0.3708	0.0035	ϕ_1	-0.4	-0.3084	0.0067	
d	0.15	0.1100	0.0018	d	0.25	0.2338	0.0016	d	0.45	0.4275	0.0021	
Ē	0	0 1220	0 0221	Ē	0	0 1219	0.0220	Ē	0	0.0785	0.0106	
ĸ	0	-0.1559	0.0231	ĸ	0	-0.1312	0.0220	ĸ	0	0.0700	0.0190	
ω_1	0.5	0.4975	0.0007	ω_1	0.5	0.4979	0.0007	ω_1	0.5	0.4980	0.0007	
ϕ_1	0.4	0.4090	0.0048	ϕ_1	0.4	0.4048	0.0056	ϕ_1	0.4	0.3543	0.0100	
d	0.15	0.1197	0.0018	d	0.25	0.2106	0.0023	d	0.45	0.3865	0.0031	
$\bar{\kappa}$	1.5	1.3602	0.0479	$ar{\kappa}$	1.5	1.2867	0.0548	$ar{\kappa}$	1.5	1.0698	0.0577	
ω_1	0.5	0.4992	0.0004	ω_1	0.5	0.4999	0.0005	ω_1	0.5	0.4967	0.0006	
ϕ_1	-0.4	-0.3882	0.0043	ϕ_1	-0.4	-0.3820	0.0054	ϕ_1	-0.4	-0.2854	0.0057	
d	0.15	0.1091	0.0021	d	0.25	0.2419	0.0023	d	0.45	0.4037	0.0029	
_		0.0505	0.0500	_		1 0000	0.0550	_		1 00 1 1	0.0510	
$\bar{\kappa}$	1.5	0.9725	0.0586	$ar{\kappa}$	1.5	1.0288	0.0552	$ar{\kappa}$	1.5	1.6044	0.0510	
ω_1	0.5	0.4979	0.0007	ω_1	0.5	0.4975	0.0008	ω_1	0.5	0.4933	0.0008	
ϕ_1	0.4	0.4065	0.0068	ϕ_1	0.4	0.4163	0.0062	ϕ_1	0.4	0.3665	0.0106	
d	0.15	0.1037	0.0023	d	0.25	0.1891	0.0031	d	0.45	0.3676	0.0037	
Ē	3.0	2 9792	0 0783	Ē	3.0	2 5281	0.0760	Ē	3.0	2 0276	0.0873	
(1)	0.5	0 4007	0.0105	(1)	0.5	0.5001	0.0004	(.).	0.5	0.4078	0.0006	
ω ₁ φ	0.5	0.4331	0.0000	ω ₁	0.5	0.0001 0.2791	0.0004	ω ₁ φ	0.5	0.4310	0.0000	
ψ_1	-0.4	-0.3991	0.0000	φ_1	-0.4	-0.3721	0.0000	φ_1	-0.4	-0.3144	0.0009	
a	0.15	0.1000	0.0022	a	0.23	0.2349	0.0025	a	0.45	0.3970	0.0054	
$\bar{\kappa}$	3.0	1.7137	0.0931	$ar{\kappa}$	3.0	2.0497	0.0903	$ar{\kappa}$	3.0	2.8007	0.0823	
ω_1	0.5	0.4978	0.0008	ω_1	0.5	0.4970	0.0008	ω_1	0.5	0.4965	0.0008	
ϕ_1	0.4	0.4094	0.0067	ϕ_1	0.4	0.4086	0.0069	ϕ_1	0.4	0.3943	0.0104	
d	0.15	0.0954	0.0023	d	0.25	0.1801	0.0034	d	0.45	0.3673	0.0040	
Parame	terizati	ion: $\beta(L;\zeta)$	$(1 - \omega_1) = (1 - \omega_1)$	$(L)^{-1}, \alpha(L)$	θ) = (1	$(-L)^{-d}(1 + 1)^{-d}$	$(-\phi_1 L)^{-1}$.	Design : ρ^{i}	$\sigma^{2} = 1, \ \sigma$	$^{2} = 1, \ \mu =$	0.	

Table 3.1										
			Sai	mple size:	1024					
Parameter	Value	Probability	Parameter	Value	Probability	Parameter	Value	Probability		
			$\bar{\kappa} = 0$		$\phi_1 = -0.8$					
d	0.15	0.957	d	0.25	1	d	0.45	0.860		
			$\bar{\kappa} = 0$		$\phi_1 = -0.4$					
d	0.15	0.846	d	0.25	0.897	d	0.45	0.931		
			$\bar{\kappa} = 0$		$\phi_1 = 0.4$					
d	0.15	0.987	d	0.25	0.770	d	0.45	0.834		
			$\bar{\kappa} = 0$		$\phi_1 = 0.8$					
d	0.15	0.960	d	0.25	1	d	0.45	0.856		
			$\bar{\kappa} = 1.5$		$\phi_1 = -0.8$					
d	0.15	0.936	d	0.25	1	d	0.45	0.879		
			$\bar{\kappa} = 1.5$		$\phi_1 = -0.4$					
d	0.15	0.989	d	0.25	0.866	d	0.45	0.895		
			$\bar{\kappa} = 1.5$		$\phi_1 = 0.4$					
d	0.15	0.982	d	0.25	0.970	d	0.45	0.824		
			$\bar{\kappa} = 1.5$		$\phi_1 = 0.8$					
d	0.15	0.959	d	0.25	1	d	0.45	0.936		
			$\bar{\kappa} = 3$		$\phi_1 = -0.8$					
d	0.15	0.943	d	0.25	1	d	0.45	0.739		
			$\bar{\kappa} = 3$		$\phi_1 = -0.4$					
d	0.15	0.980	d	0.25	0.897	d	0.45	0.858		
			$\bar{\kappa} = 3$		$\phi_1 = 0.4$					
d	0.15	0.972	d	0.25	1	d	0.45	0.773		
			$\bar{\kappa} = 3$		$\phi_1 = 0.8$					
d	0.15	0.966	d	0.25	1	d	0.45	0.783		
Parame	terizati	on: $\alpha(L;\theta)$) = (1 - L)	$)^{-d}(1 -$	$\phi_1 L)^{-1}$. De	esign: ρ^2 =	= 1, $\sigma^2 =$	1, $\mu = 0$.		
Theoret	ical pr	obability:	0.95.							

Table 3.2										
			Sai	mple size:	4096					
Parameter	Value	Probability	Parameter	Value	Probability	Parameter	Value	Probability		
			$\bar{\kappa} = 0$		$\phi_1 = -0.8$					
d	0.15	0.985	d	0.25	0.861	d	0.45	0.950		
			$\bar{\kappa} = 0$		$\phi_1 = -0.4$					
d	0.15	0.870	d	0.25	0.958	d	0.45	0.944		
			$\bar{\kappa} = 0$		$\phi_1 = 0.4$					
d	0.15	0.917	d	0.25	0.943	d	0.45	0.933		
			$\bar{\kappa} = 0$		$\phi_1 = 0.8$					
d	0.15	0.984	d	0.25	0.927	d	0.45	0.921		
			$\bar{\kappa} = 1.5$		$\phi_1 = -0.8$					
d	0.15	0.973	d	0.25	0.980	d	0.45	0.930		
			$\bar{\kappa} = 1.5$		$\phi_1 = -0.4$					
d	0.15	0.893	d	0.25	0.949	d	0.45	0.941		
			$\bar{\kappa} = 1.5$		$\phi_1 = 0.4$					
d	0.15	0.903	d	0.25	0.921	d	0.45	0.863		
			$\bar{\kappa} = 1.5$		$\phi_1 = 0.8$					
d	0.15	0.974	d	0.25	0.915	d	0.45	0.862		
			$\bar{\kappa} = 3$		$\phi_1 = -0.8$					
d	0.15	0.974	d	0.25	0.973	d	0.45	0.873		
			$\bar{\kappa} = 3$		$\phi_1 = -0.4$					
d	0.15	0.909	d	0.25	0.927	d	0.45	0.872		
			$\bar{\kappa} = 1.5$		$\phi_1 = 0.4$					
d	0.15	0.971	d	0.25	0.892	d	0.45	0.840		
			$\bar{\kappa} = 3$		$\phi_1 = 0.8$					
d	0.15	0.972	d	0.25	0.916	d	0.45	0.891		
Parame	Parameterization: $\alpha(L;\theta) = (1-L)^{-d}(1-\phi_1L)^{-1}$. Design: $\rho^2 = 1, \ \sigma^2 = 1, \ \mu = 0$.									
Theoret	ical pr	obability:	0.95.							

		power			siz€	9
Values for d	sig	nificance va	alue		significanc	e value
	10%	5%	1%	10%	5%	1 %
		Sa	ample size:	512		
0.1	0.143	0.093	0.046	0.079	0.047	0.016
0.2	0.370	0.298	0.184	0.096	0.054	0.012
0.375	0.602	0.541	0.455	0.073	0.036	0.018
0.45	0.613	0.556	0.471	0.080	0.044	0.018
0.49	0.461	0.402	0.313	0.078	0.037	0.015
		Sa	mple size:	1024		
0.1	0.184	0.124	0.066	0.082	0.052	0.014
0.2	0.424	0.343	0.214	0.084	0.041	0.009
0.375	0.692	0.647	0.541	0.094	0.045	0.010
0.45	0.673	0.625	0.534	0.092	0.048	0.016
0.49	0.529	0.475	0.380	0.095	0.056	0.017
		Sa	mple size:	2048		
0.1	0.184	0.134	0.055	0.093	0.050	0.010
0.2	0.457	0.371	0.249	0.082	0.045	0.006
0.375	0.744	0.708	0.621	0.089	0.049	0.010
0.45	0.750	0.703	0.631	0.088	0.052	0.008
0.49	0.589	0.515	0.422	0.112	0.052	0.020
Parameter	ization:	$\alpha(L;\theta) =$	$(1-L)^{-}$	d. Desig	n: $\rho^2 = 1$,	$\sigma^2 = 1, \ \mu = 0.$

Table 4: Empirical size and power of LM test Φ

	S&P500	FTSE All	FTSE100	Yen/\$ spot	Yen/\$ for	$f \le 1 $ spot	for				
Φ	16460583	1440499	1229901	4.91	81.86	93.43	104.56				
kurtos is	100.26	27.78	26.15	6.69	7.52	7.17	7.18				
skewness	-4.20	-1.72	-1.49	-0.35	-0.36	0.14	0.16				
LM tes	st statistic Φ	, presented	in first rov	v, is asympt	otically dis	tributed	like				
central	central χ_1^2 under the null hypothesis of homosked asticity.										

Table 5.1: Empirical application of LM Φ test

 Table 5.2: Semiparametric and Gaussian estimates of d

Table 0.2. Semparametric and Gaussian estimates of a										
	S&P500	FTSEAll	FTSE100	Yen/\$ spot	Yen/\$1mfor	$f \leq spot$	$f \leq 1m for$			
semiparametric	0.11	0.15	0.14	0.21	0.17	0.41	0.42			
$ARCH(\infty)$	0.33	0.35	0.37	0.32	0.24	0.54	0.56			
	(0.13)	(0.15)	(0.15)	(0.05)	(0.04)	(0.09)	(0.11)			
nonlinearMA	0.35	0.36	0.43	0.30	0.26	0.41	0.42			
	(0.07)	(0.09)	(0.10)	(0.07)	(0.07)	(0.10)	(0.08)			
	[3.06]	[3.13]	[4.07]	[3.58]	[2.25]	[1.74]	[1.81]			
Down 1 laws of 1	1 C		-+ (D-1:	1005-)	with handwi	141 1	00			

Row 1: values of local Gaussian estimator (Robinson 1995a), with bandwidth m = 100and no trimming. The asymptotic standard error is equal to $1/(2m^{1/2}) = 0.050$. Rows 2 and 3: values of Gaussian estimator pertinent to the values of p, q ($0 \le p, q \le 4$) yielding best goodness-of-fit of estimated residuals. Initial values given by grid search and semiparametric estimates of first row. Standard errors are in parentheses. Estimates of $\bar{\kappa}$ are reported in square brackets.

	ARC	$\mathrm{H}(\infty)$	nonlir	near MA	ra	w data						
	levels	squares	levels	squares	levels	squares						
S&P500	0.329	0.310	0.330	0.4047	0.319	0.196						
	[1.41]	[1.10]	[0.71]	[5.08]	[0.13]	[15.99]						
FTSEAll	0.342	0.239	0.346	0.350	0.333	0.366						
	[3.11]	[10.4]	[1.63]	[1.88]	[1.99]	[6.25]						
FTSE100	0.307	0.168	0.336	0.363	0.321	0.370						
	[1.48]	[19.7]	[1.06]	[2.67]	[0.39]	[6.82]						
$Yen/\$\ spot$	0.319	0.281	0.319	0.331	0.325	0.339						
	[0.15]	[4.83]	[0.05]	[0.76]	[0.80]	[2.72]						
Yen/\$ for	0.301	0.273	0.317	0.341	0.323	0.338						
	[2.27]	[5.95]	[-0.07]	[0.34]	[0.60]	[2.62]						
$f \leq spot$	0.322	0.318	0.320	0.477	0.318	0.471						
	[0.46]	[0.07]	[0.11]	[9.32]	[0.01]	[20.1]						
for	0.192	0.161	0.317	0.477	0.317	0.347						
	[16.5]	[20.6]	[-0.05]	[9.33]	[0.11]	[19.7]						
The speci	fication (of Milhøj	(1981) te	st for white-	noise of sa	mple of z_t is						
	W =	$=\left(\frac{2\pi}{T}\sum_{s}^{T}\right)$	$\sum_{s=1}^{n-1} (I_{zz}(\lambda_s))$	$\binom{2}{T} / \left(\frac{2\pi}{T} \sum_{s=1}^{T} \sum_{s=1}^{T} \right)^{s}$	$\left[\frac{-1}{21} I_{zz}(\lambda_s) \right]^2$							
For each s	For each series, the W test statistic, using the estimated residuals											
of best parameterization, in levels and squares, are reported.												
Under the	Under the null of perfect fit $T^{\frac{1}{2}}(W-1/\pi)$ converges in distribution											
to $\mathcal{N}(0,2/\pi)$	(π^2) . The	$\mathbf{studenti}$	zed test in	n []-brackets	is asympto	otically						
standard	normal ı	under the	null of pe	erfect fit.								

Table 5.3: Goodness-of-fit test of estimated residuals

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