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SIGN- AND VOLATILITY-SWITCHING ARCH MODELS: THEORY AND APPLICATIONS TO INTERNATIONAL STOCK MARKETS

by Fabio Fornari (*) and Antonio Mele (**)

Summary

The present work introduces three conditionally heteroscedastic models which allow an asymmetric reaction of the conditional volatility to the arrival of news. Such a reaction is induced by both the sign of past shocks and the size of past unexpected volatility. The three models are shown to converge in distribution to absolutely continuous Itô diffusion processes, as happens for other heteroscedastic formulations. Two out of the three proposed schemes differ from the existing asymmetric models, insofar as they are able to capture a particular aspect of the behaviour of the volatilities, i.e. the inversion of their asymmetric reaction to news. Empirical evidence from stock market returns in seven countries shows that Sign- and Volatility-Switching ARCH models outperform traditional asymmetric ARCH equations.

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1. Introduction[1]

Since Engle's (1982) and Bollerslev's (1986) seminal papers, ARCH (AutoRegressive Conditional Heteroscedastic) models have been widely employed in the analyses of financial variables observed at high frequencies; in this case, the effects of heteroscedasticity are more evident and witnessed by the distributions of the rates of change of the variables, which are heavy-peaked and tailed.[2]

The original ARCH model posits the existence of a relation between past squared innovations of an observation assets returns changes model and their current conditional variances. Let ε_t be the innovation of an observation model; then, the GARCH(1,1) model assumes that ε_t is conditionally normal with variance changing through time in a fashion which resembles a restricted ARMA process, i.e.:

$$\varepsilon_t \mid I_{t-1} \sim N(0,\sigma_t^2) \tag{1.1}$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 \cdot \varepsilon_{t-1}^2 + \beta \cdot \sigma_{t-1}^2, \tag{1.2}$$

where $\alpha_0 > 0$, α_1, $\beta \geq 0$ are real, non-stochastic parameters and I_{t-1} is the information set dated t-1.

A shortcoming of the GARCH model is that the sign of the forecast errors does not influence the conditional

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1 This paper builds on the initial ideas developed in our 1994 work, "The Sign Conditional GARCH Model: Theory and Applications to International Stock Markets", Quaderni di Ricerca dell'Osservatorio e Centro di Studi Monetari, LUISS University, No. 41. We wish to thank Michel Botomazava (University of Paris X), Giorgio De Santis (University of Southern California at Los Angeles), Andrew Harvey (London School of Economics) and a referee for very helpful remarks and suggestions given on an earlier version of the work. The usual disclaimer applies.

2 For an extensive review of ARCH models, see Bera and Higgins (1993).

variance, which may contradict the observed dynamics of assets returns. Black (1976), for example, noted that their volatility tends to grow in reaction to bad news (excess returns lower than expected), and to fall in response to good news (excess returns higher than expected). The economic explanation given by Black is that positive (negative) excess returns make the equity value, hence the leverage ratio, of a given firm increase (fall), thus raising (lowering) its riskness and the future volatility of its assets. This phenomenon has consequently come to be referred to as leverage effect. (see, e.g., Campbell and Hentschell, 1992, or Pagan and Schwert, 1990).

The basic attempts to include these assets returns features into a convenient econometric framework are the Exponential ARCH model of Nelson (1991), the Threshold ARCH model of Zakoïan (1991) and Rabemananjara and Zakoïan (1993), the Asymmetric Power ARCH model of Ding et al. (1993), and the Stochastic Variance model of Harvey et al. (1994), or Harvey and Shephard (1993a, 1993b). All such models include the sign of past forecast errors as a conditioning information for the current values of the conditional variance.

The main concern of the paper is to develop heteroscedastic formulations which turn out to be useful in modeling the statistical properties of financial data. It improves over previously developed models for two main reasons:

— first, it develops a class of asymmetric ARCH models in which volatility is influenced by the sign of previous shocks and the unexpected volatility induced by such shocks;

— second, it derives their asymptotic properties, useful in the estimation of recent (continuous time) theoretical models developed in finance.

With concern for the first issue, we propose three new models. In the first of them, the intercept of the volatility equation (1.2) is allowed to change according to the sign of previous shocks, so capturing asymmetries in volatility within a simple traditional GARCH structure; since the model is closely related to the Sign Conditional (SGN) Autoregressive Model in Granger and Teräsvirta (1993), it will be referred to as Sign-Switching ARCH. However, there are reasons to believe that factors other than the sign of past shocks be responsible for the asymmetric behaviour of volatilities. To examine such an opportunity, we propose two models which capture asymmetries via the impact of past shocks on the level of the volatility, rather than through the unexpected returns; they will be called Volatility-Switching ARCH. Unlike previous models, they are able to capture an already observed phenomenon, the reversal of asymmetry, which will be defined in the next paragraph.

Coming to the second point, it has been widely recognised that many of the GARCH models developed so far admit a continuous time representation, thus revealing useful in the estimation of continuous time models employed in finance. Following this stream of research, analogous results are presented for the GARCH models hereby developed, as well as for other discrete time ARCH models.[3]

3 This kind of results was pionereed by Nelson (1990), who showed that the GARCH(1,1) model of Bollerslev (1986) and the AR(1)-Exponential ARCH model of Nelson (1991) approach continuous-time AR(1) processes, as the length of the sample frequency approaches zero; later, Fornari and Mele (1994a) extended his results to the case of the Asymmetric Power ARCH model of Ding et al. (1993), and El Babsiri and Zakoïan (1994) established weak convergence theory for the Threshold ARCH model of Zakoïan (1991).

The paper is structured in the following manner. The next section deals with the Sign-Switching ARCH model and derives the expressions of its first four moments; then, two Volatility-Switching models are introduced. We conclude the section by presenting weak convergence results for the Sign-Switching ARCH, the two Volatility-Switching models and finally, the Glosten et al. (1993; henceforth GJR) model. Section 3 presents our empirical results. Evidence from seven stock market indices unambiguously shows that the Sign- and Volatility-Switching ARCH models successfully detect asymmetries in volatility. Section 4 concludes the paper.

2. Sign and Volatility Switching ARCH Models

2.1 The structure of the models

In the Sign-Switching ARCH model we capture the asymmetric reaction of the conditional variance to shocks of different sign through the sign of such shocks. Let ε_t be a (scalar) innovation of a given (unidimensional) observation model. The Sign-Switching GARCH(p,q,y) model then assumes:

(2.1) $$\varepsilon_t \equiv z_t \cdot \sigma_t, \quad \varepsilon_t | I_{t-1} \sim N(0, \sigma_t^2)$$

(2.2) $$\sigma_t^2 = w + \Sigma_{i=1,p} \beta_i \cdot \sigma_{t-i}^2 + \Sigma_{j=1,q} \alpha_j \cdot \varepsilon_{t-j}^2 + \Sigma_{x=1,y} \Phi_x \cdot s_{t-x}$$

(2.3.1) $$s_t = +1, \quad \text{if } \varepsilon_t > 0$$

(2.3.2) $$s_t = 0, \quad \text{if } \varepsilon_t = 0$$

(2.3.3) $$s_t = -1, \quad \text{if } \varepsilon_t < 0,$$

where p, q and y ≥ 0, w, α_j, j=1,..,q, β_i, i=1,..,p and Φ_x, x=1,...,y, are real, non-stochastic parameters, satisfying $w>0$, $\alpha_j \geq 0$, $\beta_i \geq 0$ and, finally, $|\Sigma_x \Phi_x \leq w|$; such constraints guarantee that the process $\{\sigma_t^2\}$ remains positive almost surely.

Throughout the paper we will confine to the special case p=q=y=1, so that (2.2) reduces to:

$$\sigma_t^2 = w + \beta\cdot\sigma_{t-1}^2 + \alpha\cdot\varepsilon_{t-1}^2 + \Phi\cdot s_{t-1}. \tag{2.4}$$

It is straight forward to see that (2.1)-(2.4) captures asymmetric responses of the volatility to positive and negative shocks;[4] in fact, when $\Phi<0$, negative (positive) shocks at t-1 will be associated with an higher (lower) level of volatility at t.

The second and fourth unconditional moments of the innovations of a Sign-Switching model are (see Appendix A):

$$E(\varepsilon^2) = w\cdot(1-\alpha-\beta)^{-1}. \tag{2.5}$$

$$E(\varepsilon^4) = 3\cdot(w^2+\Phi^2)\cdot(1-\alpha-\beta)+6w^2(\alpha+\beta)) \\ (1/((1-\alpha-\beta)\cdot(1-\alpha^2-3\beta^2 - 2\alpha\beta))). \tag{2.6}$$

While the second unconditional moment coincides with that of a GARCH(1,1) model (Bollerslev, 1986), the fourth is also a function of Φ; hence, the stronger the asymmetric effect, the higher the unconditional fourth moment. Such a feature helps capture a widely recognised characteristic of the data, i.e. high kurtoses. The coefficient of kurtosis, k, derives directly from (2.5) - (2.6) and turns out to be:

$$k = (3\cdot(1-\alpha-\beta)^2\cdot(w^2+\Phi^2) + (1-\alpha-\beta)\cdot 6w^2(\alpha+\beta))\cdot \\ (1/(w^2(1-\alpha^2-3\beta^2-2\alpha\beta))). \tag{2.7}$$

It is an increasing function of Φ, so that the Sign-Switching ARCH model interprets high kurtoses also as the consequence of the asymmetric behaviour of the volatility (beyond its persistence).

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4 Note also that (2.4) is related to the Sign-Conditional AR(1) (SGN) model in Granger and Teräsvirta (1993, pp. 137-39).

According to recent empirical evidence (Glosten et al., 1993), the Sign-Switching ARCH model might reveal unsuccessful in detecting some of the non-linear characteristics of the volatility dynamics. First, as pointed out by a referee, its news impact function, i.e. the relation between the sign of ε_{t-1} in (2.4) and σ_t^2, is discontinuous at zero.[5] Further, there is a particular feature of the data that it is not able to detect. As pointed out by Rabemananjara and Zakoïan (1993), high negative shocks increase future volatility more than high positive ones; at the same time small positive shocks too often produce a stronger impact on future volatility than negative shocks of the same size. Thus, following the occurrence of a shock of a certain size, the asymmetric behaviour of the volatility might become reversed; the modeling of this feature is the focus of the remainder of the paper.

First, we define which is the size of the shock at which the reversal occurs, which also helps clarify why the asymmetric behaviour of the volatility may eventually come to change direction.

As the first issue is concerned, the "size measure" which we employ in this paper is the level of unexpected volatility generated by a shock at t-1 (ε_{t-1}). Conditionally on the information set dated t-2, the expected value of ε_{t-1}^2 is σ_{t-1}^2; if, however, $\varepsilon_{t-1}^2 \geq \sigma_{t-1}^2$ ($\leq \sigma_{t-1}^2$), we shall say that ε_{t-1} has generated (at time t-1) a level of volatility higher (lower) than expected (at time t-2). Consider now a very small negative shock at time t-1; if it introduces a level of volatility at time t-1 lower than the volatility expected at time t-2, there should be no reasons to believe that volatility at time t will increase as a consequence of

[5] However, while such a characteristic may not be plausible in financial economics, we do not have strong a priori reasons to rule out such an occurrence working with other variables (see, e.g., Leland and Gennotte, 1990).

the leverage effect. Roughly speaking, a small negative shock which generates lower volatility than expected is a piece of good news and not a completely bad one. Also, positive shocks which generate lower volatility than expected may be regarded as relatively good news; this clarifies why reversals should occur.

Past research has generally overlooked the impact of previous (unexpected or expected) volatility on its current expected level. Engle and Ng (1993), for example, propose to analyse the impact of news on the current conditional heteroscedasticity (i.e. on σ_t^2), keeping constant the information dated t-2 and earlier, with all the lagged conditional variances evaluated at their unconditional values.

To define such issues formally, let $v_{t-1} = v_{t-1}(\varepsilon_{t-1})$ denote the (measurable) amount of unexpected volatility at time t-1, generated by a shock occurred at time t-1 (ε_{t-1}). Let $f(v_{t-1})$ be some deterministic and measurable function mapping v_{t-1} onto the current conditional volatility. Then, if $g[\varepsilon_{t-1}, |\varepsilon_{t-1}|, \mathrm{sign}(\varepsilon_{t-1})]$ is a deterministic, asymmetric and measurable response function of the current, conditional volatility with respect to both size and sign of ε_{t-1}, other things equal, all the asymmetric ARCH models so far proposed in the literature focus mainly on modeling $g(\cdot)$, rather than $f(\cdot)$.

In order to take into account the impact of past unexpected volatility on future expected volatility, one has to build plausible functional forms for $f(v_{t-1})$, thus providing a model for the "response function of the future expected volatility to past unexpected volatility"; the latter would parallel the notion of "news impact curve" of Engle and Ng (1993).

In this paper, we will assume that $f(v_{t-1})$ is proportional to v_{t-1}. Consider, for example, the following model:

$$\sigma_t^2 = w + \alpha\cdot\varepsilon_{t-1}^2 + \beta\cdot\sigma_{t-1}^2 + \Phi\cdot s_{t-1}\cdot v_{t-1}, \tag{2.8}$$

where s_{t-1} is defined by relations (2.3), and:

$$v_t \equiv \varepsilon_t^2 - \sigma_t^2. \tag{2.8.1}$$

Note, first, that v_t measures the difference between the observed conditional volatility (ε_t^2) and its estimate, based on the available set of information, thus playing the role of an error correcting variable. The term $v_t\cdot s_{t-1}$ is by construction uncorrelated with both ε_t^2 and σ_t^2, for each t; further, conditionally on the information set dated t-1, v_t has zero mean and conditional standard deviation[6] equal to 1.4142 times σ_t^2. In the remainder of the paper, we will refer to model (2.8)-(2.8.1) as Volatility-Switching Model A.

Suppose now that $\Phi < 0$ and, further, $\alpha > |\Phi|$, $\beta > |\Phi|$. If $v_{t-1}>0$, then, coeteris paribus, negative shocks generate more volatility than positive ones. However, if $v_{t-1}<0$, positive shocks increase volatility more than negative ones. Thus, model (2.8)-(2.8.1) is able to detect situations where the asymmetric behaviour of the volatility is reversed and further illustrates what has to be meant by "size" of a shock: "small" shocks are those which produce a level of volatility lower than expected, and "high" shocks are those which generate a level of volatility higher than expected.

The second and fourth moments of the Volatility-Switching model A innovations are (see Appendix A):

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6 As Engle and Mustafa (1992) remark, however, v_t is a martingale difference, but absolutely not an i.i.d. sequence.

$$(2.9) \qquad E(\varepsilon^2) = w\cdot(1-\alpha-\beta)^{-1}$$

$$(2.10) \qquad E(\varepsilon^4) = (3w^2\cdot(1-\alpha-\beta) + 6w^2(\alpha+\beta)) \\ (1/((1-\alpha-\beta)\cdot(1-\alpha^2-3\beta^2-2\alpha\beta-2\Phi^2))).$$

Since Φ measures the impact of v_{t-1} on σ_t^2, deeper asymmetries in volatility will result in more leptokurtic distributions for the unconditional innovations. The structure of the Volatility-Switching Model A much resembles that of the GJR.[7]

The second Volatility-Switching model (model B) that we propose is similar to model A, but relation (2.8.1) is replaced with:

$$(2.8.2) \qquad v_{t-1} \equiv (\varepsilon_{t-1}^2/\sigma_{t-1}^2) - 1.$$

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7 It has to be noted that the Volatility-Switching model (2.8)-(2.8.1) resembles the GJR model. The latter is specified as:

$$(i) \qquad \sigma_t^2 = w + \alpha\cdot\varepsilon_{t-1}^2 + p\cdot\sigma_{t-1}^2 + g_1\cdot S_{t-1}^-\cdot\varepsilon_{t-1}^2,$$

where S_{t-1}^- is a dummy variable which takes the value +1 if ε_{t-1} is negative, and zero otherwise. Note that S_{t-1}^- can always be written as:

$$(ii) \qquad S_{t-1}^- = (\varepsilon_{t-1} - |\varepsilon_{t-1}|)\cdot(2\cdot\varepsilon_{t-1})^{-1},$$

so that substituting (ii) into (i) and rearranging, one gets:

$$(iii) \qquad \sigma_t^2 = w + g_0\cdot\varepsilon_{t-1}^2 + p\cdot\sigma_{t-1}^2 + \Phi\cdot s_{t-1}\cdot\varepsilon_{t-1}^2,$$

where:

$$(iv) \qquad g_0 \equiv \alpha + g_1/2$$

$$(v) \qquad \Phi \equiv -(g_1/2).$$

Thus, in the Volatility-Switching model (2.8)-(2.8.1) the asymmetries of the conditional volatility are captured by s_{t-1} times v_{t-1}, while in the GJR model (iii)-(iv)-(v) they are captured by s_{t-1} times ε_{t-1}^2.

In this case v_t does not measure the amount of unexpected volatility; rather, it is defined as the difference, in basis points, from an ideal situation in which the expected volatility matches exactly its future realisations. However, Model B is not proposed for this technical reason; rather, we have introduced it for its limiting behaviour, which differs from that of Model A and, more relevantly, from that of the Power ARCH of Ding et al. (1993).

Conditionally on the information set dated t-1, v_t has zero mean and conditional standard deviation equal to 1.4142. The second and fourth moments of the Volatility-Switching Model B are (see Appendix A):

$$E(\varepsilon^2) = w\cdot(1-\alpha-\beta)^{-1} \tag{2.9'}$$

$$E(\varepsilon^4) = (3(w^2+7\Phi^2)\cdot(1-\alpha-\beta) + 6w^2(\alpha+\beta))\cdot (1/((1-\alpha-\beta)\cdot(1-\alpha^2-3\beta^2-2\alpha\beta))), \tag{2.10'}$$

so that it generates unconditional forecast errors whose kurtoses increase with Φ, the asymmetry parameter.

As already stated, our interest in the model (2.8)-(2.8.2) lies in its continuous-time representation. In fact, extending Nelson's (1990) results, we will prove that the Sign- and Volatility-Switching ARCH models converge in distribution to diffusions for σ_t^2; however, unlike Model A, whose diffusion is analogous to that of Nelson (1990), the diffusion of the Sign- and Volatility-Switching Model B are more general and collapse to the latter only when Φ is constrained to zero.

2.2 Continuous time behaviour of the models

The derivation of continuous time limits for the asymmetric models developed so far is obtained by partitioning time in (2.1)-(2.4) more and more finely, according to the following scheme, where h denotes sampling frequency:

(2.11.1) $$_h\varepsilon_{hk} \equiv {_h z_{hk}} \cdot {_h\sigma_{hk}}$$

(2.11.2) $$_h z_{hk} \sim N(0,h)$$

(2.11.3) $$s_k \equiv z_{hk}/|z_{hk}|$$

(2.11.4) $$s_k \sim i.i.d.(0,1)$$

(2.11.5) $$_h\sigma_{h(k+1)}{}^2 - {_h\sigma_{hk}}^2 = w_h + {_h\sigma_{hk}}^2(\beta_h + \alpha_h \cdot {_h z_{hk}}^2 \cdot h^{-1} - 1) + \Phi_h \cdot s_k.$$

Such a system is Markov, and we are interested in analysing the conditions under which it converges weakly (i.e. in distribution) to an Itô diffusion process, as h drops to zero. To do this, we retain Nelson's (1990) assumptions (1 through 5) omitting, for sake of simplicity, more general issues, such as the conditions under which stochastic difference equations converge to stochastic differential equations.

Before deriving the diffusion limits of the models proposed so far, we will present analogous results for the Power ARCH model of Ding et al. (1993), since it has quite a general structure and encompasses many heteroscedastic formulations, including the GJR. The latter seems the best parametric model to capture the asymmetry of volatility (Engle and Ng, 1993) and, owing to this, its empirical performance will be compared to that of both the Sign- and Volatility-Switching models.

To start with, let us replace (2.4) and (2.11.5) with the respective Power ARCH equations:

$$(2.4') \qquad \sigma_{t+1}{}^{\delta} = w' + \alpha' \cdot (|\varepsilon_t| - \tau \cdot \varepsilon_t)^{\delta} + \beta \cdot \sigma_t{}^{\delta}$$

$$(2.12) \qquad ({}_h\sigma_{h(k+1)}{}^{(h)})^{\delta} - ({}_h\sigma_{hk}{}^{(h)})^{\delta} =$$
$$w_h' + [\beta_h + h^{-0.5\delta} |{}_hz_{hk}{}^{(h)}|^{\delta} \cdot (1-\tau s_k)^{\delta} \cdot \alpha_h' - 1] \cdot ({}_h\sigma_{hk}{}^{(h)})^{\delta}.$$

Fornari and Mele (1994a) showed that the diffusion limit of (2.12) is:

$$(2.13) \qquad d\sigma_t{}^{\delta} = (w - \theta_\delta . \sigma_t{}^{\delta}) \cdot dt + \Omega_\delta \cdot \sigma_t{}^{\delta} \cdot dW_t, \qquad \theta \geq 0$$

where W_t is a standard (scalar) Brownian motion; w is the continuous time counterpart (c.t.c.) of w', and:

$$(2.14) \quad \theta_\delta = \underline{\text{c.t.c.}}\ \alpha' \frac{2^{0.5(\delta-1)}\Gamma[0.5(\delta+1);0.5]}{(2\pi)^{0.5}} [(1+\tau)^{\delta} + (1-\tau)^{\delta}] + \beta - 1$$

$$(2.15) \quad \Omega_\delta{}^2 = \underline{\text{c.t.c.}}\ \alpha' . \frac{2^{(\delta-0.5)}\Gamma[\delta+0.5);0.5]}{(2\pi)^{0.5}} \cdot [(1+\tau)^{2\delta} + (1-\tau)^{2\delta}]$$

$$- \frac{2^{0.5(\delta-1)}\Gamma[0.5(\delta+1);0.5]}{(2\pi)^{0.5}} \cdot [(1+\tau)^{\delta} + (1-\tau)^{\delta}]^2,$$

where $\Gamma[0.5(\delta+1);0.5] \equiv \int_{0,\infty} (0.5)^{0.5(\delta+1)} X^{0.5(\delta-1)} \exp(-0.5X)\,dX$.

It is now easy to verify that the diffusion limit (2.13) collapses to Nelson's (1990) standard one when $\delta=2$ and $\tau=0$; in the case of the GJR model, which is nested into the Power ARCH, (2.4') reduces to

$$(2.16) \qquad \sigma_t{}^2 = w + g_0 \cdot \varepsilon_{t-1}{}^2 + p \cdot \sigma_{t-1}{}^2 + \Phi \cdot s_{t-1} \cdot \varepsilon_{t-1}{}^2$$

once that (see also Ding et al., 1993):

(2.17.1) $$\delta = 2$$

(2.17.2) $$\alpha = \alpha'(1-\tau)^2$$

(2.17.3) $$g_1 = 4\alpha'\tau.$$

Substitution of relations (2.17) into (2.14)-(2.15) gives, after tedious but straight forward algebra:

(2.18.1) $$\theta_2^{GJR} = \underline{c.t.c.}\ p + \alpha + (g_1/2) - 1$$

(2.18.2) $$\Omega_{2GJR}^2 = \underline{c.t.c.}\ 2g_0^2 + 0.75 \cdot g_1^2,$$

suggesting that the diffusion limit of the GJR model has the following form:

(2.19) $$d\sigma_t^2 = (w - \theta_2^{GJR} \cdot \sigma_t^2) \cdot dt + \Omega_2^{GJR} \cdot \sigma_t^2 dW_t,$$

where W_t is a standard (scalar) Brownian motion.[8]

At this point, it is important to investigate whether the diffusion limit (2.13) is able to generalize the diffusion of the Sign- and Volatility-Switching ARCH models; in this case, in fact, they would be just particular cases of the Power ARCH. It turns out that the answer is negative for the Sign-Switching model and the Volatility-Switching Model B.

To show this, we first take the expected value per unit of time of $({}_h\sigma_{(k+1)}{}^2 - {}_h\sigma_{hk}{}^2)$ in (2.11.5), with $\sigma_h{}^2$ generated by (2.1), obtaining:

$$E[h^{-1}({}_h\sigma_{h(k+1)}{}^2 - {}_h\sigma_{hk}{}^2) \mid F_{hk}] =$$

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8 Glosten et al. (1993) compute the amount of persistence of their model by simply regressing the (estimated) σ_t^2 on σ_{t-1}^2. This procedure, however, lacks for a rigorous theoretical justification. Relation (2.18.2), instead, provides the analytical expression for the amount of persistence of the GJR model. Note that this is an increasing function of g_1.

$$h^{-1}w_h + h^{-1}\Phi_h . s_k + h^{-1}(\beta_h + h^{-1}\alpha_h . z_{hk}^2 - 1)\cdot\sigma_{hk}^2.$$

To avoid the drift per unit of time explosion, as $h\rightarrow 0$, we require the following Lipschitz conditions:

(2.20) $$\lim_{h\rightarrow 0} \ h^{-1}w_h = w$$

(2.21) $$\lim_{h\rightarrow 0} \ h^{-1}(\alpha_h+\beta_h-1) \equiv \lim_{h\rightarrow 0} h^{-1}\theta_h = -\theta, \quad \theta\geq 0.$$

Hence

(2.22) $$\lim_{h\rightarrow 0} \{E[h^{-1}({}_h\sigma_{h(k+1)}{}^2 - {}_h\sigma_{hk}{}^2)|F_{hk}]\} = \lim_{h\rightarrow 0} [h^{-1}w_h + h^{-1}(\alpha_h+\beta_h-1)\cdot {}_h\sigma_{hk}{}^2] = w - \theta\cdot\sigma_t^2.$$

The evaluation of $E[h^{-1}({}_h\sigma_{h(k+1)}{}^2 - {}_h\sigma_{hk}{}^2)|F_{hk}]^2$ gives:

(2.23) $$E[h^{-1}({}_h\sigma_{h(k+1)}{}^2 - {}_h\sigma_{hk}{}^2)^2|F_{hk}]$$
$$= E[h^{-1}w_h^2 + h^{-1}\Phi_h^2\cdot s_k^2 + h^{-1}\cdot(\beta_h + h^{-1}\alpha_h\cdot {}_h z_{hk}{}^2-1)^2\cdot {}_h\sigma_{hk}{}^4 + 2h^{-1}\alpha_h^2\cdot {}_h\sigma_{hk}{}^4 + 2h^{-1}w_h(\beta_h + h^{-1}\alpha_h\cdot {}_h z_{hk}{}^2 -1)\cdot {}_h\sigma_{hk}{}^2].$$

Using the Lipschitz conditions (2.20)-(2.21), and assuming the existence of the following limits:

(2.24) $$\lim_{h\rightarrow 0} \ 2h^{-1}\alpha_h^2 = \alpha^2$$

(2.25) $$\lim_{h\rightarrow 0} \ h^{-1}\Phi_h^2 = \Phi^2,$$

we get:

(2.26) $$\lim_{h\rightarrow 0} E[h^{-1}({}_h\sigma_{h(k+1)}{}^2 - {}_h\sigma_{hk}{}^2)|F_{hk}]^2 = \Phi^2 + \alpha^2\sigma_t^4.$$

Relations (2.22)-(2.25) suggest that $\{{}_h\sigma_{h(k+1)}{}^2\}_{k=0,\infty}$ converges in distribution to a diffusion limit of the following form, as $h\rightarrow 0$:

(2.27) $$d\sigma_t^2 = (w - \theta\sigma_t^2)dt + (\Phi^2 + \alpha^2\sigma_t^4)^{0.5}dW_t, \quad \theta\geq 0,$$

where dW_t denotes the increments of a (scalar) standard Brownian motion. It is straight forward to check that the structural form of such diffusion limit equals the same expression derived in Nelson (1990, eq. 2.40), when $\Phi = 0$.

Using the same arguments of the present section, it is possible to prove that the structural form of the diffusion limit for the Volatility-Switching model B (2.8)-(2.8.2) is essentially the same as (2.27). As regards the model (2.8)-(2.8.1), however, it turns out that its diffusion limit is:

$$d\sigma^2_t = (v - \theta\cdot\sigma^2_t)\cdot dt + \Omega\cdot\sigma^2_t\cdot dW_t \tag{2.28}$$

which has the same form as the standard diffusion limit of Nelson (1990), though the parameters steering the intensity at which the "volatility of volatility" is mapped onto the volatility changes are not the same (results are available on request from the authors).

2.3 The stationary distribution of the conditional variance

With concern for the GARCH(1,1), Nelson (1990) showed that the stationary distribution of the conditional variance becomes an inverted Gamma, as h drops to zero. Then, it is interesting to analyse what this distribution is in the case of both the Sign-Switching and Volatility-Switching B models. In Appendix B, we show that such a distribution is:

(2.29) $P(v|v_0)$ proportional to: $(r^2+v^2)^{-m}\cdot e^{-n\cdot \mathrm{arcotg}(q\cdot v)}$,

where $v\equiv\sigma^2$, v_0 the initial condition, $r^2\equiv(\theta/\alpha)^2$, $m\equiv(\theta/\alpha^2)+1$, $n=2w\cdot|\Phi|^{-1}$, $q\equiv\alpha\cdot(|\Phi|)^{-1}$. Note that:

(2.30) $\lim_{\Phi\to 0^+} P(v|v_0) = \lim_{\Phi\to 0^-} P(v|v_0)$ proportional to:

$$v^{-2m} \cdot e^{-k/v},$$

where $k \equiv 2w \cdot \alpha^{-2}$. The distribution (2.30) is an inverted Gamma, consistently with Nelson's (1990) results.

From (2.29), it is easy to find the stationary distribution for the standard deviation of the Sign- and Volatility-Switching model B innovations:

(2.31) $$f(\sigma|\sigma_0)\,d\sigma \text{ proportional to: } 2\sigma \cdot (r^2+\sigma^4)^{-m} \cdot e^{-n \cdot \operatorname{arcotg}(q \cdot \sigma^2)} \cdot d\sigma.$$

Relation (2.31) implies that the innovations of the Sign-Switching or, alternatively, Volatility-Switching Model B, say ε^*, have a stationary density function, $p(\varepsilon^*)$, which solves:

(2.32) $$p(\varepsilon^*) = \int_{0,\infty} N(\varepsilon^* \cdot \sigma^{-1}) \cdot f(\sigma) \cdot \sigma^{-1} \cdot d\sigma,$$

where $N(\cdot)$ is a standard normal density function. Unfortunately we did not manage to solve the integral in (2.32) analytically; hence a numerical procedure was adopted. The adaptive recursive Simpson's rule was employed (see, e.g., Abramowitz and Stegun, 1970, formula (25.4.5), p.866), and results are reported in Figure 1, which compares the density $p(\cdot)$ of (2.32) (labeled "Switching") with a standard normal density (labeled "Normal").[9] It is easy to note its fatter tails compared to the corresponding area of the normal variate; such a feature allows one to capture the high number of outliers observed in empirical distributions.

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9 All the computations were performed with the Quadrature Routine of Matlab. The parameters values used in the numerical integration procedure were $w=0 \cdot 12$, $\alpha=0 \cdot 15$, $\theta = 0.0045$, $\Phi=0.010$. The range of variation for ε^* was -8,+8.

3. Empirical Analysis

We have employed seven stock market indices to test and compare the empirical performances of the Sign- and Volatility-Switching ARCH models to the GJR formulation, indicated by Engle and Ng (1993) as the best parametric model for the conditional variance. The series are the Financial Times World Stock Indices (taken from DRI), observed at a weekly frequency from January 1, 1986 to July 16, 1993. The sample includes 390 observations for the UK, the US, Hong Kong, Italy, Singapore, Japan, and South Africa.

Let $P^j{}_t$ denote the level of the j-th stock index at time t; then we evaluate seven series of 'unpredictable' returns, u_t, which are the residuals of univariate regressions of ex-post returns $r_t{}^j$ ($r_t{}^j = \log(P^j{}_t/P^j{}_t)-1$) on a constant and on their lagged values, i.e.:

$$(3.1) \qquad r_t{}^j = \mu_0 + \Sigma_{i=1,p}\mu_i \cdot r_{t-i}{}^j + u_t, \qquad t=1,\ldots T,$$

where μ_0 and μ_i $(i=1,\ldots,p)$ are real, non stochastic parameters, i is a suitably chosen lag, and T is sample size. In all the regressions, i was chosen to be one, with the exception of the UK and South Africa, where i was two (the Schwarz criterion was employed to select the lags).

Table 1 shows a number of preliminary statistics for the seven unpredictable returns. Residual autocorrelation (of the mean) was ascertained by means of the Box and Pierce's Q test evaluated up to the fifth and tenth lag. Under the null of no autocorrelation, such statistics are asymptotically distributed as chi-squares with five and ten degrees of freedom, respectively; their 95 percent critical levels are 11.1 and 18.3. The hypothesis of autocorrelation in the second order moments was tested via the TR^2 (Engle, 1982) evaluated up to the fifth lag; it is asymptotically distributed as a chi-square with five degrees of freedom. As far as the unconditional distribution of the unpredictable returns is concerned, the coefficients of skewness and kurtosis, along with the Jarque and Bera's (JB) test are computed; the latter is asymptotically distributed as a chi-square with two degrees of freedom under the null that the data come from a normal distribution. Its 95 percent critical level is 5.99.

The tests for the presence of asymmetric behaviour of the volatility developed by Engle and Ng (1993) are also performed. These are the Sign Bias test (SBT), the Negative Sign Bias test (NSBT), the Positive Sign Bias Test (PSBT), and the Joint test (TR^2). SBT, NSBT and PSBT are the t-statistics for the coefficients of a linear regression of the squared innovations of regression (3.1) on S_{t-1}^-, $S_{t-1}^- \cdot u_{t-1}$, and $S_{t-1}^+ \cdot u_{t-1}$, respectively (where S_t^- is a dummy variable which equals plus one if $\text{sign}(u_t)=-1$, and zero otherwise; $S_t^+ = 1-S_t^-$), i.e.:

$$(3.2) \quad u_t^2 = c_0 + c_1 \cdot S_{t-1}^- + c_2 \cdot S_{t-1}^- \cdot u_{t-1} + c_3 \cdot S_{t-1}^+ \cdot u_{t-1} + z_t,$$

where c_i $(i=0,1,2,3)$ are real, non stochastic parameters and z_t is a white noise process. The Joint test is simply the TR^2 of (3.2), asymptotically chi-square distributed with three degrees of freedom; its critical 95 percent level is 7.81.

All the unpredictable returns are not significantly autocorrelated. Their unconditional distributions are not normal since the coefficients of skewness and the kurtoses diverge from those typical of a Gaussian distribution, and the JB test rejects the normality hypothesis at any reasonable level of confidence. The evidence for ARCH effects is clearly supported by the TR_5^2 test.[10] Asymmetries in

10 However, three out of the seven unpredictable returns (the UK, Singapore and Hong Kong) do not display the

volatility exist in all the series, expecially the US, Singapore and Japan.[11] As regards the Italian unpredictable returns, no evidence of asymmetric behaviour emerges from this preliminary analysis. Table 2 shows the values of the parameters of both the Sign-Switching GARCH(1,1,1) and the GJR models. All the Sign-Switching GARCH(1,1,1) models have the following structure:

$$r_t^j = \mu_0 + \mu_1 \cdot r_{t-1}^j + u_t \tag{3.3.1}$$

$$u_t \mid I_{t-1} \sim N(0, \sigma_t^2) \tag{3.3.2}$$

$$\sigma_t^2 = w + \alpha \cdot u_{t-1}^2 + \beta \cdot \sigma_{t-1}^2 + \Phi_0 \cdot s_{t-1}, \tag{3.3.3}$$

where μ_0, μ_1, w, α, β and Φ_0 are real, non stochastic parameters, r_t^j is the ex-post return of the j-th index, s_t is the sign of the forecast error (u_t) dated t. The structure of all the GJR models is the same as (3.3), but (3.3.3) is replaced with:

$$\sigma_t^2 = w + \alpha \cdot u_{t-1}^2 + \beta \cdot \sigma_{t-1}^2 + \Phi_1 \cdot s_{t-1} \cdot u_{t-1}^2, \tag{3.3.4}$$

where Φ_1 is a real, non-stochastic parameter.

Tables 3 and 4 contain the estimates of the Volatility-Switching models A and B. The structural form of these models is also the same as (3.3), but (3.3.3) is replaced with:

$$\sigma_t^2 = w + \alpha \cdot u_{t-1}^2 + \beta \cdot \sigma_{t-1}^2 + \Phi \cdot s_{t-1} \cdot v_{t-1}, \tag{3.3.5}$$

where:

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presence of ARCH effects. Nevertheless, the absence of autocorrelation in variance for such returns should be mainly due to the presence of a few outliers, which amplify considerably the range of the series. When they are removed the TR^2 become unequivocally significant.

11 Again, the loss of significance of asymmetric tests in the UK is likely to be related to the presence of two outliers; in fact, when they are excluded, the tests become highly significant.

(3.3.5.1) $$v_t \equiv u_t^2 - \sigma_t^2 \quad \text{(Model A)}$$

(3.3.5.2) $$v_t \equiv u_t^2 \cdot \sigma_t^{-2} - 1 \quad \text{(Model B)}.$$

Our empirical results confirm the outcome of the preliminary tests. All the series display both high ARCH and asymmetric effects, with the significant exception of the Italian returns (see, also, De Santis, 1991), where a Wald test, under the null of no asymmetry, rejects the four asymmetric models against the symmetric GARCH(1,1), at any level of confidence[12]. The models have successfully captured asymmetries in volatility, as highlighted by the diagnostic tests. Based on a likelihood ratio test, the Sign- and Volatility-Switching models fit the patterns of the data better than the GJR for the UK, the US and Japan, while the GJR outperforms them in the remaining cases. When the models and ranked according to the kurtosis of the standardised residuals, the Volatility-Switching B outperforms the GJR for Italy, Japan, Singapore and the US; Volatility-Switching A outperforms the GJR for Hong Kong, Singapore and the United Kingdom. Sign-Switching ARCH are not very successful for the UK and Hong Kong, if compared with Volatility-Switching models and the GJR.

However, asymmetries in volatility can be reverted, though the GJR model is not able to discriminate among such different patterns. On the contrary, Volatility-Switching models allow a more detailed analysis of the asymmetric behaviour of the volatility enabling, by construction, shifts in its direction, by switching between low and high values of past shocks.

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12 In the estimated models, the lags chosen in the conditional mean equation coincide with those employed in the preliminary analysis, and w, α and β have been estimated (and reported) in square roots. The likelihood has been maximised by means of the Berndt et al. (1974) procedure.

In this respect, the estimation results on our sample unambiguously indicate that for the UK, Hong Kong, Singapore and South Africa, high negative shocks induce more volatility than high positive shocks, while low negative shocks induce less volatility than positive shocks of the same size. Such results are fully consistent with those from the GJR ones, since Φ_1 (in equation (3.3.4)) turned out to be negative for all the analysed returns. However, Volatility-Switching models also show the opposite to hold for the US and Japan where, further, Φ_1 is not highly significant.

As regards the normality of the residuals standardized with the conditional standard deviation, a disturbing feature of the four estimated models is evidenced by high values of the JB test. At a very least, this suggests to modify the distributional assumptions made for the conditional distribution of the innovations. It may turn out that other hypotheses, such as the Generalized Error Distribution, or the Student's-t, could improve the results achieved so far.

4. Conclusions

The Sign- and Volatility-Switching models have been presented in the paper. They allow for an asymmetric behaviour of the conditional volatility with respect to negative and positive shocks, since they map the sign of past forecast errors onto the current, conditional volatility. Volatility-Switching models are also able to capture "reversions" in the asymmetric behaviour of the volatility. Weak convergence results have been presented for the Sign- and Volatility-Switching models; they converge in distribution to Itô diffusion processes, and are shown to have a stationary distribution function, for which we have found a closed form solution. Numerical procedures were employed to compute the stationary distribution of the innovations of these models. Empirical analysis has shown that the proposed models provide the same interpretative results as the GJR giving, at the same time, further insights regarding the reversion of the asymmetries in volatility.

Two issues seem to deserve further research: the hypothesis of conditional normality for the innovations of the model needs be modified; second, the response function of the two Volatility-Switching ARCH models employed in this work is just a linear specification in v_{t-1}, the size of previous unexpected volatility times the sign of past forecast errors. Other (non-linear) specifications appear plausible, though the weak convergence analysis for such specifications is likely to become more intricate.

PRELIMINARY TESTS ON THE UNPREDICTABLE RETURNS*

	Q_5	Q_{10}	TR_5^2	SKEW	KU	JB
UK	6.2190	13.6770	8.5600	-2.6500	31.8800	13794.01
US	0.6920	8.0320	42.1300	-1.2000	8.6300	599.31
HK	3.7950	9.5670	5.1200	-4.2100	44.7300	28957.02
IT	7.4360	9.9140	36.0300	-0.1900	4.4500	35.95
SI	9.4580	14.8960	8.2020	-3.4400	41.8500	24720.10
JA	4.0850	15.7680	37.3010	-0.2600	5.4600	101.15
SA	7.6970	12.2380	37.4400	-1.0500	8.0800	483.46

	SBT	NSBT	PSBT	JOINT (TR^2)
UK	0.5919	-0.6295	-0.0540	2.02
US	-1.0080	-6.8515	1.3700	47.55
HK	-0.0305	-3.2310	0.1232	11.96
IT	0.0045	1.4900	-1.4300	6.55
SI	0.6330	-2.7033	0.3482	51.94
JA	1.0200	-3.7684	3.0410	25.78
SA	-0.3420	-3.3250	0.2905	12.82

* Countries are coded as follows: UK (United Kingdom), US (USA), HK (Hong Kong), IT (Italy), SI (Singapore), JA (Japan), SA (South Africa). Q_5 and Q_{10} are the Box and Pierce's Q tests up to 5 and 10 lags, respectively; TR_5^2 the is Engle's TR^2 computed up to the fifth lag; SKEW is the coefficient of skewness; KU the coefficient of kurtosis; JB is the Jarque and Bera's test, SBT is the sign bias test; NSBT is the negative size bias test; PSBT is the positive size bias test; JOINT (TR^2) is the Lagrange Multiplier test that the squared unpredictable returns are not simultaneously explained by S^-, S^-u, and S^+u, respectively.

Table 2

SIGN-SWITCHING ARCH MODELS*

Parameters estimates (t-ratios in parentheses)

	μ_0	μ_1	w	α	β	Φ_0	Log of Likel.
UK:	2.60E-3	0.108	0.0140	0.309	0.524	-2.78E-4	1260.86
	(2.173)	(0.919)	(7.813)	(2.991)	(7.737)	(-10.405)	
US:	2.18E-3	-0.039	0.017	0.554	0.257	-7.48E-5	1333.67
	(2.096)	(-0.628)	(12.162)	(8.881)	(1.210)	(-3.481)	
HK:	6.70E-3	0.109	0.015	0.618	0.710	-4.260E-5	1155.02
	(4.163)	(1.463)	(5.239)	(25.906)	(12.520)	(-0.939)	
IT:	-1.10E-4	0.115	5.82E-3	0.384	0.980	-2.22E-5	1167.06
	(-0.080)	(2.054)	(3.523)	(6.366)	(32.412)	(-0.832)	
SI:	4.83E-3	0.101	2.516E-2	0.750	0.288	-1.95E-4	1114.17
	(2.842)	(1.513)	(16.172)	(15.980)	(3.979)	(-3.222)	
JA:	2.55E-3	-0.038	1.38E-2	0.557	0.222	-7.82E-5	1329.60
	(2.075)	(-1.681)	(12.374)	(8.839)	(12.970)	(-3.399)	
SA:	3.08E-3	0.109	1.51E-2	0.439	0.757	-1.16E-4	1160.36
	(1.910)	(1.521)	(5.944)	(5.741)	(10.938)	(-2.989)	

GJR MODELS

Parameters estimates (t-ratios in parentheses)

	μ_0	μ_1	w	α	β	Φ_0	Log of Likel.
UK:	2.27E-3	0.117	0.016	0.651	0.489	-0.477	1267.78
	(1.628)	(2.161)	(10.239)	(13.351)	(5.465)	(-8.364)	
US:	2.45E-3	-0.060	0.011	0.446	0.737	-0.087	1332.02
	(2.417)	(-1.015)	(7.946)	(7.113)	(15.296)	(-1.648)	
HK:	4.25E-3	0.095	0.018	0.534	0.659	-0.357	1169.74
	(2.614)	(1.547)	(10.318)	(13.766)	(11.994)	(-9.328)	
IT:	7.41E-5	0.117	5.64E-3	0.382	0.910	-1.69E-2	1166.89
	(0.051)	(2.058)	(3.352)	(6.450)	(33.412)	(-0.613)	
SI:	3.23E-3	0.108	0.0206	0.744	0.541	-0.419	1119.80
	(1.722)	(1.565)	(12.847)	(17.467)	(13.111)	(-7.105)	
JA:	2.43E-3	-0.051	1.13E-2	0.447	0.741	-8.85E-2	1327.94
	(2.982)	(-1.754)	(7.718)	(7.269)	(15.699)	(-1.936)	
SA:	3.52E-3	0.117	1.44E-2	0.397	0.792	-0.360	1166.66
	(2.550)	(1.921)	(5.225)	(5.648)	(12.240)	(-9.309)	

continued

TESTS ON STANDARDIZED RESIDUALS

		Sign-Switching ARCH	GJR model
UK:	Skewness:	-1.4600	-0.9400
	Kurtosis:	14.0000	10.7200
	Jarque and Bera:	2072.4000	840.5500
	Sign bias test:	0.1987	0.0688
	Negative size bias test:	-0.3482	0.4833
	Positive size bias test:	-0.2684	0.0349
	Joint test (TR2):	0.7189	0.2525
US:	Skewness:	-0.4600	-0.5500
	Kurtosis:	4.4100	4.6700
	Jarque and Bera:	45.3500	63.9800
	Sign bias test:	-0.7792	1.1369
	Negative size bias test:	-0.5389	-0.1418
	Positive size bias test:	-0.5281	0.7926
	Joint test (TR2):	0.6189	1.9444
HK:	Skewness:	-0.7600	-0.4900
	Kurtosis:	7.2200	4.3100
	Jarque and Bera:	202.8600	42.8200
	Sign bias test:	-0.0492	-0.3118
	Negative size bias test:	-1.4451	-0.3163
	Positive size bias test:	-1.8828	-0.4062
	Joint test (TR2):	6.5912	0.2664
IT:	Skewness:	-0.2500	-0.2626
	Kurtosis:	3.6000	3.7200
	Jarque and Bera:	6.0700	12.6200
	Sign bias test:	-0.6142	-0.1938
	Negative size bias test:	0.0920	0.2868
	Positive size bias test:	-0.8676	-0.6601
	Joint test (TR2):	0.3900	0.4200
SI:	Skewness:	-1.5980	-1.0600
	Kurtosis:	17.5000	13.3200
	Jarque and Bera:	3525.8000	1775.9000
	Sign bias test:	0.0591	0.7802
	Negative size bias test:	-0.3219	0.1802
	Positive size bias test:	-0.5823	0.1020
	Joint test (TR2):	0.9200	0.8400
JA:	Skewness:	-0.4500	-0.5500
	Kurtosis:	4.3800	4.6600
	Jarque and Bera:	42.3000	63.4500
	Sign bias test:	-0.0754	1.1154
	Negative size bias test:	-0.0517	-0.1479
	Positive size bias test:	-0.0535	0.7950
	Joint test (TR2):	0.6224	1.8781
SA:	Skewness:	-0.6087	-0.9500
	Kurtosis:	7.6200	8.2600
	Jarque and Bera:	365.3200	540.4400
	Sign bias test:	0.0869	1.2612
	Negative size bias test:	0.5187	0.6230
	Positive size bias test:	0.4020	0.1916
	Joint test (TR2):	0.4800	2.3000

* For country codes, see the note at Table 1.

VOLATILITY-SWITCHING MODEL A*
(t-ratios in parentheses)

	UK	US	HK	IT	SI	JA	SA
Logarithm of the likelihood:	1269.92	1332.89	1165.75	1168.11	1110.42	1331.82	1158.53
μ_0	2.72E-3	2.71E-3	6.10-3	1.66E-4	4.69E-3	2.86E-3	6.05E-4
	(3.472)	(3.117)	(4.568)	(0.129)	(2.831)	(2.895)	(4.520)
μ_1	0.112	-0.068	0.077	0.092	0.109	-0.028	0.079
	(2.772)	(-1.097)	(1.160)	(2.001)	(1.732)	(-0.427)	(1.198)
w	5.84E-3	9.74E-3	1.517E-2	8.02E-3	1.39E-2	1.518E-2	1.517
	(4.588)	(8.352)	(9.276)	(5.829)	(5.677)	(9.286)	(9.286)
α	0.321	0.510	0.5567	0.432	0.4648	0.6568	0.455
	(4.169)	(9.184)	(8.487)	(7.304)	(7.313)	(10.274)	(8.449)
β	0.932	0.738	0.746	0.874	0.803	0.743	0.746
	(43.822)	(17.166)	(19.025)	(29.136)	(13.703)	(2.673)	(19.031)
Φ	-0.108	0.042	-0.204	-0.007	-0.152	0.148	-0.203
	(-3.416)	(2.747)	(-4.263)	(-0.196)	(-3.776)	(2.630)	(-4.269)

Tests on standardized residuals

	UK	US	HK
Skewness:	-0.6112	-0.5079	-0.6453
Kurtosis:	12.5678	4.5393	5.3817
Jarque and Bera:	1114.6000	40.0900	366.9900
Sign bias test:	0.6348	0.4912	0.9182
Negative size bias test:	-0.0766	-0.7817	-0.8788
Positive size bias test:	-0.1897	-0.1616	-0.8066
Joint (TR^2):	1.3200	1.9200	6.7400

	IT	SI	JA
Skewness:	-0.2245	-2.1121	-0.5201
Kurtosis:	3.8803	15.0700	4.6072
Jarque and Bera:	11.6100	1961.9000	43.6900
Sign bias test:	-0.0708	1.0736	0.2851
Negative size bias test:	0.5315	-0.3410	-0.4316
Positive size bias test:	-0.6607	0.1576	-0.5500
Joint (TR^2):	0.6320	2.3220	1.9350

	SA
Skewness:	-0.6435
Kurtosis:	5.3948
Jarque and Bera:	82.2000
Sign bias test:	0.9275
Negative size bias test:	-0.8922
Positive size bias test:	-0.7966
Joint (TR^2):	3.5400

* For country codes, see the note at Table 1.

Table 4

Volatility-Switching Model B*
(t-ratios in parenteses)

	UK	US	HK	IT	SI	JA	SA
Logarithm of the likelihood	1263.32	1333.73	1160.61	1166.70	1110.30	1328.48	1160.65
μ_0	2.317E-3	0.003	0.006	2.902E-4	5.273E-3	2.913E-3	3.392E-4
	(2.085)	(3.139)	(4.518)	(0.218)	(3.258)	(3.025)	(2.4132)
μ_1	0.112	-0.074	0.095	0.117	0.128	-3.727E-3	0.1238
	(2.099)	(-1.160)	(1.329)	(2.097)	(1.941)	(-0.597)	(2.1636)
w	5.525E-3	0.010	0.016	5.732E-3	0.016	1.704E-2	0.0172
	(4.398)	(8.557)	(9.104)	(3.321)	(7.783)	(14.308)	5.2872
α	0.167	0.535	0.446	0.038	0.591	0.610	0.5380
	(2.506)	(11.645)	(14.390)	(6.739)	(12.056)	(11.029)	6.8889
β	0.9540	0.7030	0.7277	0.908	0.718	0.334	0.6678
	(55.2654)	(15.214)	(18.848)	(33.730)	(14.929)	(6.042)	6.0053
Φ	-3.209E-5	3.67E-5	-1.473E-4	3.50E-6	-1.50E-4	5.801E-2	-1.599E-4
	(-3.3685)	(2.000)	(-5.7552)	(0.0173)	(-2.892)	(2.4301)	(-3.6766)

Tests on standardized residuals

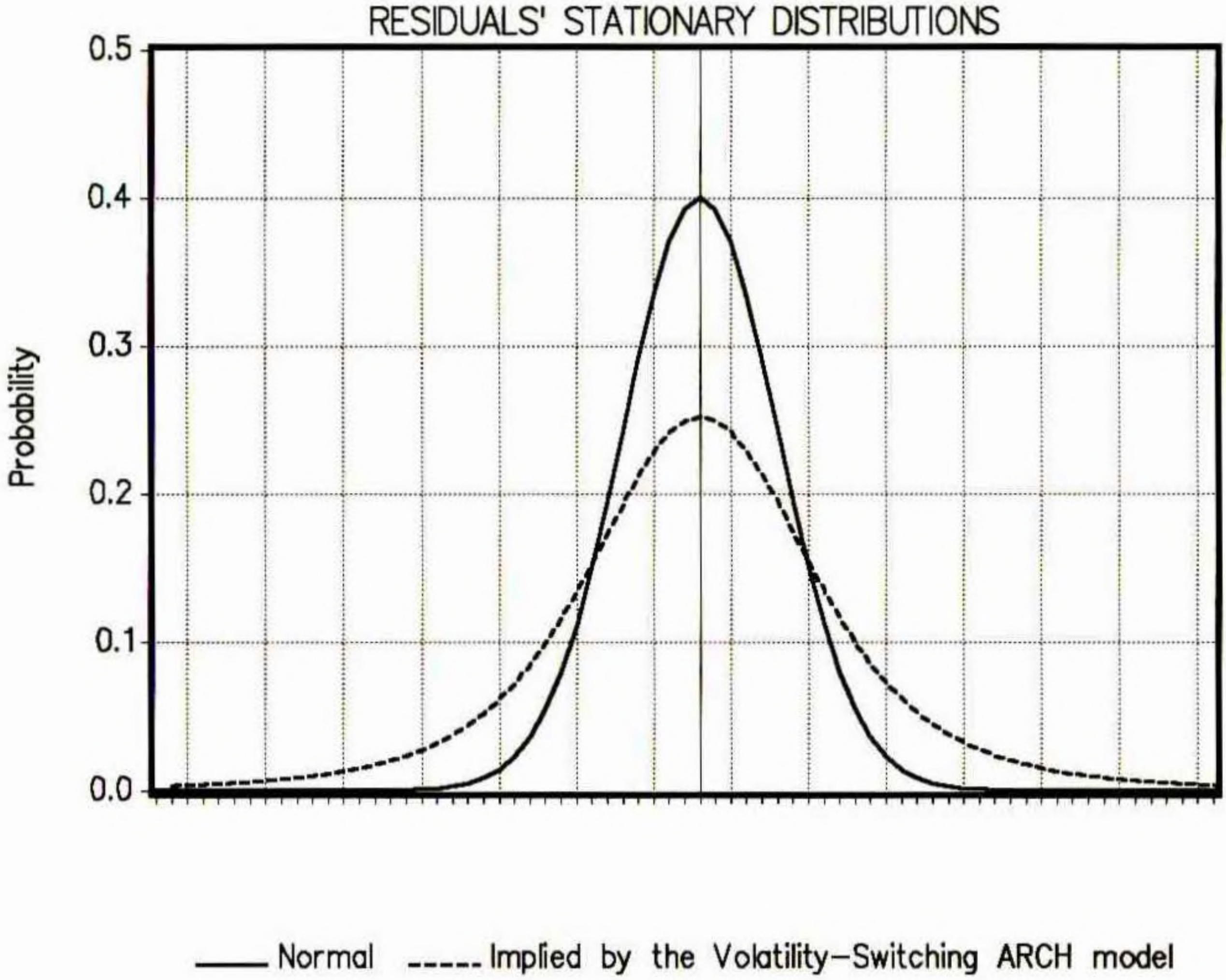
	UK	US	HK
Skewness:	-0.7416	-0.4771	-0.7435
Kurtosis:	13.1200	4.4291	6.1258
Jarque and Bera:	1673.7000	46.4000	190.8000
Sign bias test:	0.4943	0.1348	0.5084
Negative size bias test:	-0.3214	-0.8215	-1.2838
Positive size bias test:	-0.2169	-0.5701	-0.7484
Joint (TR^2):	1.3200	2.6200	4.3900

	IT	SI	JA
Skewness:	-0.2400	-1.1524	-0.4880
Kurtosis:	3.6600	15.1776	4.4300
Jarque and Bera:	10.6500	2454.4000	35.5900
Sign bias test:	-0.1244	1.1587	0.0835
Negative size bias test:	0.2777	-0.2079	-0.5144
Positive size bias test:	0.5102	0.1020	-0.7026
Joint (TR^2):	0.2800	2.6400	1.9300

	SA
Skewness:	-0.7581
Kurtosis:	3.4885
Jarque and Bera:	71.0500
Sign bias test:	0.2394
Negative size bias test:	0.3282
Positive size bias test:	-0.7589
Joint (TR^2):	1.5400

* For country codes, see the note at Table 1.

Fig. 1


RESIDUALS' STATIONARY DISTRIBUTIONS
0.5
0.4
0.3
0.2
0.1
0.0
Probability
Normal
Implied by the Volatility-Switching ARCH model

APPENDIX A

Derivation of the first unconditional moments of ε for the Sign-Switching GARCH(1,1,1)

Given that $\varepsilon_t | I_{t-1} \sim N(0, \sigma_t^2)$, and:

(A1) $$E(e_t^{2m} | I_{t-1}) = (w + \beta\sigma_{t-1}^2 + \alpha\varepsilon_{t-1}^2 + \Phi s_{t-1})^m \cdot h_{2m},$$

where $h_{2m} \equiv \Pi_{j=1,m}(2j-1)$, setting m=1 in (A1) yields:

$$E(\varepsilon_t^2 | I_{t-1}) = w + \beta\sigma_{t-1}^2 + \alpha\varepsilon_{t-1}^2 + \Phi s_{t-1},$$

when t is allowed to tend to infinity, so that the dependence of the current values on their past realisation becomes negligible, we get:

$$E(\varepsilon^2) = w + \beta E(\sigma^2) + \alpha E(\varepsilon^2) + \Phi E(s) = w/(1-\beta-\alpha).$$

Setting m=2 and letting time go to infinity in (A1) gives:

$$E(\varepsilon^4) = 3 \cdot E[w^2 + \beta^2\sigma^4 + \alpha^2\varepsilon^4 + \Phi^2 + 2\alpha\beta\sigma^2\varepsilon^2 + 2\beta w\sigma^2 + 2\alpha w\varepsilon^2]$$

$$= 3w^2 + 3\beta^2 E(\sigma^4) + 3\alpha^2 E(\varepsilon^4) + 3\Phi^2 + 6\alpha\beta E(\sigma^4) + 6\beta w E(\sigma^2) + \\ + 6\alpha w E(\varepsilon^2)$$

$$= 3w^2 + 3\beta^2 E(\varepsilon^4)/3 + 3\alpha^2 E(\varepsilon^4) + 3\Phi^2$$

$$+ 6\alpha\beta(1/3)E(\varepsilon^4) + 6\beta w(w/(1-\beta-\alpha)) + 6\alpha w(w/(1-\alpha-\beta))$$

$$= [3(w^2+\Phi^2)(1-\beta-\alpha)+6w^2(\alpha+\beta)] \cdot [(1-\beta-\alpha)(1-\beta^2-3\alpha^2-2\beta\alpha)]^{-1}.$$

The last expression collapses to the standard result of Bollerslev (1986) when Φ=0.

Derivation of the first unconditional moments of ε for the Volatility-Switching model A

Let ε_t be zero mean conditionally normally distributed, with conditional variance σ_t^2. Hence:

$$\text{(A2)} \quad E(\varepsilon_t^{2m}|I_{t-1}) = (w + \beta\sigma_{t-1}^2 + \alpha\varepsilon_{t-1}^2 + \Phi s_{t-1}v_{t-1})^m \cdot h_{2m}$$
$$= (w + (\alpha+\Phi s_{t-1})\varepsilon_{t-1}^2 + (\beta-\Phi s_{t-1})\sigma_{t-1}^2)^m \cdot h_{2m},$$

where $h_{2m} \equiv \Pi_{j=1,m}(2j-1)$. Setting m=1 in (A2) gives:

$$E(\varepsilon_t^2|I_{t-1}) = w + (\alpha+\Phi s_{t-1})\varepsilon_{t-1}^2 + (\beta - \Phi s_{t-1})\sigma_{t-1}^2.$$

Recursive substitution yields:

$$E(\varepsilon^2) = w/(1-\beta-\alpha).$$

Setting m=2, and letting time go to infinity in (A2), so that the dependence of current values on past is negligible, gives:

$$E(\varepsilon^4) = 3w^2 + 3(\alpha^2+\Phi^2)E(\varepsilon^4) + 3(\beta^2+\Phi^2)\sigma^4$$
$$+ 6w^2(\alpha+\beta)/(1-\alpha-\beta) + 6(\alpha\beta-\Phi^2)\sigma^4$$

$$= 3w^2 + 3(\alpha^2+\Phi^2)E(\varepsilon^4) + (\beta^2+\Phi^2)E(\varepsilon^4)$$
$$+ 6w^2(\alpha+\beta)/(1-\alpha-\beta) + 2(\alpha\beta-\Phi^2)E(\varepsilon^4)$$

$$= [3w^2(1-\alpha-\beta)+6w^2(\alpha+\beta)]\cdot[(1-\alpha-\beta)(1-3\alpha^2-\beta^2-2\alpha\beta-2\Phi^2)]^{-1}.$$

Derivation of the first unconditional moments of ε for the Volatility-Switching model B

As usually, ε_t is zero mean conditionally normally distributed, with conditional variance σ_t^2, and its moment of the 2m-th order is given by:

(A3) $$E(\varepsilon_t^{2m}|I_{t-1}) = (w + \beta\sigma_{t-1}^2 + \alpha\varepsilon_{t-1}^2 + \Phi s_{t-1}v_{t-1})^m \cdot h_{2m} = (w + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2 + \Phi s_{t-1}Z_{t-1}^2 - \Phi s_{t-1})^m \cdot h_{2m},$$

where $h_{2m} \equiv \Pi_{j=1,m}(2j-1)$. Setting m=1 in (A3), and substituting recursively, yields:

$$E(\varepsilon^2) = w/(1-\beta-\alpha).$$

Setting m=2, letting time go to infinity in (A3), hence making negligible the dependence of current values on past, gives:

(A4) $$E(\varepsilon^4) = 3w^2 + 3\beta^2 E(\varepsilon^4) + 3\alpha^2 E(\varepsilon^4) + 21\cdot\Phi 2 + 6w^2(\alpha+\beta)/(1-\alpha-\beta) + 2\alpha\beta E(\varepsilon^4).$$

Rearranging terms in (A4) gives relation (2.10') of the text.

APPENDIX B

The stationary distribution of σ^2 in the Sign-Switching and Volatility-Switching Model B

Let $v \equiv \sigma^2$, and rewrite (2.27) as:

$$\text{(B1)} \qquad dv = (w - \theta v)dt + (\Phi^2 + \alpha^2 \cdot v^2)^{0.5} dW.$$

Let $p(s,t,v(t)|v(0))$ be the probability density function of v, given v(0) and $0<s<t<\infty$. The Fokker-Planck-Kolmogorov forward diffusion equation (see, e.g. Papoulis (1965)) associated with (B1) is:

$$\begin{aligned} &\partial(p(s,t,v|v_0)/(\partial t) \\ &= 0.5 \cdot \partial^2\{[(\Phi^2+\alpha^2 v^2)\cdot p(s,t,v|v_0)]\}/(\partial v^2) \\ &\quad - \partial\{[(w-\theta v)\cdot p(s,t|v_0)]\}/(\partial v), \end{aligned}$$

where $\partial(\cdot)$ denotes the derivative operator.

As Nelson (1990) remarks (see also Papoulis, 1965, Wong, 1971), an invariant density function (given it exists) must satisfy:

$$\text{(B2)} \qquad 0.5 \cdot \partial\{v[(\Phi^2+\alpha^2 v^2)\cdot p(v|v_0)\}/(\partial v) = (w-\theta v)\cdot p(v|v_0),$$

where $p(v|v_0) = \lim_{t\to 0} p(s,t,v,|v_0)$.

Developing explicitely (B2) yields:

$$[(\partial P)/(\partial v)]/P = 2w\cdot(\Phi^2+\alpha^2 v^2)^{-1} - 2(\theta+\alpha^2)\cdot v\cdot(\Phi^2+\alpha^2 v^2)^{-1}.$$

Hence

$$\text{(B3)} \qquad \ln(P) \propto -2w\cdot(\alpha\cdot|\Phi|)^{-1}\cdot \mathrm{arcotg}(\alpha v/|\Phi|) - (\theta+\alpha^2)\cdot\alpha^{-2}\cdot \ln(\Phi^2\cdot\alpha^{-2}+v^2).$$

Taking the exponent of both sides of (B3) gives directly the density in (2.31) of the text.

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