BANCA D'ITALIA

Temi di discussione

del Servizio Studi

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ALTERNATIVE ESTIMATORS OF THE COX, INGERSOLL AND ROSS MODEL OF THE TERM STRUCTURE OF INTEREST RATES: A MONTE CARLO COMPARISON

by Carlo Bianchi (*), Riccardo Cesari (**)
and Lorenzo Panattoni (***)

Abstract

We evaluate, through Monte Carlo experiments, the econometric performance of seven alternative estimators (direct and indirect methods) of the basic parameters of the Cox-Ingersoll-Ross single-factor diffusion model of the term structure of interest rates. Different generating schemes are compared and the unobservability of the state-variable is taken into account. The effects of approximating interest rates, increased frequency of data and starting values are analyzed. A Monte Carlo evaluation of the effects on bond prices of biased parameter estimates is provided.

Contents

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1. Introduction [1]

Any asset has many characteristics: the issuer (with a certain probability of default), the coupon flow, the maturity date T, the taxation (of coupon and capital gains), other contractual provisions (options, etc.).

The asset price, P(t), is clearly a function of these characteristics: P(t, risk, coupon, maturity, taxes,...). The relation, ceteris paribus, of the asset price with respect to maturity is called the term structure. The ceteris paribus clause is specifically expressed as zero default risk, zero coupon, unit payment at maturity, fixed, uniform taxation. In this case the asset is a default-free zero-coupon unit discount bond i.e. a security with (money) price P(t, T) at time t, promising, with probability one, a payment of one (money) unit at time T>t. In other words, at time t, P(t, T) is the present value (discount factor) of one (sure) money unit in T. Assuming continuous time, the price function P(t, T) can be transformed into

$$R(t,T) = -\log(P)/(T-t)$$

where R(t,T) is, by definition, the continuously compounded

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[1] This research has been presented in different phases in some seminars and conferences (Erice, Ascona, Firenze, Pisa, Milano, Berlin); the participants are all gratefully thanked. A special acknowledgment has to be made to Banca d'Italia and IBM-Semea as well as to Valentine Genon-Catalot and Henri Schurz for their suggestions. Carlo Bianchi has written sections 3.2 to 3.5 and section 3.7, Riccardo Cesari sections 1, 2, 3.1, 4.1, 4.2, 6 and the Appendix, Lorenzo Panattoni sections 3.6, 4.3 and 5. Bianchi and Panattoni have also developed the Fortran procedures for main frame and personal computer. Eugene M. Cleur has provided substantial help in implementing the indirect estimator.

spot rate of return to maturity of the bond and T-t is the bond time to maturity ([2]).

As a function of maturity τ=T-t, R(t,τ) is called the term structure of interest rates.

A theory of the term structure is essentially an explanation of the difference (risk- or term-premium) between short term and long term interest rates or, equivalently, between spot and forward interest rates ([3]). Classical examples of term structure theories are Fisher (1930) pure expectations theory, Hicks (1939) liquidity preference, Modigliani and Sutch (1966) preferred habitat.

More recently, into the no-arbitrage asset pricing literature started up by Black and Scholes (1973), the theory of the term structure has known a new period of successful innovation and implementation ([4]). Not only the shape of the term structure but also the factors affecting its level and dynamics have been analyzed in a general equilibrium

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[2] In discrete time, using the discrete time interval as a natural choice for the time unit, the rate of return is $R(t,n)=(1/P)^{1/n}-1$, n being the (possibly fractional) number of intervals (periods) between current time t and the bond maturity date T.

[3] The forward rate $R^F(t,S,T)$ is the rate implied in a contract written and completely specified at time t to buy a bond at time S with maturity date T>S. The no-arbitrage condition implies that forward rates (prices) can be specified in terms of spot rates (prices).

[4] On previous models and results see e.g. Masera (1972) and Dobson, Sutch and Vanderford (1976). A recent survey is in Shiller (1990).

context, showing explicitly the link between agent preferences, uncertainty, production technology and term premia (see Cox, Ingersoll and Ross (henceforth CIR), 1981).

The new term structure models were developed in the typical Black and Scholes framework: continuous time, frictionless markets, diffusion processes i.e. intertemporal dynamics described by stochastic differential equations, no-arbitrage condition i.e. prices constrained by the maintained hypothesis that equivalent assets (or portfolios) in terms of cash flows and other characteristics must earn the same return.

These models, and in particular the CIR (1985a, 1985b) general equilibrium model ([5]), have been applied to financial data in many countries ([6]).

Considering this empirical literature, a first point to note is that notwithstanding a unified theoretical framework behind the various models, different estimation methods have been used by different authors.

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[5] The equilibrium condition (supply=demand) of the CIR model implies the no-arbitrage condition. Different models, with alternative dynamic specifications, should be mentioned: e.g. Dothan (1978), Vasicek (1977), Brennan and Schwartz (1979, 1982).

[6] See Barone and Cesari (1986), De Felice and Moriconi (1991), Barone, Cuoco and Zautzik (1989) for Italy; CIR (1979), Brennan and Schwartz (1979), Brown and Dybvig (1986), Gibbons and Ramaswamy (1986) for the US; Fischer and Zechner (1984) for the Federal Republic of Germany; Brown and Schaefer (1988) for the UK.

Secondly, in many cases, the elegant, formal derivation used in the theoretical development of the model is far from being used in the empirical counterpart, to develop and justify the adopted estimation method.

Thirdly, almost no analytical or numerical comparison has been made to evaluate the various estimators, in particular in the case of discrete and small sample data ([7]).

Our paper is addressed toward these critical points. Using the single-factor model developed by CIR (1985b) we shall consider the foundations of different estimation methods for a diffusion model of the term structure as well as the evaluation, through Monte Carlo experiments, of their relative performance in the typical case of discrete, finite samples.

The paper is as follows: in section 2 the theoretical model is sketched; in section 3 different estimation and testing methods are considered, in continuous or discrete time; section 4 draws a set up for Monte Carlo simulation of continuous (diffusion) processes; section 5 presents the main results and section 6 concludes the paper.

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[7] An important exception is Fournie and Talay (1992) who consider only the asymptotic behaviour of estimators.

2. A univariate model of the term structure

CIR (1985b) is an equilibrium model of a production/exchange economy with a single production/consumption good ([8]), in which the state variable describing the system is a univariate diffusion, agents have rational expectations and logarithmic preferences, markets are perfectly competitive and frictionless.

Given that the instantaneous rate of return r(t) on instantaneously maturing bonds is, in equilibrium, a linear function of the single state variable, it is possible to use this very short-term rate as an instrument for the system state variable.

It turns out that the short rate dynamics are given by the Feller (1951) stochastic differential equation (SDE):

$$dr(t) = \kappa(\theta - r)\,dt + \sigma\sqrt{r}\,dw(t) \tag{2.1}$$

where dw is the Itô differential of the one-dimensional standard Wiener process (Brownian motion), defined by:

$$w(t) \sim N(0,t) \tag{2.2a}$$

$$E(w(t)w(s)) = \min(t,s) \tag{2.2b}$$

$$w(0) = 0 \quad a.s. \tag{2.2c}$$

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[8] The economic system is essentially a real (non monetary) economy, with prices expressed in terms of the single good (or bundle).

Notice, in particular, that:

$$E(w(s)[w(u)-w(t)]) = 0 \text{ for every } s \le t \le u \qquad (2.2d)$$

i.e. non overlapping increments are independent.

For $\sigma, k, \theta > 0$ the process r(t), solution of (2.1) is pathwise unique (strong solution), nonnegative and it displays mean-reversion toward the long run value θ, k being the speed of the expected short term adjustment. According to Feller (1951) Lemma 4, if $2k\theta \ge \sigma^2$ the origin is inaccessible (entrance boundary) and the process is ergodic ([9]).

Solving the Kolmogorov's backward differential equation, the transition probability density from r(t) to r(s) is obtained as (Karlin and McGregor, 1960):

$$p(r(s)\,|\,r(t)) = c e^{-\frac{1}{2}(u+v)} \left(\frac{v}{u}\right)^{(q-1)/2} I_{q-1}(\sqrt{uv}) \qquad (2.3)$$

where:

$$c = \frac{2k}{\sigma^2(1-e^{-k(s-t)})}, \quad u = 2cr(t)\,e^{-k(s-t)}$$

$$v = 2cr(s), \quad q = \frac{2k\theta}{\sigma^2}$$

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[9] See Ikeda and Watanabe (1981, p. 223). On entrance boundaries and ergodicity see Karlin and Taylor (1981, p. 241).

and I_α is the modified Bessel function of the first kind of order α:

$$I_\alpha(z)=\sum_{k=0}^{\infty}\frac{(z/2)^{2k+\alpha}}{k!\Gamma(k+\alpha+1)} \tag{2.4}$$

The transition distribution is therefore a stationary [10] noncentral chi-square $\chi^2(v;\ 2q,\ u)$ with 2q degrees of freedom and noncentrality parameter u.

Conditional expected value [11] and variance are given by:

$$E(r(s)\mid r(t))=r(t)\,e^{-k(s-t)}+\theta(1-e^{-k(s-t)})$$

$$Var(r(s)\mid r(t))=r(t)\,(\frac{\sigma^2}{k})\,(e^{-k(s-t)}-e^{-2k(s-t)})+ \tag{2.5}$$

$$\theta(\frac{\sigma^2}{2k})\,(1-e^{-k(s-t)})^2$$

Note that:

[10] A Markov process with stationary transition distributions is time-homogeneous. In the case of diffusions this property follows from the independence of drift and diffusion coefficient with respect to time (autonomous SDE).

[11] The expected value solves the ordinary differential equation (deterministic version of (2.1)):

$dr=k(\theta-r)du \qquad r(t)=r_t$.

$$\lim_{k\uparrow\infty} E(r(s)\,|\,r(t)) = \theta, \quad \lim_{k\uparrow\infty} Var(r(s)\,|\,r(t)) = 0$$

and (2.6)

$$\lim_{k\downarrow 0} E(r(s)\,|\,r(t)) = r(t)$$

$$\lim_{k\downarrow 0} Var(r(s)\,|\,r(t)) = \sigma^2 r(t)(s-t)$$

The invariant (steady state) distribution is obtained as $s \uparrow \infty$ ([12]) and it is a gamma distribution with positive parameters q and q/θ and density ([13]):

$$p(r) = \frac{(q/\theta)^q}{\Gamma(q)} r^{q-1} e^{-rq/\theta} \tag{2.7a}$$

The first moments are:

$$\bar{E}(r) = \theta, \quad \overline{Var}(r) = \frac{\sigma^2\theta}{2k}, \quad \bar{E}(1/r) = \frac{2k}{2k\theta - \sigma^2} \tag{2.7b}$$

where $\bar{E}$ denotes expectations with respect to the invariant distribution. Note that, in steady state, k and σ^2 cannot be separately identified.

In this economy any asset price with no coupon or dividend flow must satisfy the following partial differential (valuation) equation (PDE):

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[12] It solves the stationary form of the Kolmogorov's forward differential equation.

[13] Time-homogeneity and the existence of an invariant distribution are necessary and sufficient conditions for the Markov process r(t) to be stationary (Arnold, 1974, p. 33).

$$\frac{1}{2}\sigma^2 r P_{rr} + [\kappa(\theta - r) - \lambda r] P_r + P_t - rP = 0 \qquad (2.8)$$

where λr is the "market price of risk" (covariance between percentage change in wealth and change in the short interest rate) ([14]).

If P(r,t,T) is the time t price of a unit discount bond, the appropriate terminal condition is:

$$P(r,T,T) = 1 \qquad (2.9)$$

so that the solution, dependent only on r and τ=T-t, is ([15]):

$$P(r,\tau) = F(\tau)\, \exp(-rG(\tau)) \qquad (2.10)$$

where:

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[14] If λ is negative (positive) the rate of return contains a risk premium (discount) the market requires to compensate for holding a bond whose price has a positive (negative) covariance with wealth.

[15] See Friedman (1975, p. 147) for a mean-value representation of the unique solution of a general parabolic boundary value problem.

$$F(\tau) = \left[\frac{\phi_1 \exp(\phi_2 \tau)}{\phi_2(\exp(\phi_1 \tau)-1)+\phi_1}\right]^{\phi_3} \in \,]0,1[$$

$$G(\tau) = \left[\frac{\exp(\phi_1 \tau)-1}{\phi_2(\exp(\phi_1 \tau)-1)+\phi_1}\right] > 0 \qquad (2.11)$$

$$\phi_1 = \sqrt{(k+\lambda)^2+2\sigma^2}, \quad \phi_2 = (k+\lambda+\phi_1)/2$$

$$\phi_3 = 2k\theta/\sigma^2 > 0$$

Note that:

$$\sigma^2 = 2\phi_2(\phi_1-\phi_2), \quad \phi_1 > \phi_2 > 0 \qquad (2.12)$$

The term structure is given by:

$$R(r,\tau) \equiv -\log(P)/\tau = r(t)\; G(\tau)/\tau - \log(F(\tau))/\tau \qquad (2.13)$$

with a constant infinite-maturity interest rate (consol rate):

$$\lim_{\tau \to \infty} R(r,\tau) = R_\infty = \frac{2k\theta}{k+\lambda+\phi_1} = (\phi_1-\phi_2)\phi_3 \qquad (2.14)$$

If:

$$\frac{2k\theta}{k+\lambda+\phi_1} < r(t) < \frac{k\theta}{k+\lambda} \qquad (2.15)$$

the term structure is humped. It is rising for values of r(t) below this interval, falling for values above it [16].

Clearly:

$$\lim_{\tau \to 0} R(r,\tau) = r(t) \qquad (2.16)$$

Note that $P(r,\tau)=\exp(-R(r,\tau)\tau)$ so that a change in the time unit (e.g. time from years to months) implies a corresponding change in the interest rates (e.g. from annual to monthly rates: $\tau*12$, $R/12$). The price formula is invariant to a scale change in time and rates if τ is multiplied (divided) and r, k, θ, σ and λ are divided (multiplied) by the scale constant (e.g. 12) [17].

3. Estimation methods

3.1 Continuous time ML estimation of SDE

Let us assume, for the moment, that the process r(t) is observable and that a continuous record of observations is available: $\{r(t),\ t \in [0,T]\}$. Our purpose is to estimate the SDE (2.1).

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[16] Using the approximation $R_\infty \approx \theta$, for $\lambda<0$, Barone and Cuoco (1991) obtain $\phi_1=k-\lambda$, $\phi_2=k$, $\phi_3=-\theta/\lambda$.

[17] To show this take the differential of $r^\circ=r/m$ and make a deterministic time change $t'=mt$ using $dt'=mdt$ and $dz(t')=\sqrt{m}\,dz(t)$, obtaining:

$$dr^\circ(t')=k^\circ(\theta^\circ-r^\circ)dt'+\sigma^\circ\sqrt{r^\circ}dz(t').$$

Alternatively, make a change of variables in the PDE (2.8).

From the quadratic variation property of semimartingales (Shiryayev, 1981) we obtain:

$$\int_0^T dr(t)\,dr(t) = \int_0^T \sigma^2 r(t)\,dt$$

that is:

$$\sigma^2 = \frac{\int_0^T [dr(t)]^2}{\int_0^T r(t)\,dt} \qquad (3.1.1)$$

The integral in the numerator ([18]) is defined through the constructive characterization of the quadratic variation process: if $T_n=(t_0^{(n)}, t_1^{(n)}, \ldots, t_{Nn}^{(n)})$ is an increasing (i.e. nested) partition of [0,T], i.e. $T_n \subset T_{n+1}$, $0=t_0^{(n)}<t_1^{(n)}<\ldots<t_{Nn}^{(n)}=T$ and:

$$\lim_{n\to\infty} \max_p (t_p^{(n)} - t_{p-1}^{(n)}) = 0$$

then:

[18] Remember that a diffusion is a Markov process with continuous but nowhere differentiable (and therefore of unbounded variation) sample paths. The Stieljes definition of integrals does not apply.

$$\lim_{n\to\infty}\sum_{p=1}^{N_n}\left[r(t_p^{(n)})-r(t_{p-1}^{(n)})\right]^2 = \int_0^T [dr(t)]^2 = \qquad (3.1.2)$$

$$\sigma^2\int_0^T r(t)\,dt = \sigma^2\lim_{n\to\infty}\sum_{p=1}^{N_n} r(t_{p-1}^{(n)})\,(t_p^{(n)}-t_{p-1}^{(n)})$$

and the convergence is almost sure as n goes to infinity for any given T ([19]).

This means that considering more and more frequent observations over a given interval [0,T] the diffusion coefficient σ becomes known with probability one.

In order to estimate the drift coefficients we use the probability measure induced by $r(t,\alpha)$, where $\alpha=(k,\theta)$ is the parameter vector. If $L_T(r,\alpha)$ is the likelihood ratio (Liptser and Shiryayev, 1974, ch. 7), the maximum likelihood (ML) estimators are obtained by setting the score vector $q_T(r,\alpha)$ equal to 0:

$$q_T(r,\alpha) \equiv \frac{\partial}{\partial\alpha}\log L_T(r,\alpha)$$

$$=\int_0^T\left(\frac{\partial}{\partial\alpha}A(r,\alpha)\right)G^{-2}(r)\,dr_s-\int_0^T\left(\frac{\partial}{\partial\alpha}A(r,\alpha)\right)G^{-2}(r)\,A(r,\alpha)\,ds=0 \qquad (3.1.3)$$

where

$$A(r,\alpha) = k(\theta-r) \qquad G(r) = \sigma\sqrt{r}$$

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[19] An example of nested partitions is $T_n=(pT/2^n, p=0,1,\ldots,2^n)$. If partitions are not nested as for $T_n=(pT/n, p=0,1,\ldots,n)$ the first convergence is only in quadratic mean. See Wong and Hajek (1985, p. 53).

It can be easily seen that, in the case of Wiener noises (exponential families), the ML estimator is equivalent to the minimum distance (weighted least squares) estimator:

$$\min_{\alpha} \int_0^T [dr-A(r,\alpha)\,ds]' G^{-2}(r)\,[dr-A(r,\alpha)\,ds] \tag{3.1.4}$$

Using martingale limit theorems it can be shown that the MLE $\hat{\alpha}_T$ is strongly consistent for α and asymptotically normal in the sense that [20]:

$$I_T^{1/2}(r,\alpha)\,(\hat{\alpha}_T-\alpha) \underset{T\to\infty}{\overset{aD}{\sim}} N(0,\mathbf{1}) \tag{3.1.5}$$

where the process I_T (quadratic covariation of q_T) is given by:

$$I_T(r,\alpha)=\int_0^T\left(\frac{\partial}{\partial\alpha}A(r,\alpha)\right)G^{-2}(r)\left(\frac{\partial}{\partial\alpha'}A(r,\alpha)\right)ds \tag{3.1.6}$$

and, under regularity conditions, it can be considered a random Fisher conditional information matrix for dependent

[20] McKeague (1984) shows that for stationary ergodic processes strong consistency and asymptotic normality (with a loss in efficiency) still hold even in the case of a misspecified diffusion function ($\sigma\sqrt{r}$ in our case).

observations [21].

In the case of an ergodic process with "smooth" drift function we have (see Kutoyants, 1978):

$$\sqrt{T}(\hat{\alpha}_T-\alpha) \underset{T\to\infty}{\overset{aD}{\sim}} N(0,D^{-1}(\alpha))$$

where (3.1.7)

$$D(\alpha)=\overline{E}\left(\frac{\partial}{\partial\alpha}A(r,\alpha)\,G^{-2}(r)\,\frac{\partial}{\partial\alpha'}A(r,\alpha)\right)$$

$$=\lim_{T\to\infty}\frac{1}{T}I_T(\alpha)\ \text{a.s.}\quad\text{(by ergodicity)}$$

In our case the ML estimators are [22]:

$$\hat{k}=\frac{\int_0^T\frac{ds}{r(s)}\int_0^T dr(s)-T\int_0^T\frac{dr(s)}{r(s)}}{T^2-\int_0^T r(s)\,ds\int_0^T\frac{ds}{r(s)}} \qquad (3.1.8)$$

and

---

[21] See Feigin (1976). The optimal asymptotic results depend on the factorization of the score as (conditional exponential families):

$q_T(\alpha)=I_T(\alpha)\,(\hat{\alpha}_T-\alpha)$

in which case $I^{-1}{}_T(\alpha)$ identifies a Rao-Cramér minimum variance bound.

[22] We used the fact that max $L(k,\theta)=$ max $L(k,\ k\theta/k)\equiv$ max $M(k,\ k\theta)$.

$$\theta = \frac{T\int_0^T dr(s) - \int_0^T r(s)\,ds\int_0^T \frac{dr(s)}{r(s)}}{\int_0^T \frac{ds}{r(s)}\int_0^T dr(s) - T\int_0^T \frac{dr(s)}{r(s)}} \tag{3.1.9}$$

where:

$$\int_0^T dr(s) = r(T) - r(0)\,, \quad \int_0^T \frac{dr(s)}{r(s)} = \log\left(\frac{r(T)}{r(0)}\right) + \frac{\sigma^2}{2}\int_0^T \frac{ds}{r(s)}$$

and the (symmetric) Fisher matrix is given by:

$$I_T(k,\theta) = \begin{bmatrix} \int_0^T \frac{(\theta - r(s))^2}{\sigma^2 r(s)}ds & \int_0^T \frac{k(\theta - r(s))}{\sigma^2 r(s)}ds \\ & \int_0^T \frac{k^2}{\sigma^2 r(s)}ds \end{bmatrix} \tag{3.1.10}$$

Strong consistency of $\hat{\alpha}_T$ can be easily checked using (2.7b). Moreover, the Rao-Cramér bound, for large T, is obtained using (3.1.7) as:

$$I_T^{-1}(\alpha) \approx \frac{1}{T}D^{-1}(\alpha) = \frac{1}{T}\begin{bmatrix} 2k & -\sigma^2/k \\ -\sigma^2/k & \theta\sigma^2/k^2 \end{bmatrix} \tag{3.1.11}$$

going to 0, by consistency, as T goes to infinity.

Notice that, from Itô formula, the transformation

$y=\sqrt{r}$ has dynamics ([23]):

$$dy(t) = [\frac{(4k\theta-\sigma^2)}{8y} - ky/2]\,dt + \sigma/2\,dw \tag{3.1.12}$$

so that

$$\sigma^2 = \frac{4}{T}\int_0^T [dy(t)]^2$$

Comparing the last equation with (3.1.1) we obtain:

$$T = \frac{4\int_0^T [d\sqrt{r(t)}]^2 \int_0^T r(t)\,dt}{\int_0^T [dr(t)]^2} \tag{3.1.13}$$

In practice, the assumption of a continuous record of observations is not satisfied and, usually, only a set of discrete observations is available at the partition $0=t_0<t_1<\dots<t_n=T$ of $[0,T]$, where $\delta_n=t_{p+1}-t_p$ is independent of p (constant step) and refining to 0, i.e. $\lim_n \delta_n = 0$ for $n\to\infty$. The suggestion is then to replace continuous integrals with discrete sums (e.g. Cauchy approximations).

In particular, take $t_p=pT/n \equiv p\delta_n$, $p=0,1,\dots,n$ and define $\hat{\sigma}^2_{n,T,r}$ as the discretized version of (3.1.1):

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[23] It can be shown that, for regular transformations $y=h(r)$ the ML estimators are unaffected, being $q_T(r,\alpha)=q_T(h(r),\alpha)$ and $I_T(r,\alpha)=I_T(h(r),\alpha)$.

$$\hat{\sigma}^2_{n,T,r} = \frac{\sum_{p=1}^{n} [r(t_p) - r(t_{p-1})]^2}{\delta_n \sum_{p=1}^{n} r(t_{p-1})} \tag{3.1.14}$$

We have:

$$\underset{n\to\infty}{l.i.m.} \ (\hat{\sigma}^2_{n,T,r} - \sigma^2) = 0$$

where l.i.m. means limit in mean square (and therefore in probability). Moreover, the following limit in distribution can be proved (Genon-Catalot and Jacod, 1993):

$$\underset{n\to\infty}{l.i.d.} \ \sqrt{n}(\hat{\sigma}^2_{n,T,r} - \sigma^2) = \sqrt{2}\sigma^2 \frac{\int_0^T r(s)\, d\hat{W}(s)}{\int_0^T r(s)\, ds}$$

where $\hat{W}$ is a Brownian motion independent of $\{r(s),\ s \le T\}$.

At the same time, from (3.1.12):

$$\hat{\sigma}^2_{n,T,\sqrt{r}} = \frac{4}{T}\sum_{p=1}^{n} [\sqrt{r(t_p)} - \sqrt{r(t_{p-1})}]^2 \tag{3.1.15}$$

which is consistent (in mean square) and recursive (Banon, 1978, p. 392):

$$\hat{\sigma}^2_{n+1,T+\delta,\sqrt{r}} = \frac{n}{n+1}\hat{\sigma}^2_{n,T,\sqrt{r}} + \frac{4}{T+\delta}(\sqrt{r(t_{n+1})} - \sqrt{r(t_n)})^2 \tag{3.1.16}$$

Analogously, define $\hat{\alpha}_{n,T}$ as the discretized version of the ML estimator $\hat{\alpha}_T$ (Cauchy approximating sums instead of integrals). Then, as in Le Breton (1976):

$$\underset{n\to\infty}{plim}\ (\hat{\alpha}_{n,T}-\hat{\alpha}_T)=0, \qquad (\hat{\alpha}_{n,T}-\hat{\alpha}_T)=O_p(\delta_n^{1/2}) \tag{3.1.17}$$

where $O_p(\ )$ means "same order in probability as".

This means that as $n \to \infty$, for given T, i.e. as the frequency of observations per unit time increases ($\delta_n \downarrow 0$) toward the limit of a continuous record, the discretized estimator $\hat{\alpha}_{n,T}$ is consistent for $\hat{\alpha}_T$ (but not for α) ([24]).

Note that in the case of sample-path integrals, more refined approximations (trapezoidal, Simpson's, etc.) can be used. In particular, for any smooth function f( ):

$$\int_0^T f(t)\,dt \approx \begin{cases} \delta_n \sum_{p=0}^{n-1} f(t_p) & \text{Cauchy} \\ \delta_n \sum_{p=0}^{n-1} \dfrac{f(t_p)+f(t_{p+1})}{2} & \text{trapezoidal} \\ \delta_n \sum_{\substack{p=0 \\ p\ \text{even}}}^{n-2} \dfrac{f(t_p)+4f(t_{p+1})+f(t_{p+2})}{3} & \text{Simpson} \end{cases}$$

In the first case the approximation error is of order $O(\delta)$ (we omit the subscript n of δ_n), in the second case it

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[24] Marsh and Rosenfeld (1983, p. 639) observe that in most markets data are not generated in real time so that as the interval δ shrinks observation or missing errors increase. Non-trading time is a leading example.

is of order $O(\delta^2)$ and for Simpson's formula it is of order $O(\delta^4)$ [25].

3.2 Estimation of the discrete equivalent of linearized SDE

It is well known that the linear SDE:

$$dx(t) = (Ax + a)\,dt + b\,dw(t) \tag{3.2.1}$$

has solution, for $x(s)=x_s$ (Arnold, 1974, p. 130):

$$x(t) = x_s e^{A(t-s)} + a\frac{e^{A(t-s)}-1}{A} + b\int_s^t e^{A(t-u)}\,dw(u) \tag{3.2.2}$$

so that, for $t=p\delta$, $s=p\delta-\delta$ (Sargan, 1974):

$$x_p = e^{A\delta}x_{p-1} + a\frac{e^{A\delta}-1}{A} + b\int_{p\delta-\delta}^{p\delta} e^{A(p\delta-u)}\,dw(u) \tag{3.2.3}$$

$$= c_1 x_{p-1} + c_0 + u_p$$

where:

$$c_1 = e^{A\delta}, \quad c_0 = a\frac{e^{A\delta}-1}{A}, \quad u_p \sim N(0, v^2), \quad v^2 = b^2\frac{e^{2A\delta}-1}{2A}$$

that is

$$A = \frac{\log c_1}{\delta}, \quad a = \frac{c_0 \log c_1}{\delta(c_1-1)}, \quad b^2 = v^2\frac{2\log c_1}{\delta(c_1^2-1)}$$

[25] See McShane (1983, p. 66). In the case of stochastic integrals, the limit of trapezoidal approximations is used in the definition of the Stratonovich integral.

The error term is normal i.i.d. and the ordinary least squares (OLS) estimator has optimal properties in large as well as small samples.

For a nonlinear SDE such as (3.1.12) a popular estimation procedure (Fischer and Zechner, 1984; Barone and Cesari, 1986; De Felice and Moriconi, 1991) is to linearize the drift around a mean value $\bar{y}$ obtaining a linear SDE (approximation of (3.1.12)) for which the exact discrete equivalent (3.2.3) is known.

Linearizing 1/y, the following SDE is obtained:

$$dy(t) \approx \left[\left(-\frac{(4k\theta-\sigma^2)}{8\bar{y}^2}-\frac{k}{2}\right)y+\frac{4k\theta-\sigma^2}{4\bar{y}}\right]dt+\frac{\sigma}{2}dw(t) \qquad (3.2.4)$$
$$\equiv [\ Ay+a\ ]dt+b\,dw(t)$$

and the above procedure can be applied giving:

$$\hat{\sigma}^2=4\hat{b}^2, \quad \hat{k}=-2\hat{A}-\frac{\hat{a}}{\bar{y}}, \quad \theta=\frac{4\hat{a}\bar{y}+\hat{\sigma}^2}{4\hat{k}} \qquad (3.2.5)$$

3.3 Conditional-mean estimation

Using the conditional expected value in (2.5) we can write:

$$r_p = \theta(1-e^{-k\delta})+e^{-k\delta}r_{p-1}+\epsilon_p \qquad (3.3.1)$$
$$E(\epsilon_p)=0, \quad E(\epsilon_p^2\,|\,r_{p-1})=r_{p-1}\sigma^2\delta$$

where the innovation term is not normal (its conditional distribution is in fact a non central χ^2) and its variance, from (2.5), is approximately $r_{p-1}\sigma^2\delta$ so that a weighted LS transformation gives [26]:

$$\frac{r_p}{\sqrt{r_{p-1}}} = \theta(1-e^{-k\delta})\frac{1}{\sqrt{r_{p-1}}} + e^{-k\delta}\sqrt{r_{p-1}} + u_p$$

$$\equiv A\frac{1}{\sqrt{r_{p-1}}} + B\sqrt{r_{p-1}} + u_p \qquad (3.3.2)$$

$$E(u_p)=0, \quad E(u_p^2)\simeq\sigma^2\delta$$

$$\hat{k}=-\frac{\log\hat{B}}{\delta}, \quad \hat{\theta}=\frac{\hat{A}}{1-\hat{B}}, \quad \hat{\sigma}^2=\frac{Var(\hat{u})}{\delta}$$

3.4 OLS estimation of naïve discretization of SDE

Let us consider an approximate discrete-time specification of the basic SDE (2.1):

$$r_p - r_{p-1} = (\alpha+\beta r_{p-1})\delta + \epsilon_p \qquad (3.4.1)$$

$$\alpha\simeq k\theta, \quad \beta\simeq -k, \quad E(\epsilon_p)=0, \quad E(\epsilon_p^2 \mid r_{p-1})\simeq\sigma^2 r_{p-1}\delta$$

---

[26] The parameter δ represents the observation frequency in terms of the chosen time unit. For example, for annual rates (i.e. time unit = 1 year) and monthly observations δ=1/12; for monthly rates (time unit = 1 month) and monthly observations δ=1.

where the disturbance term is not necessarily normal.

Using a weighted LS transformation we obtain (see for example Brennan and Schwartz, 1982):

$$\frac{r_p - r_{p-1}}{\sqrt{r_{p-1}}} = \frac{\alpha\delta}{\sqrt{r_{p-1}}} + \beta\delta\sqrt{r_{p-1}} + u_p \equiv A\frac{1}{\sqrt{r_{p-1}}} + B\sqrt{r_{p-1}} + u_p$$
$$E(u_p) = 0, \quad E(u_p^2) = \sigma^2\delta \tag{3.4.2}$$
$$\hat{k} = -\hat{B}/\delta, \quad \hat{\theta} = -\hat{A}/\hat{B}, \quad \hat{\sigma}^2 = \frac{Var(\hat{u})}{\delta}$$

Note that the regression coefficients are the linear approximation of the coefficients in the previous case.

Moreover the estimates for θ and σ^2 are clearly numerically equivalent.

3.5 GMM estimation of naïve discretization of SDE

Under the assumption of stationary, ergodic processes Hansen's (1982) generalized method of moments (GMM) can be used to estimate the above given naïve discretization:

$$r_p - r_{p-1} = (\alpha + \beta r_{p-1})\delta + \epsilon_p \tag{3.5.1}$$
$$\alpha = k\theta, \quad \beta = -k, \quad E(\epsilon_p) = 0, \quad E(\epsilon_p^2 - \sigma^2 r_{p-1}\delta) = 0$$

The idea under the GMM procedure is to impose a set of orthogonality conditions on the variables involved, choosing as estimates the values that minimize (in a certain optimal metric) the relevant distance. Various forms of LS, quasi-ML and nonlinear instrumental variables (IV) estimators can be viewed as special cases of GMM.

In our case (see Chan, Karolyi, Longstaff and Sanders, 1992) the orthogonality condition is [27]:

$$E(f_p(\zeta))=0, \qquad \text{where } f_p(\zeta)=\begin{bmatrix} \epsilon_p \\ \epsilon_p r_{p-1} \\ \epsilon_p^2-\sigma^2 r_{p-1}\delta \\ (\epsilon_p^2-\sigma^2 r_{p-1}\delta)\, r_{p-1} \end{bmatrix} \tag{3.5.2}$$

$$\zeta\equiv(\alpha\delta,\beta\delta,\sigma^2\delta)$$

and the GMM estimator is obtained as:

$$\hat{\zeta}_n \equiv \underset{\zeta}{argmin}\; g_n'(\zeta)\, D_n g_n(\zeta) \tag{3.5.3}$$

that is:

$$\frac{\partial g_n(\zeta)}{\partial \zeta'} D_n\, g_n(\zeta)=0 \qquad \text{(f.o.c.)}$$

$$\text{where} \quad g_n(\zeta)\equiv\frac{1}{n}\sum_{p=1}^{n} f_p(\zeta)$$

and D_n is a positive semi-definite weighting matrix.

---

[27] The number of orthogonality conditions must be not less than the number of parameters. As Hansen (1982, p. 1048) observes, his result "is limited in that it takes the specification of the orthogonality conditions as given and does not discuss how to construct optimally orthogonality conditions".

The optimal GMM estimator (Hansen, 1982, theorem 3.2) is obtained using $D_n = \hat{S}_n^{-1}$ where $\hat{S}_n$ is a consistent estimator of the covariance matrix $S(\zeta)=E(f_p(\zeta)f'_p(\zeta))$ so that:

$$\sqrt{n}(\hat{\zeta}_n-\zeta) \underset{n\to\infty}{\overset{aD}{\sim}} N\left(0, \left[\frac{\partial g_n}{\partial \zeta'} S_n^{-1} \frac{\partial g_n}{\partial \zeta}\right]^{-1}\right) \tag{3.5.4}$$

A consistent, positive semi-definite estimator of $S(\zeta)$ is given by Newey and West (1987):

$$\hat{S}_n=\hat{\Omega}_0+\sum_{j=1}^{m} \frac{m+1-j}{m+1} [\hat{\Omega}_j+\hat{\Omega}'_j] \quad \text{with } \hat{\Omega}_j= \frac{1}{n} \sum_{p=j+1}^{n} \hat{f}_p \hat{f}'_{p-j} \tag{3.5.5}$$

where m is the number of nonzero autocorrelations of $f_p(\zeta)$ ([28]).

A goodness-of-fit test for the model is obtained by the result (Hansen, 1982, lemma 4.2):

$$n\, g'_n(\hat{\zeta})\ \hat{S}_n^{-1} g_n(\hat{\zeta}) \underset{n\to\infty}{\overset{aD}{\sim}} \chi^2_v \qquad v=dim(f(\zeta))-dim(\zeta) \tag{3.5.6}$$

3.6 Discrete-time ML estimation

Let us consider a set of n observations, possibly not

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[28] In general (Newey and West, 1987, theorem 2) m(n) may be a function of n going to infinity with n more slowly than $n^{0.25}$. In our Monte Carlo experiments we used the largest integer less than $n^{0.24}$.

equally spaced (29), of the process $r(t)$, i.e. $(r(t_1), r(t_2),..,r(t_p),..,r(t_n))$. The likelihood function, given an initial value $r(t_0)$ is:

$$\begin{aligned} &L(r(t_1),r(t_2),..,r(t_n)\,|\,r(t_0),\zeta)= \\ &\quad p(r(t_1)\,|\,r(t_0),\zeta)\ p(r(t_2)\,|\,r(t_1),r(t_0),\zeta)\,..... \\ &\qquad .....p(r(t_n)\,|\,r(t_{n-1}),r(t_{n-2}),..,r(t_0),\zeta) \\ &\quad =\prod_{p=1}^{n} p(r(t_p)\,|\,r(t_{p-1}),\zeta) \end{aligned} \tag{3.6.1}$$

where the last equality comes from Markov's property and $p(r(t_p)\ |r(t_{p-1}))$ is the transition density from $r(t_{p-1})$ to $r(t_p)$, in our case a non central χ^2 (see (2.3)).

It is well known that the ML estimator has optimal asymptotic properties. It is given by:

$$\begin{aligned} &\hat{\zeta}_n \equiv \underset{\zeta}{argmax}\ L(\zeta) = \\ &\quad \underset{\zeta}{argmax} \prod_{p=1}^{n} c e^{-\frac{1}{2}(u_{p-1}+v_p)}\left(\frac{v_p}{u_{p-1}}\right)^{(q-1)/2} I_{q-1}(\sqrt{u_{p-1}v_p}) \end{aligned} \tag{3.6.2}$$

where

$$\zeta=(k,\theta,\sigma^2),\quad c=\frac{2k}{\sigma^2(1-e^{-k\delta_p})},\quad u_{p-1}=2cr(t_{p-1})e^{-k\delta_p}$$

$$v_p=2cr(t_p),\quad q=\frac{2k\theta}{\sigma^2},\quad \delta_p=t_p-t_{p-1}$$

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[29] See Marsh and Rosenfeld (1983), Lo (1986, 1988), Robinson (1977). When the observation interval is small (day, hour, etc.) not equally spaced data is often due to non-trading time which generates missing observations.

and it is consistent and asymptotically normal ([30]):

$$\underset{n\to\infty}{plim}\ \hat{\zeta}_n = \zeta \quad \text{(true value)}$$

$$\sqrt{n}\,(\hat{\zeta}_n-\zeta) \overset{aD}{\underset{n\to\infty}{\sim}} N(0, I^{-1}(\zeta)),$$

$$I(\zeta) = \lim_{n\to\infty}\frac{1}{n}\sum_{p=1}^{n} -E_{p-1}\left[\frac{\partial^2 \log p(r(t_p)\,|\,r(t_{p-1}),\zeta)}{\partial\zeta\partial\zeta'}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{p=1}^{n} E_{p-1}\left[\frac{\partial \log p(r(t_p)\,|\,r(t_{p-1}),\zeta)}{\partial\zeta}\right]^2 \qquad (3.6.3)$$

For the discount bond price $P(r,t,T,\zeta,\lambda)$, if λ were known:

$$\hat{P}_n = P(r,t,T,\hat{\zeta}_n,\lambda) \quad \text{(MLE)}$$

$$\sqrt{n}\,(\hat{P}_n-P) \overset{aD}{\underset{n\to\infty}{\sim}} N\left(0, \frac{\partial P}{\partial\zeta'} I^{-1}(\zeta)\frac{\partial P}{\partial\zeta}\right) \qquad (3.6.4)$$

3.7 Indirect estimation by simulation ([31])

A different estimation approach has been recently proposed by Gourieroux, Monfort and Renault (1993) (see also Gourieroux and Monfort, 1993).

[30] See Bar-Shalom (1971) and Bhat (1974) where the result is obtained for the general case of dependent observations.

[31] We would like to thank Eugene M. Cleur for his help in the implementation of this method.

Let us suppose that the true model is given by:

$$r_p = f(r_{p-1}, u_p, \zeta_0) \tag{3.7.1}$$

where f is a known function, u_p is a noise with known probability distribution and ζ_0 is a parameter vector.

In general, because of the function f, the likelihood of r_p is untractable or even unknown and one has either to approximate the exact likelihood or (equivalently) to use the exact likelihood of an approximated model.

Let (3.7.2) be the approximated or auxiliary model (for example a naïve discretization of SDE):

$$r_p = g(r_{p-1}, \epsilon_p, \beta) \tag{3.7.2}$$

where ϵ_p is a residual noise and β is a parameter vector, generally different, for value and interpretation, from ζ_0.

The indirect method is given by four steps:

i) using the observed values r_p estimate β in (3.7.2) obtaining $\hat{\beta}_n$;

ii) generate the noise u_p from the known distribution and simulate (3.7.1) assuming a particular ζ, obtainibg the simulated data $r^s_p(\zeta)$;

iii) using the simulated data estimate β in (3.7.2) obtaining $\beta^\circ_n(\zeta)$;

iv) calibrate ζ in order to have $\beta^{\circ}_n(\zeta)$ close to $\hat{\beta}_n$:

$$\hat{\zeta}_n = \underset{\zeta}{\operatorname{argmin}}[\beta^{\circ}_n(\zeta)-\hat{\beta}_n]'\hat{H}_n[\beta^{\circ}_n(\zeta)-\hat{\beta}_n] \tag{3.7.3}$$

where $\hat{H}_n$ is a positive definite matrix (the identity matrix in our case).

Essentially, we know that $\hat{\beta}_n$ is "wrong" and we find a value for ζ making $\beta^{\circ}_n(\zeta)$ as "wrong" as $\hat{\beta}_n$.

4. Monte Carlo simulations of continuous processes

4.1 Introduction

Quoting Talay (1990) "the numerical analysis of stochastic differential systems is at its very beginning". A major critical point is the fact that the concept of white noise (posing no problem in discrete time) in continuous time means a process independent at every time point t with respect to any other time point s however close to t:

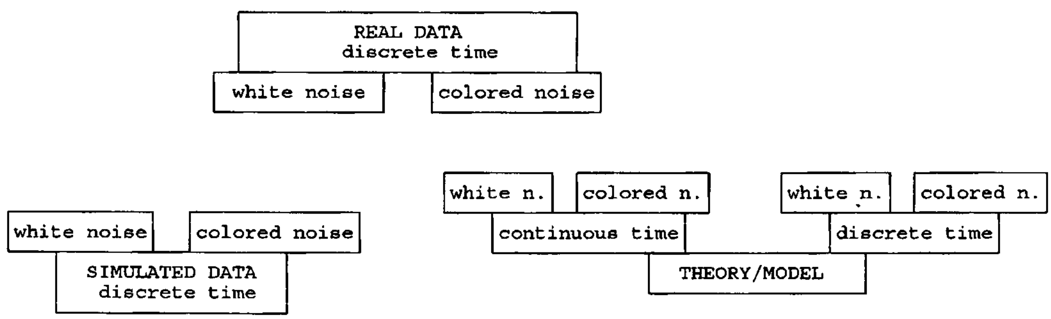
$$\mathrm{Cov}(\xi(t),\xi(s)) = \begin{cases} \sigma^2 & \text{for } t=s \\ 0 & \text{for } t\neq s \end{cases}$$

This discontinuity of $\xi(t)$ (for example in the mean square sense) suggests the definition of stochastic processes of a more general type (see for example Yaglom, 1952, p. 210) for which a generalized differential and a generalized calculus can be defined.

These general processes are abstract concept with no direct real counterpart ([32]) and this implies a second trouble whenever the theory has to be compared with data, real or empirically simulated. Roughly speaking, one could say that real data imply only discrete, colored noises and the link with a model driven by continuous, white noises is not clear.

A trade-off, however, comes out due to the fact that continuous time is theoretically very appealing and manageable and the white noise hypothesis (if not a theoretical result ([33])) is a very economic parametrization of the model.

Graphically, the situation may be depicted as follow:

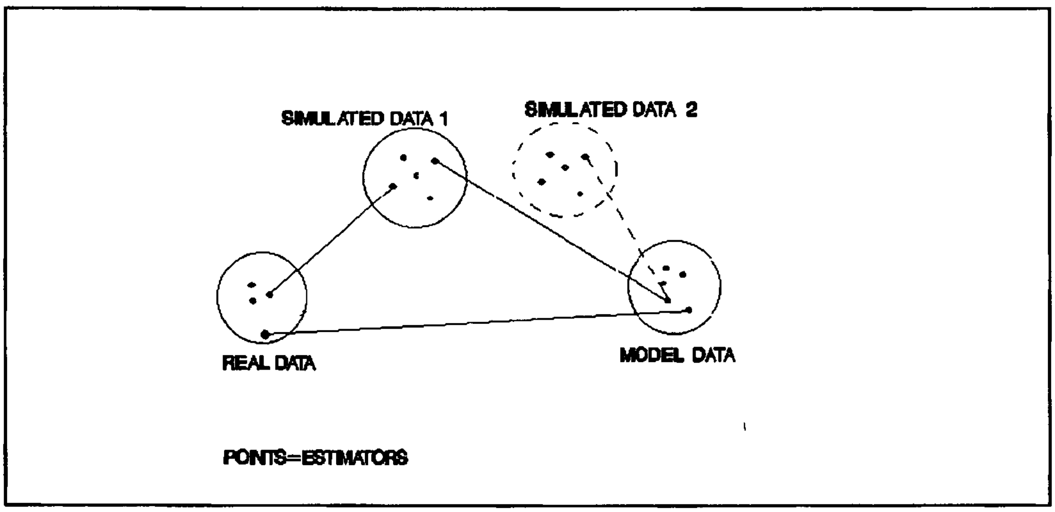


[32] "The point is that in practice one must always use some physical device to measure $\xi(t)$, and since the device always has 'inertia' (or 'memory'), corresponding to its nonzero 'time constant', the input process $\xi(t)$ will inevitably be subjected to some time averaging." (Yaglom, 1952, p. 208).

[33] A fundamental result of finance is that under the no-arbitrage condition the asset price vector is a martingale (orthogonal increments). See for example Harrison and Kreps (1979) and Harrison and Pliska (1981). This is also a property of Itô integrals.

There are many possible combinations between real data, simulated data and the theoretical or model data. In a performance analysis, if the real data generating mechanism (real DGM) was known (its features, not its parameters) the simulation task would be just to replicate it "in vitro".

However, the real DGM is unknown and the question is whether the student has to follow his or her "a priori" on it or, instead, just to accurately replicate the theoretical DGM assumed in the model, or also to take an average position between the two alternatives. Depending on the taken position, (more near to the "a priori" real DGM or to the model DGM) the distance between simulated and theoretical points may change significantly as in the figure below [34].



[34] In principle, the best position should be where an isometry exists between simulated and theoretical data on one hand and real and theoretical data on the other.

Moreover, when the theoretical model is time continuous a problem is raised concerning the convergence of discrete (simulated) data to a continuous theoretical process.

It is well known, for example, that differential equations of linear interpolations of brownian motion paths at discrete time points converge (in mean square) to Stratonovich (not Itô) stochastic differential equations (Stratonovich calculus).

4.2 Some simulation data-generating mechanisms

Let $[0,T]$ be a time interval and $0=t_0<t_1<....<t_n=T$ a partition of $[0,T]$ such that $\delta_n=t_{p+1}-t_p$ is independent of p (constant step) and refining to 0, i.e. $\lim_n \delta_n = 0$ for $n\to\infty$. For simplicity take $t_p=pT/n \equiv p\delta_n$, $p=0,1,...,n$ and consider the one dimensional diffusion $x(t)$ solution of the Itô SDE:

$$dx(t) = \alpha(x(t))\ dt + \sigma(x(t))\ dw(t) \tag{4.2.1}$$

or the corresponding equivalent Itô stochastic integral equation:

$$x(t) = x(v) + \int_v^t \alpha(x(s))\, ds + \int_v^t \sigma(x(s))\, dw(s) \tag{4.2.2}$$

A simple simulation approach (strong approximation) is to approximate the diffusion x(t) by a discrete process $x^{\circ}_{p}=x^{\circ}(p\delta_n)$, p=0,1,...,n such that, omitting the subscript n for δ_n:

$$E\,[\sup_{p} |x(p\delta)-x^{\circ}_{p}|\,] \leq c\delta^{\gamma} \tag{4.2.3}$$

where γ is the order of strong convergence.

A different approach (moment or weak approximation) is suggested by a different intuition: given that x(t) is a stochastic process, its moments or certain quantities depending on the law of x(t) may be more interesting than some particular path (Talay, 1990).

This implies to approximate the mean of a smooth function g() of the process according to the criterion:

$$\max_{p} |E(g(x(p\delta)))-E(g(x^{\circ}_{p}))| \leq c_g\delta^{\beta} \tag{4.2.4}$$

where β is the order of weak convergence.

Let us now consider the following simulation schemes. Euler scheme (Maruyama, 1955):

$$x^{\circ}_{p+1} = x^{\circ}_{p} + \alpha(x^{\circ}_{p})\delta + \sigma(x^{\circ}_{p})u_{p+1} \qquad p=0,1,\ldots,n-1$$

where $u_{p+1} \equiv w(p\delta+\delta)-w(p\delta)$ is normal $N(0,\delta)$

Milstein (1974) scheme:

$$x^{\circ}_{p+1} = \text{Euler} + \sigma'(x^{\circ}_p)\sigma(x^{\circ}_p) \; ½[u_{p+1}^2-\delta]$$

where a prime ′ means first derivative.

Talay (1984) scheme:

$$x^{\circ}_{p+1} = \text{Milstein} + [\alpha'\alpha+½\alpha''\sigma^2] \; ½\delta^2 + [\alpha'\sigma+\sigma'\alpha+½\sigma''\sigma^2] \; ½u_{p+1}\delta$$

where a double prime ″ means second derivative and, as before, the functions are evaluated at x°_p.

The stochastic foundation of the schemes and their approximation errors are sketched in the Appendix, using a stochastic Taylor formula. It can be shown (Milstein, 1974) that the Euler scheme is of order $\gamma=0.5$ of strong convergence and the Milstein scheme is of order $\gamma=1$. In terms of weak convergence they are both of order $\beta=1$ while Talay scheme is of order $\beta=2$ ([35]).

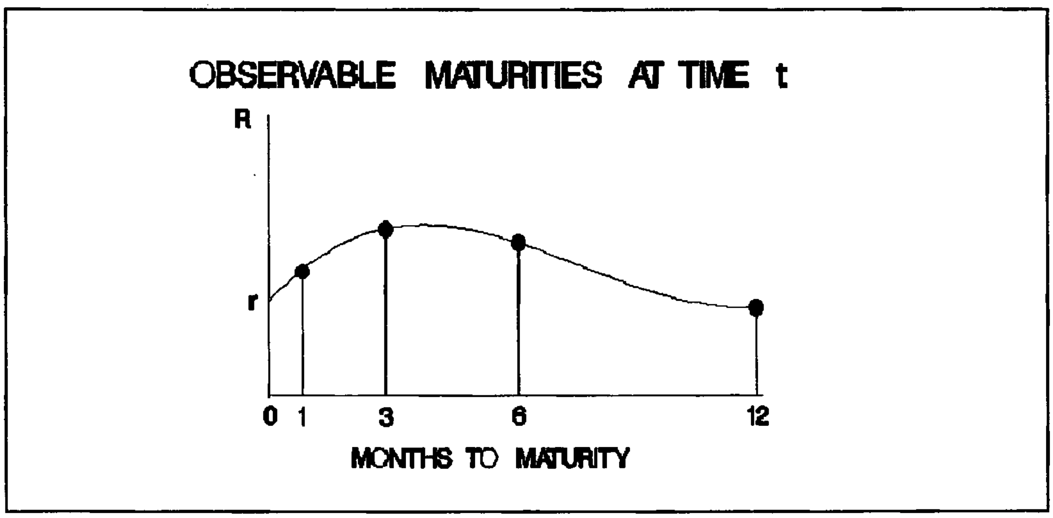
4.3 The Monte Carlo experiment

The first point to stress is that in actual markets the instantaneous interest rate r(t) is not observable. In fact, the rate r(t) is defined as the interest rate on a loan obtained at time t and maturing in the next instant, i.e. at

[35] See Kloeden and Platen (1992, p. 465 and chapters 10-15) for more refined strong and weak approximation schemes.

time t+dt. It is, therefore, a limit concept (see (2.16)), not observable in real markets.

What is in fact observable is a particular, short maturity interest rate $R(t,\tau^{o})\equiv R^{o}(t)$, for τ^{o} fixed. More precisely, at any given time t, the term structure is known at a small number of maturities, for example 1 day to maturity (so called overnight rate), 2, 3 days, 1, 2, 3 weeks, 1, 3, 6, 12 months as in the following picture.



We tried to replicate this situation in which only interest rates with positive maturities are available at discrete-time observation intervals.

The basic set up is given as follows. We assumed the time unit is equal to 1 year. The instantaneous rate r(t) has been generated using the three given DGMs assuming δ equal to 1 hour (δ=1/8760) and parameter values k=0.3, θ=0.1, σ=0.06.

The observable process R(t) has been obtained using definition (2.13), λ=-0.03 and assuming a positive fixed maturity ($\tau°$ equal to 3 months). Finally, the data have been sampled at fixed interval Δ (1 observation every month: Δ=1/12). The parameter values are similar, by and large, to real data estimates ([36]). The starting value r(0) is set to θ, the long run mean.

The usual estimation procedures do not recognize that available observations concern discount bonds with a given non-zero maturity $\tau°$ and use these data as an approximation of the instantaneous rate (τ=0). Our experiment evaluates the effects of this approximation.

We considered 200 simulations over a sample time span up to 200 years (2, 5, 10, 20, 50, 100, 200 years) ([37]). A few examples of generated interest rates $R(t,\tau°)$ over 20 years of sample time span, for $\tau°$= 3 months, are given in Fig. 1. The dotted line is the long run mean θ of the instantaneous rate r(t). A real time series of 3 months Italian Treasury Bill rates over the last 20 years is displayed in Fig. 2 and compared with a similar simulated

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[36] Fournie and Talay (1992) use δ=1/365 (one day), k=0.75, θ=0.1 (10%), σ=0.105 and T up to 110 years (365x 110= 40150 simulated data). They are interested only in the large sample properties of estimators and simulate just one sample path.

Barone and Cesari (1986) and De Felice and Moriconi (1991) use 3-months Treasury bills and estimate for the period 8106-8411 (monthly data) k=0.39341, θ=0.146844, σ=0.05611 and λ=0.065296.

Barone, Cuoco and Zautzik (1989) for daily data 8312-8903 find k=0.25118, θ=0.1117, σ=0.0627 and λ=-0.00783 (calculated under the assumption $\theta=R(\infty)$).

[37] In the case of small samples (2 years of monthly observations) 13 simulations were rejected because of the influence of large outliers.

path (Fig. 2 bis).

5. Monte Carlo results

In this section we shall present first our results for the direct estimation methods (sections 3.1 to 3.6 above) and then some preliminary outcomes concerning the indirect method (section 3.7 above). The main results, in the given Monte Carlo set up, are the following ([38]).

A) The three simulation schemes, for the chosen parameter values, generates very similar processes in terms of interest rates and estimates (equal up to the fifth decimal figure). We may, therefore, confine ourselves to the simpler Euler scheme. Of course, this does not imply that the differences are irrelevant also in the case of different parameter values and in particular in the case of larger diffusion coefficient σ and non ergodic processes. This point may be object of further investigation.

B) The performance of estimators is different for different estimated parameters (k, θ, σ).

[38] We use the abbreviations: CTML for continuous time maximum likelihood estimator (§3.1), LDE for linearized discrete equivalent estimator (§3.2), CME for conditional mean estimator (§3.3), OLS for ordinary least squares estimator of crude discretization (§3.4), GMM for generalized method of moments (§3.5), DTML for discrete time maximum likelihood estimator (§3.6), IES for indirect estimation by simulation (§3.7).

In the case of our basic set up of monthly observations (see Table 1 and Figures 3 to 5) ([39]), the best estimators for σ are LDE and DTML which, in the case of small samples (in particular 5 to 20 years), gives estimates not significantly different from the true values ([40]). The bias is about 2-3 per cent of the true value. The root mean square error (rmse, in percentage of the true value) is below 10 per cent (16 per cent in 2 year samples). In samples up to 5 years the GMM for σ performs poorly with a bias over 10 per cent.

In the case of the lung-run mean θ the best estimator (in terms of rmse) in small as well as large samples is CTML. The bias is 1.3 per cent, not significantly different from zero, and the rmse is 13 per cent. The GMM has a statistically not significant bias of 2.7 per cent and it is the second best estimator. There is no significant bias also asymptotically ([41]).

For the parameter k, all the estimators are grossly upward biased for small samples and converge to the true values only for very large sample time span: with 20 years of monthly observations the bias is 80 per cent of the true

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[39] In Figures 3 to 11 the time span axis is in logs. The values displayed correspond to 1, 2, 5, 10, 20, 50, 100, 200 years.

[40] To judge significance we use a Student's t test given by the bias over the standard error of the estimates.

[41] Only the LDE bias for θ is increasing with the time span, indicating that for this method there is a cumulating error effect in the long run mean estimation.

value (0.3); with 50 years of monthly observations the estimators reach from above the first decimal figure of the true parameter value, with a percentage bias as large as 30 per cent.

The biases and rmse of σ and θ are small and decrease only slightly as the time span increases. The bias (and rmse) of k is viceversa quite large and decreasing to zero over very large time periods.

Asymptotically, also the standard errors of the estimates decrease so that the bias significance for k is decreasing and not significant in large samples. For σ, viceversa, the bias becomes more and more significant.

As the number of observations increases ([42]), the costs of DTML in terms of computation time becomes too high with respect to its performance. In particular, for 100 replications of a 20 years sample of monthly data the DTML requires 7530 seconds of cpu time (on a IBM 3090) with respect to 2/100 sec. for CTML, 4/100 sec. for LDE, 7/100 sec. for CME, 1.38 sec. for GMM. For large samples this method has been dropped.

C) The increased frequency of observations per unit time (from monthly to daily data) for a given time span and a given maturity $\tau°$ implies a general reduction of the bias for σ to 3-4 per cent as well as a reduced dispersion of

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[42] The bias for k is also important in the computation of the modified Bessel function (2.4).

results for different methods in small samples. In particular, LDE displays a slight increase of the bias (from 2.2 to 3.4 per cent in samples up to 5 years) whilst GMM presents a large bias reduction from above 10 to below 4 per cent (see Table 2 and Figures 6 to 8). LDE is still the best estimator for σ (DTML is too costly with daily data) but now the larger number of observations leads to smaller standard errors and to significant bias already in 5 years samples. The asymptotic bias is persistent. Moreover, the difference with the other estimators is negligible.

The best estimator for θ is again CTML, with a not significant bias of 1.6 per cent in 2 year samples, half as large as the bias (3.2 per cent) of the GMM. This one and OLS are the second best estimators, whilst LDE displays again an increasing significant bias (2.8 per cent for a 20 years sample against 1.3 per cent for CTML). The effect on k of higher frequency observations is a worsening of the finite sample bias.

D) In order to assess the effect of approximating $r(t)$ with $R(t,\tau°)$ we have directly used monthly data of the instantaneous rate $r(t)$ (see Table 3 and Figures 9 to 11). For σ the best estimator is now DTML with a bias of about 2 per cent for 2 years samples, practically negligible (-0.2 per cent) from 5 years samples onward. Quite good results are obtained with CTML and, when samples are not too short (10 years or more), with LDE. In any case the decreasing bias is never significant, indicating that the asymptotic bias was due to the assumed approximation.

For k all the estimators display a large bias in small samples. The same results for CTML can be found in Fournie and Talay (1992).

The best estimator for θ is CTML with a (not significant) bias of 1 per cent and a rmse of 14 per cent (2 years of data). LDE has a bias of 2.6 per cent in 20 years samples. GMM and OLS have similar performance in large samples (the rmse gets 9.7 per cent for 50 year samples) and biases are always not significant.

If, instead of monthly data, we use high frequency observations for r(t), for example hourly observations, with an observation step Δ (1 hour) equal to the generation step δ, we obtain, for a 2 years sample, a rmse of 0.55 per cent for σ using LDE and of 13.8 per cent for θ using CTML. Even in this case, the bias and standard error for k are quite large for every method, showing definitely that the relevant information to efficiently estimate the speed of adjustment k can be obtained only from a large sample time span (the range [0,T] in continuous time), almost independently from the frequency of observation: we could say that in order to correctly estimate k, 100 monthly observation are better than 1000 daily data.

E) In order to better appreciate the effect of the approximation $R(\tau^{\circ})\approx r$ we have used different values for τ°: 0, 1 day, 1 week, 1 month, 3 months, 6 months, 1 year. The results for the various methods (DTML excluded) are displayed in Fig. 12 ([43]). For $\tau°$= 1 month or greater the estimation bias is increasing for σ and θ. For k, only CTML is sensitive to an increasing $\tau°$ whilst the other estimators are not.

A similar exercise was made to assess the effect of the "price of risk" parameter λ (see equation (2.8)). For $\tau°$=3 months and λ=-0.1, -0.03, 0, 0.03 we obtained the results in Fig. 13. We can see that an increasing λ has small reducing effect (i.e. increasing bias in this case) on the estimates for σ, almost no effect for k and sensible reducing effect for θ (decreasing bias). The simultaneous effects of $\tau°$ and λ are one-sided and reciprocally reinforcing for σ and k whilst they tend to compensate for θ when the maturity $\tau°$ is relatively large (3 months or greater).

F) As expected, the staring point r(0) has important effects in small samples. A starting point 25 per cent below the long run value (0.1) implies a bias for θ as large as 17 per cent in 2 year samples and 11 per cent in 5 years samples. The estimators are not equally affected. CTML is no longer the best estimator when the starting point is out of equilibrium. For this method the bias is still at 8 per cent when the sample has been increased to 10 years of monthly data. Better estimators are LDE and OLS.

We also note that the effects of initial values above or below the long run value are not symmetric. The bias is larger and more persistent starting the process from above θ

[43] The exercise consists of 200 replications of monthly data over 10 years of time span.

(1.25 θ) than from below (0.75 θ): for LDE with a 2 years sample we have in the former case a bias of 17.3 per cent and in the latter case a bias of -12,7 per cent; with a 5 years sample the two biases are 11.8 and -4.8 per cent respectively; with a 10 years sample they reach 4.1 and -0.9 per cent, a value not significant at the 5 per cent level (see the following table).

Time span	**r(0)=0.75 θ**	**r(0)=1.25 θ**
CTML (bias per cent)		
2 years	-17.0	19.3
5 years	-11.0	14.3
10 years	-7.3	8.2
20 years	-2.7	5.4
LDE (bias per cent)		
2 years	-12.7	17.3
5 years	-4.8	11.8
10 years	-0.9 (*)	4.1
20 years	1.9 (*)	3.8

(*) not significant at 5 per cent level.

The estimation for σ is not affected by r(0) whilst the identification of the speed of adjustment k, always very poor in finite samples, seems slightly improved by a staring value out of long run equilibrium.

G) The preliminary results concerning the indirect estimation are promising. First of all, we have found a substantial reduction of the bias for k (from 76 to 29 per cent) even in small samples (25 years of monthly data); secondly, the bias for σ has been eliminated:

Parameters	OLS ESTIMATES (*)	IES ESTIMATES (*)
k (=0.3)	0.529 (0.25)	0.386 (0.28)
θ (=0.1)	0.100 (0.01)	0.099 (0.02)
σ (=0.06)	0.057 (0.00)	0.061 (0.01)

(*) standard errors in parenthesis.

This method, however, requires further investigation, in particular because of convergence problems during the simulation and calibration phases. It may also gain in efficiency from the expected enhancement in computational levels (parallel computers) so that it could really outperform all direct methods.

H) Finally, given the distortion of the considered estimators, we have investigated the effect of simulating the bond price dynamics using wrong parameters, in order to assess the financial effects (for example in portfolio strategy) of inferential problems.

We have, therefore, simulated the bond price (2.10) for a single maturity τ= 3 months, using alternatively the "true" parameter set (control series) and a different set with biased parameter values (biased series) ([44]). According to our previous results in the case of 20 years of time span, σ has been reduced by -5 per cent, k increased by

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[44] We have run 500 replications using hourly data over one year of time span. Note that simulation (direct and indirect effects) is required instead of partial derivatives $\partial P/\partial \alpha$ because r(t) is endogenous, function of the basic parameters. If r(t) were known the effect of a bias would be smaller.

+100 per cent and θ increased by +2 per cent. The price difference, calculated as biased price less control price, results to be small and not significant for σ and k: after 1 month it is, in absolute value, about 0.0000013 (1.3 $ per million); after 12 month it reaches -0.0000096 for σ and 0.000031 (31 $ per million) for k.

In the case of θ the difference is larger and stable over time at about -0.00049 (490 $ per million). The negative sign indicates that increasing θ reduces the bond price. The long run mean θ seems therefore a crucial parameter to estimate, whilst biased values for σ and k have minor effect on the asset price.

6. Conclusions

We have investigated the relative performance of alternative estimators of a popular diffusion model of the term structure of interest rates, characterized by a single state-variable with Feller (square root) dynamics (see section 2).

A number of different estimators, reviewed in section 3, has been proposed in the empirical literature: the continuous-time ML estimator (CTML), the linearized discrete equivalent (LDE), the conditional mean estimator (CME) and the estimator of the naïve discretization of the SDE (OLS), the generalized method of moments estimator (GMM), the discrete-time ML estimator (DTML), the indirect estimator

(IES). However, no general evaluation of their econometric performance and relative behaviour, particularly with small samples of data, was available.

In order to attack this problem we built a set of Monte Carlo experiments (see section 4) designed to simulate continuous-time (diffusion) processes and replicate the typical situation of empirical applications. In particular, we set the time unit to 1 year, the generation time interval to 1 hour, the observation time frequency from 1 to 30 days, the time span from 1 to 200 years. The state-variable parameters (k, θ, σ) were set to values approximately equal to published empirical estimates. Different simulation schemes (Euler, Milstein and Talay schemes) were used without finding important differences in the estimators results.

Particular attention has been paid to the unobservability of the state variable, the instantaneous (zero maturity) interest rate r(t), and the consequences of using observable rates $R^{\circ}(t)$, for a given positive maturity τ°, as a proxy variable for r(t).

For σ, the diffusion parameter, we have found a general downward distortion of the direct estimators. This persistent asymptotic bias is due to the approximating rate R° and it would disappear if the (unobservable) instantaneous rate r(t) (or a very short maturity rate: see Fig. 12) were used. Using data of higher frequency (e.g. from monthly to daily) the estimator variance is reduced, not the bias. LDE and DTML appears to be the best estimators, but the latter is computationally more demanding. The worst performance, in small samples, comes from GMM. Viceversa, IES seems able to eliminate the bias.

For k, the speed of adjustment parameter, there is no bias asymptotically but large upward bias in finite samples. No estimator clearly outperforms the others but IES seems to present the smaller bias. Increasing the frequency of the observations has a worsening effect on the already bad finite sample results. It appears that to confidently estimate the speed of adjustment a time series is needed much longer than usually available data.

The parameter θ, the steady state mean of the instantaneous rate, is in general estimated with no significant bias. The best estimator is CTML and the worst is LDE. This result, however, depends on the starting point $r(0)=\theta$. Starting values out of equilibrium reduce the relative performance of CTML and make LDE more efficient. Moreover, starting values below equilibrium are more quickly absorbed than starting values above θ.

Finally, we have investigated the financial effect (in terms of bond price) of using biased parameters. Considering the typical distortion found in the case of data over 20 years of time span, we obtain no significant effect from σ and k but a more sensible price gap even from small bias (2 per cent) for θ. This parameter seems therefore a crucial one to estimate.

APPENDIX

A stochastic Taylor formula

In this appendix we shall give an euristic derivation of a stochastic Itô-Taylor formula (Platen and Wagner, 1982; Kloeden and Platen, 1992, ch. 5) which generalizes Itô formula and is useful to understand the discrete data schemes used in the text (see section 4.2).

Let us consider the one dimensional diffusion x(t) solution of the Itô stochastic integral equation:

$$x(t) = x(v) + \int_v^t \alpha(x(s))\,ds + \int_v^t \sigma(x(s))\,dw(s) \tag{A.1}$$

Let f(x) be a real function with continuous partial derivatives f', f'', etc. Writing x_s for x(s), α_s for $\alpha(x_s)$, σ_s for $\sigma(x_s)$, Itô formula, in integral form, is:

$$f(x_t) = f(x_v) + \int_v^t \left[f'(x_s)\,\alpha_s + \frac{1}{2} f''(x_s)\,\sigma_s^2\right] ds + \int_v^t f'(x_s)\,\sigma_s\,dw_s \tag{A.2}$$

$$\equiv I_{t,x}(f_v)$$

For f(x)=x, (A.2) reduces to (A.1). For f(x)=α(x) and f(x)=σ(x) we obtain two expressions for the drift and diffusion functions:

$$\alpha_s = \alpha_v + \int_v^s \left[\alpha'_u \alpha_u + \frac{1}{2}\alpha''_u \sigma^2_u\right] du + \int_v^s \alpha'_u \sigma_u dw_u = I_{s,x}(\alpha_v) \tag{A.3}$$

$$\sigma_s = \sigma_v + \int_v^s \left[\sigma'_u \alpha_u + \frac{1}{2}\sigma''_u \sigma^2_u\right] du + \int_v^s \sigma'_u \sigma_u dw_u \equiv I_{s,x}(\sigma_v)$$

so that, substituting in (A.1):

$$x_t = x_v + \alpha_v \int_v^t ds + \int_v^t \int_v^s \left[\alpha'_u \alpha_u + \frac{1}{2}\alpha''_u \sigma^2_u\right] du\, ds + \int_v^t \int_v^s \alpha'_u \sigma_u dw_u ds + \tag{A.4}$$

$$\sigma_v \int_v^t dw_s + \int_v^t \int_v^s \left[\sigma'_u \alpha_u + \frac{1}{2}\sigma''_u \sigma^2_u\right] du\, dw_s + \int_v^t \int_v^s \sigma'_u \sigma_u dw_u dw_s$$

This is the simplest stochastic Taylor formula for x(t) and could be written as:

$$x_t = x_v + \alpha_v \int_v^t ds + \sigma_v \int_v^t dw_s + R_1 \tag{A.5}$$

where R_1 is the remainder. This formula suggests the Euler scheme given in the text.

More elaborate expressions can be obtained substituting again the integrand functions in (A.4) with their Itô expansions $I_{u,x}$, obtaining expressions with deterministic and stochastic triple integrals. In particular:

Using Itô formula for w^2_t we notice that:

$$x_t = x_v + \alpha_v \int_v^t ds + \sigma_v \int_v^t dw_s + \sigma'_v \sigma_v \int_v^t \int_v^s dw_u dw_s +$$

$$\left[\alpha'_v \alpha_v + \frac{1}{2} \alpha''_v \sigma^2_v\right] \int_v^t \int_v^s du ds + \tag{A.6}$$

$$\alpha'_v \sigma_v \int_v^t \int_v^s dw_u ds + \left[\sigma'_v \alpha_v + \frac{1}{2} \sigma''_v \sigma^2_v\right] \int_v^t \int_v^s du dw_s + R_2$$

$$\int_v^t \int_v^s dw_u dw_s = \frac{(w_t - w_v)^2 - (t-v)}{2} \tag{A.7}$$

which is the term in the Milstein scheme.

Moreover, using Itô formula for tw_t:

$$tw_t = vw_v + \int_v^t w_s ds + \int_v^t s dw_s \tag{A.8}$$

we have:

$$\int_v^t \int_v^s dw_u ds = (t-v) \int_v^t dw_s - \int_v^t \int_v^s du dw_s$$

$$\text{with } E\left(\int_v^t \int_v^s du dw_s\right) = 0, \quad Var\left(\int_v^t \int_v^s du dw_s\right) = \frac{1}{3}(t-v)^3 \tag{A.9}$$

$$Cov\left(\int_v^t dw_s, \int_v^t \int_v^s du dw_s\right) = \frac{1}{2}(t-v)^2$$

so that the two double integrals in (A.9) have the same mean, variance and covariance with the Wiener increments.

The last two integrals in (A.6) justify the last component in the Talay scheme.

The remainder of the stochastic Taylor formula allows to assess the order of strong and weak convergence of different approximating schemes.

Applying the same token to (A.2) we can obtain a stochastic Taylor formula for $f(x_t)$ as an extension of Itô formula.

PERCENTAGE RELATIVE BIAS, ROOT MEAN SQUARE ERROR AND STUDENT T: SIMULATED MONTHLY DATA FOR 3 MONTH T-BILLS

	bias			rmse			t student		
years=2									
187	sigma	k	theta	sigma	k	theta	sigma	k	theta
CTML	-5,76	732,69	1,34	14,25	842,39	13,14	-0,44	1,76	0,10
LDE	2,16	843,91	5,72	15,95	1034,72	36,84	0,14	1,41	0,16
CME	-8,68	845,49	4,92	15,76	1036,55	33,65	-0,66	1,41	0,15
OLS	-8,68	707,28	4,92	15,76	846,88	33,65	-0,66	1,52	0,15
GMM	-14,05	723,62	2,71	19,65	859,88	26,72	-1,02	1,56	0,10
DTML	-2,04	848,20	5,34	15,25	1031,80	35,20	-0,13	1,44	0,15
years=5	bias			rmse			t student		
199	sigma	k	theta	sigma	k	theta	sigma	k	theta
CTML	-6,38	366,95	1,74	10,20	465,64	15,82	-0,80	1,28	0,11
LDE	-2,18	397,27	3,15	8,95	535,04	31,36	-0,25	1,11	0,10
CME	-7,85	396,32	2,47	11,23	533,98	30,03	-0,98	1,11	0,08
OLS	-7,85	353,23	2,47	11,23	462,86	30,03	-0,98	1,18	0,08
GMM	-10,53	370,27	2,15	13,58	483,36	25,64	-1,23	1,19	0,08
DTML	-3,81	395,92	3,14	9,33	533,36	30,20	-0,45	1,11	0,10
years=10	bias			rmse			t student		
198	sigma	k	theta	sigma	k	theta	sigma	k	theta
CTML	-4,89	181,39	0,64	8,24	236,86	14,34	-0,74	1,19	0,04
LDE	-2,40	177,30	2,97	7,26	247,01	25,33	-0,35	1,03	0,12
CME	-5,60	177,27	1,81	8,60	246,57	21,49	-0,86	1,03	0,08
OLS	-5,60	164,61	1,81	8,60	227,36	21,49	-0,86	1,05	0,08
GMM	-6,88	172,07	1,56	9,62	235,80	19,81	-1,02	1,07	0,08
DTML	-3,21	177,23	1,96	7,52	246,95	22,85	-0,47	1,03	0,09
years=20	bias			rmse			t student		
200	sigma	k	theta	sigma	k	theta	sigma	k	theta
CTML	-4,79	82,91	1,34	6,56	122,94	12,51	-1,07	0,91	0,11
LDE	-3,36	80,30	2,90	5,53	127,46	17,07	-0,77	0,81	0,17
CME	-5,43	79,57	1,82	6,92	127,07	16,32	-1,27	0,80	0,11
OLS	-5,43	74,51	1,82	6,92	119,06	16,32	-1,27	0,80	0,11
GMM	-6,13	81,75	1,84	7,52	126,41	18,26	-1,41	0,85	0,10
DTML	-3,75	79,79	1,80	5,76	127,23	16,50	-0,86	0,81	0,11
years=50	bias			rmse			t student		
200	sigma	k	theta	sigma	k	theta	sigma	k	theta
CTML	-4,44	31,66	0,48	5,24	51,77	8,80	-1,59	0,77	0,05
LDE	-3,53	31,05	1,86	4,47	52,60	9,46	-1,28	0,73	0,20
CME	-5	29,58	0,69	5,70	51,53	9,18	-1,83	0,70	0,08
OLS	-5	27,32	0,69	5,70	49,09	9,18	-1,83	0,67	0,08
GMM	-5,38	32,64	0,51	6,09	54,29	9,26	-1,89	0,75	0,05
DTML	-3,69	29,69	0,69	4,59	51,68	9,18	-1,34	0,70	0,08
years=100	bias			rmse			t student		
200	sigma	k	theta	sigma	k	theta	sigma	k	theta
CTML	-4,16	14,52	0,45	4,63	33,01	6,59	-2,04	0,49	0,07
LDE	-3,52	14,74	1,81	4,07	34,54	7,09	-1,73	0,47	0,26
CME	-4,82	13,84	0,53	5,22	33,55	6,79	-2,41	0,45	0,08
OLS	-4,82	12,14	0,53	5,22	32,01	6,79	-2,41	0,41	0,08
GMM	-4,98	15,41	0,28	5,38	35,37	6,79	-2,47	0,48	0,04
DTML									
years=200	bias			rmse			t student		
200	sigma	k	theta	sigma	k	theta	sigma	k	theta
CTML	-3,93	5,50	0,74	4,17	18,78	4,66	-2,79	0,31	0,16
LDE	-3,39	6,01	2,09	3,67	19,65	5,17	-2,42	0,32	0,44
CME	-4,60	5,49	0,74	4,79	19,36	4,73	-3,40	0,30	0,16
OLS	-4,60	4,09	0,74	4,79	18,54	4,73	-3,40	0,23	0,16
GMM	-4,68	6,24	0,51	4,87	19,90	4,63	-3,43	0,33	0,11
DTML									

TABLE 2

PERCENTAGE RELATIVE BIAS, ROOT MEAN SQUARE ERROR AND STUDENT T: SIMULATED DAILY DATA FOR 3 MONTH T-BILLS

years=1	bias			rmse			t student		
196	sigma	k	theta	sigma	k	theta	sigma	k	theta
CTML	-3,76	1823,80	1,38	4,90	2248,97	9,88	-1,19	1,39	0,14
LDE	-3,26	1845,73	2,61	4,52	2306,06	20,15	-1,04	1,32	0,13
CME	-4,03	1844,59	2,58	5,12	2313,97	20,43	-1,27	1,32	0,13
OLS	-4,03	1821,31	2,58	5,12	2277,08	20,43	-1,27	1,33	0,13
GMM	-4,61	1888,54	3,31	5,70	2361,90	23,37	-1,37	1,33	0,14
DTML									
years=2	bias			rmse			t student		
193	sigma	k	theta	sigma	k	theta	sigma	k	theta
CTML	-3,62	909,39	1,65	4,36	1123,21	12,70	-1,49	1,38	0,13
LDE	-3,37	934,59	4,61	4,15	1179,29	26,46	-1,39	1,30	0,18
CME	-3,78	934,22	4,31	4,49	1179,48	26,55	-1,57	1,30	0,16
OLS	-3,78	927,74	4,31	4,49	1168,77	26,55	-1,57	1,31	0,16
GMM	-4,09	963,45	3,18	4,77	1210,49	22,33	-1,66	1,31	0,14
DTML									
years=5	bias			rmse			t student		
198	sigma	k	theta	sigma	k	theta	sigma	k	theta
CTML	-3,53	401,30	1,66	3,86	518,73	15,85	-2,27	1,22	0,11
LDE	-3,41	400,22	3,24	3,75	530,86	27,75	-2,16	1,15	0,12
CME	-3,60	398,59	2,63	3,93	529,97	26,32	-2,29	1,14	0,10
OLS	-3,60	397,07	2,63	3,93	527,39	26,32	-2,29	1,14	0,10
GMM	-3,77	425,81	2,51	4,09	549,63	23,21	-2,39	1,16	0,11
DTML									
years=10	bias			rmse			t student		
199	sigma	k	theta	sigma	k	theta	sigma	k	theta
CTML	-3,62	187,10	0,35	3,78	245,86	14,33	-3,34	1,17	0,02
LDE	-3,61	171,39	1,69	3,76	238,35	19,75	-3,38	1,03	0,09
CME	-3,71	170,80	0,94	3,86	237,92	19,46	-3,50	1,03	0,05
OLS	-3,71	170,38	0,94	3,86	237,29	19,46	-3,50	1,03	0,05
GMM	-3,78	177,87	1,14	3,93	246,40	20,06	-3,53	1,04	0,06
DTML									
years=20	bias			rmse			t student		
200	sigma	k	theta	sigma	k	theta	sigma	k	theta
CTML	-3,56	88,27	1,31	3,66	129,88	12,51	-4,24	0,93	0,11
LDE	-3,59	80,71	2,84	3,68	126,81	15,96	-4,37	0,83	0,18
CME	-3,66	79,90	1,83	3,75	126,39	15,47	-4,46	0,82	0,12
OLS	-3,66	79,72	1,83	3,75	126,11	15,47	-4,46	0,82	0,12
GMM	-3,71	84,40	1,73	3,79	130,70	15,17	-4,52	0,85	0,11
DTML									
years=100	bias			rmse			t student		
200	sigma	k	theta	sigma	k	theta	sigma	k	theta
CTML	-3,49	31,87	0,42	3,53	54,10	9,04	-6,61	0,73	0,05
LDE	-3,55	28,41	1,70	3,59	52,54	9,56	-6,94	0,64	0,18
CME	-3,60	27,48	0,51	3,64	52,04	9,31	-7,06	0,62	0,06
OLS	-3,60	27,40	0,51	3,64	51,96	9,31	-7,06	0,62	0,06
GMM	-3,61	30,42	0,45	3,65	54,53	9,35	-7,03	0,67	0,05
DTML									
years=100	bias			rmse			t student		
200	sigma	k	theta	sigma	k	theta	sigma	k	theta
CTML	-3,49	16,24	0,43	3,51	34,21	6,56	-9,02	0,54	0,07
LDE	-3,56	14,82	1,80	3,58	34,19	7,06	-9,62	0,48	0,26
CME	-3,60	13,94	0,52	3,62	33,12	6,76	-9,74	0,46	0,08
OLS	-3,60	13,89	0,52	3,62	33,07	6,76	-9,74	0,46	0,08
GMM	-3,60	15,50	0,49	3,62	35,45	6,74	-9,81	0,49	0,07
DTML									

PERCENTAGE RELATIVE BIAS, ROOT MEAN SQUARE ERROR AND STUDENT T: SIMULATED MONTHLY DATA FOR INSTANTANEOUS T-BILLS

years=2	bias			rmse			t student		
187	sigma	k	theta	sigma	k	theta	sigma	k	theta
CTML	-2,03	751,30	1,10	13,35	877,61	13,84	-0,15	1,66	0,08
LDE	6,70	901,56	6,27	17,79	1103,18	38,16	0,41	1,42	0,17
CME	-5,30	903,48	5,47	14,29	1105,92	35,39	-0,40	1,42	0,16
OLS	-5,30	749,39	5,47	14,29	888,34	35,39	-0,40	1,57	0,16
GMM	-10,91	764,95	3,46	17,72	899,99	27,22	-0,78	1,61	0,13
DTML	2,21	902,37	5,21	15,86	1101,91	37,04	0,14	1,43	0,14
years=5	bias			rmse			t student		
199	sigma	k	theta	sigma	k	theta	sigma	k	theta
CTML	-2,96	363,20	1,51	8,72	462,62	16,62	-0,36	1,27	0,09
LDE	1,48	400,72	3,50	9,03	536,14	32,06	0,17	1,13	0,11
CME	-4,43	399,79	2,79	9,37	535,12	30,49	-0,54	1,12	0,09
OLS	-4,43	356,48	2,79	9,37	463,79	30,49	-0,54	1,20	0,09
GMM	-7,28	373,91	2,43	11,44	484,30	25,49	-0,82	1,21	0,10
DTML	-0,23	398,54	2,89	8,75	534,43	31,31	-0,03	1,12	0,09
years=10	bias			rmse			t student		
198	sigma	k	theta	sigma	k	theta	sigma	k	theta
CTML	-1,40	179,09	0,13	6,96	234,73	14,92	-0,20	1,18	0,01
LDE	1,24	177,95	2,60	7,16	247,20	25,92	0,18	1,04	0,10
CME	-2,08	177,99	1,39	7,04	246,79	22,03	-0,31	1,04	0,06
OLS	-2,08	165,31	1,39	7,04	227,57	22,03	-0,31	1,06	0,06
GMM	-3,47	172,73	1,10	7,75	236,01	20,40	-0,50	1,07	0,05
DTML	0,41	177,74	1,54	7,01	247,02	23,45	0,06	1,04	0,07
years=20	bias			rmse			t student		
200	sigma	k	theta	sigma	k	theta	sigma	k	theta
CTML	-1,35	81,67	0,98	4,84	121,94	12,93	-0,29	0,90	0,08
LDE	0,21	80,33	2,65	4,56	127,49	17,62	0,05	0,81	0,15
CME	-1,94	79,66	1,49	4,86	127,13	16,81	-0,43	0,80	0,09
OLS	-1,94	74,60	1,49	4,86	119,12	16,81	-0,43	0,80	0,09
GMM	-2,69	81,97	1,51	5,28	126,75	19,06	-0,59	0,85	0,08
DTML	-0,20	79,65	1,46	4,55	127,16	17,07	-0,04	0,80	0,09
years=50	bias			rmse			t student		
200	sigma	k	theta	sigma	k	theta	sigma	k	theta
CTML	-0,72	28,92	0,04	2,93	51,73	9,35	-0,25	0,67	0
LDE	0,21	28,37	1,44	2,85	52,76	9,83	0,07	0,64	0,15
CME	-1,30	27,30	0,15	3,08	52,14	9,62	-0,46	0,61	0,02
OLS	-1,30	25,08	0,15	3,08	49,67	9,62	-0,46	0,59	0,02
GMM	-1,70	31,28	-0,16	3,42	54,53	9,70	-0,57	0,70	-0,02
DTML	0,05	27,45	0,16	2,85	52,37	9,62	0,02	0,62	0,02
years=100	bias			rmse			t student		
200	sigma	k	theta	sigma	k	theta	sigma	k	theta
CTML	-0,70	13,96	0,08	2,22	32,70	6,80	-0,33	0,47	0,01
LDE	0,07	14,79	1,55	2,12	34,53	7,26	0,03	0,47	0,22
CME	-1,29	13,86	0,17	2,44	33,52	7,01	-0,62	0,45	0,02
OLS	-1,18	12,17	0,17	2,44	31,99	7,01	-0,62	0,41	0,02
GMM	-1,49	15,47	-0,14	2,57	35,44	7,02	-0,71	0,49	-0,02
DTML									
years=200	bias			rmse			t student		
200	sigma	k	theta	sigma	k	theta	sigma	k	theta
CTML	-0,47	5,01	0,38	1,55	18,65	4,78	-0,32	0,28	0,08
LDE	0,19	6,07	1,84	1,48	19,66	5,22	0,13	0,32	0,38
CME	-1,06	5,53	0,39	1,77	19,38	4,85	-0,75	0,30	0,08
OLS	-1,06	4,13	0,39	1,77	18,56	4,85	-0,75	0,23	0,08
GMM	-1,17	6,23	0,11	1,84	19,83	4,75	-0,82	0,33	0,02
DTML									

Fig. 1

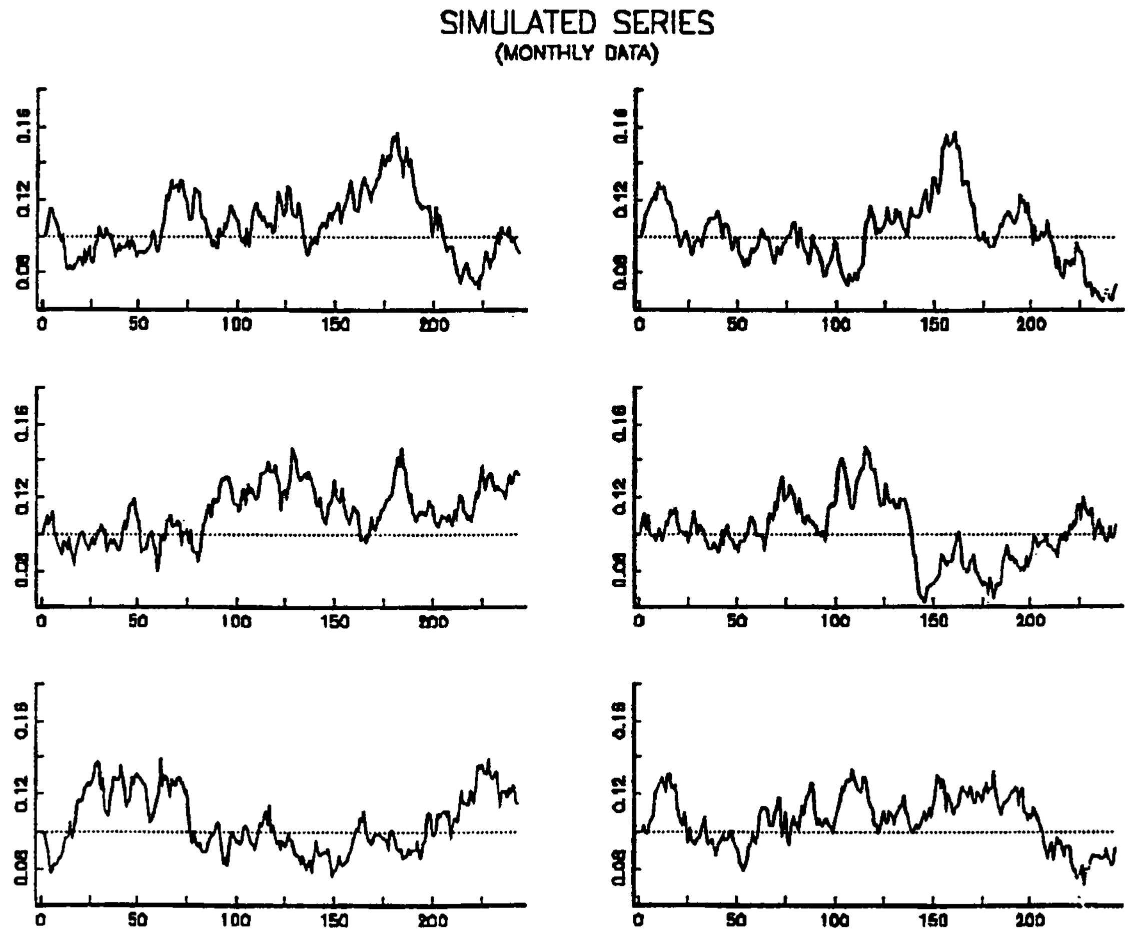


Fig. 2

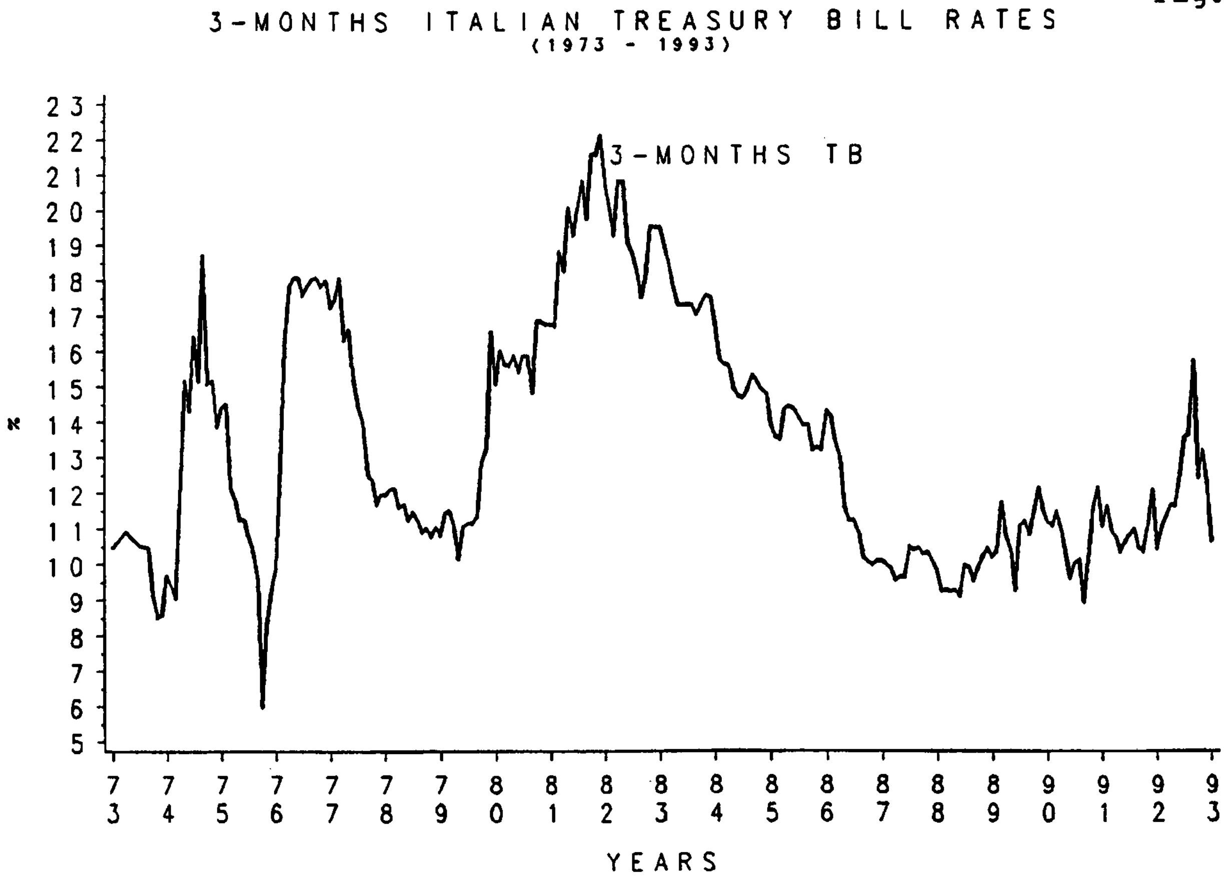

3-MONTHS ITALIAN TREASURY BILL RATES
(1973 - 1993)
3-MONTHS TB
%
23
22
21
20
19
18
17
16
15
14
13
12
11
10
9
8
7
6
5
73
74
75
76
77
78
79
80
81
82
83
84
85
86
87
88
89
90
91
92
93
YEARS

Fig. 2bi

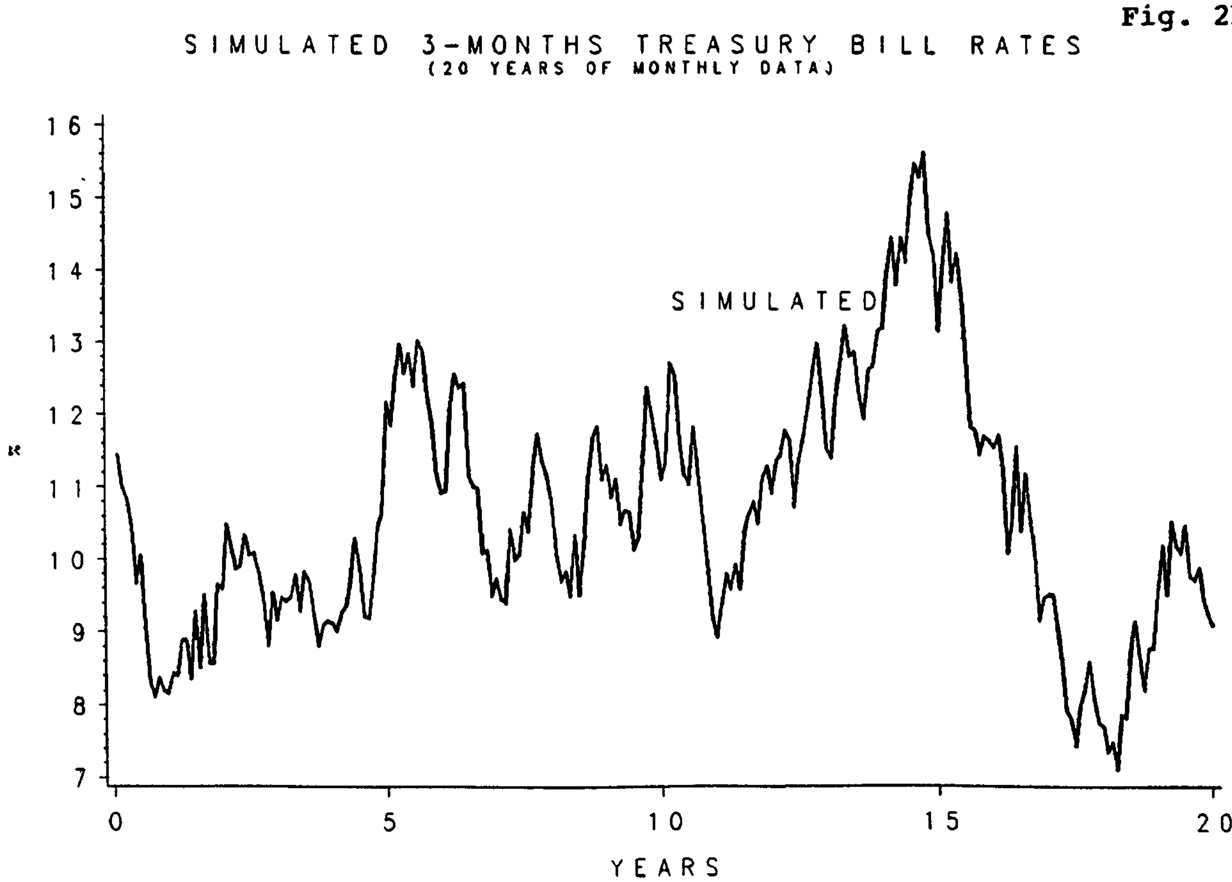

SIMULATED 3-MONTHS TREASURY BILL RATES
(20 YEARS OF MONTHLY DATA)
SIMULATED
%
16
15
14
13
12
11
10
9
8
7
0
5
10
15
20
YEARS

Fig. 3

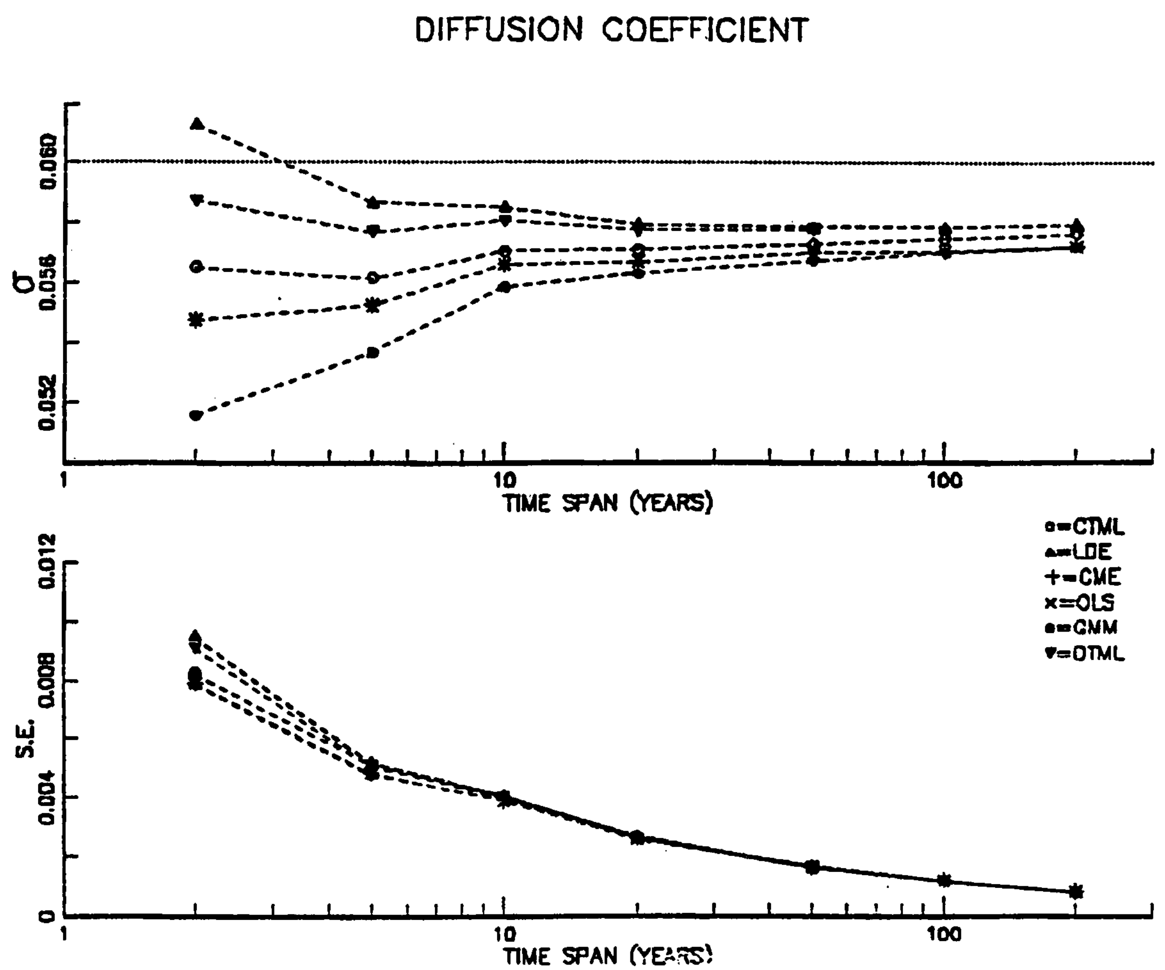

DIFFUSION COEFFICIENT
σ
0.060
0.056
0.052
1
10
100
TIME SPAN (YEARS)
o=CTML
▲=LDE
+=CME
x=OLS
●=GMM
▼=DTML
S.E.
0.012
0.008
0.004
0
1
10
100
TIME SPAN (YEARS)

Fig. 4

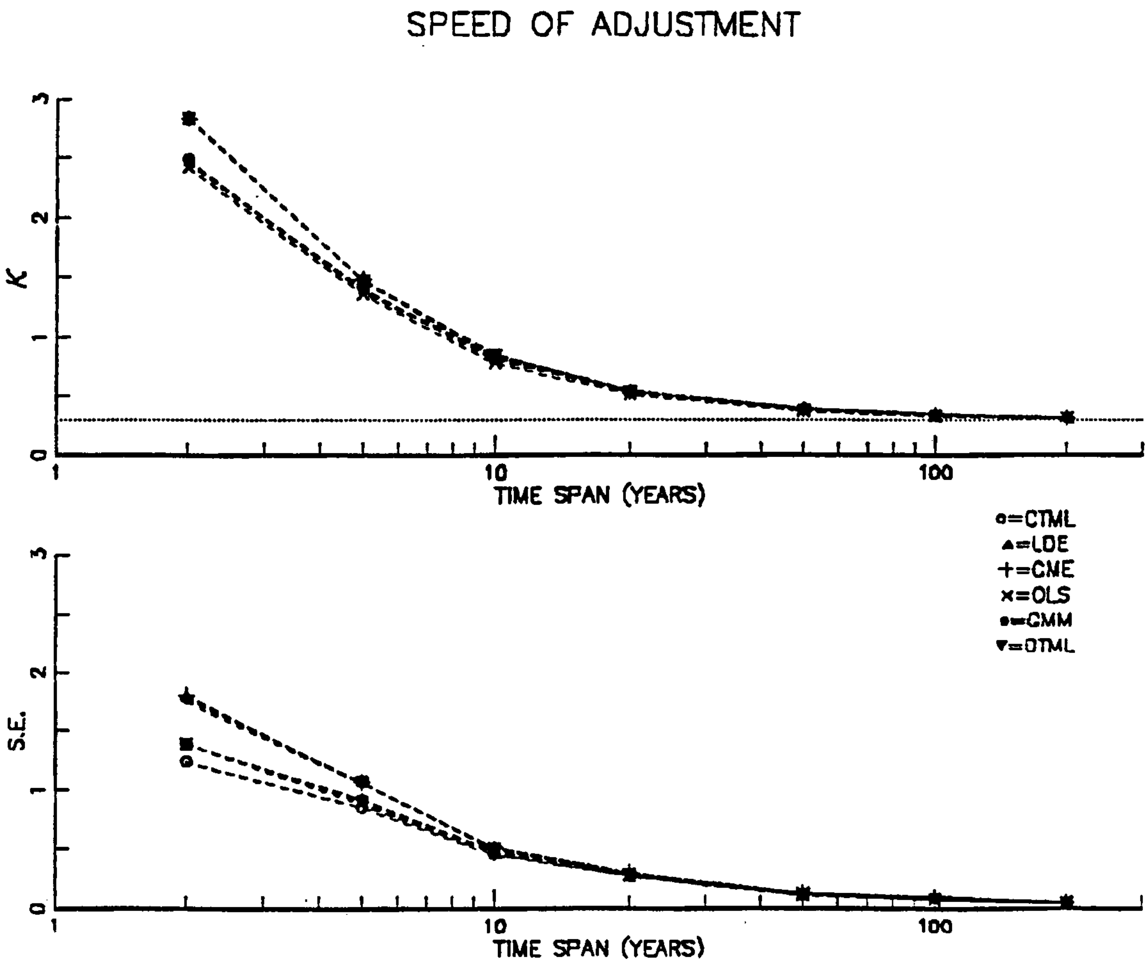
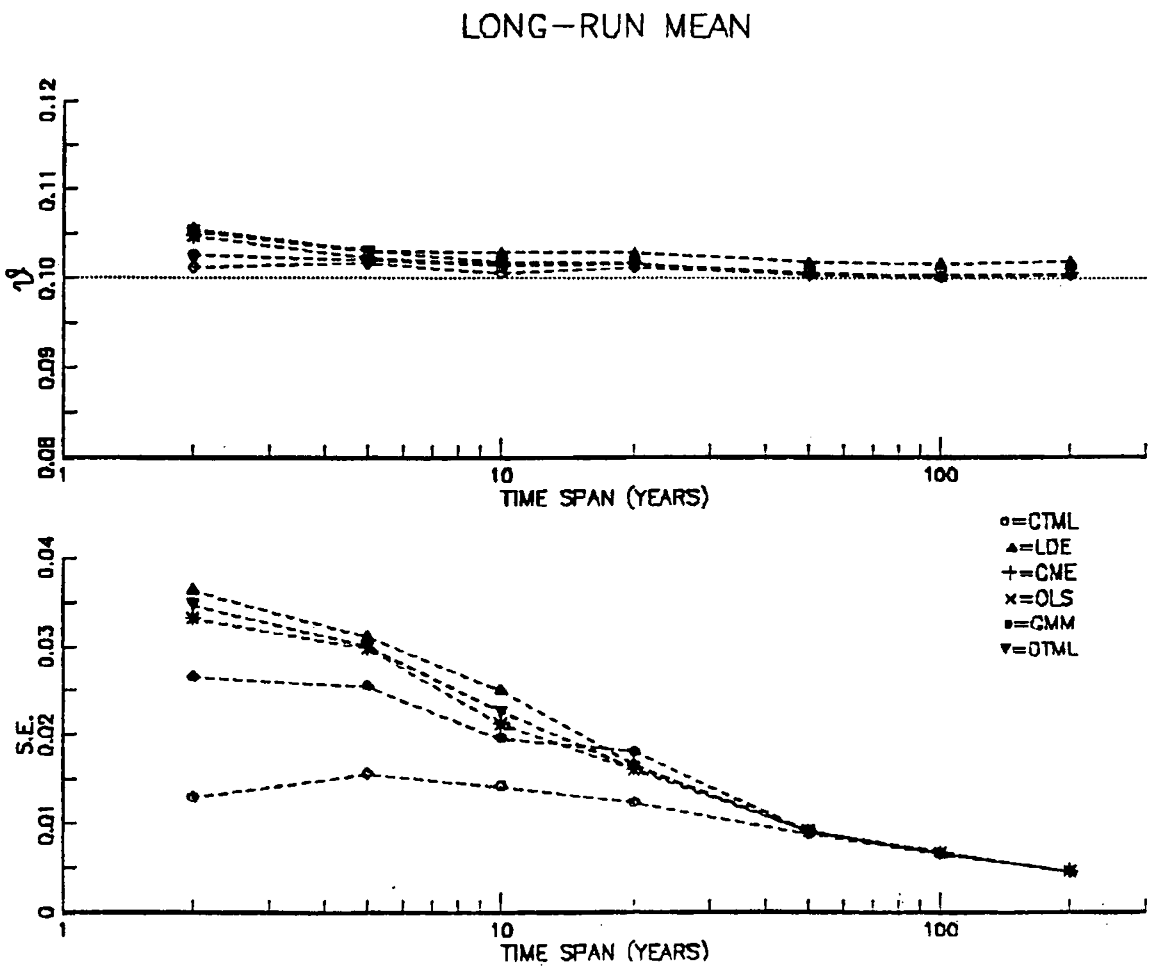

SPEED OF ADJUSTMENT
K
3
2
1
0
1
10
100
TIME SPAN (YEARS)
o=CTML
▲=LDE
+=CME
×=OLS
■=GMM
▼=DTML
S.E.
3
2
1
0
1
10
100
TIME SPAN (YEARS)

Fig. 5



LONG-RUN MEAN
0.12
0.11
0.10
0.09
0.08
ϑ
1
10
100
TIME SPAN (YEARS)
0.04
0.03
0.02
0.01
0
S.E.
1
10
100
TIME SPAN (YEARS)
o=CTML
▲=LDE
+=CME
×=OLS
•=GMM
▼=DTML

Fig. 6

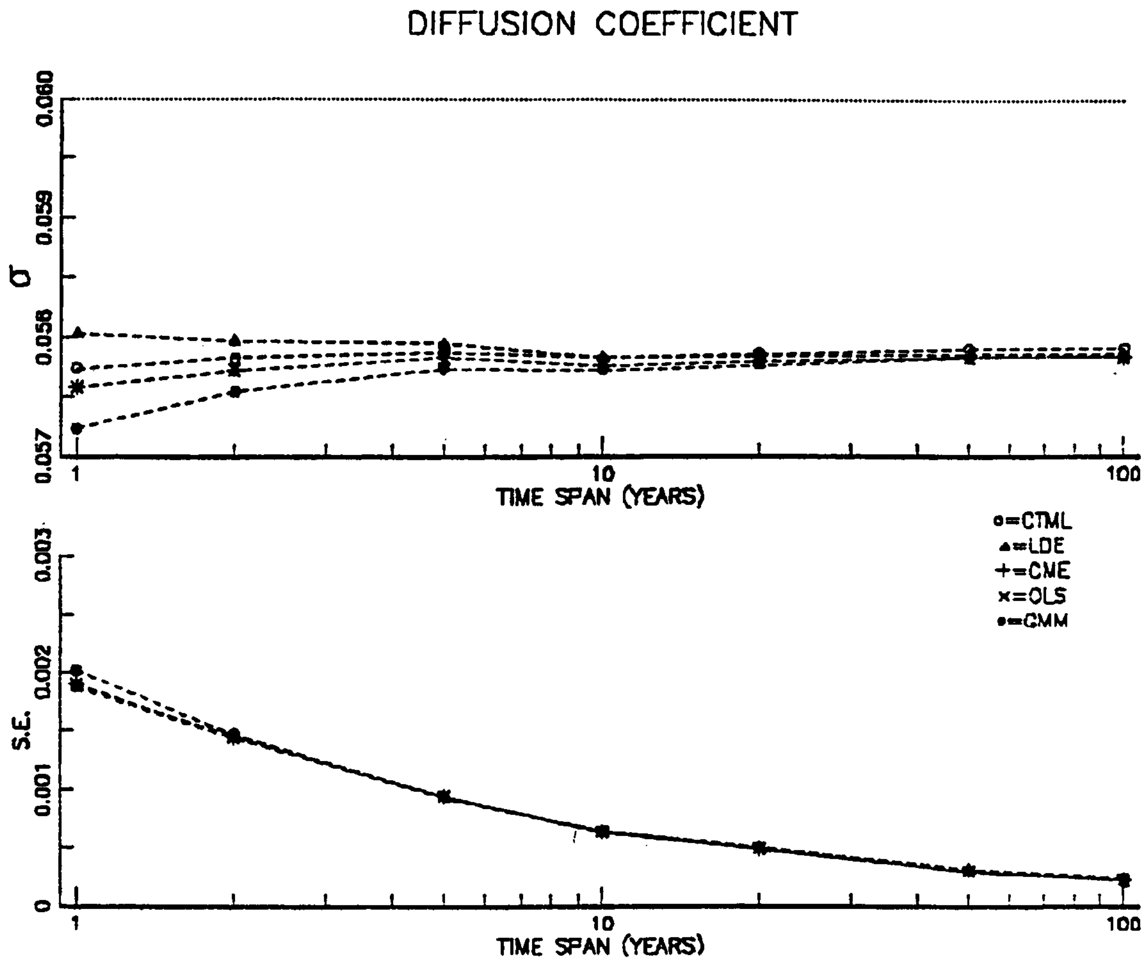

DIFFUSION COEFFICIENT
0.060
0.059
0.058
0.057
σ
1
10
100
TIME SPAN (YEARS)
o=CTML
▲=LDE
+=CME
×=OLS
•=GMM
0.003
0.002
0.001
0
S.E.
1
10
100
TIME SPAN (YEARS)

Fig. 7

SPEED OF ADJUSTMENT

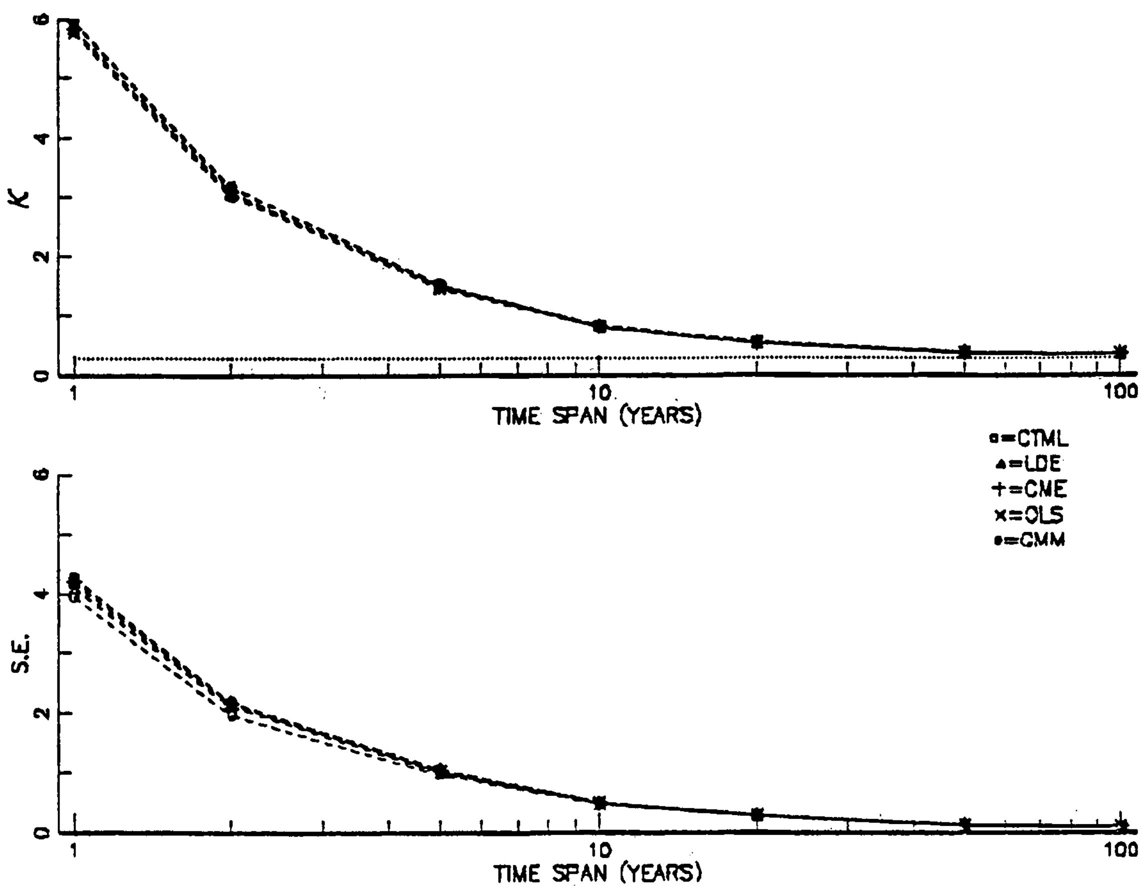


Fig. 8

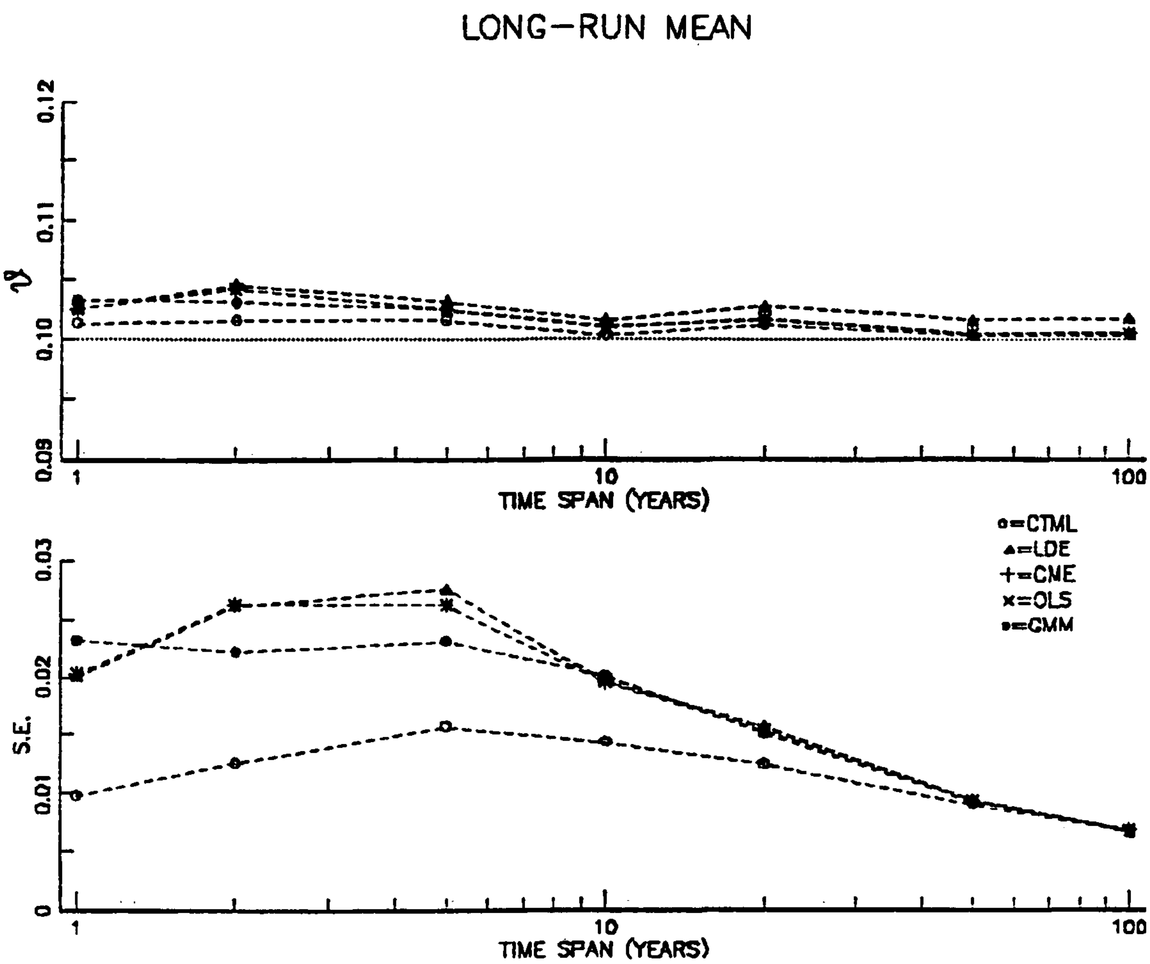
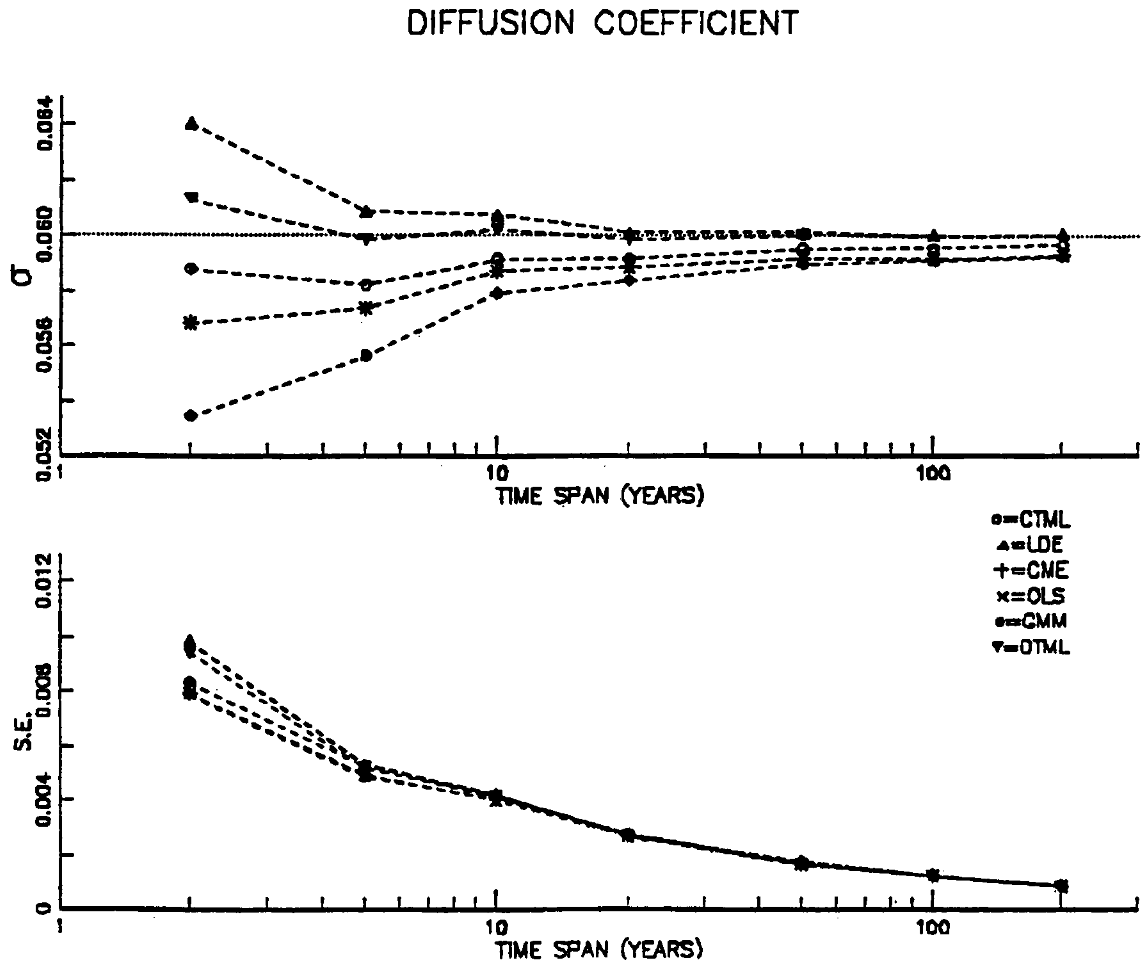
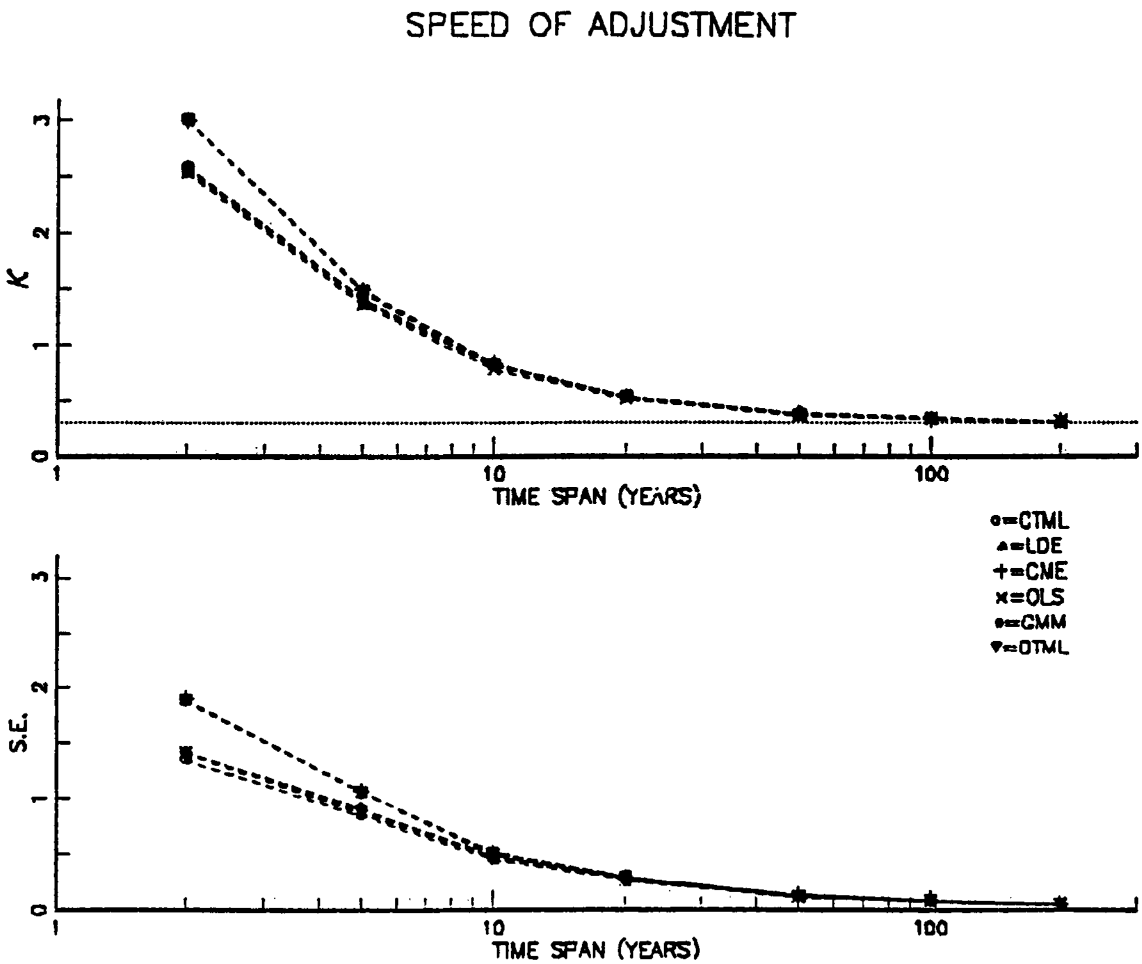

LONG-RUN MEAN
0.12
0.11
0.10
0.09
1
10
100
TIME SPAN (YEARS)
o=CTML
▲=LDE
+=CME
×=OLS
*=CMM
0.03
0.02
S.E.
0.01
0
1
10
100
TIME SPAN (YEARS)

Fig. 9



DIFFUSION COEFFICIENT
σ
0.064
0.060
0.056
0.052
1
10
100
TIME SPAN (YEARS)
S.E.
0.012
0.008
0.004
0
1
10
100
TIME SPAN (YEARS)
o=CTML
▲=LDE
+=CME
×=OLS
●=CMM
▼=DTML

Fig. 10



SPEED OF ADJUSTMENT
K
3
2
1
0
1
10
100
TIME SPAN (YEARS)
o=CTML
▲=LDE
+=CME
×=OLS
•=CMM
▼=DTML
S.E.
3
2
1
0
1
10
100
TIME SPAN (YEARS)

Fig. 11

LONG-RUN MEAN

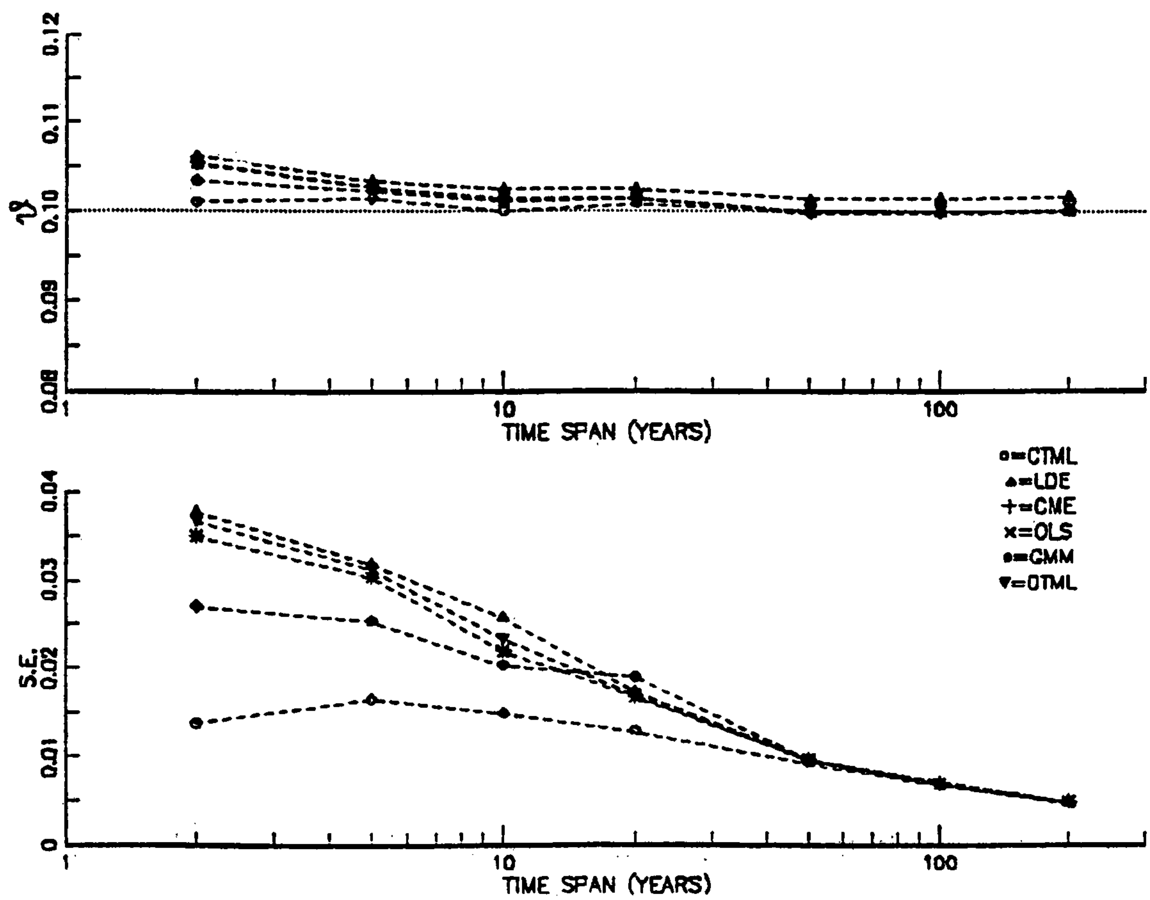


Fig. 12

MATURITY EFFECT

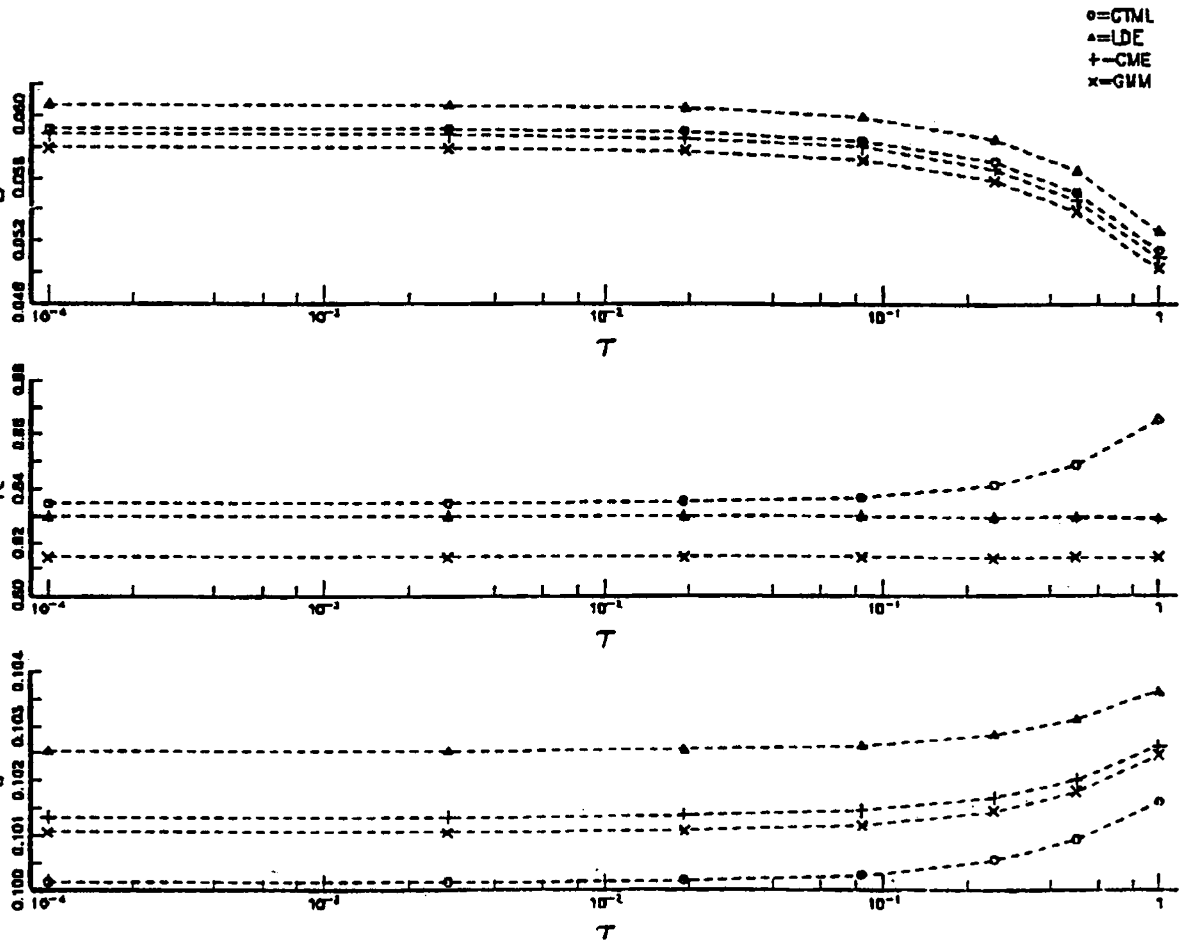
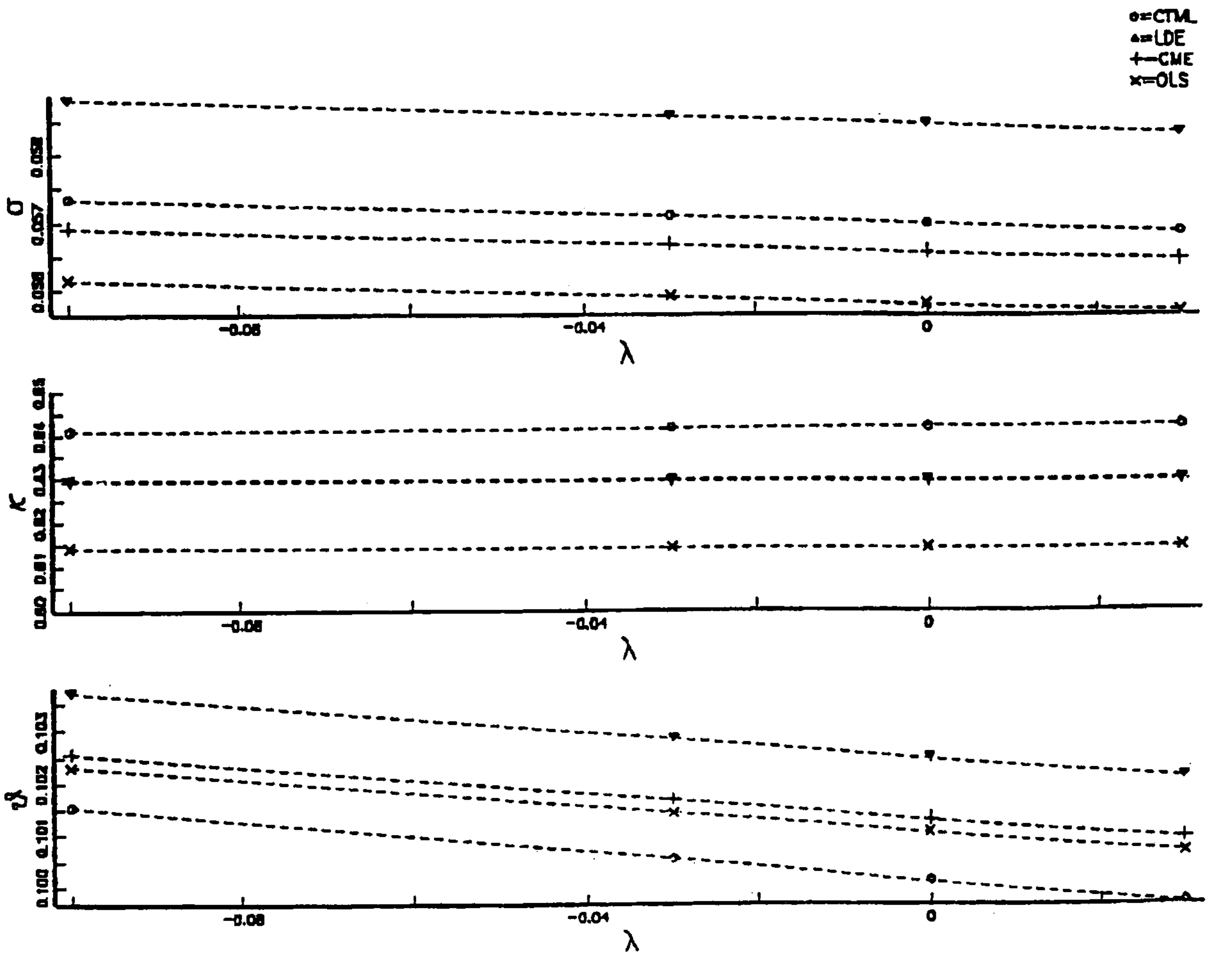


Fig. 13

PRICE OF RISK EFFECT



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Printed by the
Printing Office of the Banca d'Italia
Rome, November 1994