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MEAN REVERSION TESTS WITH REFLECTING BARRIERS: AN APPLICATION TO EUROPEAN MONETARY SYSTEM EXCHANGE RATES

Clifford A. Ball (*) and Antonio Roma (**)

Abstract

This paper derives a statistical test, based on the first order autocorrelation, to ascertain whether a stochastic process evolving within reflecting barriers is mean reverting. Under these conditions the standard unit root analysis does not apply. Since the presence of reflecting barriers per se will induce mean reverting behaviour, the detection of mean reversion inside the two boundaries requires that the effect of reflection be properly accounted for. The test is applied to the exchange rate in terms of Deutsche Marks of five currencies participating in the European Monetary System. Our methodology is helpful in deciding whether the behaviour of these exchange rates inside the barriers may be modeled as a simple reflected Brownian motion, or whether a more complex model is warranted.

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1 - Introduction[1]

Dickey and Fuller [1979,1981] investigated the statistical properties of the sample autocorrelation coefficient under the null hypothesis of a random walk and under the alternative hypothesis of an autoregressive mean reverting process. Their analysis becomes increasingly important as the autocorrelation approaches unity and we have a unit root. This research has spurred a literature on unit root or mean reversion tests, see for example, Dickey, Bell, and Fuller [1986], Evans and Savin [1981,1984], Phillips [1987, 1988], Schwert [1989], and many others. These tests have been applied to US dollar exchange rates see, for example, Meese and Singleton [1982]. Also, Doukas and Rahman [1987] examine five different freely-floating currency futures against the US dollar over the period 1977-1983. They perform Monte Carlo simulation experiments to investigate the sensitivity of unit root tests to changing volatility. They find no mean reversion in these exchange rates. Perron [1989] examines unit root tests when the underlying process is subject to a single shift in regime. Subject to this misspecification, he devises alternative tests which are robust.

We focus, in this paper, on the sensitivity of mean reversion tests to misspecification of the null hypothesis of the random walk, or Brownian Motion in continuous time, when the underlying process is subject to reflecting barrier behavior. The exchange rate mechanism of the European Monetary System (EMS) motivates our study of this problem. The institutional framework of the EMS requires the bilateral exchange rates of participating countries to lie within intervention limits which act as reflecting barriers for the exchange rate. The

[1]Part of this research was carried out while A. Roma was visiting the Anderson Graduate School of Management of the University of California at Los Angeles. C.A. Ball acknowledges support from a grant from the Dean's fund at the Owen School of Management, Vanderbilt University. We thank Gordon Bodnar, Daniele Terlizzese, and an anonymous referee for useful comments. The Authors are solely responsible for errors.

basics of the mechanism are quite simple: a grid of central bilateral parities is established between the participating currencies, and a maximum fluctuation of ±2.25% around the central parity is enforced through the obligation of both currencies' central banks to intervene whenever a bilateral exchange rate reaches a boundary[2]. These are called marginal interventions. An important characteristic of the system is that central banks' direct market interventions increasingly occur when the exchange rate is still within the EMS band. Autoregressive behavior within the reflecting barriers may correspond to such intramarginal intervention by the central banks, to stabilizing speculation inside the fluctuation band, or to other factors. However, when a fluctuation band cannot be mantained, a 'realignment' occurs[3]. The modeling of the exchange rate process, including realignments, requires the use of a jump-diffusion process, see for example Ball and Roma [1991] or Svensson [1991a]. Here we focus on the exchange rate behavior conditional on no realignment, i.e. the diffusion component of such a model.

Within the EMS we do not expect the standard unit root analysis for detection of autoregressive behavior to remain valid. In fact, the whole structure of the problem is altered by the presence of reflecting barriers which provide the added regularity of stationarity of limiting distributions. Recall that it is precisely the nonstationarity of Brownian Motion which causes the unit root problem. However, recent authors, for example, Edison and Fisher [1991] and Svensson [1991b], have applied the standard Dickey Fuller analysis to EMS exchange rates. Svensson [1991b], applying Dickey Fuller tests, rejects a random walk for these exchange rates. We propose a new statistical test which is appropriate for the detection of mean reversion inside reflecting barriers, against the null hypothesis of Brownian Motion

[2]A wider fluctuation band (±6%) was enforced for the Italian Lira until January 1990, and it is currently enforced for the Spanish Peseta, and Portoguese Escudo.

[3]See Van Hypersele and Koeune [1985] for details of the European Monetary System.

with reflecting barriers.

In section 2, we examine the Dickey-Fuller analysis when the underlying process is misspecified by reflecting barrier behavior. For plausible parameter values, we argue that the standard analysis is no longer valid. Section 3 establishes the asymptotic normality of the sample autocorrelation coefficient under reflecting behavior and also establishes a numerical algorithm to compute the asymptotic moments. In section 4, we discuss some of the estimation problems in implementing our statistical methods. We introduce in section 5 a new statistic for detecting mean reversion beyond that caused by reflecting barrier behavior. We also apply our analysis to EMS exchange rate data and discuss the economic implications. Section 6 contains our conclusions.

2 - Misspecification of the Unit Root Tests

In a Markov framework, the presence of mean reverting behavior is often tested by examining the first order autocorrelation. Such tests commonly include a deterministic trend and/or an estimated mean of the process, see Fuller [1976]. However, for EMS exchange rates, due to the presence of bilateral limits, a time trend is inappropriate and the mean of the process, the official central parity, has been assumed known. Under a null hypothesis of Brownian Motion against an alternative hypothesis of the Ornstein-Uhlenbeck (OU) process in continuous time, based on discrete observations, a standard statistical test may be performed. The usual test statistic is the sample first order autocorrelation $\hat{\rho}_n$ defined by $\hat{\rho}_n = \frac{\sum_{m=1}^{n-1} X_{m+1}X_m}{\sum_{m=1}^{n-1} X_m^2}$ where for EMS rates X_m is the deviation of the logarithm of the exchange rate from its mean at period m and n is the number of observations. When there is no ambiguity we will drop the n subscript on $\hat{\rho}_n$. Dickey and Fuller [1979, 1981] and others, have examined this statistical problem in some detail. In summary, for values of ρ

less than unity the distribution of $\sqrt{n}(\hat{\rho} - \rho)$ is asymptotically normal with mean zero and variance $\sigma_\rho^2 = (1 - \rho^2)$. For $\rho = 1$, when the underlying process is no longer stationary, the asymptotic normality breaks down and the order of convergence changes. In fact, $n(\hat{\rho} - \rho)$ has a limiting distribution and we have a unit root problem. This distribution is nonnormal (Fuller [1976]) and markedly skewed to the left. Furthermore, for values of ρ close to one, Phillips [1988] has shown that the sample autocorrelation coefficient behaves very much as in the nonstationary $\rho = 1$ case.

Consider, however, representing the EMS exchange rate behavior as a Brownian Motion between reflecting barriers. In this case the Dickey-Fuller analysis, which assumes unbounded Brownian Motion as the null hypothesis, will no longer be valid. Suppose we perform a standard Dickey-Fuller unit root test when the true process is Brownian Motion with reflecting barriers. For parameter values similar to those estimated from real EMS exchange rate data (i.e. a volatility $\sigma \simeq 0.003$ on a weekly basis and a $\pm 2.25\%$ fluctuation band, see for example Ball and Roma [1991]), Monte Carlo simulation experiments indicate that the probability of observing $\hat{\rho} < 1$ with samples of more than 100 observations is practically 1.0 rather than 0.6826, the number determined under the standard unit root analysis.

Despite the problems with the standard Dickey-Fuller analysis, extensive simulations show that it may still be useful for moderate sample sizes (less than 100) when the barriers are relatively wide (15-20 standard deviations apart) and the likelihood of reaching a limit is small. In general, however, for the modelling of exchange rates within the EMS both the null and alternative hypotheses should recognize the barrier conditions. Paralleling the unit root analysis, the simplest candidate for the alternative hypothesis would be the OU process with reflecting barriers. Significantly, this alternative hypothesis would imply the existence of additional mean reversion inside the EMS bands over and above that induced by the barriers imposed on a Brownian Motion process. Considering explicitly this alternative hypothesis is computationally challenging since the transition density of reflected OU is not known and can be represented only as the solution of a particular partial differential equation with specific boundary conditions. More general alternatives would be computationally more demanding. In response to these difficulties, we propose a simple test based on the sample autocorrelation coefficient $\hat{\rho}$. The test is not designed specially for a particular alternative specification but, rather, to detect mean reversion not already accounted for by reflecting barrier behavior.

3 - Asymptotic Normality of the Sample Autocorrelation

In this section we examine carefully the asymptotic properties of the sample autocorrelation under the misspecification of Brownian Motion with reflecting barriers. The Dickey-Fuller analysis, which assumes unbounded Brownian Motion as the null hypothesis, will no longer be valid. We show that the asymptotic distribution of the scaled autocorrelation coefficient, $\hat{\rho}$, is normal. Moreover, we compute the asymptotic mean and variance of this coefficient without recourse to Monte Carlo simulation experiments. In addition, our precise calculations are supported by simulation analysis. For ease of exposition, we assume the process is in probabilistic equilibrium, that is, the initial distribution of the process is its equilibrium distribution which is uniform. However, the results follow regardless of the initial distribution. We observe the process in discrete time.

Theorem: Let $X(.)$ be reflected Brownian Motion with zero drift, variance rate σ^2, and barriers at $\pm b$, $b > 0$. The scaled sample autocorrelation coefficient based on n observations, $\sqrt{n}(\hat{\rho}_n - \rho)$, converges in distribution to $N(0, \sigma_\rho^2)$, where the asymptotic mean

2

$\rho = \frac{96}{\pi^4} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} \cdot e^{-\frac{(2k+1)^2 \sigma^2 \pi^2}{8b^2}} < 1$. Both ρ and σ_ρ^2 are functions of $\frac{\sigma}{b}$ only.

Proof: First note that the stationary distribution of $X(.)$ is uniform $[-b, b]$. Define the three functions:

$$\begin{aligned} A_n &= n^{-0.5} \sum_{m=1}^{n} (X_{m+1} - \rho X_m) X_m \\ B_n &= n^{-1} \sum_{m=1}^{n} X_m^2, \\ Q_n &= n^{-1} \sum_{m=1}^{n} X_{m+1} X_m. \end{aligned}$$

By the strong law of large numbers for stationary Markov processes, see Doob [1953] Theorem 6.1, B_n converges almost surely to the constant $\mathbf{E}[X_1^2] = b^2/3$, and Q_n converges almost surely to the constant, $\mathbf{E}[X_1 X_2]$. Note $\mathbf{E}[X_1 X_2] = \mathbf{E}[X_1 \mathbf{E}[X_2 \mid X_1]]$ Hence,

$$\mathbf{E}[X_1 X_2] = \int_{x=-b}^{b} (x/2b) \int_{y=-b}^{b} y f(y \mid x) dy dx. \tag{1}$$

Here, $f(. \mid .)$ is the one-period transition density of reflected Brownian Motion. We have, according to Feller [1971], a Poisson expansion for the $(j-i)$-period transition probability density function given by:

$$f(y_j \mid x_i) = (2b)^{-1} + b^{-1} \sum_{k=1}^{\infty} exp(\frac{-k^2}{8b^2} \sigma^2 (j-i) \pi^2) cos \frac{k\pi}{2b} (y_j + b) cos \frac{k\pi}{2b} (x_i + b). \tag{2}$$

Observe that the term $\sigma^2(j-i)$ reflects the underlying Brownian volatility over a time interval of length $j-i$. Recall, $\hat{\rho_n} = \frac{\sum_{m=1}^{n-1} X_{m+1} X_m}{\sum_{m=1}^{n-1} X_m^2}$. Clearly, as $n \to \infty$ $\mathbf{E}[\hat{\rho_n}] \to \mathbf{E}[X_1 X_2]/\mathbf{E}[X_1^2]$. Integrating expression (1) by parts using the transition density (2), and dividing by $b^2/3$, we find the asymptotic mean: $\rho = \frac{96}{\pi^4} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} \cdot e^{-\frac{(2k+1)^2 \sigma^2 \pi^2}{8b^2}}$. This function is of interest in its own right since it captures the reversion induced by reflection at the barriers.

To establish normality, under regularity conditions we may apply standard central limit theorem results for strictly stationary Markov processes. Since the transition density is bounded and converges geometrically to its limiting distribution, we may apply Doob [1953], Theorem 7.5':

Let $f_m = f(X_{m+1}, X_m)$ where $X(.)$ is reflected Brownian motion with zero drift, variance rate σ^2, and barriers at $\pm b$. Let $\mathbf{E}[f_m] = E_m$.

If

$$\lim_{n \to \infty} \mathbf{E}[n^{-1} \sum_{m=1}^{n} (f_m - E_m)]^2 = \sigma_1^2$$

exists; if σ_1^2 is positive and for any initial distribution,

$$n^{-0.5} \sum_{m=1}^{n} (f_m - E_m) \Longrightarrow N(0, \sigma_1^2).$$

Set

$$f_m = f(X_{m+1}, X_m) = (X_{m+1} - \rho X_m) X_m$$

with $\rho = \mathbf{E}[X_1 X_2]/\mathbf{E}[X_1^2]$.

Under the uniform distribution and the specification of ρ, $\mathbf{E}[f_m] = E_m = 0$. By Doob's theorem, A_n converges in distribution to a normal random variable if the limiting variance exists and it is positive. Since,

$$\sqrt{n}(\hat{\rho} - \rho) = A_n / B_n,$$

we can establish the asymptotic normality of the scaled autocorrelation by determining the limiting variance σ_1^2. We now furnish the necessary technical analysis.

By choice of ρ, $E[A_n] = 0$ for all n, and,

$$Var[A_n] \quad = \quad n^{-1} Var[\sum_{m=1}^{N} (X_{m+1} X_m - \rho X_m^2)]$$

$$
\begin{aligned}
&= n^{-1}\mathbf{E}[(\sum_{m=1}^{n} X_{m+1}X_m - \rho \sum_{m=1}^{n} X_m^2)^2] \\
&= n^{-1}\mathbf{E}[(\sum_{m=1}^{n} X_{m+1}X_m)^2 - 2\rho(\sum_{m=1}^{n} X_{m+1}X_m)(\sum_{m=1}^{n} X_m^2) + (\rho)^2(\sum_{m=1}^{n} X_m^2)^2].
\end{aligned}
$$

Define

$$
\begin{aligned}
C(j) &= \mathbf{E}[X_1X_2X_{1+j}X_{2+j}], \\
D(j) &= \mathbf{E}[X_1^2X_{1+j}^2], \\
F_1(j) &= \mathbf{E}[X_1X_2X_{1+j}^2], \\
F_2(j) &= \mathbf{E}[X_1^2X_{1+j}X_{2+j}],
\end{aligned}
$$

for all j. Using the strict stationarity of the process and expanding the quadratic form, with some algebra we can show that:

$$
\begin{aligned}
\mathbf{E}[(\sum_{m=1}^{n} X_{m+1}X_m)^2] &= nC(0) + 2\sum_{j=1}^{n-1}(n-j)C(j), \\
\mathbf{E}[(\sum_{m=1}^{n} X_m^2)^2] &= nD(0) + 2\sum_{j=1}^{n-1}(n-j)D(j), \\
\mathbf{E}[\sum_{m=1}^{n} X_{m+1}X_m \sum_{m=1}^{n} X_m^2] &= nF_1(0) + \sum_{j=1}^{n-1}(n-j)[F_1(j) + F_2(j)].
\end{aligned}
$$

Combining these results,

$$
\begin{aligned}
Var[A_n] &= n^{-1}(n[C(0) - 2\rho F_1(0) + (\rho)^2 D(0)] + 2\sum_{j=1}^{n-1}(n-j)[C(j) - \rho(F_1(j) + F_2(j)) + (\rho)^2]) \\
&= L(0) + 2\sum_{j=1}^{n-1}(1 - j/n)L(j),
\end{aligned}
$$

where for each j,

$$
L(j) = C(j) - \rho(F_1(j) + F_2(j)) + (\rho)^2 D(j).
$$

For fixed j, consider $C(j)$ more carefully. Note that

$$
\mathbf{E}[X_1X_2X_{1+j}X_{2+j}] =
$$

$$\int_{x_{2+j}=-b}^{b}\int_{x_{1+j}=-b}^{b}\int_{x_2=-b}^{b}\int_{x_1=-b}^{b} x_{2+j}x_{1+j}x_2x_1 f(x_{2+j}x_{1+j}x_2x_1)dx_1dx_2dx_{1+j}dx_{2+j},$$

where $f(x_{2+j}x_{1+j}x_2x_1)$ represents the joint density of the four random variables $(X_1, X_2, X_{1+j}, X_{2+j})$ at $(x_{2+j}x_{1+j}x_2x_1)$. However, due to the Markov property of the underlying process we may rewrite this as:

$$\int_{x_{2+j}=-b}^{b}\int_{x_{1+j}=-b}^{b}\int_{x_2=-b}^{b}\int_{x_1=-b}^{b} x_{2+j}f(x_{2+j}\mid x_{1+j})x_{1+j}f(x_{1+j}\mid x_2)$$
$$x_2 f(x_2\mid x_1)x_1 f(x_1)dx_1dx_2dx_{1+j}dx_{2+j}. \tag{3}$$

Note that $f(x_{1+j},|x_2)$ represents the transition density over $(j-1)$ time periods. As j increases this transition density converges to the uniform distribution geometrically fast. According to (2), the transition density $f(x_{1+j}|x_2)$ is given by a constant plus a sum of terms for $k = 1, 2, .., \infty$. We can therefore write the quadruple integral $C(j)$ as a sum of quadruple integrals, each term of the sum corresponding to a term of the transition density $f(x_{1+j}|x_2)$. However, for k odd the contribution of the term to $C(j)$ is 0. For j sufficiently large, we may truncate the expansion at $k = 2$ with higher order terms being arbitrarily small and hence neglected.

We have

$$f(x_{1+j}|x_2) = (2b)^{-1} + b^{-1}\cos\tfrac{\pi}{b}(x_{1+j}+b)\cos\tfrac{\pi}{b}(x_2+b)\delta^j$$

where $\delta = exp(\frac{-\sigma^2\pi^2}{2b^2})$.

Substituting this expansion into (3) and integrating we obtain

$$C(j) = E[X_1X_2]^2 + K\delta^j,$$

where K is a bounded constant which can be determined as a simple quadruple integral, independent of j. Consequently, $C(j)$ converges to its easily calculated limit at the geometric rate δ. Directly similar arguments can be made for the limiting behavior of $D(.), F_1(.)$, and $F_2(.)$. As $j \to \infty$, $D(j) \to \mathbf{E}[X_1^2]^2$, $F_i(j) \to \mathbf{E}[X_1 X_2]\mathbf{E}[X_1^2]$, $i = 1, 2$. Each of the limits is achieved at the same geometric rate δ. Combining these results and noting the choice of ρ, we see that as $j \to \infty$, $L(j) \to 0$ at the geometric rate δ. Recall

$$Var[A_n] \;=\; L(0) + 2\sum_{j=1}^{n-1}(1 - j/n)L(j),$$

Hence

$$\lim_{n\to\infty} Var[A_n] \;=\; L(0) + 2\sum_{j=1}^{N} L(j) + 2\sum_{j=N+1}^{\infty}(1 - j/n)L(j).$$

However, we see that by summing the appropriate geometric series, the second sum is bounded by $\mid L(N)\frac{1}{1-\delta} \mid$. In summary, the limiting variance exists, and furthermore, we may compute the limiting value to any fixed level of accuracy. □

The standard convergence results we employ are useful only when we can establish that the limiting variance exists. For this application not only do we establish the existence of the limiting variance, but we are also able to compute it to an arbitrary degree of accuracy. The result hinges on the geometric convergence of fourth order moments of this process. Furthermore, due to the Markovian structure of the process, we can compute the quadruple integral by means of a string of double integrals where each double integral is determined by the extended Simpson's rule. In effect, the computation of the integral reduces to a string of matrix multiplications where the dimension of the matrix corresponds to the number of steps in the Simpson rule integration.

Table 1 examines, by simulation analysis using 10,000 replications, the statistical properties of $\hat{\rho_n}$ for various sample sizes, based on the null hypothesis of reflected Brownian Motion with different σ. For each replication, we generated a time series beginning at $X_0 = 0.0$

subject to the null hypothesis, and computed $\hat{\rho}_n$. We took advantage of the reflection principle to perform the simulation of reflected Brownian motion, see Karlin and Taylor [1975]. Specifically, we generate unrestricted Brownian Motion from its increments which are independent and normally distributed. For reflected Brownian Motion about a single barrier at b say, we generate the increments as in the unrestricted case except when the new value of the process X exceeds b. In this case, the value of the reflected process becomes $b-(X-b)$, its reflection about b. We have two reflecting barriers in this application. From the simulation, we see that very large samples are needed for convergence to normality to take place. Also, the convergence is faster for larger σ/b.

Table 2 computes $\lim_{n\to\infty} \mathbf{E}[\hat{\rho_n}] = \mathbf{E}[X_1X_2]/\mathbf{E}[X_1^2] = \rho$ and by numerical integration σ_ρ, for various plausible values of σ and barriers at $b = 0.0225$ which we see in practice. Of course, the results depend on the ratio $\frac{\sigma}{b}$ only, so that the Table may be applied for alternative values of b and σ by simple scaling. For example, Lindberg and Soderlind [1991] analyze exchange rate target zones for Swedish exchange rates where the band limit is set at $b = 0.015$. Table 3 describes some calculations when $\sigma = 0.003$. In this case, $\delta = 0.91600833$. The quadruple integrals are computed using a 101 point Simpson extended double integration rule which, for these parameter settings, gives accuracy to at least 8 significant digits. For $j = 1,300$, we computed $L(j)$ and from these the estimate of the asymptotic variance of A_n, $L(0) + 2\sum_{j=1}^{N} L(j)$, as well as a bound on the error of the approximation $\mid L(N)\frac{1}{1-\delta} \mid$. Finally, we compute the approximation to σ_ρ, the limiting standard deviation of $\sqrt{n}\hat{\rho}_n$. The Table clearly identifies the accuracy of our approximations. Inspection of Tables 1 and 3 also reveals that the simulated standard error of $\sqrt{n}\hat{\rho_n}$ matches its computed asymptotic value for large sample sizes.

4 - Estimation Problems

Table 2 may be useful for carrying out statistical tests on EMS exchange rates. For a fixed barrier b, both ρ and σ_ρ are functions of σ only. In empirical applications, the true standard deviation of the process under the null hypothesis of reflected Brownian Motion, σ, is not known, and hence must be estimated. Given this parameter, the mean value of $\hat{\rho}$ and its scaled standard error can be computed. The residuals of the OLS regression provide one estimator of σ^2:

$$\hat{\sigma}^2 = \frac{1}{n-2}\sum_{t=1}^{n-2}(X_{t+1} - \hat{\rho}X_t)^2.$$

In fact, we may apply maximum likelihood estimation (MLE) on the volatility parameter under the reflective barrier hypothesis. The corresponding transition density may only be represented by an infinite series expansion but, following Lo [1988], with appropriate truncation MLE may be implemented numerically[4]. See Ball and Roma [1991] for details. Table 4 reports the two estimates of volatility for selected EMS bilateral exchange rates. Although the estimate based on the residuals is slightly biased it is easier to compute which makes it quite useful in practical applications. Given an estimate $\hat{\sigma}$, Table 2 may be used to obtain the mean and scaled standard deviation of $\hat{\rho}$ when the sample is sufficiently large.

[4]The transition density of Brownian motion $\{X(t) : t \geq 0\}$ with reflecting barriers at a and $-b$ and instantaneous variance σ^2 is given also by Feller [1971, p.341] as:

$$f(X(t) \mid X(0)) = (2\pi\sigma^2 t)^{-0.5} \sum_{k=-\infty}^{+\infty} [exp\{-[X(t) - X(0) + 2k(a+b)]^2/2\sigma^2 t\}$$

$$+exp\{-[X(t) + X(0) + 2b + 2k(a+b)]^2/2\sigma^2 t\}].$$

However, for our parameter values, the following approximation, based on a mixture of densities for the case of a single reflecting barrier, proves extremely accurate:

$$f(X(t) \mid X(0)) \simeq (2\pi\sigma^2 t)^{-0.5}[exp\{-[X(t) - X(0)]^2/2\sigma_1^2 t\}$$

$$+exp\{-[X(t) + X(0) - 2a]^2/2\sigma^2 t\} + exp\{-[X(t) + X(0) + 2b]^2/2\sigma^2 t\}].$$

Suppose that the standard error of $\sqrt{n}\hat{\rho}$ is computed as $\sqrt{1-\hat{\rho}^2}$, that is, directly from the OLS regression. This approach, based on an inappropriate assumption, will lead to a strongly upward biased standard error. From Table 2 we see that when $\sigma = 0.003$ the expected value of $\hat{\rho} = 0.9752$ and the standard error of $\sqrt{n}\hat{\rho} = 0.1206$. The estimate of this standard error as $\sqrt{1-0.9752^2} = 0.2213$ would be almost two times too large.

Given reflecting barrier behavior, one should not use the usual standard error of $\hat{\rho}$ but, rather, an estimate based on the null hypothesis. We demonstrate our mean reversion tests on EMS exchange rate data in the following section.

5 - Application to EMS Exchange Rates

Our data source is the Italian Stock Exchange in Rome and Milan which has a daily fixing of exchange rates against the Italian Lira. These rates are actual transaction prices and the whole fixing process for a variety of foreign exchange rates lasts approximately 15 to 30 minutes. Our bilateral exchange rate data are obtained by crossing the rates in terms of the Italian Lira. The fixing process creates a slight error in variables problem due to lack of perfect synchronization of cross rates[5]. There may also be day of the week effects and weekend effects in the foreign exchange markets. For these reasons, we decided to use weekly data and to concentrate on the most heavily traded exchange rates[6]. We used 5 exchange rates in terms of the Deutsche Mark and time series of weekly observations spanning the period March 1979 when the EMS began through November 1991. In total, we have 662 weekly observations for each bilateral exchange rate. We perform our test on the deviation

[5]This latter problem amounts to an increase of cross rates standard deviations, σ, of the order of magnitude of the lira exchange rates standard deviations over 30 minutes. Compared to the weekly standard deviation, this is negligible. The effect on ρ may infact be evaluated by increasing σ appropriately, i.e., with weekly data, by using $\sigma(336+1)/336$ instead of σ.

[6]We used exchange rates in terms of the Deutsche Mark which are subject to a 2.25% fluctuation band. The Italian Lira moved from the ±6% band to the ±2.25% band in January 1990.

of the logarithm of the exchange rate from the central parity, X, which is constrained to the $\pm 2.25\%$ band. Figures 1-5 display the deviations from central parity graphically. During this period there were a small number (at most seven for any particular bilateral exchange rate) of realignments of central parities. We eliminated any observations corresponding to theses shifts in parity since the focus here is mean reversion of exchange rates within the EMS bands.

Table 4 reports the least squares analysis including the standard T-Statistics for testing the null hypothesis $\rho = 1$. Observe that $\hat{\rho}$ is always significantly < 1. However, this result is not surprising since due to the presence of reflecting barriers on X, following our theorem we would expect $\hat{\rho} < 1$. From Table 2, we may compute the expected value of $\hat{\rho}$ and its scaled standard error, given the estimated volatility of the process. A sharper question is whether $\hat{\rho}$ is sufficiently small to indicate additional mean reversion beyond that induced by reflection at the bilateral limits, i.e. by marginal central bank intervention. To detect this additional mean reversion, we propose the statistic $Z = \frac{\sqrt{n}(\hat{\rho}-\rho)}{\sigma_\rho}$ which has, according to our theorem under the null hypothesis of reflecting Brownian motion, an asymptotic standard normal distribution. However, as Table 1 shows, in a finite sample of 662 observations Z may be left-skewed. Therefore, we also performed a small sample analysis (Small Sample Analysis). In Table 4 we report the estimates of volatility, σ, based on the null hypothesis of reflected Brownian motion and based on the least squares residuals. Based on this value, we computed by simulation the empirical distribution of $\sqrt{n}\hat{\rho}_n$ under the null hypothesis of Brownian motion with reflecting barriers in a sample of 662 observations. We then evaluated the p-value of the Z statistic estimated from empirical data based on this small sample distribution.

By applying incorrectly the standard unit root tests with the critical values for the T-Statistic (Fuller [1976], Table 8.5.2) we see evidence at at least the 5% significance level of autoregressive behavior for the French Franc, Danish Kroner, Dutch Guilder and Irish Pound. The results for the Belgian Franc are not significant. In contrast, when we apply our modified analysis using the Z Statistic and account for reflection at the barriers, we see evidence of additional mean reversion only for the Dutch Guilder. In summary, except for the Dutch Guilder, the autocorrelation we see in these time series may be accounted for by the reflecting barrier behavior alone.

6 - Conclusions

We have examined tests of mean reversion on time series which are subject to reflecting barrier behavior. The institutional arrangements of the EMS place such restrictions on the bilateral exchange rates of its member currencies. Under these circumstances, the standard Dickey Fuller analysis of the unit root problem no longer applies. We establish the asymptotic distribution of the sample autocorrelation coefficient, $\hat{\rho}$ under these conditions. Significantly, we provide a simple expression for the mean and a numerical algorithm, which provides arbitrary accuracy, for the standard deviation of $\hat{\rho}$. Based on this analysis, we introduce a simple Z-Statistic test for detecting mean reversion beyond the effects of reflecting barriers. Furthermore, we provide a table which allows the implementation of this test without extensive numerical calculations.

We apply this analysis to EMS exchange rate data. We might expect additional mean reversion when the central banks engage in intramarginal intervention or perhaps when market participants expect the exchange rate band to be fully credible and engage in stabilizing speculation. However, reflection at the intervention limits induces an autocorrelation below unity even when the underlying process is Brownian Motion and our statistical analyses confirm this result in the observed exchange rates. In 4 out of 5 cases examined, our statistical test rejects the hypothesis that the process for this exchange rate is subject to additional mean reverting behavior inside the fluctuation band. In summary, the empirical observed mean reversion can, in most cases, be accounted for by the presence of reflecting barriers alone.

Table 1

Simulation of $\hat{\rho_n}$ for various sample sizes n, and standard deviation σ[7].

$\sigma = 0.002$

Sample Size (n)	Mean $\hat{\rho}_n$	Std. Deviation $\sqrt{n}\hat{\rho}_n$	Skewness $\sqrt{n}\hat{\rho}_n$	Kurtosis $\sqrt{n}\hat{\rho}_n$
250	0.9860	0.165	-2.799	12.768
500	0.9877	0.111	-1.885	6.859
1000	0.9882	0.095	-1.158	2.498
5000	0.9886	0.090	-0.509	0.510
10000	0.9887	0.082	-0.482	0.397

$\sigma = 0.003$

Sample Size (n)	Mean $\hat{\rho}_n$	Std. Deviation $\sqrt{n}\hat{\rho}_n$	Skewness $\sqrt{n}\hat{\rho}_n$	Kurtosis $\sqrt{n}\hat{\rho}_n$
250	0.9733	0.162	-1.704	5.938
500	0.9744	0.139	-1.139	2.732
1000	0.9748	0.130	-0.793	1.560
5000	0.9752	0.122	-0.287	0.226
10000	0.9752	0.121	-0.224	0.095

$\sigma = 0.004$

Sample Size (n)	Mean $\hat{\rho}_n$	Std. Deviation $\sqrt{n}\hat{\rho}_n$	Skewness $\sqrt{n}\hat{\rho}_n$	Kurtosis $\sqrt{n}\hat{\rho}_n$
250	0.9553	0.195	-1.068	2.251
500	0.9564	0.177	-0.693	0.910
1000	0.9567	0.177	-0.495	0.474
5000	0.9571	0.176	-0.198	0.144
10000	0.9571	0.166	-0.101	0.014

[7]This table provides sample mean and sample standard deviation of $\hat{\rho_n}$ for various sample sizes n and standard deviation σ. The experiment is based on 10,000 replications. $b = 0.0225$.

Table 2

Computation of ρ and σ_ρ for Various Parameter Settings.

Standard Deviation: σ	Mean: ρ	σ_ρ
0.0020	0.9887	0.0778
0.0022	0.9864	0.0861
0.0024	0.9839	0.0945
0.0026	0.9812	0.1031
0.0028	0.9783	0.1118
0.0030	0.9752	0.1206
0.0032	0.9720	0.1296
0.0034	0.9685	0.1386
0.0036	0.9649	0.1477
0.0038	0.9611	0.1570
0.0040	0.9571	0.1664

Table 3

Asymptotic Behavior of $L(j)$[8]

N	$\mid L(N)\frac{1}{1-\delta} \mid$	$L(0)+2\sum_{j=1}^{N} L(j)$	Estimated σ_ρ
50	$7.235*10^{-12}$	$4.2092244*10^{-10}$	0.12157863
100	$9.004*10^{-14}$	$4.1437697*10^{-10}$	0.12062964
150	$1.120*10^{-15}$	$4.1429551*10^{-10}$	0.12061778
200	$1.394*10^{-17}$	$4.1429450*10^{-10}$	0.12061763
250	$1.699*10^{-19}$	$4.1429449*10^{-10}$	0.12061763
300	$3.939*10^{-21}$	$4.1429449*10^{-10}$	0.12061763

[8]For σ =0.003, this table provides for different truncation points N, $L(0)+2\sum_{j=0}^{N} L(j)$, the approximation to the limiting variance of A_n, $\mid L(N)\frac{1}{1-\delta} \mid$, the bound on the error of the approximation, and the estimate of σ_ρ, the limiting value of the standard deviation of $\sqrt{n}\hat{\rho}_n$.

Table 4: Analysis by Exchange Rate[9]

Currency	$X_0^{*10^3}$	$\hat{\sigma}_{OU}^{*10^3}$	$\hat{\rho}$	$se(\hat{\rho})^{*10^2}$	$T-Stat$
DM.FFR	2.569	3.123	0.963	1.070	-3.458
DM.DKR	3.913	3.589	0.955	1.172	-3.840
DM.BFR	7.334	2.604	0.990	0.805	-1.242
DM.DFL	4.105	2.268	0.933	1.460	-4.589
DM.IRP	0.257	3.079	0.963	1.179	-3.138

Asymptotic Analysis

Currency	$\hat{\sigma}_{BM}^{*10^3}$	$\hat{\rho}$	ρ	$n^{-0.5}\sigma_{\rho}^{*10^3}$	$Z-Stat$
DM.FFR	3.156	0.963	0.9732	0.5016	-2.034
DM.DKR	3.644	0.955	0.9651	0.5855	-1.725
DM.BFR	2.634	0.990	0.9811	0.4108	2.167
DM.DFL	2.223	0.933	0.9856	0.3538	-14.87
DM.IRP	3.085	0.963	0.9739	0.4939	-2.207

Small Sample Analysis

Currency	$\hat{\sigma}_{BM}^{*10^3}$	$\hat{\rho}$	ρ	$n^{-0.5}\sigma_{\rho}^{*10^3}$	$Z-Stat$
DM.FFR	3.156	0.963	0.9721	0.5441	-1.68
					(0.062)
DM.DKR	3.644	0.955	0.9635	0.6218	-1.37
					(0.092)
DM.BFR	2.634	0.990	0.9801	0.4702	2.09
					(0.99)
DM.DFL	2.223	0.933	0.9855	0.4158	-12.6
					(0.00001)
DM.IRP	3.085	0.963	0.9733	0.5324	-1.93
					(0.044)

[9]In this Table, all currencies have 662 weekly observations. Here, X_0 is the initial value of the deviation from central parity measured as the natural logarithm of the exchange rate minus the natural logarithm of the central parity in Deutsche Marks. The symbols $\hat{\sigma}_{OU}$ and $\hat{\sigma}_{BM}$ designate the estimates of the volatility assuming an OU process and a Brownian Motion with reflection and $\hat{\rho}$ is the estimated autocorrelation parameter. The standard error of $\hat{\rho}$ ignoring reflecting barriers is designated $se(\hat{\rho})$ while the mean and standard error of $\hat{\rho}$ accounting for reflecting behavior are ρ and $n^{-0.5}\sigma_{\rho}$ The $T-Stat$ is the standard T-statistic for testing the null hypothesis that $\rho = 1$ ignoring reflecting behavior while the $Z-Stat$ is the Z-transform of $\hat{\rho}$ accounting for reflection. Small sample statistics are also reported, where ρ, σ_{ρ} and Z are computed according to the appropriate small sample distributions, derived by simulation. In parenthesis are reported the P-values of the small sample Z statistic (one-sided test).

Figure 1: Deviation from Central Parity

Weekly Data: DM.FFR

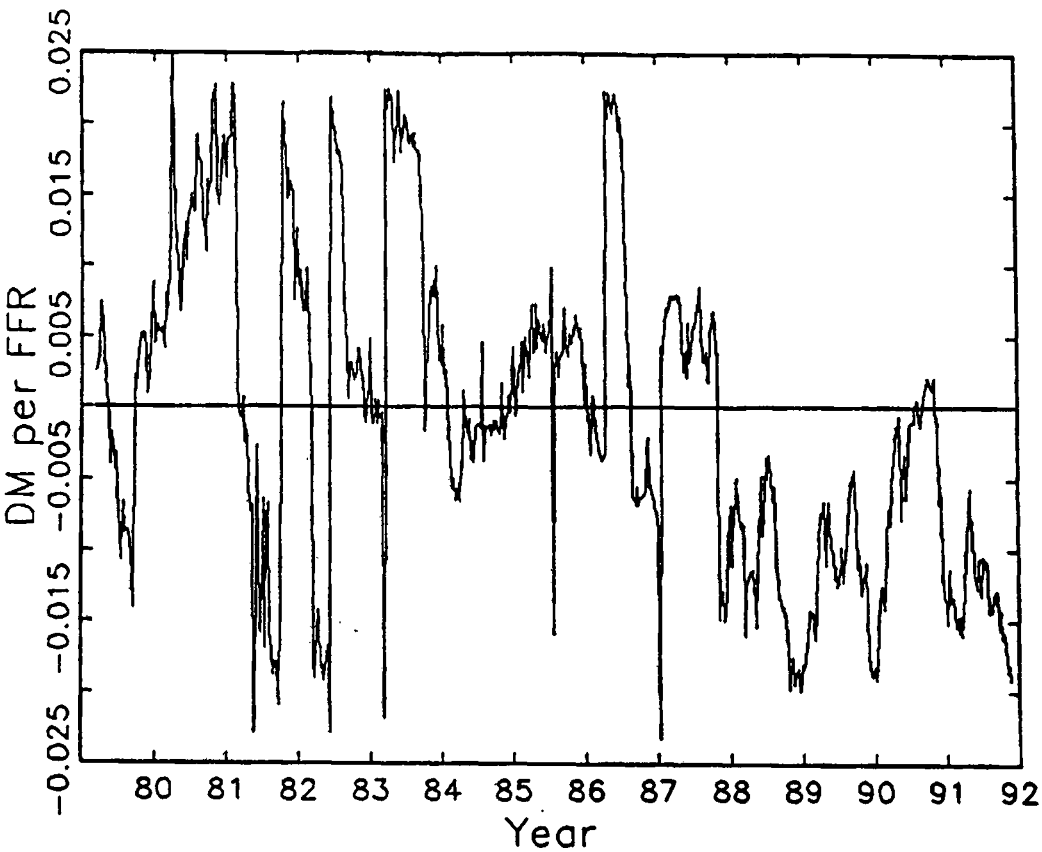


Figure 2: Deviation from Central Parity

Weekly Data: DM.DKR

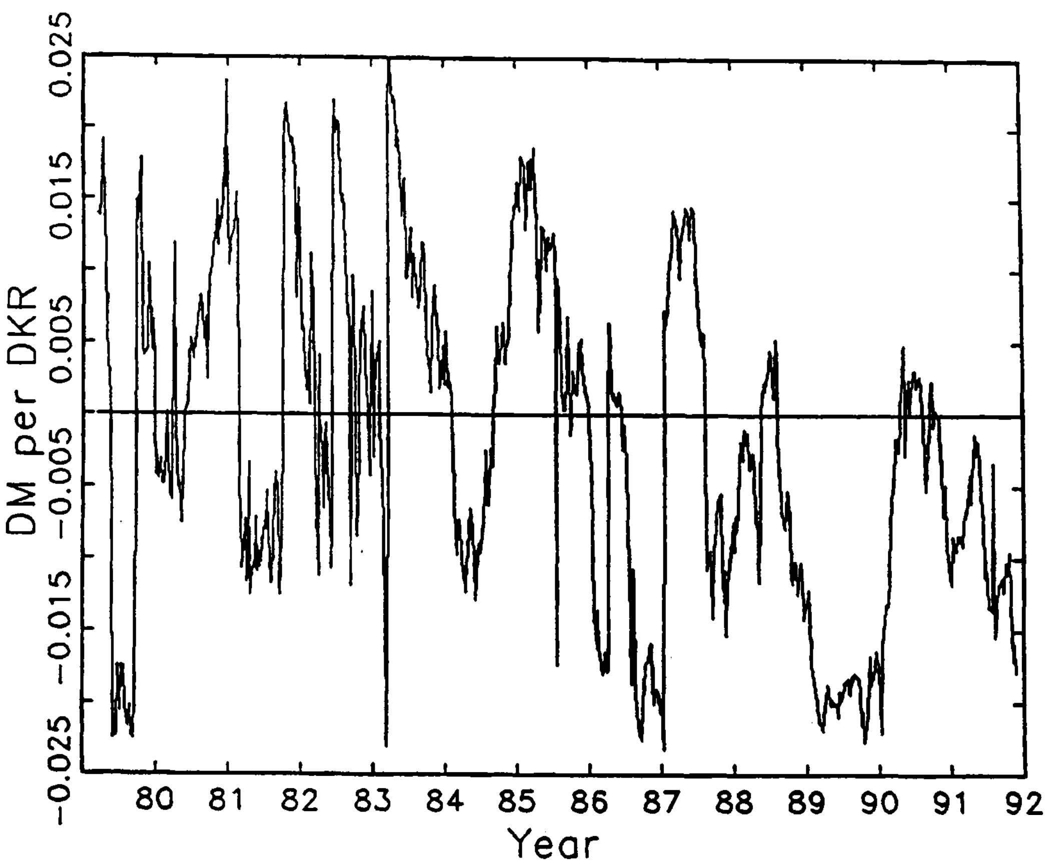


Figure 3: Deviation from Central Parity

Weekly Data: DM.BFR

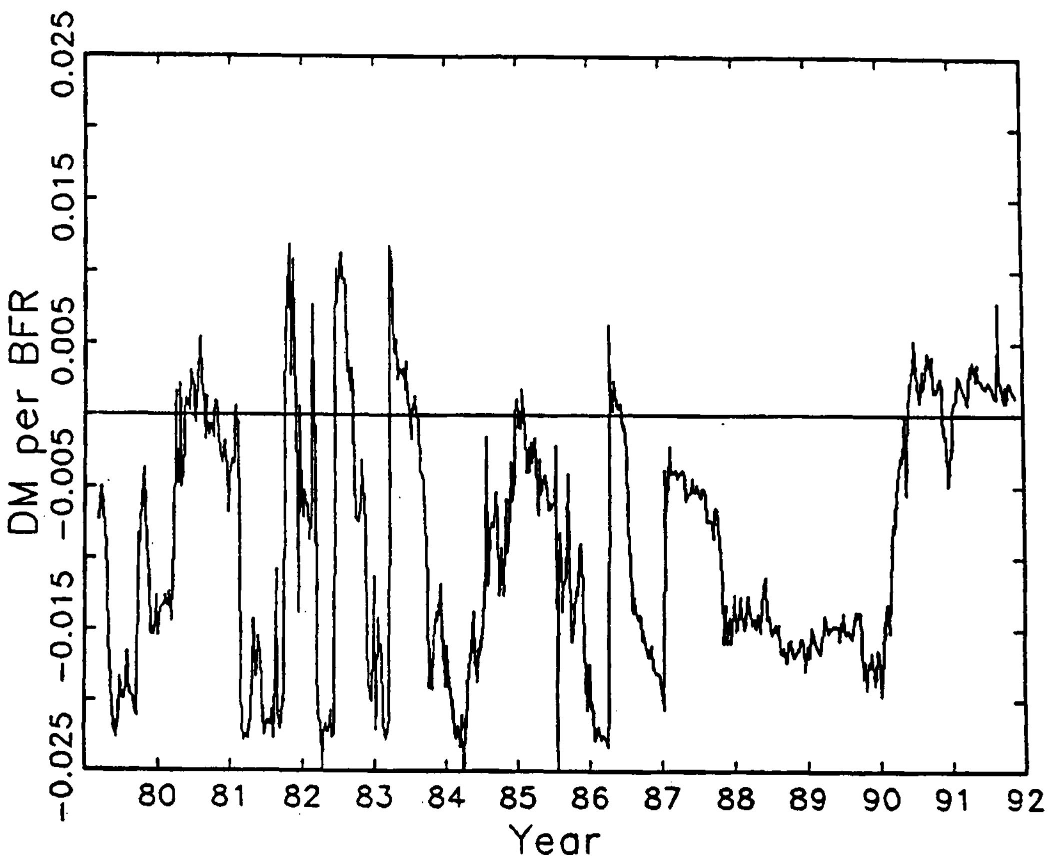


Figure 4: Deviation from Central Parity

Weekly Data: DM.DFL

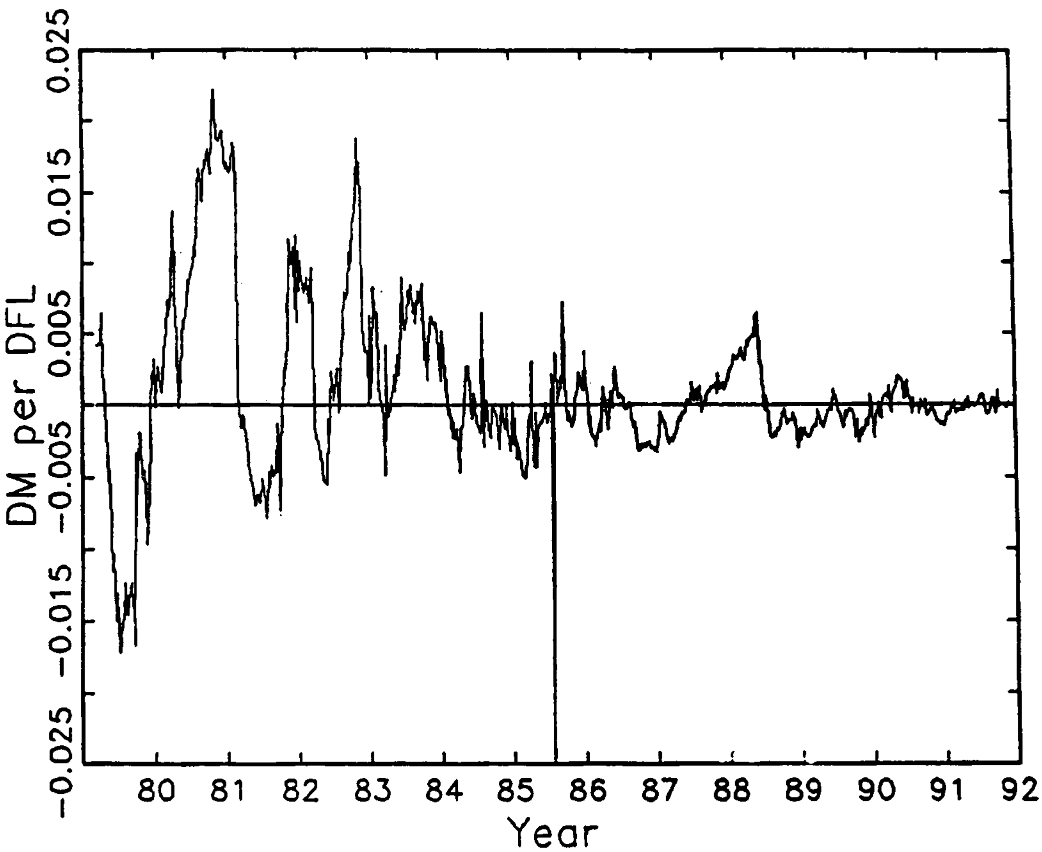
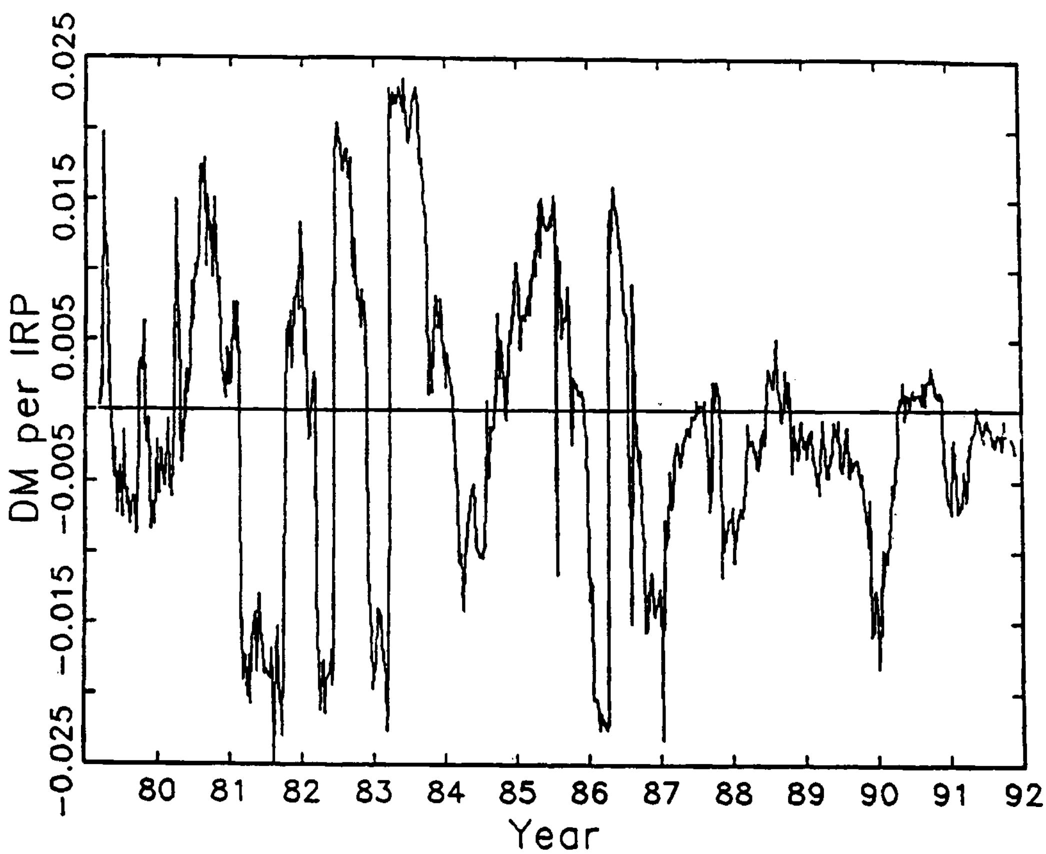


Figure 5: Deviation from Central Parity

Weekly Data: DM.IRP



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