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Summary

On the basis of the difference between promises and threats, we offer an intuitive explanation of the difference between time consistency and subgame perfection that complements the formal analysis of the two concepts.

"If ye walk in my statutes, and keep my commandments ... I will give you rain in the season ... But if ye will not hearken unto me ... I will even appoint over you terror, consumption, and a burning ague, that shall consume the eyes, and cause sorrow of heart."

(Leviticus, XXVI)

Introduction

Time consistency is sometimes equated with subgame perfection[1] (for instance, Barro, 1986 and Blanchard and Fischer, 1989, ch. 11 pag. 621). Recent papers (Fershtman, 1989; McTaggart and Salant, 1989; Guiso and Terlizzese, 1989) have made clear that this is improper, as subgame perfection is a stronger refinement of the equilibrium concept than time consistency.[2] However, we still lack an intuitively clear explanation of the difference. Also, the general principle that leads to time inconsistency,

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1. A confusion between the two concepts can be found even in the paper by Kydland and Prescott (1977), which originated the interest of (macro)economists on the issue. As the following analysis will make clear, their example regarding flood control must be interpreted as showing lack of subgame perfection rather than time inconsistency. Had the agents believed in the government's announcement that no dam would be built, they would never have discovered that, once established in the area, the government would renege on the announcement and build a dam.

2. The paper by McTaggart and Salant (1989), though provides a correct distinction between subgame perfection and time consistency, analyzes the problem in a Nash game in which, by construction, inconsistency cannot arise.

though well understood mathematically (Sargent, 1987), has not yet been furnished with any simple behavioural content. The present paper, relying on the difference between promises and threats, offers an interpretation of time consistency and subgame perfection which makes both the mechanism leading to time inconsistency and the distinction between the latter and the requirement of subgame perfection readily comprehensible.

1. A homely example

Suppose a boy is struggling in school and his parents promise to buy him a car if he studies hard and is promoted. When he does, though, the parents' worry over his school problems may easily be overshadowed by worry over his safety, and they may renege on the promise and deny him the car. Conversely, if they threaten not to take him to the seashore on holiday if he fails, when he is promoted he will not discover that, had he failed, they would have felt sorry, relented on the threat, and taken him along anyway. A game theorist, indifferent to the petty drama, would pontificate that in the first instance the parents' behaviour was time inconsistent, whereas in the second only lack of subgame perfection could be detected.

The point here is that, by its very nature, a promise is meant to be kept. When effective, a promise induces behaviour that calls for its fulfillment. Threats, by contrast, are not meant to be carried out. An effective threat induces behaviour that obviates the need for its implementation.

Both promises and threats are announcements that, for the purpose of analyzing strategic interaction, we take to be conditional on other agents' actions.[3] We base the difference between them on whether or not it is necessary to implement the

3. This accords with the standard dictionary definition of "threat". By contrast, "promises" are often defined as being unconditional and our usage is, therefore, somewhat restrictive.

announcement if it is acted on.[4]

The time inconsistency originates from the simple fact that in a leader-follower, multiperiod game (this is how the game theorist would formalize our example), the leader uses promises to manipulate the current behaviour of the follower. When this succeeds, the leader can bank on the follower's behaviour so obtained and renege on his promise.

Smart followers should anticipate that the promise will not be kept and neutralize the manipulation, taking into account only credible promises. This leads to the notion of time consistency.

By the same token, smart players should realize when a threat is in fact a bluff and should call it. This leads to the requirement of subgame perfection, which excludes equilibria sustained by non-credible threats.

Of course, the analogy with the requirement of time consistency is strong. In both cases the announcements of some of the players are constrained to be credible (i.e. optimal ex-post). Nevertheless, the difference is clear, and it amounts to the difference between promises and threats. A time-inconsistent policy relies on a non-credible promise, and this is bound to become manifest when time comes for keeping the promise. A non-perfect equilibrium relies on a non-credible threat, but the lack of credibility will never surface unless some player, as a

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4. This distinction holds for the case of certainty. Introducing uncertainty greatly complicates the issue. In particular it becomes possible for the announcement of future actions to be conditional not only on other agents' actions but also on uncertain events. Verifying whether the announcement is credible is no longer a straightforward implication of behaving in accord with a given strategy, but involves chance. It must be noted, however, that introducing incompleteness in the information structure of the game is one way to "cure" the time inconsistency (Backus and Driffill, 1985). To put it differently, when the information is not complete it is no longer possible to define a time inconsistency problem unambiguously: the latter arises when, believing in the announcement, the agents act so as to determine a situation in which implementing the announced strategy is not optimal. When the information is incomplete, in given configurations of the (uncertain) players' pay-offs, implementing the announced strategy could well be optimal.

result of sophisticated reasoning (or by mistake), chooses actions off the equilibrium path.

This implies that a player needs to be smarter to detect lack of perfection than to detect time inconsistency, since the former requires a conceptual experiment and the latter only observation.

The rest of the paper simply makes these arguments more precise.

2. The origin of time inconsistency

Consider a two period leader-follower game. Let A and B represent the sets of admissible actions of the leader and the follower, respectively, and let $A^2 = A \times A$ and $B^2 = B \times B$ be the sets of all possible sequences of actions chosen by the two players. A generic element of A^2 is $\alpha = (\alpha_1, \alpha_2)$, representing the actions chosen by the leader in periods 1 and 2. Similarly, define $\beta = (\beta_1, \beta_2) \varepsilon B^2$.

The payoffs of the two players are given by $U^L(\alpha, \beta)$ for the leader and $U^F(\alpha, \beta)$ for the follower[5].

As usual, given a sequence of actions for the leader, $\bar{\alpha}$, the follower computes his best reply by maximizing $U^F(\bar{\alpha}, \beta)$.

Let $\beta^*(\bar{\alpha}) = [\beta_1^*(\bar{\alpha}), \beta_2^*(\bar{\alpha})]$ represent the reaction function so determined. The leader then chooses his action by solving

$$(1) \quad \max_{\alpha} U^L(\alpha_1, \alpha_2, \beta_1^*(\alpha_1, \alpha_2), \beta_2^*(\alpha_1, \alpha_2))$$

5. The very general pay-off function that we use allows us not to refer explicitly to a state variable, as must Sargent (1987) because of the time separability of the pay-off, and Fershtman (1989), because of the restriction to Markovian strategies. It might be useful, however, to interpret β_1 directly as a state variable for the leader decision problem. Note that a link between the follower's current pay-off and the leader's future action, as provided by non-separability or by the presence of the state variable, is essential for time inconsistency problems to arise.

Let $\alpha^* = (\alpha_1^*, \alpha_2^*)$ be the chosen sequence of actions.

Notice that whereas α_1^* is the action chosen by the leader in in the present, α_2^* is merely the announcement that action α_2^* will be chosen when period 2 arrives. We interpret α_2^* as a promise (this will be justified in the next section).

The selection of α^* trades off a direct effect on U^L with an indirect effect, coming through the influence that the promise of α_2^* exerts on the choices of the follower.

Thus the selection of α^* incorporates a manipulative component. The leader promises some action for tomorrow (α_2^*) in order to induce the follower to react today $(\beta_1(\cdot, \alpha_2^*))$ in the fashion that the leader prefers.

Period 2 then comes about. The follower, manipulated by the promise, chose β_1. This is a bygone, and there is no longer need for the leader to keep the promise. In general, even taking the reaction $\beta_2(\cdot)$ into account there will be actions, different from α_2^*, which yield the leader a higher pay-off, since the reaction $\beta_1(\cdot)$, which would have taken place if something different from α_2^* had been announced at the outset, can no longer materialize. A problem of time inconsistency is present when there exists a feasible action that is preferred to α_2^*.

This formal analysis corroborates the intuitive insight of the example in section 1. When the leader uses promises to manipulate the current behaviour of the follower, he will find that if the promise is believed, whatever its objective be, it has been obtained even before the promise itself is kept. This generates the incentive to renege on the promise.

3. The difference between promises and threats

To provide a precise distinction between promises and threats we need to nest our previous open-loop formalization of the game into a more complex formulation which takes strategies into explicit consideration.[6]

In addition to the previously defined symbols, let $h = (\alpha_1, \beta_1)$ represent a possible history of first-period actions and H be the space of all possible histories. We can define a strategy as a function mapping histories of the game into the action set. More precisely, let the leader's strategy be given by the function

$\lambda = [\lambda_1, \lambda_2] \varepsilon\Lambda$, with $\lambda_1 \equiv \alpha_1 \varepsilon A$ and $\lambda_2(h) = \alpha_2 \varepsilon A$, for all $h \varepsilon H$,

where $\Lambda = \Lambda_1 \times \Lambda_2$, $\Lambda_1 \equiv A$, Λ_2 is a set of functions $H \to A$.

The follower's strategy is given by the function

$\phi=[\phi_1,\phi_2] \varepsilon\Phi$, with $\phi_1(\alpha_1)=\beta_1 \varepsilon B$, $\phi_2(h,\alpha_2)=\beta_2 \varepsilon B$, for all $h \varepsilon H$,

where $\Phi = \Phi_1 \times \Phi_2$, Φ_1 is a set of functions $A \to B$ and Φ_2 a set of functions $H \times A \to B$.

The appearance of the leader's actions α as arguments of strategy ϕ indicates that the follower, at each date, takes the leader's actions as given (i.e. acts as a follower)[7].

Let $\hat{h} = (\hat{\alpha}_1, \hat{\beta}_1)$ represent a particular history of the game. A leader's strategy is said to agree with $\hat{h}$ when it prescribes action $\hat{\alpha}$, in times 1. We denote by $\hat{h}°\lambda_2$ a strategy $\bar{\lambda}$ that agrees with $\hat{h}$ and "continues" with λ_2, i.e. $\bar{\lambda}_1 = \hat{\alpha}_1$, $\bar{\lambda}_2 = \lambda_2$. Let $\hat{h}°\phi_2$ have a similar interpretation.

For the two-period, leader-follower game analyzed we have the following definitions.

<u>Definition 1</u>. A leader-follower equilibrium is a couple of strategies $[\lambda^*, \phi^*]$ such that

6. We adopt the formalization presented in Gale (1982).

7. The formulation does not preclude that β_1 is influenced by the announcement α_2. As definition 1 makes clear, β_1 is chosen given the action α_1^* and the strategy λ_2^* (α_1, β_1).

(i) $U^F(\lambda^*, \phi^*) \geq U^F(\lambda^*, \phi)$ for all $\phi \in \Phi$

(ii) $U^L(\lambda^*, \phi^*) \geq U^L(\lambda, \phi^*)$ for all $\lambda \in \Lambda$

(iii) $U^F(\hat{h}\circ\lambda_2^*, \hat{h}\circ\phi_2^*) \geq U^F(\hat{h}\circ\lambda_2^*, \hat{h}\circ\phi_2)$ for all $\phi_2 \in \Phi_2$

(iv) $U^L(\hat{h}\circ\lambda_2^*, \hat{h}\circ\phi_2^*) \geq U^L(\hat{h}\circ\lambda_2, \hat{h}\circ\phi_2^*)$ for all $\lambda_2 \in \Lambda_2$

where, according to the appropriate specification of the sets Φ_2 and Λ_2, we obtain different characterizations of the equilibrium.

<u>Definition 1.1</u> - An open-loop leader-follower equilibrium is obtained from def. 1 when

(i) Φ_2 and Λ_2 are set of functions constant over H

(ii) $\Lambda_2 = \{\lambda_2^*\}$, $\Phi_2 = \{\phi_2^*\}$. (This implies that requirements (iii), and (iv) in definition 1 are immaterial)

<u>Definition 1.2</u> - A closed-loop leader-follower equilibrium is obtained from def. 1 when

(i) $\Lambda_2 = \{\lambda_2^*\}$, $\Phi_2 = \{\phi_2^*\}$

<u>Definition 1.3</u> - A time-consistent leader-follower equilibrium is obtained from def. 1 when

(i) $h' = h^*$, $h^* = (\alpha_1^*, \beta_1^*)$, $\alpha_1^* = \lambda_1^*$, $\beta_1^* = \phi_1^*(\alpha_1^*)$

(ii) Λ_2 is the set of functions $h^* \to A$, Φ_2 is the set of all functions $h^* x A \to B$

<u>Definition 1.4</u> - A leader-follower perfect equilibrium is obtained from def. 1 when

(i) Λ_2 is the set of all functions $H \to A$, Φ_2 is the set of all functions $H x A \to B$

We now provide a formal definition of promises and threats.

<u>Definition 2</u>. An announcement $\alpha_2^p \in A$ is a promise, relative to a strategy $\hat{\lambda}$, if

(i) there exist $\hat{\beta}_1^c \varepsilon B$ such that $\hat{\beta}_1^c \varepsilon \text{argmax } U^F(\beta_1, \hat{\alpha}_1, \hat{\lambda}_2(\hat{\alpha}_1, \beta_1), \beta_2)$, where $\hat{\alpha}_1 = \hat{\lambda}_1$

(ii) $\hat{\lambda}_2 (\hat{\alpha}_1, \hat{\beta}_1^c) = \alpha_2^p$

Part (i) of the definition says that the promise is effective, i.e. that it manipulates the follower choice inducing a particular behaviour.

Part (ii) says that the promise is expected to be kept when the follower complies with its (implicit) suggestion of choosing $\hat{\beta}_1^c$, provided that the strategy $\hat{\lambda}$ is adhered to.

Definition 3. An announcement $\alpha_2^T \varepsilon A$ is a threat, relative to a strategy $\hat{\lambda}$, if

(i) there exist $\hat{\beta}_1^c \varepsilon B$ such that, $\hat{\beta}_1^c \varepsilon \text{argmax } U^F(\beta_1, \hat{\alpha}_1, \hat{\lambda}_2(\hat{\alpha}_1, \beta_1), \beta_2)$, where $\hat{\alpha}_1 = \hat{\lambda}_1$

(ii) there exist $\hat{\beta}_1^{nc} \varepsilon B$ such that $\hat{\lambda}_2(\hat{\alpha}_1, \hat{\beta}_1^{nc}) = \alpha_2^T$

(iii) $\hat{\lambda}_2 (\hat{\alpha}_1, \hat{\beta}_1^c) \neq \alpha_2^T$

Part (iii) of the definition says that the threat induces a behaviour which does not call for the threat to be carried out.

Definition 4. A promise, relative to $\hat{\lambda}$, is credible if there exists no other strategy $\lambda' = [\hat{\lambda}_1, \lambda_2]$, $\lambda_2: \hat{\alpha}_1 \times \hat{\beta}_1^c \rightarrow A$ which is preferred to $\hat{\lambda}$.

Similarly, a threat, relative to $\hat{\lambda}$, is credible if there exists no other strategy $\lambda' = [\hat{\lambda}_1, \lambda_2]$, $\lambda_2: \hat{\alpha} \times \hat{\beta}_1^{nc} \rightarrow A$ which is preferred to $\hat{\lambda}$.

With these definitions, it is simple to prove the following claims (since all equilibria refer to the leader-follower game, we omit the explicit qualification).

Claim 1 The announcement α_2^* in the open-loop equilibrium is a promise

Proof — From def. 1.1 and using the notation introduced in section 2 it follows that $\lambda_1^* \equiv \alpha_1^*$, $\lambda_2^* \equiv \alpha_2^*$, $\phi_1^*(\alpha_1^*) = \beta_1^*$, $\phi_2^*(h, \alpha_2^*) = \beta_2^*$. In definition 2, set $\hat{\lambda} = \lambda^*$, $\hat{\beta}_1^c = \beta_1^*$. Then $\alpha_2^p = \alpha_2^*$.

Claim 2 — In a closed-loop equilibrium both promises and threats are present

Proof — Consider $h^* = [\lambda_1^*, \phi_1^*(\lambda_1^*)]$ and set $\hat{\lambda} = \lambda^*$, $\hat{\beta}_1^c = \beta_1^* = \phi_1^*(\lambda_1^*)$. Then $\lambda_2^*(h^*) = \alpha_2^*$ is, according to definition 2, a promise, relative to λ^*. Consider any other $h \neq h^*$ and set $\lambda_2^*(h)=\alpha_2^T$. α_2^T is, according to definition 3, a threat, relative to λ^*.

Claim 3 — A time-consistent equilibrium is obtained from a closed-loop equilibrium by eliminating non-credible promises. Non-credible threats are still possible.

Proof — Consider definition 1.3. Take $\hat{\lambda} = \lambda^*$ and $\hat{\beta}_1^c = \beta_1^*$. Because of (ii) (together with (iv) of definition 1), $\alpha_2^* = \lambda_2^*(\alpha_1^*, \beta_1^*)$ satisfies the condition of definition 4. Hence α_2^* is a credible promise, relative to λ^*. No constraint is imposed by definition 1.3 on the threats that might be included in λ^*.

Claim 4 — A perfect equilibrium eliminates both non-credible threats and non-credible promises.

Proof — Consider definition 1.4. Condition (ii) (together with (iv) of definition 1) rules out both promises and threats which are not credible.

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