Macroeconomics After Lucas[*]

Thomas J. Sargent[†]

July 1, 2024

Abstract

This sequel to Lucas and Sargent (1978) tells how equilibrium Markov processes underlie much applied dynamic economics today. It recalls how Robert E. Lucas, Jr. saw Keynesian and rational expectations revolutions as interconnected transformations of economic theories and econometric practices. It describes rules that Lucas used to guide and constrain his research by restricting himself to equilibrium Markov processes and to conserving quantitative successes achieved by previous researchers, including those attained by quantitative Keynesian macroeconometric modelers.

Keywords: simultaneous equations, rational expectations, cross-equation restrictions, Markov processes, dynamic programming, causality, direct problem, inverse problem, equilibrium Markov process, master equation.

1 Introduction

For more than a decade, most economists ignored two papers (Muth (1960, 1961)) that showed how to use optimal linear prediction theory to model economic agents' beliefs about the future within coherent probabilistic settings. In the early 1970s, Robert E. Lucas, Jr. used Muth's ideas to make the artificial people who live inside the dynamic simultaneous equations econometric models of Koopmans (1950), Hood and Koopmans (1953), Marschak (1950), and Hurwicz (1966) solve well-posed intertemporal optimization problems. Lucas resolved pressing theoretical issues, reduced dimensions of parameter spaces, and created a research program that has been pursued fruitfully in macroeconomics, industrial organization, public finance, labor economics, and other applied fields.[1]

Section 2 describes Lucas's tools and prejudices. Section 3 describes rules that constrained and guided Lucas's research. Section 4 describes how Lucas interpreted the Keynesian revolution. Section 5 describes how he started another revolution by formulating equilibrium Markov processes. Section 6 recalls what Lucas meant by "rational expectations" and how other uses of that phrase annoyed him. Section 7 explains how economists who want to advise monetary and fiscal policy makers think about causality, and also how the artificial people who live inside an equilibrium Markov process think about it. Section 8 lists examples of equilibrium Markov processes. Section 9 describes how equilibrium Markov processes are accompanied by non-linear impulse response functions, many uses of which Lucas found uninteresting. Section 10 describes the "rational expectations econometrics" intrinsic in the likelihood function associated with an equilibrium Markov process. Section 11 describes how planners who choose among alternative equilibrium Markov processes assume a communism of statistical models. Section 12 describes connections between techniques for approximating equilibrium Markov processes numerically and limiting behaviors of models in which agents inside a model statistically learn about objects that agents in an equilibrium Markov model already know. Section 13 describes how, like Copernicus, Lucas thought that a beautiful simple model that fits less well than a more complicated ugly model is somehow closer to the truth. It also describes how his preference for simplicity along with constraints imposed by his section 3 rules for research limited Lucas's use of rational expectations econometrics. Section 14 illustrates commotions that Lucas's writings provoked by citing his opinions about price rigidities, macro-labor models, Samuelson's neoclassical synthesis, reconciling Phelps islands and Arrow-Debreu complete markets models, and ways to implement Ramsey plans. Section 15 concludes with some remarks about how Lucas responded to economists who

---

[1]I have written other papers about how Lucas accomplished that. See Lucas and Sargent (1978), Lucas and Sargent (1981, pp.xi–xl), Manuelli and Sargent (1988), and Sargent (1981, 1982, 1996, 2015, 2022, 2024).

didn't like equilibrium Markov processes.

Lucas was an extraordinarily gifted writer, not just for an economist. I quote Lucas (1987) often.

2   Influences and Studies

Milton Friedman's tools, research questions, and prejudices influenced Lucas. Milton Friedman accomplished much partly because when young he had mastered much of what had then been known about probability theory and statistics. He thought hard about uses and limits of Neyman and Pearson's frequentist approach to testing hypotheses and about parameter identification as exclusion restrictions in system of simultaneous equations. He was cautious about interpreting "causality" in general equilibrium settings. Through his interactions with Harold Hotelling and Abraham Wald, he helped invent sequential likelihood ratio tests for statistical model selection. He created models of decisions makers having both subjective and objective expected utilities. He thought about decision theoretic consequences of misspecified statistical models. He worked on stochastic approximation and learning. He appealed to survival of the fittest to justify what later came to be called rational expectations. In work with Savage, he laid foundations of "machine learning" when he proposed an early version of stochastic approximation to maximize an unknown function by statistical sampling. He foresaw possibilities for spectral analysis of economic time series.

Armed with those techniques, Friedman approached macroeconomics with set of prejudices, i.e., personal prior probabilities over models, that included an affection for Burns-Mitchell NBER reference-cycle techniques; a present-value-equalization model of professional incomes that he deployed in his PhD thesis and that he eventually published jointly with Simon Kuznets; consumption-smoothing models and associated Euler equations he had learned from reading Irving Fisher; a plan to assemble US data that would let him complete Irving Fisher's statistical verification of the quantity theory of money; the principle that intertemporal government budget balance means that monetary and fiscal policies are either consolidated or coordinated that he acquired from Simons and Mints; and an exponential smoothing statistical model for forecasting, i.e., adaptive expectations.

Constrained by his tools and prejudices, Friedman proceeded to interpret Burns-Mitchell business cycle patterns with statistical models whose parameters encode the demand and supply curves of Marshall's "representative agents;" to extend Irving Fisher's "Statistical Verification" of Quantity Theory of Money by using the accounting framework of Appendix B of Friedman and Schwartz to measure monetary aggregates; to formalize "short-run" versus "long-run" distinctions; to convert "perfect foresight" models into statistical models of

vector stochastic processes by using adaptive expectations and imposing long run restrictions; to put micro-foundations underneath Phillips curve; to take randomness and model ambiguity into account in framing monetary and fiscal policies; to acknowledge "long and variable" distributed lags while professing ignorance about their sources; to practice a "neoclassical synthesis" that separates redistribution and social insurance from macroeconomic stabilization; and to express ambiguity about "narrow banking" versus "free banking" in his work on the optimal quantity of money and paying interest on reserves.[2]

Lucas learned the mathematical tools that had empowered Milton Friedman, adopted many of Friedman's prejudices, and worked on many of the same topics. He deepened and altered Friedman's findings. To help him do that, Lucas learned tools that Friedman either hadn't known about or had chosen not to use. These included dynamic programming and optimal control theory; Markov chains and optimal prediction theory; general competitive equilibria and separating hyperplanes (a.k.a. "welfare theorems"); stochastic discount factors; Samuelson's overlapping generations model; the Cass-Koopmans optimal growth model; game theory; Chicago-Yale-Cowles Commission econometric methods for estimating systems of simultaneous linear difference equations that rest on sharp distinctions between structural statistical models, on the one hand, and the reduced forms models whose parameters are functions of the parameters of a particular structural model, on the other hand; the Phelps island model; and a communism of statistical models called the rational expectations hypothesis. To create the equilibrium Markov processes that we'll describe in section 5, Lucas used Markov decision problems (MDPs), the max-min separating hyperplane theorem,[3] and a rational expectations assumption.[4] In using those tools to remake macroeconomics as well as much of applied economic dynamics, Lucas followed rules.

3   Research Rules

Lucas constrained himself (1) to preserve quantitative successes of earlier theories, (2) to construct equilibrium stochastic processes, and (3) to make a theory and an econometrics fit together. Other scientists and artists had used similar rules.

> ... the constraints that artists and theoretical physicists have to respect, how
> they make our craft difficult, and how they also make it possible. ... often the

---

[2] Friedman emphasized that it matters how those interest payments were to be financed.

[3] The max-min theorem implies the two fundamental theorems of welfare economics as well as related useful results in implementation theory.

[4] As remarked in section 2, Milton Friedman either hadn't known these tools or hadn't used them in ways that Lucas did.

most important constraint on a new theory is not that it should survive this or that new experimental test, but that it should agree with the body of past observations, as crystallized in former theories. ... New theories ... must not throw out all the successes of former theories. This sort of thing makes the work of the theorist far more conservative than is often thought. The wonderful thing is that the need to preserve successes of the past is not only a constraint, but also a guide.

Weinberg (2018, ch. 24)

Lucas insisted on preserving past successes that included cross-country and historical evidence about inflation that quantity theory of money fit well; apparent money supply and price level"non-neutralities;" Burns-Mitchell NBER reference cycle characterizations of business cycles; Friedman-Schwartz evidence pointing to monetary shocks as sources of business cycles; good fits to US business cycles of Klein-Goldberger and other Keynesian econometric models; and statistical evidence about stock prices and expectations theories of the term structure of interest rates.[5]

Lucas confined himself to building models that contain artificial people who solve constrained optimization problems; binding those artificial people together with an equilibrium concept that enforces coherence; and, to economize on free parameters, assuming that the agents share joint probability distributions with each other and the model builder.[6]

4 Two Revolutions

Lucas emphasized that, despite their differences, protagonists of the Keynesian and Rational Expectations Revolutions agreed about many important things.

The Keynesian Revolution was, in the form in which it succeeded in the United States, a revolution in method. ... if one does not view the revolution in this way, it is impossible to account for some of its most important features: the evolution of macroeconomics into a quantitative, scientific discipline, the development of explicit statistical descriptions of economic behavior, the increasing reliance of government officials on technical economic expertise, and the introduction of the

---

[5]Aspects of rational expectations and optimal prediction theory were implicit in regression equations that Meiselman (1962) used to implement the expectations theory of the term structure of interest rates. Bob Lucas told me that the term structure was an ideal laboratory for rational expectations. When I first met him in his office at Carnegie Tech in November 1966, Bob was reading a preprint of Wallace (1967).

[6]They don't necessarily share information sets.

use of mathematical control theory to manage an economy. It is the fact that Keynesian theory lent itself so readily to the formulation of explicit econometric models which accounts for the dominant scientific position it attained by the 1960s. As a consequence of this, there is no hope of understanding either the success of the Keynesian Revolution or its eventual failure at the purely verbal level at which Keynes himself wrote. It will be necessary to know something of the way macroeconometric models are constructed and the features they must have in order to "work" as aids in forecasting and policy evaluation.

"After Keynesian Macroeconomics," 1978.

Keynesian and rational expectations revolutions shared objects of interest and purposes. A shared object of interest was a system of simultaneous stochastic difference equations.

... economic data are generated by systems of relations that are, in general, stochastic, dynamic, and simultaneous. ... these very relations constitute economic theory and knowledge of them is needed for economic practice. ... Hypotheses about economic structure are also known as economic theories. They try to state relations that describe the behavior and environment of men and determine the values taken at any time by economic variables such as prices, output, and consumption of various goods and services, and the prices and amounts of various assets. As there are several variables the economic structure must involve several simultaneous relations to determine them.

Marschak (1950)

A shared purpose was to identify parameters that are invariant to a set of historically unprecedented possible government policies.[7]

The economist's objectives are similar to those of an engineer but his data are like those of a meteorologist. The economist is often required to estimate the effects of a given (intended or expected) change in the "economic structure," i.e., in the very mechanism that produced his data. None of these changes can he produce beforehand, as in a laboratory experiment; and since some of the changes envisaged have never happened before, the economist often has to estimate the results of changes he has never observed. ... The economist can do this if his past

---

[7]Footnote 14 below describes Lucas's opinion about Christopher Sims's opinion about this "utopian" project.

observations suffice to estimate the relevant structural constants prevailing before the change. Having estimated the past structure the economist can estimate the effects of varying it. He can thus help to choose those variations of structure that would produce – from a given point of view – the most desirable results. That is, he can advise on policies (of a government or a firm).

> Marschak (1950, p. 2)

The flaw "fatal to the purposes of the empirical study of economic time series" was that Keynesian statistical models weren't equilibrium Markov models, a class of models that now transcends much of applied dynamic economics.[8]

5 Equilibrium Markov Processes

In various papers, Lucas defined and formulated a class of statistical models suitable for analyzing the types of macroeconomic policy interventions that Marschak (1953) and his colleagues at the Cowles Commission had wanted to study.

Definition 5.1. *An equilibrium Markov process contains: (1) a collection of decision makers, (2) associated Markov decision problems defined over a common state space, and (3) budget and resource constraints that bind decision makers' MDP's together.*

In equilibrium Markov models, parameters of the dynamic demand and supply curves that Keynesian macroeconometric models wanted to be invariant to interventions are themselves functions of some of the parameters that an historically unprecedented government policy intervention would alter. Analyzing those interventions requires knowing how parameters of those dynamic demand and supply curves depend on deeper parameters describing preferences, technologies, and timing protocols. An equilibrium Markov model pins down functions that describe those dependencies.

Recent applied researchers represent an equilibrium Markov model with a single "master equation." Carefully designed "deep neural networks" can approximate solutions of master equations for some interesting equilibrium Markov models well (see Bilal (2023) and Gu et al. (2024)). Coaxing neural networks to approximate solutions of stand-alone master equations has proven to be substantially easier than for master equations supplemented with the auxiliary equations that some equilibrium Markov models also impose. HANK models, models with non-redundant long term assets, and models with adjustment costs augment a master equation with auxiliary equations. See Gu et al. (2024).

---

[8]Lucas (1987, Sect. I) and Lucas and Sargent (1981, pp.xi–xl) described components and features of this equilibrium concept.

6 Rational Expectations

Equilibrium Markov processes assign a rational expectations assumption critical roles in (1) building in coherence, and (2) economizing on free parameters.

> The term 'rational expectations', as Muth used it, refers to a consistency axiom for economic models, so it can be given precise meaning only in the context of specific models. I think this is why attempts to define rational expectations in a model-free way tend to come out either vacuous ('People do the best they can with the information they have') or silly ('People know the true structure of the world they live in').

> *Models of Business Cycles*, 1987, p. 13.

> What is "wrong" with [adaptive expectations] is not [expressing] forecasts of future variables as distributed lags of current and lagged variables. The future must be forecast on the basis of the past, and it is surely acceptable to simplify things by modeling agents as using linear forecasting rules. (These points are obvious enough, but are so widely misunderstood as to warrant emphasis here.) The difficulty lies not in postulating forecasts which are linear functions of history but rather in introducing the coefficients in these linear functions as so many additional "free parameters," unrestricted by theory. That this practice is unnecessary, and in an important way fatal to the purposes of the empirical study of economic time series, is the message of [Muth (1961).

> Lucas and Sargent (1981, pp. xv–xvi)

To economize on parameters, a rational expectations assumption imposes a communism of statistical models that manifests itself in constructing models and in inferring parameters. Thus, a rational expectations assumption makes all decision makers inside a model share a vector stochastic process with each other and with the theorist who built the model. Decision makers use that stochastic process to form the conditional distributions that appear in Euler equations that restrict their decision rules. Rational expectations econometrics extends communism of statistical models to include a "sharing with nature" that is an essential input into making maximum likelihood or generalized method of moments be good estimators.

7 Causality

Statements about causality boil down to assertions that some parameters of a statistical model are invariant with respect to a class of possible interventions.[9] Which parameters are invariant depends on the class of interventions. When a model's author and the people inside it are concerned about different interventions, they want different sets of parameters to be invariant. Being a coherent collection of Markov decision problems, an equilibrium Markov model reconciles those distinct ideas about about "causality" – i.e, about which parameters are invariant.

A well posed Markov Decision Problem (MDP) includes a specification of (1) vectors of states and decisions (a.k.a. controls), and (2) a partition of a state space into controllable and uncontrollable subspaces. Each such specification forms a theory of causes and effects. An MDP describes how decisions shape trajectories through a controllable subspace. It does so by fixing parameters in a controlled Markov transition equation that tells the decision maker how future payoffs are affected by alternative feasible choices of controls. A controlled Markov transition law is "causal" in the sense that it is invariant across a set of admissible controls. An MDP also implies a joint probability density over sequences of states in an uncontrollable subspace and an associated theory of optimal prediction.[10]

Thus, an equilibrium Markov process contains as many assumptions about causality – i.e., about invariance of parameters – as there are decision makers. These include the author of the model and the agents who live inside it.

Other meanings of "causality"

Economists use other senses of "causality", including a concept of Wiener, Granger, and Sims that is about the structure of a joint conditional distribution of a fixed stationary vector stochastic process. That concept differs from the control-theoretic sense of causality that applies to MDPs within an equilibrium Markov model and is instead about comparisons across alternative vector stochastic processes indexed by alternative choices of rules for a control vector.[11]

---

[9]My personal conversation with Leo Hurwicz after a 1975 Minneapolis Fed seminar at which Neil Wallace and I presented a preprint of Sargent and Wallace (1976) convinces me that my account here is compatible with Hurwicz (1966). Like Marschak (1953), Hurwicz wanted some parameters (states?) to be invariant under some hypothetical policy interventions, for example, parameters that describe agents' preferences, technologies, information sets, market structures, and timing protocols.

[10]To bring out links to rational expectations, Lucas and Sargent (1981, pp.xi–xl) used the "certainty equivalence" property that linear quadratic MDP's possess to highlight the theory of optimal prediction that MDP's provide.

[11]Lucas and Sargent (1981, pp. 405-452) offer an example from Germany during the early 1920s in which inflation Granger caused money growth according to the joint probability distribution that emerged

The dynamic control theoretic sense of "causality" also differs from "causal inferences" like those drawn from R.A. Fisher's hypothesis tests of agricultural fertilizer treatments that assume fixed regressors in repeated samples. Inapplicability of Fisher's assumptions motivated Koopmans, Marschak, and Hurwicz to invent an econometric theory applicable to systems of stochastic difference equations that are interrelated in ways that policy interventions alter. [12] Fisher's assumptions don't hold for the dynamic economic systems that they wanted to interpret and control. Many recent applications of techniques for "causal inference" study much more limited "treatments" than the historically unprecedented policy interventions that Koopmans, Marschak, and Hurwicz wanted to learn about.

8   Examples of Equilibrium Markov Processes

Equilibrium Markov processes pervade modern applied dynamic economics. They include representative agent recursive competitive equilibria with their "Big K, little k" distinctions; Markov perfect equilibria; Ramsey (a.k.a. Stackelberg) equilibria in which the state variables of a leader's problem include followers' continuation values; models of credible public policies like ones studied by Stokey (1989, 1991); Atkeson and Lucas (1992) models of redistribution dynamics in which the state includes joint cross section distribution of continuation values and a Markov operator $T_\#$ that maps a cross section at t into a cross section at $t + 1$; Kantorovich optimal transportation models; Hopenhayn models of firm dynamics; mean field games in which states include cross section distributions of wealth or consumption or continuation values (these can be viewed as extensions of Lucas and Prescott (1971) "Big K-little k" models); as well as single-agent robust decision problems that include adversarial control and actor-critic systems.

That all of these are equilibrium Markov models extends Lucas's 1989 observation that

> Complete market economies are all alike but each incomplete market economy is incomplete in its own individual way.

> Robert E. Lucas, Jr. (1989)

"All alike" means that each complete market model belongs to a class of models; a particular model, i.e., an "instance of the class," is pinned down by a commodity space, a price system, a list of decision makers together with their preferences and technologies, and a definition

---

from an equilibrium Markov model. But that joint distribution was not invariant to monetary-fiscal policy interventions that altered the money growth process.

[12] See Marschak (1950), Koopmans (1950), Hood and Koopmans (1953).

of equilibrium that applies to all members of the class. One gets a new complete markets model by specifying a new set of components.[13] You cannot get an incomplete markets model simply by redesigning those standard components; and a model can fail to have those components in many ways. But when they can be cast as equilibrium Markov models, incomplete markets models can be viewed as alike too, being constructed just by redefining their components. That a large number of superficially different models are all equilibrium Markov models has helped to organize applied dynamic economics.

9 Impulse Response Functions

Lucas framed macroeconomic policy choices in a way that made him skeptical of many applications of fixed impulse response functions.

> ... one cannot usefully think about economic policy - about the strategies of government, another 'player' in this game - in terms of current policy decisions only. Private agents necessarily have to make inferences about the way future fiscal and monetary policy will be conducted. If we discuss policy as though it involved only what government does today - that is, if we discuss policy in the terms that dominate current political discussion - then we are leaving the most important aspects of policy undiscussed and their consequences unanalyzed.
>
> *Modeling Business Cycles*, 1987, pp. 102

Associated with every equilibrium Markov process is a (non-linear) stochastic vector impulse response process that records transient and enduring responses to surprises. Many of the questions that impulse response functions answer didn't interest Lucas. Thus, for fixed impulse response functions, one can study dynamic responses of many variables to an innovation in one variable. This approach is followed by "event studies", e.g., about central banks' "quantitative easing." Many equilibrium Markov models tell us that responses to large and small shocks differ. Because surprises can't be systematically chosen *ex ante*, fixed impulse response functions are of little use in designing improved policies. Nevertheless, they are salient features of Gallant and Tauchen (1996) auxiliary models for constructing moments for Generalized Methods of Moments estimators of free parameters of equilibrium Markov models whose likelihood functions cannot be written down.

A more ambitious application characterizes impulse response functions as functionals of (parameterized) government policy rules for a manifold of equilibrium Markov models.

---

[13]Hansen and Sargent (2013) deploy this insight repeatedly. Lucas's remark illustrates Poincare's dictum that "Mathematics is the art of giving the same name to different things."

Such characterizations are essential inputs to evaluating outcomes under the historically unprecedented policies that a "utopian" Ramsey planner wants to understand.[14]

10   Rational Expectations Econometrics

Rational expectations econometrics requires solving two interconnected problems. A "direct problem" takes a vector of parameters as known and computes an equilibrium Markov process. A solution of the direct problem lets you simulate the model, i.e., draw random samples from a joint probability distribution, thus generating artificial data sets. An "inverse problem" exchanges knowns and unknowns. It takes an observed data set as known and extracts information about unknown parameters. Thus, a direct problem takes known parameters as inputs and computes fake data. An inverse problem takes genuine data set as input and infers parameter values.

Via a direct problem, an equilibrium Markov process induces a joint probability distribution over sequences of prices, quantities, and information sets indexed by a vector of parameters. "Likelihood function" is a synonym for "equilibrium Markov process." Therefore, an equilibrium Markov process makes possible two varieties of "rational expectations econometrics." An econometrician can pretend to be a frequentist and use maximum likelihood to infer parameters. An econometrician can instead pretend to be a Bayesian, put a prior over the parameter vector, merge the prior and the likelihood to form a joint distribution, and then use laws of inverse probability to approximate a posterior distribution for parameters.

As emphasized in section 4, the econometric parts of the rational expectations revolution owe much to the Koopmans-Marschak-Hurwicz Cowles Commission approach to macroeconometrics, sharing purposes and objects of interest. A shared purpose is to estimate structural parameters that are invariant to the proposed macroeconomic policy changes that the model is designed to study. Reduced form parameters aren't invariant with respect to the interventions that Keynesian macroeconometrians and rational expectations macroeconometricians both wanted to study.[15] A shared object of interest is a system of simultaneous stochastic

---

[14] "Sims criticizes the 'rational expectations revolution' for 'destroying or discarding much that was valuable in the name of utopian ideology." (Lucas (1987, footnote 1, p. 8)). (See Sims (1982).) Section 4 above indicates that Sims's characterization also applies to Koopmans's and Marschak's aspiration to use structural stochastic dynamic simultaneous equation models models to analyze consequences of historically unprecedented policies. I still think that designing a macroeconomics that can tell us how to avoid big depressions and big inflations is a noble enterprise. For Lucas's perspective on tensions between positive and normative economics, read all of Lucas (1987, footnote 1).

[15] Similarly, impulse response functions aren't invariant to the interventions that a Ramsey planner contemplates.

difference equations with (reduced form) parameters that are functions of deeper parameters that govern aspects of behavior invariant to a range of possible policy interventions. Pre and post rational expectations structural models also share R.A. Fisher's notion of "parameter identification" as statements about the condition number of the Hessian of a log likelihood function evaluated at parameter values that maximize log likelihood function.

Nevertheless, Lucas (1976) pointed out that a rational expectations equilibrium subverts many Cowles Commission (Koopmans (1950), Hood and Koopmans (1953)) exclusion restrictions for parameter identification.[16] As a system of stacked Bellman equations, an equilibrium Markov process imposes few exclusive restrictions but instead imposes extensive "cross-equation" restrictions across equilibrium decision rules and agent-specific conditional probability densities for agent-specific uncontrollable state variables. This technical point about invariant parameters is the "revolution" part of rational expectations that helped make Lucas unpopular at Martha's Vineyard in 1978 (see Solow (1978) and section 15 below).

11 Optimal Government Policies

To design an optimal policy, a Ramsey planner in Lucas and Stokey's (1983) model compares outcomes across many putative joint distributions that generated by alternative possible government decision rules. While searching across alternative plans, the Ramsey planner makes extensive use of cross-equation restrictions brought by a rational expectations assumption. Those restrictions describe how the plan influences that the joint distributions that the Ramsey planner wants to shape.

To appreciate how a Ramsey planner relies on a rational expectations assumption, it is enlightening to think about how government policies might be designed in a setting in which rational-expectations is abandoned, for example in a self-confirming equilibrium in which agents don't share a statistical model with nature. Instead, each type of agent has its own manifold of statistical models, each indexed by a distinct vector of parameters. Meanwhile, the data are generated by nature's statistical model, indexed by yet another vector of parameters. The parameters of each agent's model take values that among all models in that agent's manifold of models, best fit data generated by nature's model. Technically,

---

[16] I wrote Sargent (2024) to describe the vast ramifications of Lucas (1976) for macroeconomics on the occasion of the 50th anniversary of Lucas's presentation of it at the inaugural Carnegie-Rochester conference in 1973. Sims (1980) and Sargent and Sims (1977) also challenged the plausibility of the Cowles Commission exclusion restrictions that Lucas criticized. Instead of seeking substitutes for those restrictions, Sims recommended not pursuing the Marschak-Koopmans dream of using quantitative macro models to analyze the alternative historically unprecedented monetary and fiscal policy rules that Marschak (1953) wanted to study. Sims's doubts about structural macroeconomic models extended beyond rational expectations models. Again see Lucas (1987, footnote 1, p. 8).

this means that each agent's statistical model is an "information projection" from nature's model onto that agent's manifold of statistical models.[17]

Lucas (1986) explored consequences of replacing rational expectations with the assumption that decision makers form conditional distributions of future variables by recursively updating least squares estimates of parameters of an arbitrary and presumably misspecified statistical model. Under suitable conditions, a Law of Large Numbers makes parameter estimates and associated decision rules converge to become parts of a self-confirming equilibrium in which decision maker's beliefs are statistically confirmed for events that occur infinitely often within the equilibrium.[18] In a self-confirming equilibrium, an agent's statistical model can be verified along an observed sample path generated by nature's model, but nevertheless misrepresent outcomes along sample paths associated with historically unprecedented government policies. Misunderstandings about policies that haven't been tried lead to inferior government policies in a self-confirming equilibrium.[19]

A self-confirming equilibria helps us appreciate how thoroughly a Ramsey planner relies on a rational expectations assumption. In a self-confirming equilibrium, a government adopts inferior policies because its misspecified statistical model misleads it about off-equilibrium outcomes associated with historically unprecedented policies.[20] The data contain no "treatments" that would raise statistical misspecification alarms bells .

In addition to bringing insights about designing macroeconomic policies, self-confirming equilibria are important in models of learning that we shall discuss in section 12. Information projections and self-confirming equilibria will lurk in the shadows when we discuss Lucas's preferences for simplicity in section 13.

---

[17] Csiszár and Matus (2003) and Nielsen (2018) describe information projections. Let $\{f_\theta(x)\}_{\theta \in \Theta}$ and $\{g_\delta(x)\}_{\delta \in \Delta}$ be two collections (manifolds) of probability distributions for outcomes $x \in X$. When model $g_{\delta_o}(x)$ governs the data, a population maximum likelihood estimator of parameter vector $\theta \in \Theta$ of misspecified statistical model $f_\theta(x)$ minimizes the Kullback-Leibler divergence

$$\text{KL}(g_{\delta_o}, f_\theta) = \int \log\left(\frac{g_{\delta_o}(x)}{f_\theta(x)}\right) g_{\delta_o}(x) dx = -H(g_{\delta_o}) - E_{g_{\delta_o}} \log f_\theta(x)$$

where $H(g_{\delta_o}) = \int \log\left(\frac{1}{g_{\delta_o}(x)}\right) g_{\delta_o}(x) dx$ is the Shannon information of nature's probability distribution g_{δ_o} and $E_{g_{\delta_o}}$ is the mathematical expectation under $g_{\delta_o}(x)$. The information projection of g_{δ_o} onto $\{f_\theta(x)\}_{\theta \in \Theta}$ is the particular distribution $f_{\theta_o}(x)$ in the manifold $\{f_\theta(x)\}_{\theta \in \Theta}$ selected by maximum likelihood when nature's model g_{δ_o} generates the data, i.e., $\theta_o = \arg\max_{\theta \in \Theta} E_{g_{\delta_o}} \log f_\theta(x)$. In some formulations, parameters δ_o in nature's model g_{δ_o} are functions $\delta_o = \delta(\vec{\alpha}_o, \eta_o)$ where $\vec{\alpha}_o$ are maximum likelihood estimators of parameters of agents' models and η_o is another vector of parameters in nature's model. This dependence emerges because agents' statistical models influence their decision rules, which in turn influence an equilibrium joint distribution.

[18] A Law of Large Numbers brings that "infinitely often" qualification.

[19] Fudenberg and Levine (2009) relate self-confirming equilibria to Lucas (1976).

[20] Self-confirming equilibria appear in contending accounts of US inflation dynamics in the 1970s and 1980s. See Sargent (1999, 2008).

14

12   Equilibrium Computation and Learning

Using an equilibrium Markov model to do quantitative macroeconomic analysis requires computing an equilibrium for a vector of fixed parameter values. Solving the section 10 direct and inverse problems requires doing that, and the faster, the better.

I use "compute" as a synonym for "approximate." A fixed point of a mapping from perceived laws of motion to actual laws of motion is associated with an equilibrium. That fact brings connections between equilibrium computation algorithms and non rational expectations models in which agents inside a model learn about laws of motion and perhaps also price functions. Such models differ in terms of who is learning and what they are learning. Sometimes the person learning is a model builder who is outside the model who wants to learn about a fixed point. In other settings, agents inside a model learn about transition equations that govern evolution of the uncontrollable states that they have misspecified.[21]

Techniques for analyzing convergence of models with least squares learners to a rational expectations equilibrium have contributed algorithms for approximating equilibrium Markov models. Connections between models of learning and equilibrium computation are intermediated through a mathematical tool called "stochastic approximation", early contributions to which were made by Milton Friedman (see Friedman and Savage (1947)) and his teacher Harold Hotelling (Hotelling (1941)). Sean Meyn (2022, ch. 5) links recent developments to "machine learning" algorithms for approximating functions.

13   Approximating Models

Lucas agreed with Copernicus that

> ... a simple and beautiful theory that agrees well with observation is often closer
> to the truth than a complicated ugly theory that agrees better with observation.
>
> Weinberg (2015, ch. 6)

That "a simple and beautiful theory that agrees well with observation is often closer to the truth than a complicated ugly theory that agrees better with observation" collides with rational expectations econometrics. Bayesian and frequentist statisticians know a manifold of parameterized joint probability distributions (i.e., likelihood functions); they just don't know

---

[21]See Lucas (1986) for an early analysis in which an agent inside a model is learning about the model. Bray and Kreps (1987) draw a distinction between models of learning "within" a rational expectations equilibrium and models of learning "about" a rational expectations equilibrium. There is a connection with Hansen's (2014) distinction between uncertainties "outside" and "inside" models.

parameter values. Regarding an equilibrium Markov process (a.k.a. a likelihood function) as an approximation forces a model's author to think about inference and decision making in the presence of misspecified statistical models. It also raises questions about how to evaluate approximating models.

> Kydland and Prescott do not say much about which questions they hope their model could simulate accurately, or with what level of accuracy. ... Whether [Kydland and Prescott's] results are viewed as 'good' or 'bad' is a difficult question, as is the related question of which comparisons of theoretical to sample moments are most interesting.[22] One could obtain a formal sharpening of these questions by using the discipline of classical hypothesis testing ... ... but the interesting question raised by the Kydland and Prescott model is surely not whether it can be accepted as 'true' when nested within some broader class of models. Of course the model is not 'true': this much is evident from the axioms on which it is constructed. We know from the outset in an enterprise like this (I would say, in any effort in positive economics) that what will emerge - at best - is a workable approximation that is useful in answering a limited set of questions.

> *Modeling Business Cycles*, 1987, p. 91

Rational expectations econometrics offers little guidance to a quantitative economist who professes an unknown gap between his model and nature's.[23] Macroeconomists have responded to this difficulty in various ways. Calibrators who follow Kydland and Prescott (1982, 1996) still rely heavily on the section 10 direct problem, but much less on the inverse problem. Instead they condition on known parameters, adopt assumptions sufficient to make an equilibrium Markov model induce a stationary and ergodic process, and use associated laws of large numbers.[24] After importing some parameters from extraneous sources, they "calibrate" other parameters by finding other parameters that make their model's population moments match particular sample moments. Before computing those moments, calibrators

---

[22]Lucas (1987, p. 72) noted that Kydland and Prescott (1982) abandoned Solow's method of inferring the conditional variance and persistence of technological change by fitting an aggregate production function. To fit US business cycle fluctuations, they substantially increased Solow's calibration of the variability of technical change. Lucas (1987, Sec. VII) indicated that by neglecting monetary shocks as sources of cycles, Kydland and Prescott's procedure for setting technology change process parameters overstated their role in generating aggregate fluctuations.

[23]Bob told me that "anything is an approximation to anything else." A model can be wrong, i.e., an approximation, in an infinite number of ways. When don't know what you're trying to approximate, you don't even have an approximation criterion.

[24]For generations of calibrators, Stokey et al. (1989) has been a source of such assumptions.

sometimes decide that their model is designed to be a better approximation to some frequencies than others, so they filter data to attenuate some frequencies and amplify others.[25] Sometimes they "filter" data by conditioning only on events that they had designed the model to explain, for example by excluding data during "sales" for a model in which firms set prices, as in Golosov and Lucas (2007).

Rather than ignoring particular frequencies or events, sometimes a calibrator discriminates among variables, e.g., focusing on quantities and ignoring prices in an equilibrium Markov model that jointly determines them. To measure benefits from attenuating post WWII US business cycles, Lucas (1987, Sec. III) and Lucas (2003) used the value function for his asset pricing model Lucas (1978) . Asset prices are subgradients of that value function.[26] An economist who regards Lucas (1978) as an adequate approximation to a joint quantity-price process would use that information. But Hansen and Singleton (1982, 1983) had convinced Lucas that Lucas (1978) was not useful for understanding asset prices. Hansen and Singleton had combined inverse problems for the Lucas (1978) model with US data on consumption and asset prices to construct specification test statistics that forced Lucas into unpleasant compromises.[27] He imported an extraneous estimate of a coefficient of relative risk aversion and of the parameters of an exogenous consumption process and used them to quantify a value function that measures the costs of business cycles.[28] To justify that calibration strategy, Lucas (2003) said that it is implausible to impute big equity premia to a representative agent's high aversion to risk, and that sources of behavior other than risk aversion not included in his model are required to explain the equity premium and other asset pricing facts that, from the perspective of the Lucas (1978) model, appear to be anomalies. Can other types of behavior preserve most of the quantity implications of Lucas (1978) that Lucas had relied on to measure costs of business cycles, while realigning asset prices closer to data? Yes.

Hansen et al. (1999) and Tallarini (2000) found that adding concerns about model misspecification to a representative agent's aversion to risk improves fits to equity risk premia while

---

[25]Doing that alters information content of the theories and disrupts rational expectations cross-equation restrictions. By distinguishing "parameters of interest" and "nuisance" parameters, Hansen and Sargent (1993) and Sims (1993) convert that into an advantage. They construct examples in which seasonal adjustment improves estimates of preference and technology parameters – the parameters of interest – while degrading "nuisance parameters" that describe evolution of information variables in agents' uncontrollable subspaces. Their analysis could be extended to other frequencies. Hansen and Singleton (1991) describe how a partitioned inverse formula obeyed by covariances takes nuisance parameters into account when inferring parameters of interest.

[26]See Hansen et al. (1999) and Alvarez and Jermann (2004).

[27]Lucas (1976) had advocated imposing the cross-equation and cross-frequency restrictions brought by an equilibrium Markov model. A model brought a package of quantitative implications, among which its author was not free to pick and choose.

[28]Kuh and Meyer (1957) assessed the pros and cons of importing parameters from extraneous sources.

leaving implications about quantities unaltered. Agents inside the equilibrium Markov model of Hansen et al. (2008) regard it as an approximation. Hansen et al. use robust control and filtering techniques to represent how those agents express concerns about statistical model specifications and also about appropriate priors to put on alternative statistical models.[29] Doing that requires finding a practical substitute for the rational expectations assumption that a common, statistical model is shared by a model's authors, the decision makers inside the model, and nature. Is it possible to replace that "communism" assumption with one that does not increase the number of free parameters fatally? An approach described by Hansen (2014) assumes that a model builder presents to the decision makers inside the model a statistical model of variables that those decision makers want to forecast in order to make good decisions.[30] The decision makers solve robust Markov decision problems to protect themselves from their concerns that the statistical model is misspecified. Although agents inside the models of Hansen (2014) share their model builders' approximating models, they distrust them. That brings to life a market price of model uncertainty that is distinct from the market price of risk in the Lucas (1978) model and that helps to explain the asset pricing anomalies that made Lucas abandon some of his model's quantitative implications even though he had relied on other of its implications to measure costs of business cycles.

When people who share a common model respond to their specification doubts by solving robust Markov decision problems, *ex post* they can appear to have different statistical models.[31] Although they share a common approximating model, each decision maker behaves "as if" he or she puts probability 1 on a "worst-case model." Because they have different purposes, "worst-case" models of different decision makers differ. This situation opens disciplined ways of modeling apparent belief heterogeneity.[32]

---

[29]Hansen distinguished between concerns about model misspecification, which he called uncertainty, and doubts about a prior to put over alternative statistical models, which he called "ambiguity". Also see Hansen and Sargent (2022).

[30]Gallant and Tauchen (1996) call such a good-fitting model an "auxiliary model." It plays a different role in the analysis of Hansen et al. (2008), Hansen (2014) than it does in Gallant and Tauchen's simulation procedure for estimating parameters of a Markov decision model whose likelihood function cannot be written down analytically.

[31]After I presented a joint paper with Lars Hansen about robustness at the Minneapolis Fed, Bob asked me "why should the people in our models be like us?" According to the Muth (1961) paper that got Bob started, shouldn't they be like us?

[32]Assuming a common approximating model provides "discipline" in the sense of economizing on free parameters.

14 Lucas's Opinions

The following subsections recall how Lucas thought about nominal price rigidities; macro-labor; reconciling Phelps island and Arrow-Debreu models; and implementing Ramsey plans.

Price rigidities

> ... the term rigidity does not refer to some characteristic of nominal price or wage series by themselves, but rather to the behavior of these series relative to the way they would have been predicted to behave under a particular class of models. ... The problem with price rigidities is that they seem to come and go. Sometimes monetary changes that 'ought' to be pure units effects seem to be just that; sometimes they seem to have large non neutral effects. ... the futility of theorizing by postulating that the behavior of agents is what it is without trying to locate the reasons for this behavior in preferences, technology, or the structure of the underlying game.

> *Modeling Business Cycles*, 1987, pp. 89, 91

Turning first to models that don't "locate the reasons for this behavior in preferences, technology, or the structure of the underlying game," Calvo (1983) and Rotemberg (1982) constructed models that explain observed individual firms' price, quantity paths within settings in which monetary rules and shocks affect allocations.[33] To do that, they imposed socially improvable price-setting policies on firms, then proceeded to deduce monetary-fiscal policy functions that correct collateral damage from firms' price-setting policies In contrast to Calvo-Rotemberg models, firms inside the models of Golosov-Lucas and Alvarez-Lippi choose how sticky to make prices. Impulse responses are non-linear and depend partly on shock volatilities.

Does a Golosov and Lucas (2007) or Alvarez-Lippi model look more like a Calvo-Rotemberg model or a flexible price model. "More like" in response to what? To small shocks? To big shocks? To changes in the monetary-fiscal policy functions that equilibrium Markov models are designed to study?

Answers are that Golosov-Lucas or Alvarez-Lippi models look more like Calvo-Rotemberg models for small shocks, more like flexible price model for large shocks, and more like flexible

---

[33]People inside the sticky price models compared in this subsection set prices. In general equilibrium models in the Arrow-Debreu tradition, nobody inside the model chooses prices: they are set by someone outside the model, perhaps by a Walrasian auctioneer or by a robust algorithm (e.g. Scarf (1982)). Lucas (1972) is a model of "sticky" prices in which no one inside the model sets prices.

price model for change in systematic monetary-fiscal policies. Thus, in models in which firms choose stickiness:

> ... for small shocks the nature of the friction is irrelevant, that is, the propagation of the nominal shock is the same in state- and time-dependent models provided that the models are fit to the same steady-state moments. ... the inherent nonlinear nature of decision rules of SD models implies that for aggregate shocks above a minimum size, the economy displays full price flexibility. Thus, for SD models the impact effect of the shock depends on their size.
>
> "Are State- and Time-Dependent Models Really Different?" Alvarez, Lippi, and Passadore, NBER Macro Annual, 2017, pp. 380- 381.

In the spirit of Stephen Weinberg's rules as guides for research, models in which firms choose stickiness preserve the following past successes:

- Cross-country and historical evidence about inflation that the quantity theory of money fit well
- Apparent money supply "non-neutralities"
- Friedman-Schwartz evidence that points to monetary shocks as sources of business cycles

Macro-labor

Lucas preferred to use models without jobs to study aggregate prices, wages, interest rates, and employment. He preferred to use models with jobs to study unemployment. For modeling aggregate employment, aggregate inflation, interest rates, and GDP and its composition, he said that modeling flows into and out of unemployment is a side show.

> What we mean, in ordinary usage, by 'unemployment' is exactly disruptions in, or difficulties in forming, employer-employee relationships. Simply hamstringing the auctioneer in a Walrasian framework that assigns no role at all to such a relationship is not going to give us the understanding we want. If we are serious about obtaining a theory of unemployment, we want a theory about unemployed people, not unemployed 'hours of labor services about people who look for jobs, hold them, lose them, people with all the attendant feelings that go along with these events. Walras's powerfully simple scenario, at least with the most obvious choice of 'commodity space', cannot give us this, with cleared markets or without them.

Nevertheless, Lucas asked

> ... whether modeling aggregative employment in a competitive way as in the Kydland and Prescott model (and hence lumping unemployment together with 'leisure' and all other non-work activities) is a serious strategic error in trying to account for business cycles.
>
> *Modeling Business Cycles*, 1987, p. 66.

Lucas answered

> I see no reason to believe that it is. If the hours people work - choose to work - are fluctuating it is because they are substituting into some other activity. For some purposes - designing an unemployment compensation scheme, for example - it will clearly be essential to break non-work hours into finer categories, including as one 'activity' unemployment. But such a finer breakdown need not substantially alter the problem Kydland and Prescott have tried to face by finding a parameterization of preferences over goods and hours that is consistent with observed employment movements.

Many macroeconomists have agreed with Lucas that to understand aggregate employment, aggregate inflation, interest rates, and GDP and its composition, modeling flows into and out of unemployment is a side show. Lucas and Rapping (1969), Hansen (1985), Prescott (2002) and many real and monetary business cycle models include no employer-employee relationships interpretable as jobs. Neither did most pre-rational-expectations models that also assumed spot markets (e.g., "hiring halls") that continuously equate supplies and demands for labor.

Combining features of Phelps Islands and Arrow-Debreu models

A 20th century macro tradition that Paul Samuelson called a "neoclassical synthesis" reconciled macroeconomics with microeconomics by directing microeconomic policies to redistribute and provide social insurance and directing macro polices to attenuate business cycles. Lucas adopted and modernized that synthesis by merging components of Arrow-Debreu and Phelps island models.

In a real general equilibrium model like Kydland and Prescott's, exchange occurs in centralized markets, so that goods are valued only if they are valued in use (consumption or production) by someone. To model a monetary economy, one thus needs to imagine that trading is decentralized in some way. My preference is to do this in a way that does minimal violence to the original, real theory that is being modified, so as not to discard altogether the theory's considerable ability to account for important real observations. ... By postulating an individual with specific preferences over cash and credit goods, and by being specific as well about the timing with which information gets revealed, we can derive all of classical monetary theory by just thinking through the margins on which an agent operates in this world of centralized/ decentralized markets. ... Everything that is valid in the traditional quantity theory of money can be extracted from these two marginal conditions, as can much that is new.

Models of Business Cycles, 1987, pp. 76, 78, 88

Lucas combined components of Arrow-Debreu and search-island models while insisting on preserving versions of most complete markets Euler equations for consumption, labor supplies, and asset prices. Examples of combined models include Lucas and Prescott (1974) and Alvarez and Veracierto (1999, 2012) island search models as well as Lucas and Stokey (1987) cash-in-advance models. Each of those structures incorporates a version of Samuelson's neoclassical synthesis.[34]

Implementations

It is challenging to motivate governments to adhere to an optimal plan.[35] Here Lucas made important contributions. Examples include (1) Lucas and Stokey's (1983) implementation of a Ramsey plan by requiring governments to service carefully designed continuation debt maturity structures, and (2) Atkeson and Lucas's (1992) implementations of incentive compatible social insurance arrangements that feature barriers to entry, contract exclusivities, and pecking orders among insurance contracts.

---

[34]By making social insurance and redistribution be important parts of a central bank's mandate, some heterogeneous agent New Keynesian (HANK) models subvert Samuelson's neoclassical synthesis. Lucas criticized that aspect of HANK models when he discussed an early version of Bhandari et al. (2021) at a 2012 Minneapolis Fed conference. In that model, social insurance motives make Taylor rules become much more aggressive than they are in representative agent New Keynesian models..

[35]This is the message of Kydland and Prescott (1977) and Calvo (1978).

15 Concluding Remarks

Although I celebrate them here, not everybody likes the equilibrium Markov processes that Lucas promoted. Summers (1991) did not. He asserted that "progress is unlikely as long as macroeconomists require the armor of a stochastic pseudo-world before doing battle with the real one." It puzzles me why some technically sophisticated economists also didn't like the way Lucas practiced macroeconomics.

> Deep down I really wish I could believe that Lucas ... is right, because the one thing I know how to do well is equilibrium economics. The trouble is I feel so embarrassed at saying things that I know are not true.
>
> It is plain as the nose on my face that the labor market and many markets for produced goods do not clear in any meaningful sense.
>
> Solow (1978)

Solow was responding to the following statements:

> In recent years, the meaning of the term equilibrium has changed so dramatically that a theorist of the 1930s would not recognize it. An economy following a multivariate stochastic process is now routinely described as being in equilibrium, by which is meant nothing more than that at each point in time, postulates (a) [markets clear] and (b) [agents act in their self interest] are satisfied. This development, which stemmed mainly from work by Arrow (1964) and Debreu (1959), implies that simply to look at any economic time series and conclude that it is a disequilibrium phenomenon is a meaningless observation. Indeed, a more likely conjecture, on the basis of recent work by Sonnenschein (1973), is that the general hypothesis that a collection of time series describes an economy in competitive equilibrium is without content.
>
> Lucas and Sargent (1978, p. 304)

Equilibrium Markov processes acquire content only by looking at more data or by imposing more restrictions on prices and quantities than Sonnenschein (1973) had.[36] For over 35 years, Stokey et al. (1989) has been our handbook for constructing stationary and ergodic equilibrium Markov processes amenable to econometric implementations. That book tells us

---

[36]Brown and Matzkin (1996), Chiappori et al. (2004), and Kübler and Polemarchakis (2024) are just some of many important papers that have responded to Sonnenschein (1973) by describing data sets and specifications of primitives of general equilibrium models that restrict data on prices, quantities, and endowments.

how to economize on free parameters and expand data sets enough to make an equilibrium Markov process become econometrically restrictive.

Before being too hard on Solow (1978), we should remember how Lucas (1987, Sec. III), Lucas (2003) tempered his initial enthusiasm about rational expectations econometrics after likelihood ratio and Chi-square specification test statistics rejected some of his favorite equilibrium Markov models. Such adverse findings pushed Lucas into the wilderness of calibration. Informal Kydland and Prescott (1982, 1996) techniques that Lucas came to favor resembled the informal parameter selection methods that Solow had used.

Despite Lucas's misgivings, the coherence between economic theory and econometric practice that rational expectations econometrics brings remains attractive today. Rational expectations econometrics flourishes with the Central Banks and Treasuries at which the critique in Lucas (1976) was aimed. Herbst and Schorfheide (2016). Dynare manuals are bibles at many economics departments and research organizations. New applications of deep neural nets to solving master equations extend the types of models and data sets for which rational expectations econometrics is practical.[37] For example, by treating parameters as additional state variables, Friedl et al. (2023) approximate a manifold of master equations. They compute a "look up table" that represents a manifold of equilibrium Markov process swept out as a parameter vector is varied. This is what Lucas (1976) and Lucas (1987, Sect. I) wanted quantitative macroeconomists to present to government policy designers. Recent advances like these make it possible to practice rational expectations econometrics today with much better machinery than we had in the 1970s.

Kreps and Porteus (1978, 1979), Lucas and Stokey (1984), Epstein and Zin (1991), Gul and Pesendorfer (2001, 2004) broadened the range of behaviors that Markov decision problems can incorporate. These can be cast as what Bertsekas (2022) calls "abstract dynamic programs." Abstract dynamic programming exploits connections between conditions for existence of solutions of Bellman equations and the stability of controlled Markov processes. It extends our Stokey et al. (1989) toolkit and the phenomena that we can use equilibrium Markov models to interpret.[38]

Section 2 described how Milton Friedman had influenced Lucas's choices about topics to study in mathematics, statistics, and economics. Backward induction instructed me to describe Milton Friedman's technical tools and the questions that he studied. I could also have told how Irving Fisher influenced Milton Friedman's tools and questions. That

---

[37] "A Markov process that solves a master equation" is today a synonym for "an equilibrium Markov model". In HANK models and also models with non-redundant long term assets and some types of adjustment costs, this synonym should be amended to "a Markov process that solves a master equation coupled with some auxiliary equations." See section 5.

[38] Sargent and Stachurski (2024, ch. 9) provides an elementary account confined to finite state spaces.

would have strengthened my message about how their mathematical tools constrained and empowered Friedman and Lucas.[39] Lucas got farther than Friedman partly because he knew more math and probability theory. Lucas confronted more constraints and had better guides.

References

Alvarez, Fernando and Urban J Jermann. 2004. Using asset prices to measure the cost of business cycles. *Journal of Political economy* 112 (6):1223–1256.

Alvarez, Fernando and Marcelo Veracierto. 1999. Labor-market policies in an equilibrium search model. *NBER macroeconomics annual* 14:265–304.

———. 2012. Fixed-term employment contracts in an equilibrium search model. *Journal of Economic Theory* 147 (5):1725–1753.

Atkeson, Andrew and Robert E Lucas. 1992. On efficient distribution with private information. *The Review of Economic Studies* 59 (3):427–453.

Bertsekas, Dimitri. 2022. *Abstract dynamic programming*. Athena Scientific.

Bhandari, Anmol, David Evans, Mikhail Golosov, and Thomas J Sargent. 2021. Inequality, Business Cycles, and Monetary-Fiscal Policy. *Econometrica* 89 (6):2559–2599.

Bilal, Adrien. 2023. Solving heterogeneous agent models with the master equation. National Bureau of Economic Research.

Bray, Margaret and David M Kreps. 1987. Rational learning and rational expectations. In *Arrow and the ascent of modern economic theory*, edited by George Feiwell, 597–625. Springer.

Brown, Donald J and Rosa L Matzkin. 1996. Testable restrictions on the equilibrium manifold. *Econometrica: Journal of the Econometric Society* 1249–1262.

Calvo, Guillermo A. 1978. On the Time Consistency of Optimal Policy in a Monetary Economy. *Econometrica* 46 (6):1411–1428.

---

[39]As in some modern labor and macro growth models, during the early part of his career as a student and practitioner of economics, Lucas spent many hours and much effort mastering ideas that Friedman and other great economists of the generation before him had used productively. A macro growth theorist might describe this as the "imitation" phase of Lucas's growth process. What Lucas had learned in that phase constrained and empowered his achievements during the subsequent "innovation" phase that we celebrate here.

———. 1983. Staggered prices in a utility-maximizing framework. *Journal of monetary Economics* 12 (3):383–398.

Chiappori, P-A, Ivar Ekeland, Felix Kübler, and Herakles M Polemarchakis. 2004. Testable implications of general equilibrium theory: a differentiable approach. *Journal of Mathematical Economics* 40 (1-2):105–119.

Csiszár, Imre and Frantisek Matus. 2003. Information projections revisited. *IEEE Transactions on Information Theory* 49 (6):1474–1490.

Epstein, Larry G and Stanley E Zin. 1991. Substitution, risk aversion, and the temporal behavior of consumption and asset returns: An empirical analysis. *Journal of political Economy* 99 (2):263–286.

Friedl, Aleksandra, Felix Kübler, Simon Scheidegger, and Takafumi Usui. 2023. Deep uncertainty quantification: with an application to integrated assessment models. Tech. rep., Working Paper University of Lausanne.

Friedman, Milton and L.J. Savage. 1947. Planning Experiments Seeking Maxima. In *Selected Techniques of Statistical Analysis for Scientific and Industrial Research, and Production and Management Engineering*, edited by Millard W. Hartay Churchill Eisenhart and W. Allen Wallis, 363–372. New York and London: McGraw-Hill.

Fudenberg, Drew and David K Levine. 2009. Self-confirming equilibrium and the Lucas critique. *Journal of Economic Theory* 144 (6):2354–2371.

Gallant, A Ronald and George Tauchen. 1996. Which moments to match? *Econometric theory* 12 (4):657–681.

Golosov, Mikhail and Robert E Lucas. 2007. Menu costs and Phillips curves. *Journal of Political Economy* 115 (2):171–199.

Gu, Zhouzhou, Mathieu Lauriere, Sebastian Merkel, and Jonathan Payne. 2024. Global Solutions to Master Equations for Continuous Time Heterogeneous Agent Macroeconomic Models. Princeton University.

Gul, Faruk and Wolfgang Pesendorfer. 2001. Temptation and self-control. *Econometrica* 69 (6):1403–1435.

———. 2004. Self-control and the theory of consumption. *Econometrica* 72 (1):119–158.

Hansen, Gary D. 1985. Indivisible labor and the business cycle. *Journal of monetary Economics* 16 (3):309–327.

Hansen, Lars Peter. 2014. Nobel lecture: Uncertainty outside and inside economic models. *Journal of Political Economy* 122 (5):945–987.

Hansen, Lars Peter and Thomas J. Sargent. 1993. Seasonality and approximation errors in rational expectations models. *Journal of Econometrics* 55 (1-2):21–55.

———. 2013. *Recursive Linear Models of Dynamic Economics*. Princeton, New Jersey: Princeton University Press.

———. 2022. Risk, ambiguity, and misspecification: Decision theory, robust control, and statistics. *Journal of Applied Econometrics* .

Hansen, Lars Peter and Kenneth J Singleton. 1982. Generalized instrumental variables estimation of nonlinear rational expectations models. *Econometrica* 1269–1286.

———. 1983. Stochastic consumption, risk aversion, and the temporal behavior of asset returns. *Journal of political economy* 91 (2):249–265.

———. 1991. Computing semi-parametric efficiency bounds for linear time series models. *Nonparametric and Semiparametric Methods in Econometrics and Statistics* 387–412.

Hansen, Lars Peter, Thomas J Sargent, and Thomas D Tallarini Jr. 1999. Robust permanent income and pricing. *The Review of Economic Studies* 66 (4):873–907.

Hansen, Lars Peter, John C Heaton, and Nan Li. 2008. Consumption strikes back? Measuring long-run risk. *Journal of Political economy* 116 (2):260–302.

Herbst, Edward P and Frank Schorfheide. 2016. *Bayesian estimation of DSGE models*. Princeton University Press.

Hood, William C and T C Koopmans. 1953. *Studies in econometric method*. Wiley.

Hotelling, Harold. 1941. The Experimental Determination of the Maximum of a Function. *Annals of Mathematical Statistics* 12:20–45.

Hurwicz, Leonid. 1966. On the structural form of interdependent systems. In *Studies in Logic and the Foundations of Mathematics*, vol. 44, 232–239. Elsevier.

Koopmans, Tjalling C. 1950. *Statistical inference in dynamic economic models*. New York: Wiley.

Kreps, David M and Evan L Porteus. 1978. Temporal resolution of uncertainty and dynamic choice theory. *Econometrica: journal of the Econometric Society* 185–200.

———. 1979. Dynamic choice theory and dynamic programming. *Econometrica: Journal of the Econometric Society* 91–100.

Kübler, Felix and Heraklis M Polemarchakis. 2024. Identification in general equilibrium .

Kuh, Edwin and John R Meyer. 1957. How extraneous are extraneous estimates? *The Review of economics and Statistics* 39 (4):380–393.

Kydland, Finn E. and Edward C. Prescott. 1977. Rules Rather than Discretion: The Inconsistency of Optimal Plans. *Journal of Political Economy* 106 (5):867–896.

Kydland, Finn E and Edward C Prescott. 1982. Time to build and aggregate fluctuations. *Econometrica* 1345–1370.

———. 1996. The computational experiment: An econometric tool. *Journal of economic perspectives* 10 (1):69–85.

Lucas, Robert E Jr. 1972. Expectations and the Neutrality of Money. *Journal of economic theory* 4 (2):103–124.

———. 1976. Econometric policy evaluation: A critique. *Carnegie-Rochester Conference Series on Public Policy* 1 (1):19–46.

———. 1978. Asset prices in an exchange economy. *Econometrica* 1429–1445.

———. 1986. Adaptive behavior and economic theory. *journal of Business* S401–S426.

———. 1987. *Models of business cycles*, vol. 26. Oxford Blackwell.

———. 2003. Macroeconomic Priorities. *American Economic Review* 93 (1):1–14.

Lucas, Robert E Jr. and Edward C Prescott. 1971. Investment under uncertainty. *Econometrica: Journal of the Econometric Society* 659–681.

———. 1974. Equilibrium search and unemployment. *Journal of Economic Theory* 7 (2):188–209.

Lucas, Robert E Jr. and Leonard A Rapping. 1969. Real wages, employment, and inflation. *Journal of political economy* 77 (5):721–754.

Lucas, Robert E Jr. and Thomas J Sargent. 1978. After Keynesian Economics. In *After the Phillips Curve: Persistence of High Inflation and High Unemployment*, 47–72. Boston, MA: Federal Reserve Bank of Boston. Proceedings of a Conference Held at Edgartown, Massachusetts, June 1978.

Lucas, Robert E Jr and Thomas J Sargent. 1981. *Rational expectations and econometric practice.* Minneapolis, Minnesota: University of Minnesota Press.

Lucas, Robert E Jr. and Nancy L Stokey. 1983. Optimal Fiscal and Monetary Policy in an Economy without Capital. *Journal of monetary Economics* 12 (3):55–93.

———. 1984. Optimal growth with many consumers. *Journal of Economic Theory* 32 (1):139–171.

———. 1987. Money and Interest in a Cash-in-Advance Economy. *Econometrica* 55 (3):491–513.

Manuelli, Rodolfo and Thomas J Sargent. 1988. Models of business cycles: A review essay. *Journal of Monetary Economics* 22 (3):523–542.

Marschak, Jacob. 1950. Statistical inference in economics: an introduction. In *Statistical inference in dynamic economic models*, vol. 10 of *Cowles Commission for Research in Economics*, edited by T C Koopmans, 1–50. New York: John Wiley & Sons, Inc.

———. 1953. Economic Measurement for Policy and Prediction. In *Studies in econometric method*, edited by William C Hood and T C Koopmans, 1–26. Wiley.

Meiselman, David. 1962. *The Term Structure of Interest Rates.* Englewood Cliffs, N. J.: Prentice-Hall.

Meyn, Sean. 2022. *Control systems and reinforcement learning.* Cambridge University Press.

Muth, John F. 1960. Optimal properties of exponentially weighted forecasts. *Journal of the american statistical association* 55 (290):299–306.

———. 1961. Rational expectations and the theory of price movements. *Econometrica: Journal of the Econometric Society* 315–335.

Nielsen, Frank. 2018. What is an information projection. *Notices of the AMS* 65 (3):321–324.

Prescott, Edward C. 2002. Prosperity and depression. *American Economic Review* 92 (2):1–15.

Rotemberg, Julio J. 1982. Sticky prices in the United States. *Journal of political economy* 90 (6):1187–1211.

Sargent, Thomas J. 1981. Interpreting economic time series. *Journal of Political Economy* 89 (2):213–248.

———. 1982. Beyond demand and supply curves in macroeconomics. *The American Economic Review* 72 (2):382–389.

———. 1996. Expectations and the nonneutrality of Lucas. *Journal of Monetary Economics* 37 (3):535–548.

———. 1999. *The conquest of American inflation*. Princeton University Press.

———. 2008. Evolution and intelligent design. *American Economic Review* 98 (1):5–37.

———. 2015. Robert E. Lucas Jr.'s Collected Papers on Monetary Theory. *Journal of Economic Literature* 53 (1):43–64.

———. 2022. Learning from Lucas. *Journal of Economic Methodology* 29 (1):17–29.

———. 2024. Critique and consequence. *Journal of Monetary Economics* 141:2–13.

Sargent, Thomas J and Christopher A Sims. 1977. Business cycle modeling without pretending to have too much a priori economic theory. In *New methods in business cycle research*, vol. 1, 145–168. Federal Reserve Bank of Minneapolis.

Sargent, Thomas J. and John Stachurski. 2024. *Dynamic Programming, Vol I*. Cambridge, England: Cambridge University Press.

Sargent, Thomas J and Neil Wallace. 1976. Rational expectations and the theory of economic policy. *Journal of Monetary economics* 2 (2):169–183.

Scarf, Herbert E. 1982. The computation of equilibrium prices: an exposition. *Handbook of mathematical economics* 2:1007–1061.

Sims, Christopher A. 1980. Macroeconomics and reality. *Econometrica* 1–48.

———. 1982. Policy analysis with econometric models. *Brookings papers on economic activity* 1982 (1):107–164.

———. 1993. Rational expectations modeling with seasonally adjusted data. *Journal of Econometrics* 55 (1-2):9–19.

Solow, Robert M. 1978. Summary and Evaluation. In *After the Phillips Curve: Persistence of High Inflation and High Unemployment*, 203–209. Boston, MA: Federal Reserve Bank of Boston. Proceedings of a Conference Held at Edgartown, Massachusetts, June 1978.

Sonnenschein, Hugo. 1973. Do Walras' identity and continuity characterize the class of community excess demand functions? *Journal of economic theory* 6 (4):345–354.

Stokey, Nancy L. 1989. Reputation and time consistency. *The American Economic Review* 134–139.

———. 1991. Credible public policy. *Journal of Economic Dynamics and Control* 15 (4):627–656.

Stokey, Nancy L, Robert E Lucas, and Edward C Prescott. 1989. *Recursive Methods in Economic Dynamics*. Harvard University Press.

Summers, Lawrence H. 1991. The Scientific Illusion in Empirical Macroeconomics. *Scandinavian Journal of Economics* 93 (2):129–48.

Tallarini, Thomas D. 2000. Risk-sensitive real business cycles. *Journal of Monetary Economics* 45 (3):507–532.

Wallace, Neil. 1967. The term structure of interest rates and the maturity composition of the federal debt. *The Journal of Finance* 22 (2):301–312.

Weinberg, Steven. 2015. *To explain the world: The discovery of modern science*. Penguin UK.

———. 2018. *Third Thoughts*. Harvard University Press.