Managing Public Portfolios*

Léo Aparisi de Lannoy, University of Chicago
Anmol Bhandari, University of Minnesota
David Evans, University of Oregon
Mikhail Golosov, University of Chicago
Thomas Sargent, NYU

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Abstract

We study optimal public portfolios in a class of macro-finance models that includes widely-used specifications of households' risk and liquidity preferences, market structures for financial assets, and trading frictions. An optimal portfolio hedges fluctuations in interest rates, primary surpluses, and income inequalities. We express an optimal portfolio in terms of statistics that are functions only of macro and financial market data. An application to U.S. data shows that hedging interest rate risk plays a dominant role in shaping an optimal maturity structure of U.S. government debt.

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1 Introduction

We characterize main forces that shape optimal public portfolios of financial assets in a large class of stochastic economies in which a government uses distortionary taxes to raise revenues and finance expenditures. We provide formulas for optimal portfolios in terms of a small number of statistics that are functions of observables. For U.S. data, an optimal portfolio's bond shares decrease approximately exponentially with increases in their maturity.

We begin by studying an environment that shares many features with the Ramsey literature on optimal taxation and debt management. We consider an economy with a representative, infinitely lived household that derives utility from consumption and leisure. We abstract from income effects on labor supplies but allow various attitudes about risk, model ambiguity, and intertemporal substitution. A benevolent government uses distortionary taxes to finance exogenous public expenditures. Households and the government trade an exogenous set of financial assets. Our benchmark model is a small open economy in which large foreign investors trade these financial and assets determine their prices.

We develop a new approach to study optimal government policies that builds on two key ideas. The first idea, inspired by the "sufficient statistics" approach in public finance, is to study consequences of perturbing government policies along histories of a competitive equilibrium allocation. Welfare impacts of such policies can be isolated using the envelope theorem. The second idea is to use "small noise" expansions to simplify formulas, making them analytically tractable and convenient for empirical applications. By combining these two ideas, we derive explicit formulas that characterize an optimal public portfolio in terms of population moments with sample counterparts in macro and financial market data.

In our benchmark economy, fluctuations in government spending directly affect primary surpluses. Fluctuations in future interest rates make costs of rolling over debt obligations uncertain. Costs of distortionary taxation motivate the government to structure its portfolio to make the return on the portfolio offset these fluctuations, thereby attenuating variations in tax rates. Our formulas express an optimal government portfolio as a sum of two terms. One captures hedging of fluctuations in primary surpluses. Another captures hedging of fluctuations in risk-free interest rates on zero coupon bonds of various maturities. These terms depend on determinants with empirical counterparts, such as covariances of excess returns of each security with excess returns of other securities, government expenditures, and interest rates on bonds of various maturities. We also show that, up to orders of approximation under consideration here, these variables are independent of a government's portfolio choices, so that even if actual portfolio choices are suboptimal, we can construct these statistics from the data and plug them into our formula for optimal public portfolios.

Our formulas bring significant insights. First, our formula for an optimal portfolio includes no terms that summarize risk premia or risk aversion, thereby indicating a key difference between classical portfolio theory for private investors (e.g., Samuelson (1970), Merton (1971)) and our prescriptions for public portfolios. This difference casts doubt on the practice of many treasury departments and finance ministries of exploiting differences in borrowing costs across bonds of different maturities to reduce debt financing costs. Our model asserts that this practice is suboptimal because households can exploit those differences themselves without bearing deadweight losses from taxation.

Although our formulas apply to all securities that a government can trade, they offer specific additional insights when those securities are bonds of various maturities. The return on a bond of maturity k co-moves mechanically with a k-period interest rate, implying that the government can use bonds to hedge interest rate risk perfectly. Such a portfolio embodies a simple "maturity-matching" principle that prescribes that the quantity of bonds of maturity k should be proportional to expected primary surpluses k periods ahead.

We apply our approach to determinants of optimal public portfolios. We examine roles of uncertainty about tax revenues, additional liquidity services that government bonds may provide, and household heterogeneities that include inabilities of some households to participate in asset markets. We derive statistics that express influences of these determinants and that can be estimated from data. We also move beyond a small open economy and consider several models of asset price determination, including preferred habitat models. Our analysis shows that when the government faces downward-sloping demand curves for its debt, it tilts its portfolio to avoid circumstances that call for large rebalancings.

We apply our framework to prescribe an optimal composition of U.S. debt. We use data on returns of U.S. government, taxes, and primary surpluses to construct each component of an optimal portfolio. As a starting point, we restrict our attention to bonds with up to 30 years maturity. We find debt portfolio shares decline approximately geometrically with maturity. The optimal portfolio has a shape qualitatively similar to the actual U.S. portfolio but with longer duration.

Interest rate risk contributes most to the shape of the optimal portfolio. Empirically, covariances of bond returns with government expenditures and revenues are small, implying small scope for using these bonds to hedge such risks. Scope of trading bonds to hedge other risks, such as inequality risks, also appears to be small. If the demand for U.S. bonds were perfectly elastic, an optimal portfolio of bonds would be very similar to one that adheres to the maturity matching principle. This portfolio would have a duration of about 9.6 years; much longer than the duration of the U.S. debt which is about 5 years. Such a portfolio, however, requires substantial reissuances of 30-year bonds each year, which is costly when demand for those bonds is downward sloping. Using demand elasticities gathered from the literature, the optimal portfolio tilts toward shorter maturities but not as much the U.S. portfolio.

Our analysis suggests that issuing bonds with maturities beyond 30 years has several benefits. First, it improves hedging of interest rate risk over longer time horizons. Second, it reduces the fraction of debt that needs to be reissued each period, mitigating adverse price impact. We find that the portfolio of bonds with maturities up to 50 years is very close to the one obtained under the maturity matching principle.

In the final part of the paper, we study connections between our findings and the widely-cited results in the Ramsey literature on optimal debt management. A striking finding in that literature highlighted by Buera and Nicolini (2004) is that the optimal portfolio of bonds in a calibrated neoclassical model is extreme: holdings of bonds with specific maturities can equal hundreds or thousands times annual GDP, and portfolio shares of bonds with similar maturities often take opposite signs. Using a calibrated economy similar to Buera and Nicolini's (2004), we construct two portfolios: the exact optimal bond portfolio prescribed by formulas of Angeletos (2002), and the approximate portfolio implied by our formulas. The two portfolios are very similar; both exhibit the extreme pattern noted by Buera and Nicolini (2004). We use our formulas to investigate sources of that pattern. We find that simulated data from the calibrated economy have variances of bond excess returns that are substantially lower, correlations of bond returns with macroeconomic variables that are substantially higher, and often of different signs than their empirical counterparts. We show that augmenting this model with discount factor shocks can bring these statistics closer to their empirical counterparts, at which point the optimal portfolio becomes similar to the one we constructed via our sufficient statistics for U.S. data.

Related Literature Our paper is related to an extensive Ramsey literature on the optimal composition of government debt, such as Lucas and Stokey (1983), Bohn (1990), Zhu (1992), Chari et al. (1994), Angeletos (2002), Buera and Nicolini (2004), Farhi (2010), Faraglia et al. (2018), Lustig et al. (2008), Bhandari et al. (2017a). Those authors used closed economy neoclassical growth models to characterize optimal public portfolios. However, those models don't fit empirical relationships among asset prices, asset supplies, and macroeconomic variables, key objects that determine how well alternative securities hedge risks. We overcome that deficiency by assuming more general specifications of preferences and asset demands that includes multiple forces that can account for the observed asset pricing behavior.

Our paper builds on a literature in finance that focuses on asset price determination, such as Ai and Bansal (2018), Bansal and Yaron (2004), Albuquerque et al. (2016), Krishnamurthy and Vissing-Jorgensen (2012), Greenwood and Vayanos (2014). Those authors modified the standard neoclassical environment in ways designed to make it do a better job of fitting asset prices. By setting up a framework broad enough to include all of these structures and by deriving obtaining expressions for optimal portfolios that depend on only a small number of statistics that are functions of aggregates and asset returns, we sidestep taking a stand on details of those structures.

We obtain formulas for optimal government portfolio that are related to the formulas for private portfolios that appear in classic portfolio theory contributions of Samuelson (1970), Merton (1969, 1971), Campbell and Viceira (1999, 2001), and Viceira (2001). Although individual investors in that theory and the government in our model both choose portfolios to hedge their risks, we show that substantially different forces determine portfolio compositions in the two settings.

Our findings are also related to some recent work by Debortoli et al. (2017, 2022). For a deterministic version of Lucas and Stokey (1983), they find that issuing a consol aligns incentives across successive governments and eliminates time inconsistency. We study a timing protocol in which government commits to a plan but nevertheless find that in a stationary world the optimal portfolio is well approximated by a (growth-adjusted) consol—a security that implements the maturity matching principle and eliminates needs to rollover or rebalance the portfolio.[1]

In recent papers, Jiang et al. (2019, 2020) document a number of puzzling facts about market values of total debt and primary surpluses in the U.S. These facts are puzzling when debt valuation is viewed through the lens of an arbitrage-free and frictionless asset pricing framework. Our setting departs from such a framework by incorporating market segmentation as well as a broad notion of liquidity services that U.S debts provide. However, in this paper we focus on how the market value of government debt is optimally allocated across various securities, and not on determinants of the level itself.

Methodologically, this paper relates to two strands of literature. We borrow our approach

[1] Our work is also related to Bigio et al. (2023) who study the optimal composition of government portfolios of bonds of different maturities. They mostly abstract from the interest rate risk and primary surplus risk that we emphasize, and focus on understanding how price impacts from debt issuance affect portfolio composition. Because they impose an exogenous cap on the maturities, the government in their setup wants to rebalance its portfolio even in the absence of all risks.

of using a small number of statistics to characterize an optimal government portfolio from a recent applied public finance literature, notably Saez (2001) and Chetty (2009). That literature typically focuses on settings in which a government faces no risk. When applied to our problem, that approach yields no clear and transparent insights. We make progress by deploying small-noise approximations. Small noise approximations have been used frequently both in finance (e.g., Samuelson (1970), Devereux and Sutherland (2011)) and computational economics (e.g., Guu and Judd (2001), Schmitt-Grohe and Uribe (2004), Bhandari et al. (2021)). The particular class of expansions that we use does not require us to assume stationarity or to ignore heteroskedasticity. That makes it particularly suitable to study portfolio problems in dynamic stochastic economies.

Outline The rest of the paper is organized as follows. To demonstrate our approach and convey main ideas, Sections 2 and 3 start with our simplest setting ("benchmark economy"). In Section 4, we consider several extensions of the benchmark economy. In Section 5, we apply our theory to infer an optimal portfolio for the U.S. and compare it to the observed portfolio. In Section 6, we contrast our findings to neoclassical settings studied in Angeletos (2002) and Buera and Nicolini (2004). Section 7 concludes. Proofs of all statements in the main text are in the online appendix.

2 A benchmark economy

We consider a discrete time infinite horizon economy. Uncertainty is described by a stochastic process $\{s_t\}_t$, where $s_t \subset \mathbb{R}^N$ for some $N \leq \infty$. We use s^t to denote the history of shocks $(s_0, ..., s_t)$ and $s^{t+k} \succeq s^t$ to denote histories s^{t+k} in which first $t+1$ elements are equal to s^t. $\Pr\left(s^{t+k}\right)$ and $\Pr\left(s^{t+k}|s^t\right)$ denote probabilities of s^{t+k} conditional on information in periods 0 and s^t, respectively.

We use z_t to denote the vector of all exogenous shocks that affect agents. These shocks are functions of s_t, i.e., $z_t = z_t\left(s_t\right)$, so the stochastic process for s_t determines z_t. For technical reasons, we assume that z_t is bounded. The conditional expectation in history s^t is denoted by $\mathbb{E}_{s^t}$, or $\mathbb{E}_t$ if the specific history s^t is clear from the context.

The economy is inhabited by three groups of agents: households, the government, and foreign investors. All agents trade a given set of securities. A security i is characterized by an exogenous stochastic stream of dividends $\{D_t^i\}_t$. We use Q_t^i to denote the price of security i and $R_{t+1}^i = (Q_{t+1}^i + D_{t+1}^i)/Q_t^i$ to denote its holding period return from t to $t+1$. A risk-free bond is a security that pays one unit of dividend next period. Let Q_t^{rf} be the price of a risk-free bond and $R^{rf}_{t+1} = 1/Q^{rf}_t$ be the risk-free interest. The excess return of security i is $r^i_{t+1} = R^i_{t+1} - R^{rf}_{t+1}$.

In the benchmark economy, there is measure one of identical, infinitely-lived households who supply effort to earn income Y_t, pay proportional taxes τ_t, and allocate after-tax income between consumption C_t and investing into a portfolio of securities. We use $\left\{b^i_t\right\}_i$ to denote date t market values of household investments. The household's problem is

$$V = \max_{\{C_t, Y_t, \{b^i_t\}_i\}_t} \mathbb{E}_0 \sum_t \beta^t u\left(C_t - v\left(Y_t\right)\right) \tag{1}$$

where maximization is subject to

$$C_{t+1} + \sum_i b^i_{t+1} = \left(1 - \tau_{t+1}\right) Y_{t+1} + \sum_i R^i_{t+1} b^i_t, \tag{2}$$

given an initial portfolio $\{b^i_{-1}\}_i$ and natural borrowing limits. Here β is a discount factor, functions u and v are strictly increasing, twice differentiable, and u and $-v$ are strictly concave.

The government sets tax rates $\{\tau_t\}_t$ and trades portfolios of securities to finance exogenous stochastic expenditures $\{G_t\}_t$. We use $T_t = \tau_t Y_t$ to denote tax revenues. For all dates $t \geq -1$, the period-by-period government budget constraint is

$$T_{t+1} - G_{t+1} + \sum_i B^i_{t+1} = \sum_i R^i_{t+1} B^i_t,$$

where a positive value of B^i_t indicates the market value of government's liability in security i. We adopt this sign convention so that positive values indicate values of outstanding liabilities.

Government budget constraints play an important roles in our analysis of optimal public portfolios. It will be helpful to re-write using portfolio shares. Let $B_t := \sum_i B^i_t$ be the total market value of the government portfolio and $\omega^i_t = B^i_t / B_t$ be the portfolio share of security i. We refer to B_t as the government's debt level, and to vector $\omega_t = \{\omega^i_t\}_{i \neq rf}$ as the public portfolio. The portfolio share of the risk-free bond is $\omega^{rf}_t = 1 - \sum_{i \neq rf} \omega^i_t$.

Using this notation, the period-by-period government budget constraint is

$$T_{t+1} - G_{t+1} + B_{t+1} = (R^{rf}_{t+1} + \sum_{i \neq rf} \omega^i_t r^i_{t+1}) B_t. \tag{3}$$

Let $\mathcal{R}_{t+1} := (R^{rf}_{t+1} + \sum_{i \neq rf} \omega^i_t r^i_{t+1})$ be the realized return on the public portfolio from date t to $t+1$, and $\mathcal{Q}_{t,t+k+1} = (\mathcal{R}_{t+1} \times ... \times \mathcal{R}_{t+k+1})^{-1}$ be the inverse of the accumulated return on the public portfolio between periods t and $t+k+1$, with convention that $\mathcal{Q}_{t,t} = 1$. Summing

(3) forward from date t gives[2]

$$\mathbb{E}_{t+1}\sum_{k=1}^{\infty}\mathcal{Q}_{t+1,t+k}\left(T_{t+k}-G_{t+k}\right)=(R^{rf}_{t+1}+\sum_{i\neq rf}\omega^i_t r^i_{t+1})B_t. \tag{4}$$

A third group of agents consists of foreign investors. Our benchmark model is a small open economy, in which the foreign investors are wealthy and that their (exogenous) stochastic discount factor determines prices of securities. Specifically, we assume that there exists a strictly positive stochastic process $\{S_t\}_t$ that prices all assets:

$$S_t Q^i_t = \mathbb{E}_t S_{t+1}\left(Q^i_{t+1}+D^i_{t+1}\right) \text{ for all } i,t. \tag{5}$$

A government policy is a triple of stochastic processes $\{B_t,\omega_t,\tau_t\}_t$. A competitive equilibrium consists of a government policy, exogenous stochastic processes $\{G_t,S_t\}_t$, and initial conditions $\left\{b^i_{-1},B^i_{-1}\right\}_i$ such that asset prices satisfy (5), the allocation of goods and labor to consumers satisfy (1), and the government policy satisfies budget constraints (3).

Our benchmark economy is closely related to a large Ramsey literature that studies optimal debt management (e.g., Lucas and Stokey (1983), Angeletos (2002), Buera and Nicolini (2004), Farhi (2010), Faraglia et al. (2018)). Like those papers, we focus on interactions between a government that uses distortionary taxes to finance exogenous expenditures and households who supply labor. We make two departures from those papers. First, we temporarily focus on a small open economy because that allows us to develop many of our results most simply. We extend our analysis beyond open economies in Section 4.5. Second, we assume that there are no income effects on labor supplies. This assumption helps us to derive simple formulas describing optimal government portfolios. We show in Section 6 that these formulas to provide excellent approximations to optimal public portfolios in a calibrated neoclassical model with preferences that do have income effects on labor supplies.

3 A class of perturbations

We are interested in characterizing the properties of an optimal portfolio. As a starting point, we first characterize the optimal *Ramsey* portfolio, which is a part of government policy $\{B_t,\omega_t,\tau_t\}_t$ that maximize household welfare (1) across all competitive equilibria. We then discuss a more general notion of an optimal portfolio without restricting government policies to be Ramsey optimal.

[2]Throughout the paper, we will be focusing on equilibria in which $\{\mathcal{Q}_{t+1,t+k}\}_{t,k}$ decays sufficiently fast so that terms such as $\lim_{s\to\infty}\mathbb{E}_{t+1}\mathcal{Q}_{t+1,t+s}\|\left(T_{t+s}-G_{t+s}\right)\|=0$ for all t.

We start with a household's maximization problem. In a competitive equilibrium, a household's first-order necessary conditions for optimality with respect to labor supply are

$$v'(Y_t) = 1 - \tau_t \text{ for all } t; \tag{6}$$

and with respect to asset holding are

$$1 = \mathbb{E}_t \frac{\beta M_{t+1}}{M_t} R^{rf}_{t+1}, \quad 0 = \mathbb{E}_t \frac{\beta M_{t+1}}{M_t} r^i_{t+1} \text{ for all } i, t, \tag{7}$$

where $\beta^t M_t$ is a Lagrange multiplier on date t budget constraint (2).

A standard way to study Ramsey optimal government policies (see, e.g., Lucas and Stokey (1983), Chari and Kehoe (1999)) expresses M_t in terms of the marginal utility of consumption of households, substitutes optimality conditions (6) and (7) into the budget constraint (2) to form the so-called implementability constraints, and then analyze a problem in which the planner maximizes household utility by choosing allocations subject to those implementability constraints. This approach can be difficult to use. Apart from a few special cases, it yields few insights about the forces that determine an optimal public portfolio. So instead papers in the Ramsey literature often use numerical methods to find optimal allocations and portfolios. Implementing those numerical methods can be challenging when agents trade even a moderate number of securities. So authors often simplify their environments either by assuming that markets are complete (Lucas and Stokey (1983), Angeletos (2002), Buera and Nicolini (2004)) or by allowing only a small numbers of securities (Farhi (2010), Lustig et al. (2008), Faraglia et al. (2018)).

In this section we develop an alternative approach that is transparent, easy to use, and does not require simplifying assumptions about securities that agents trade. Take a competitive equilibrium associated with some, not necessarily optimal, government policy. Next consider welfare effects of two classes of perturbations of government policies at a given history s^t. In the first class of perturbations, we consider the effect of increasing the market value of government debt $B_t(s^t)$ an infinitesimal amount ε, while keeping portfolios in *all* histories and market values of debts in all histories *other* than s^t unchanged. In the second class of perturbations, we increase in $B^j_t(s^t)$ for some security j by ε and reduce $B^{rf}_t(s^t)$ by the same amount, keeping the market values of debts in *all* histories and portfolios in all histories *other* than s^t unchanged. In both types of perturbations, the government adjusts taxes so that its budget constraint (3) is satisfied at all histories. We refer to these two classes of perturbations as the *debt level* and the *portfolio* perturbations.

To understand welfare consequences of these perturbations, it is helpful to define the tax revenue elasticity $\xi_t := \frac{\partial \ln T_t}{\partial \ln \tau_t}$. Its inverse $\frac{1}{\xi_t} = \frac{\partial \tau_t}{\partial T_t} Y_t$ captures how much tax rates must change if the government wants to increase tax revenues normalized by output by one unit. Using (6), it is easy to see that ξ_t equals $1 - \frac{v'(Y_t)}{v''(Y_t)Y_t}\frac{\tau_t}{1-\tau_t}$. Since output Y_t is implicitly a function of τ_t, the elasticity ξ_t is a transformation of τ_t that we can write as $\xi_t = \xi(\tau_t)$ for some function ξ.

The debt level perturbation decreases tax revenues $T_t(s^t)$ by ε and increases them by $\mathcal{R}_{t+1}(s^{t+1})\varepsilon$ in all $s^{t+1} \succeq s^t$, leaving taxes in all other histories unchanged.[3] The portfolio perturbation increases taxes only in histories $s^{t+1} \succeq s^t$ by $r^j_{t+1}(s^{t+1})\varepsilon$. Using the envelope theorem and the definition of the tax revenue elasticity, the welfare effects of these two perturbations are

$$\partial_{debt} V = \beta^t \Pr\left(s^t\right)\left[M_t(s^t)\frac{1}{\xi_t(s^t)} - \mathbb{E}_{s^t}\beta M_{t+1}\mathcal{R}_{t+1}\frac{1}{\xi_{t+1}}\right], \tag{8}$$

and

$$\partial_{prfl,j} V = -\beta^t \Pr\left(s^t\right)\mathbb{E}_{s^t} M_{t+1} r^j_{t+1}\frac{1}{\xi_{t+1}}, \tag{9}$$

respectively.

So far we have considered perturbations of arbitrary government policies. If government policies are optimal, there exist no welfare improving perturbations. Therefore, at an optimum we must have

$$\xi_t^{-1} = \mathbb{E}_t \frac{\beta M_{t+1}}{M_t}\mathcal{R}_{t+1}\xi_{t+1}^{-1} \text{ for all } t, \tag{10}$$

$$\mathbb{E}_t \frac{\beta M_{t+1}}{M_t} r^j_{t+1}\xi^{-1}_{t+1} = 0 \text{ for all } j, t. \tag{11}$$

Ramsey optimal government policies can be determined by combining government optimality conditions (10) and (11), household optimality (6) and (7), and government budget constraints (4). As discussed before, it is difficult to "invert" those non-linear stochastic equations to express optimal policies in terms of primitives. To make progress, we assemble a family of approximations.

Fix any history s^t and for all $s^{t+k} \succeq s^t$ we can write without loss of generality the vector of exogenous variables as

$$z_{t+k}(s^{t+k}) = \overline{z}_{t+k}(s^t) + \hat{z}_{t+k}(s^{t+k}),$$

where $\overline{z}_{t+k} := \mathbb{E}_t z_{t+k}$ and $\hat{z}_{t+k} := z_{t+t} - \mathbb{E}_t z_{t+k}$. Let $\sigma \geq 0$ be a scalar and consider a richer family of exogenous stochastic processes $\{z_t(\sigma)\}_t$, defined by $z_k\left(s^k;\sigma\right) = z_k\left(s^k\right)$ if $s^k \not\succeq s^t$ and $z_k\left(s^k;\sigma\right) = \overline{z}_k\left(s^t\right) + \sigma\hat{z}_k\left(s^k\right)$ if $s^k \succeq s^t$. That is, we scale all exogenous shocks arriving

[3] The assumption of no income effects is being used here. With income effects, households may adjust labor supplies in other histories, which would require tax adjustments in those histories as well.

after history s^t by σ, leaving other shocks unchanged. We refer to the economy with exogenous shocks $\{z_t(\sigma)\}_t$ as the σ-economy.

Let $\{y_k(\sigma)\}_k$ be the stochastic process for the endogenous variables, namely, government policies, prices, and household choices in a σ-economy; $\sigma = 1$ is the original economy, and $\sigma = 0$ is an economy in which all uncertainty "switches off" after history s^t. Using Taylor expansions, we can write

$$y_k \simeq \overline{y}_k + \partial_\sigma y_k + \frac{1}{2}\partial_{\sigma\sigma} y_k \text{ for all } k, \tag{12}$$

where $\overline{y}_k = y_k(0)$; $\partial_\sigma y_k$, $\partial_{\sigma\sigma} y_k$ are first and second derivatives of $y_k(\sigma)$ with respect to σ evaluated at $\sigma = 0$; and $\simeq$ denotes approximation up to the order $o\left(\{\|\hat{z}_{t+k}\|^2\}_{k\geq 1}\right)$. We refer to $\overline{y}_k$, $\overline{y}_k + \partial_\sigma y_k$, and the right hand side of equation (12) as zeroth, first, and second order approximations of process $\{y_k(\sigma)\}_k$, respectively.

The following result significantly simplifies our analysis.

Lemma 1. *$\overline{r}^i_{t+k} = 0$, $\mathbb{E}_t \partial_\sigma r^i_{t+k} = 0$, and $\mathbb{E}_t \partial_\sigma r^j_{t+1} \partial_\sigma r^i_{t+k+1} = 0$ for all i, j, and $k \geq 1$.*

Proof. Applying the law of the iterated expectations to equation (5), we get

$$\mathbb{E}_{t+1} S_{t+k+1} r^i_{t+k+1} = 0 \text{ for all } t, i, k \geq 1. \tag{13}$$

The zeroth order expansion of equation (13) yields $\overline{r}^i_{t+k+1} = 0$ since $\overline{S}_{t+k+1} > 0$. Using this, the first order expansion yields $\mathbb{E}_{t+1} \partial_\sigma r^i_{t+k+1} = 0$. Multiply (13) by r^j_{t+1} and compute the expectation at time t to get $\mathbb{E}_t r^j_{t+1} S_{t+k+1} r^i_{t+k+1} = 0$. The second order expansion of this equation, using zeroth and first order implications, implies that $\mathbb{E}_t \partial_\sigma r^j_{t+1} \partial_\sigma r^i_{t+k+1} = 0$. □

An implication of Lemma 1 is that intertemporal covariances of returns with each other must be zero to the second order, that is, $cov_t\left(r^j_{t+1}, r^i_{t+k+1}\right) \simeq 0$ for all i, j, and $k \geq 1$. This leads to important simplifications. The optimal portfolio composition in period t depends, in general, on all cross-covariances $\{cov_t(r^j_{t+1}, r^i_{t+k+1})\}_{i,j,k}$. However, only the intratemporal cross-covariances $\{cov_t(r^j_{t+1}, r^i_{t+1})\}_{i,j}$ are of the second order, while the remaining ones are of higher orders. Thus, intratemporal covariances play the dominant role in formation of the optimal portfolio, while the intertemporal ones can be ignored to the second order.

Using Lemma 1 we can derive useful implications of optimality conditions (10) and (11).

Lemma 2. *(a) Optimal debt level condition (10) implies that $\overline{\tau}_{t+k} = \overline{\tau}_t$ and that $\mathbb{E}_t \partial_\sigma r^j_{t+1} \partial_\sigma \tau_{t+1} = \mathbb{E}_t \partial_\sigma r^j_{t+1} \partial_\sigma \tau_{t+k+1}$ for all j, $k \geq 1$.*

(b) Optimal portfolio condition (11) implies that $\mathbb{E}_t \partial_\sigma r^j_{t+1} \partial_\sigma \tau_{t+1} = 0$ for all j.

Proof. We shall establish part (b). Because the proof of part (a) is very similar, we put it in online Appendix A. Take the second order approximation of (11) and use Lemma 1 to rewrite it as

$$\frac{1}{2}\overline{\frac{\beta M_{t+1}}{M_t}\frac{1}{\xi_{t+1}}}\mathbb{E}_t\partial_{\sigma\sigma}r^j_{t+1}+\overline{\frac{1}{\xi_{t+1}}}\mathbb{E}_t\partial_\sigma r^j_{t+1}\partial_\sigma\frac{\beta M_{t+1}}{M_t}+\overline{\frac{\beta M_{t+1}}{M_t}}\mathbb{E}_t\partial_\sigma r^j_{t+1}\partial_\sigma\frac{1}{\xi_{t+1}}=0. \tag{14}$$

A second order expansion of households' optimality condition (the second equation in (7)) implies

$$\frac{1}{2}\overline{\frac{\beta M_{t+1}}{M_t}}\mathbb{E}_t\partial_{\sigma\sigma}r^j_{t+1}+\mathbb{E}_t\partial_\sigma r^j_{t+1}\partial_\sigma\frac{\beta M_{t+1}}{M_t}=0. \tag{15}$$

Combine (14) and (15) to obtain $\mathbb{E}_t\partial_\sigma r^j_{t+1}\partial_\sigma\frac{1}{\xi_{t+1}}=0$. But $\partial_\sigma\frac{1}{\xi_{t+1}}=-\frac{\xi'(\overline{\tau}_{t+1})}{\xi(\overline{\tau}_{t+1})^2}\partial_\sigma\tau_{t+1}$, which establishes part (b). □

Part (a) of Lemma 2 is related to some celebrated tax smoothing results. Formula (10) for optimal debt levels imply that, to the first order, expected tax rates $\mathbb{E}_t\tau_{t+k}$ for any $k\geq 1$ should be equal to τ_t. This is reminiscent of a version of Barro's (1979) tax smoothing result. While optimal tax rates need not follow a random walk in our economy, departures from this tax smoothing result are of the second order. Part (b) of Lemma 2 implies that the optimal portfolio in time t sets $cov_t(\tau_{t+1}, r^j_{t+1})$ equal to zero to the second order. Thus, the government chooses portfolios in time t so that fluctuations in excess returns reduce fluctuations in tax rates. Versions of Part (b) of Lemma 2 also appear in papers on tax smoothing (see Bohn (1990), Farhi (2010), or Bhandari et al. (2017a))

Lemma 2 is informative about terms that do and don't appear in our approximate optimality conditions. For example, a portfolio perturbation has three effects, captured by the three terms in equation (14). First, in period $t+1$ it earns excess returns $\partial_{\sigma\sigma}r^j_{t+1}$ that are rebated back to households by adjusting tax rates. Second, because these rebates are stochastic, they may amplify or reduce households' marginal utility risk. Third, the perturbation amplifies or reduces fluctuations in tax rates and associated deadweight losses. It is reasonable to anticipate that the optimal portfolio should take all three of these effects into account. But equation (15) reminds us that households already make personal portfolio choices that trade off risk premia captured by $\mathbb{E}_t\partial_{\sigma\sigma}r^j_{t+1}$ and hedging motives captured by covariances of excess returns with marginal utilities. Therefore, the government's portfolio decisions focus solely on risks that households cannot hedge, namely, on reducing fluctuations in the deadweight losses of taxation.

To use these results to characterize the optimal Ramsey portfolio, it is helpful to introduce prices of notional bonds.[4] Let Q_t^k be the period t price of a zero coupon bond that matures in $t+k$ periods. The long interest rate between periods t and $t+k$ is $1/Q_t^k$. Under this convention, $Q_t^1 = Q_t^{rf}$ is the price of a one-period risk-free bond. We call the sequence $\left\{Q_t^k\right\}_k$ a date t *bond price curve* and it captures the relationship between bond maturities and their prices at a given period t. Lemma 1, implies that prices Q_{t+1}^k and discount rates $\mathcal{Q}_{t+1,t+k+1}$ coincide up to the first order. Also, to the second order we have

$$cov_t\left(\mathcal{Q}_{t+1,t+k+1}, r_{t+1}^j\right) \simeq cov_t\left(Q_{t+1}^k, r_{t+1}^j\right) \text{ for all } j,k \geq 1.$$

We now use this observation to derive the optimal portfolio at s^t. Multiply (4) by r_{t+1}^j, take expectations at history s^t, and apply the second order expansion using the preceding observations to get:

Theorem 1. *The optimal Ramsey portfolio satisfies,* for all t, j

$$\sum_{i\neq rf}\overline{\omega}_t^i \mathbb{E}_t \partial_\sigma r_{t+1}^i \partial_\sigma r_{t+1}^j = \sum_{k=1}^{\infty}\frac{\overline{Q}_t^{k+1}\overline{X}_{t+k+1}}{\overline{Q}_t^1 \overline{B}_t}\mathbb{E}_t\partial_\sigma \ln Q_{t+1}^k \partial_\sigma r_{t+1}^j - \sum_{k=1}^{\infty}\frac{\overline{Q}_t^k \overline{G}_{t+k}}{\overline{Q}_t^1 \overline{B}_t}\mathbb{E}_t \partial_\sigma \ln G_{t+k}\partial_\sigma r_{t+1}^j, \tag{16}$$

where $\overline{X}_{t+k} = \overline{T}_t - \overline{G}_{t+k}$ *and* $\overline{T}_t = \frac{\overline{B}_t + \sum_{k=1}^{\infty}\overline{Q}_t^k \overline{G}_{t+k}}{\sum_{k=1}^{\infty}\overline{Q}_t^k}$.

Equation (16) determines optimal public portfolios ω_t as functions of debt levels B_t and exogenous objects (recall that returns and asset prices are pinned down by a representative foreigner's stochastic discount factor $\{S_k\}_k$). Remarkably, future portfolio choices do not appear in the optimal portfolio formula because effects of those future portfolios are of the third order. The sequence $\left\{\overline{X}_{t+k}\right\}_k$ is a zeroth order approximation of primary surpluses $X_{t+k} = T_{t+k} - G_{t+k}$ when debt levels are set optimally to the zeroth order.

It is enlightening to re-write equation (16) in terms of equilibrium objects rather than their approximating counterparts. For a pair of variables y'_{t+k}, y''_{t+k}, we have the following relationship

$$\mathbb{E}_t y'_{t+k} cov_t\left(y''_{t+k}, r_{t+1}^j\right) \simeq \overline{y}'_{t+k}\mathbb{E}_t \partial_\sigma y''_{t+k}\partial_\sigma r_{t+1}^j. \tag{17}$$

The left hand side of this equation entails some population moments for a competitive equilibrium, while the right hand side is their second-order approximation. Using this equation, we re-write (16). Let Σ_t, Σ_t^Q and Σ_t^G be covariance matrices with elements $\{cov_t(r_{t+1}^i, r_{t+1}^j)\}_{i,j}$, $\{cov_t(\ln Q_{t+1}^k, r_{t+1}^j)\}_{j,k}$, and $\{cov_t(\ln G_{t+k}, r_{t+1}^j)\}_{j,k}$ for all j and $k \geq 1$, and let s_t^Q and s_t^G be

[4] We call this bond 'notional' because we don't require it to be traded; we only need to be able to price such a bond. In particular, notional bond prices are given by $S_t Q_t^k = \mathbb{E}_t S_{t+k}$.

vectors with elements $\{\frac{Q_t^{k+1}\mathbb{E}_t X_{t+k+1}}{Q_t^1 B_t}\}_k$ and $\{\frac{-Q_t^k \mathbb{E}_t G_{t+k}}{Q_t^1 B_t}\}_k$. Then equation (16) can be written succinctly as

$$\Sigma_t \omega_t \simeq \Sigma_t^Q s_t^Q + \Sigma_t^G s_t^G. \tag{18}$$

Equation (18) expresses optimal Ramsey portfolios in terms of objects that have meaningful interpretations. Here Σ_t, Σ_t^Q, and Σ_t^G are covariance matrices of excess returns with other excess returns, with long interest rates, and with government expenditures at different time horizons. Vectors s_t^Q and s_t^G are constructed by computing mathematical expectations of primary surpluses and expenditures at different horizons, $\mathbb{E}_t X_{t+k}$ and $\mathbb{E}_t G_{t+k}$, converting them into date t units using the date t bond price curve $\{Q_t^k\}_{k\geq 1}$, and dividing them by the market value of the date t outstanding debt level. If there are redundant securities then there are multiple optimal portfolios, all of which satisfy (18). If matrix Σ_t is invertible, then an optimal portfolio is unique and approximately equals to[5]

$$\omega_t^* := \Sigma_t^{-1}\Sigma_t^Q s_t^Q + \Sigma_t^{-1}\Sigma_t^G s_t^G. \tag{19}$$

Formula (19) expresses two goals sought by an optimal public portfolio: hedging interest rate risk and hedging expenditure risk. These two hedges are captured by matrices $\Sigma_t^{-1}\Sigma_t^Q$ and $\Sigma_t^{-1}\Sigma_t^G$. Vectors s_t^Q and s_t^G provide quasi-weights that determine relative importance of hedging these two risks.

Formula (19) applies to any set of traded securities. Bonds of various maturities are among the most common securities that governments trade. Let r_{t+1}^k be the $t \to t+1$ excess return of the bond that matures in period $t+k+1$. By definition, $r_{t+1}^k = Q_{t+1}^k / Q_t^{k+1}$, so fluctuations in the excess return of a k-period bond are proportional to fluctuations in k-period interest rate. Specifically, Lemma 1 implies that $Q_t^1 cov_t(r_{t+1}^k, r_{t+1}^j) \simeq cov_t(\ln Q_{t+1}^k, r_{t+1}^j)$, so that to the second order the two covariances are the same up to the price of the risk-free bond Q_t^1.

One implication of this observation is as follows. Suppose that the government trades a full set of bond maturities. In this case, interest rate risk is hedged by a portfolio

$$\Sigma_t^{-1}\Sigma_t^Q s_t^Q \simeq \left[\frac{Q_t^2 \mathbb{E}_t X_{t+2}}{B_t}, \frac{Q_t^3 \mathbb{E}_t X_{t+3}}{B_t}, ...\right]^\mathsf{T}. \tag{20}$$

This interest rate hedging portfolio takes a simple form. Let $\widetilde{B}_t^k$ denote the quantity (or the face value) of the bonds of maturity k. By definition, the relationship between bond quantities and market values is given by $B_t^k = Q_t^k \widetilde{B}_t^k$. For each maturity k, if the government

[5]Note that this approximation is in the zeroth order sense, as $\sigma \to 0$. In the zeroth order economy all securities are risk-free and the optimal portfolio composition is undetermined, but ω_t^* is the limit of the optimal portfolios in the stochastic economy as risk shrinks to zero.

sets quantities of k-period bonds equal to the expected primary surpluses at the time of the maturity, $\widetilde{B}_t^k = \mathbb{E}_t X_{t+k}$, then the portfolio shares automatically satisfy the right hand side of (20). We refer to the construction of this interest rate hedging portfolio as the *maturity matching* principle. Note that this principle implies that any changes in bond prices that are orthogonal to expected primary surpluses may affect portfolio shares of bonds of different maturities but not their quantities. The extent to which the government departs from maturity matching depends on bonds' abilities to hedge expenditure risks. In Section 5, we show that covariances of returns with expenditures in the U.S. data are low, implying that optimal deviations from interest rate hedging are small.

While the optimal Ramsey portfolio in equation (19) depends on covariances that measure various types of risk, it does not include terms that capture risk premia or risk aversion, even though those terms play the central role in the classical portfolio theory (e.g., Samuelson (1970), Merton (1971), Campbell and Viceira (1999), Viceira (2001)). This is because a benevolent government has the same attitudes towards risks and returns as households. Thus, there is no motive for the government to chase higher excess returns on securities – households can get those same excess returns for themselves without bearing deadweight losses from taxation.

This result is at odds with common practices of some Treasury departments to play the yield curve (see, e.g., Missale (1999)). Since yield curves are usually steeper than the expected path of short rates, it seems cheaper to borrow using short-term debt and many Treasury departments tilt their portfolios towards shorter maturities to reduce borrowing costs. In our benchmark economy, this practice is misguided. Lower yields on short bonds are a compensation for bearing future interest rate risk. A strategy that tilts public portfolios towards shorter maturities would force households to bear additional and unwanted risk of tax rate fluctuations.

3.1 Sufficient statistics and government policies

We derived our expressions for the optimal portfolio under the assumption that government policy $\{B_t, \omega_t, \tau_t\}_t$ is Ramsey optimal. We now discuss how our approach can be extended when we relax this assumption. In what follows, we consider equilibria associated with government policies $\{B_t, \omega_t, \tau_t\}_t$ in which the debt levels are set so that equation (10) holds to the first order approximation. The latter assumption is helpful in separating two conceptually distinct issues: the optimal choice of total debt level and the optimal allocation of that debt into different securities.[6,7]

Consider a competitive equilibrium associated with such a policy. Suppose that the government finds itself in some history s^t with some debt level $B_t\left(s^t\right)$. How should it allocate this debt into a portfolio of different securities? To answer this equation, consider our portfolio perturbation (9) in that history s^t. The portfolio $\omega_t(s^t)$ is optimal if there are no welfare gains from such perturbations, i.e., equation (11) holds. Households optimality conditions (6) and (7), asset pricing conditions (5), and government budget constraints (4) must hold in any equilibrium, irrespective of how government policies are set. Using these equations, together with the assumption that (10) holds to the first order, it is easy to verify that all arguments behind proofs of Lemmas 1 and 2 are unchanged and equation (16) in Theorem 1 holds given history s^t.

We note a few properties of the optimal portfolio that follow from the extended version of Theorem 1. First, the optimal portfolio given history s^t is expressed in terms of only exogenous variables and the debt level $B_t(s^t)$. Second, equation (17) that expresses the optimal portfolio using population moments still holds and it implies that the dependence of moments on the left hand side of (17) on government policies after s^t is of the third order and can be ignored in our approximations.

This has several implications. First, equations (19) and (20) continue to describe the optimal public portfolio even if government policies are not Ramsey optimal. While some of these statistics, such as s_t^Q, depend on future government policies, this dependence is of the order that is smaller than our approximation error. From a practical perspective, it means that one can estimate all the sufficient statistics Σ_t, Σ_t^Q, Σ_t^G, s_t^G, s_t^Q under the equilibrium data generating process, and plug them directly into the right hand side of (19) to construct the optimal portfolio in any history s^t with debt level $B_t\left(s^t\right)$. Second, the same reasoning implies that we can use these equations to study the direction in which any given portfolio $\omega_t(s^t)$ should be adjusted to improve welfare.

Corollary 1. *Consider an equilibrium in which the debt level is chosen optimally to the first order and let* ω_t^* *be constructed using moments from the equilibrium data-generating process.*

[6] If the debt level is not chosen optimally then portfolio perturbations (9) would try, in addition to the forces we emphasized, to also compensate for the suboptimal debt levels. While it is possible to characterize such portfolios, they are hard to interpret. Debt levels is a superior policy instrument for the intertemporal transfer of resources than the portfolio choice, and those levels should be first order optimized before deciding on the portfolio allocations.

[7] This assumption that the debt level is approximately optimal implies that taxes are approximately a random walk. This is consistent with the behavior of taxes in the U.S. data, see, e.g. Barro (1981), Kingston (1987), Mankiw (1987), and Marcet and Scott (2009) who document little departures from tax smoothing at business cycle or higher frequencies.

Then

$$\sum_{i \neq rf} \left(\omega_t^{*,i} - \omega_t^i\right) cov_t\left(r_{t+1}^i, r_{t+1}^j\right) \simeq const_t \times \partial_{prfl,j} V \text{ for any } j, \tag{21}$$

where $const_t > 0$ *provided that* $B_t > 0$, $\xi(\tau_t) > 0$, $\xi'(\tau_t) < 0$.

This corollary provides a method to determine how government portfolio ω_t can be improved. We focus on the case of $B_t > 0$ and $\xi, -\xi' < 0$, which with a constant elasticity of labor supply is equivalent to assuming that taxes τ_t are to the left of the peak of the Laffer curve. Recall from our discussion of equation (21) that welfare can be improved by increasing the portfolio share of security j if $\partial_{prfl,j} V > 0$ and decreasing it if $\partial_{prfl,j} V < 0$. The left hand side of (21) provides a way to compute the sign of $\partial_{prfl,j} V$.

To understand implications of this expression, first consider an economy in which agents trade only one risky security. In that case, we immediately get that the sign of $\partial_{prfl,j} V$ is the same as the sign of $\omega_t^{*,j} - \omega_t^j$, so welfare is improved by rebalancing portfolio weights closer to $\omega_t^{*,j}$. When agents trade multiple securities, the welfare effect of moving ω_t^j closer to $\omega_t^{*,j}$ also depends on how far portfolio shares for other securities $i \neq j$ are from their optima, as well as on covariances of returns of those securities with r_{t+1}^j. As can be deduced from (21), moving ω_t^j closer to $\omega_t^{*,j}$ is welfare improving provided that covariances of returns of other securities with r_{t+1}^j are sufficiently low relative to the variance of r_{t+1}^j, or that portfolio shares of other securities are sufficiently close to the optimum.

4 Extensions

In this section, we discuss the effects of alternative preference specifications, additional shocks that affect primary deficits, liquidity services of government bonds, and household heterogeneities on compositions of optimal public portfolios. We also relax the assumption that asset prices are not affected by government policies.

4.1 Role of preferences

In the benchmark economy, households have fixed time-separable preferences. The only property of preferences that we used in studying our benchmark economy is the absence of income effects. Consequently, all our results from Section 3 apply in economies in which households' preferences are ordered by $V_0(\{C_t, Y_t\}_t)$ which is recursively defined by

$$V_t = u_t(C_t - v(Y_t)) + \beta \mathbb{W}_t(V_{t+1}), \tag{22}$$

where utility function u_t can depend on exogenous shocks, and $\mathbb{W}_t$ is twice continuously differentiable, strictly increasing functional that is increasing in the first- and second-order stochastic dominance and that has a property that $\mathbb{W}_t\left(x'_{t+1}\right) = x'_{t+1}$ for any time-t measurable random variable x'_{t+1}.

Preference specification (22) is widely used in asset pricing papers designed to explain stock and bond risk premia. As shown by Ai and Bansal (2018), it includes preferences with recursive preferences of Epstein and Zin (1989), variational preferences of Maccheroni et al. (2006a) and Maccheroni et al. (2006b), multiplier preferences of Hansen and Sargent (2008) and Strzalecki (2011), second-order expected utility of Ergin and Gul (2009), smooth ambiguity preferences of Klibanoff et al. (2005), Klibanoff et al. (2009), disappointment aversion preference of Gul (1991), and the recursive smooth ambiguity preference of Hayashi and Miao (2011). The stochastic function u_t for period utilities can represent a discount factor shock used in Albuquerque et al. (2016).

Preferences in equation (22) allow for many differences in how households evaluate risks and return of different securities. But those do not affect the structure of optimal public portfolios. This situation extends our Section 3 finding that optimal public portfolios are independent of households' attitudes about risks.

4.2 Tax revenue risks

In our benchmark economy, tax revenues T_t depend only on the tax rate τ_t. In more general settings, other variables like exogenous productivity shocks that influence the tax base also affect tax revenues. In this section, we extend our approach to study the optimal public portfolio choice in such settings.

We assume that the disuility of effort takes the form $v_t\left(Y_t\right) = \Theta_t^{-1/\gamma}\frac{Y_t^{1+1/\gamma}}{1+1/\gamma}$ where Θ_t is an exogenous stochastic process and $\gamma > 0$ is the elasticity of labor supply. Under this preference specification, household earnings and tax revenues are given by $Y_t = \Theta_t\left(1-\tau_t\right)^{\gamma}$ and $T_t = \Theta_t\tau_t\left(1-\tau_t\right)^{\gamma}$ and they depend both on the tax rate τ_t and the exogenous shock Θ_t.

We consider the same perturbations as in Section 3. The debt level and portfolio optimality conditions (10) and (11) still hold in this economy. Under constant elasticity preferences, the relationship between the tax revenue elasticity and the tax rate is given by $\xi_t = 1 - \gamma\frac{\tau_t}{1-\tau_t}$, which implies that smoothing of tax distortions ξ_t^{-1} is equivalent to smoothing of tax rates τ_t.[8]

[8]This is the sole reason that we assume constant elasticity preferences in this section. Our approach can be extended to an arbitrary disutility of effort $v(Y_t/\Theta_t)$. Under such preferences, distortion smoothing implies a particular relationship between realizations of Θ_{t+k} and τ_{t+k} that should hold in the optimum. One can

We can decompose fluctuations of tax revenues (see online Appendix B) into policy and non-policy components as

$$cov_t\left(\ln T_{t+k}, r^j_{t+1}\right) \simeq cov_t\left(\ln\left(\tau_t\left(1-\tau_t\right)^\gamma\right), r^j_{t+1}\right) + cov_t\left(\ln\Theta_{t+k}, r^j_{t+1}\right), \tag{23}$$

The portfolio optimality condition (11) calls for setting the first covariance on the right hand side of (23) to zero. This, in turn, implies that portfolios need to be chosen to additionally hedge Θ_{t+k}, captured by the second covariance on the right hand side of (23).

The shock Θ_{t+k} is hedged similarly to the government expenditure shock G_{t+k}. Let Σ^Θ_t be a matrix with elements $\{cov_t(\ln\Theta_{t+k}, r^j_{t+1})\}_{j,k}$ and s^Θ_t be a vector with elements $\{\frac{Q^k_t\mathbb{E}_t T^{tax}_{t+k}}{Q^1_t B_t}\}_k$, with $T^{tax}_{t+k} = \Theta_{t+k}\tau_t\left(1-\tau_t\right)^\gamma$ which are the analogues of Σ^G_t and s^G_t. If the covariance matrix Σ_t is invertible then the optimal portfolio is unique and is approximately equal to

$$\omega^*_t = \Sigma^{-1}_t\Sigma^Q_t s^Q_t + \Sigma^{-1}_t\Sigma^G_t s^G_t + \Sigma^{-1}_t\Sigma^\Theta_t s^\Theta_t. \tag{24}$$

As can be seen from this equation, the optimal portfolio is chosen to hedge fluctuations in interest rates, expenditures and tax revenues shocks, with quasi-weights s^Q_t, s^G_t and s^Θ_t aggregating these risks in the optimal portfolio formula. The rest of the discussion in Section 3, such as Corollary 1, extends directly as well.

4.3 Liquidity premia on government bonds

A large theoretical and empirical literature that has emphasized that governments appear to be able to borrow at cheaper rates than private sector because of convenience or liquidity benefits of government-issued debt.[9] In this section, we explore the implication of liquidity premia enjoyed by government bonds for optimal portfolio formation.

We introduce liquidity premia by assuming that holding of government bonds has direct utility benefit to households. For concreteness, we assume that government only issues bonds and trades no other other securities. We use our earlier convention that superscript k refers to a bond that matures in k periods so Q^k_t and R^{k-1}_{t+1} are prices and $t \to t+1$ holding period returns of a bond that matures in period $t+k$. We use notation $\{\}_k$ to denote the collection of government bonds of available maturities. Our approach applies in the same way to economies

explicitly characterize this relationship and use it to construct optimal portfolios, which take a form similar to equation (24) but with a slightly different definition of s^Θ_t. We opt for the constant elasticity specification because the analysis is more transparent and because it is the specification that is most commonly used in the quantitative Ramsey literature.

[9] Theoretical contribution that emphasize convenience yields for government debts include Woodford (1990), Aiyagari and Gertler (1991), Aiyagari (1994), Bansal and Coleman (1996), Holmstrom and Tirole (1998), and Lagos (2010). Empirical papers that measure convenience yields for the U.S. include Longstaff (2004), Krishnamurthy and Vissing-Jorgensen (2012), and Jiang et al. (2019).

in which the government can issue the full set of bond maturities and to economies in which maturities are capped or restricted in some other way. Let $\{b_t^k\}_k$ be households' holdings of government bonds. We assume that these holdings give household pecuniary utility $w_t(\{b_t^k\}_k)$, where w_t is a function that is strictly increasing and differentiable in each b_t^k, with derivatives denoted by $w_{t,k}$. We allow w_t to be subject to exogenous shocks. Households' intratemporal utility is assumed to be

$$u\left(C_t - v\left(Y_t\right) + w_t\left(\left\{b_t^k\right\}_k\right)\right),$$

with the rest of the economy as in Section 2.[10]

The welfare effects of debt level and portfolio perturbations that we considered in Section 3 remain unchanged in this economy, and equations (10) and (11) still hold. The main difference from the benchmark economy is that household optimality conditions for government bonds are given by

$$1 - w_{t,1} = \frac{1}{Q_t^1}\mathbb{E}_t\frac{\beta M_{t+1}}{M_t}, \quad w_{t,k} - w_{t,1} = -\mathbb{E}_t\frac{\beta M_{t+1}}{M_t}r_{t+1}^{k-1}. \tag{25}$$

These equations show that the price of a one period government bond Q_t^1 depends both on households' rate of discount $\mathbb{E}_t\frac{\beta M_{t+1}}{M_t}$ and the liquidity premium $w_{t,1}$ that this bond offers. Similarly, the excess return of the government bond that matures in period $t+k$ is depends on the excess liquidity premium of that bond, $w_{t,k} - w_{t,1}$, i.e., on the difference between liquidity premia of a k- and a one-period bond.

Before taking approximations of these conditions, it is useful to think about their empirical counterparts. Let $Q_t^{1,pr}$ be the price of a notional privately-issued one-period risk-free bond, i.e., the price of a bond at which domestic households are willing to borrow and lend from each other. This price satisfies $1 = \frac{1}{Q_t^{1,pr}}\mathbb{E}_t\frac{\beta M_{t+1}}{M_t}$. Combine it with (25) to find that the liquidity premium of the one-period bond can be written as

$$1 - \frac{Q_t^{1,pr}}{Q_t^1} = w_{t,1}. \tag{26}$$

Thus, the liquidity premium can be obtained by comparing yields of government- and privately-issued bonds.

This observation is not surprising: the empirical finance literature typically uses a similar logic to estimate the liquidity premia (see for instance Longstaff (2004) or Krishnamurthy and Vissing-Jorgensen (2012)). But equation (26) contains lessons about good ways to approximate

[10] When government bonds provides additional liquidity services, one may also want to include the non-negativity constraints $b_{i,t} \geq 0$ on government bonds. For simplicity, we ignore these constraints in our analysis, but most of the discussion in this section can be extended by explicitly incorporating these constraints and adding their corresponding Lagrange multipliers into the definition of liquidity premium.

this economy. It is easy to obtain empirical analogues of prices Q_t^1 and $Q_t^{1,pr}$ and use equation (26) to construct the liquidity premium $w_{t,1}$. In the data this liquidity premium is of the same order of magnitude, if not smaller, than risk premia of bonds of different maturities, $\mathbb{E}_t r_{t+1}^k$.[11] If we were naively to apply small noise approximation only to the exogenous disturbances, $z_{t+k}(\sigma) = \overline{z}_{t+k} + \sigma \hat{z}_{t+k}$, we would be scaling all risk premia with σ^2 while keeping liquidity premia intact. This approach would implicitly approximate around an economy in which the liquidity premia is infinitely large relative to risk premia, which is unrealistic. A much better approach is to the use a small-noise expansion that scales w_t with σ^2 in the same way that it scales $\{\hat{z}_{t+k}\}_k$ with σ. This approach ensures that relative magnitudes of liquidity and risk premia remain unchanged at all $\sigma > 0$.

This observation has several immediate implications. Suppose that government bonds of different maturities are perfect substitutes for households, so that $w_t(\{b_t^k\}_k) = w_t(\sum_k b_t^k)$. In this case, all bonds have the same liquidity premium and portfolio optimality conditions are the same as in the benchmark economy. The optimal dynamics of the debt level, which is characterized by combining equation (10) and the first equation in (25), is affected by the liquidity premium but this effect is of the second order. Since we only used first order approximations of equations governing debt level dynamics to prove Theorem 1, the conclusions of that theorem and the rest of the discussion in Section 3 remain unchanged.

If government bonds are imperfect substitutes then the optimal portfolio also depends on the excess liquidity premium $w_{t,k} - w_{t,1}$. Let μ_t be the vector $\{w_{t,k} - w_{t,1}\}_{k>1}$. This vector can be constructed from notional prices of government-issued and privately-issued pure discount bonds using the relationship

$$\ln Q_t^k - \ln Q_t^{k,pr} \simeq w_{t,k} + \mathbb{E}_t w_{t+1,k-1} + ... + \mathbb{E}_t w_{t+k-1,1}. \tag{27}$$

Assuming that Σ_t is invertible, the optimal portfolio in the economy with liquidity premia is given by

$$\omega_t^* = \Sigma_t^{-1} \Sigma_t^Q s_t^Q + \Sigma_t^{-1} \Sigma_t^G s_t^G + a_t \Sigma_t^{-1} \mu_t, \tag{28}$$

where $a_t = \frac{Y_t \sum_{k=1}^{\infty} Q_t^k}{\left(Q_t^1\right)^2 B_t} \frac{\xi^2(\tau_t)}{-\xi'(\tau_t)}$. Comparing this equation to equation (19), we see that portfolio weights increase (decrease) relative to the benchmark economy if the covariance adjusted vector of excess liquidity premia $\Sigma_t^{-1} \mu_t$ is positive (negative).

Equation (28) was derived under the assumption that government policies are Ramsey optimal. This assumptions can be relaxed along the same lines that we discussed in Section

[11] For instance, Krishnamurthy and Vissing-Jorgensen document a liquidity premium of about 73 basis points per year for long maturity bonds. This is lower than the average excess returns on long bonds that are in the range of 100 to 150 basis points per year.

3.1. In particular, if government debt is set approximately optimally then equation (28) still holds and all the terms on the right hand side other than μ_t depend only on debt level B_t and exogenous shocks to the order of approximation we consider. The vector of excess liquidity premia μ_t is an implicit function of portfolio ω_t^*. In order to use (28) to construct optimal portfolios using empirical data, one would need to estimate the dependence of μ_t on ω_t and then use equation (28) to compute ω_t^*.

4.4 Household heterogeneity

We now include heterogeneous households who differ in their skills and their access to asset markets and consider its consequences for the composition of an optimal portfolio.

Suppose that household h has household-specific productivity $\theta_{h,t}$ and disutility of labor takes the same form as in Section 4.2. Also suppose that households can be partitioned into two sets: $\mathbb{T}$, a set of households who can trade securities, and $\mathbb{N}$, a set of households who cannot trade securities. The government objective is a Pareto-weighted sum across households with weights $\{\varpi_h\}_h$. We maintain all other assumptions from our benchmark economy.

We then consider our Section 3 perturbations. The welfare effects are given by

$$\partial_{debt} V = \beta^t \Pr\left(s^t\right) \sum_h \varpi_h \left[M_{h,t}(s^t) \frac{1}{\xi_t(s^t)} \frac{Y_{h,t}(s^t)}{Y_t(s^t)} - \mathbb{E}_{s^t} \beta M_{h,t+1} \mathcal{R}_{t+1} \frac{1}{\xi_{t+1}} \frac{Y_{h,t+1}}{Y_{t+1}} \right], \tag{29}$$

and

$$\partial_{prfl,j} V = -\beta^t \Pr\left(s^t\right) \sum_h \varpi_h \left[\mathbb{E}_{s^t} M_{h,t+1} r_{t+1}^j \frac{1}{\xi_{t+1}} \frac{Y_{h,t+1}}{Y_{t+1}} \right], \tag{30}$$

where $M_{h,t}$ is the Lagrange multiplier on the date t budget constraint of household with productivity $\theta_{h,t}$ and $Y_{h,t}$ is that households date t pre-tax income and ξ_{t+1} is the elasticity of aggregate tax revenues with respect to tax rates.

Comparing equation (29) and (30) to its representative agent counterparts (8) and (9), there are two new terms highlighting the new forces that are present in heterogeneous agent settings. The first is that the inverse tax revenue elasticities are weighted by $\frac{Y_{h,t}}{Y_t}$, which is the share of household type h's income. To the extent these shares fluctuate, there is a motive for the government to use the returns on its portfolio to hedge those fluctuations. The second is the presence of the Lagrange multipliers $\{M_{h,t}\}$ on budget constraints for all households. In the representative agent counterpart, we used household optimality in security markets, that is, equation (7) to "net out" the implications on government optimality. With heterogeneous agents, the counterpart of equation (7) holds only for $h \in \mathbb{T}$. Thus, fluctuations in the wedge between the Lagrange multipliers on budget constraints of the traders and non-traders (a measure of deviation from perfect risk-sharing) capture a planners' desire to trade on behalf of agents who have trouble trading.

These two forces are summarized by two new statistics. Movements in inequality are summarized by a measure $\sum_h \mu_{h,k} \ln\left(s_{h,t+k}^{-1}\right)$, where $s_{h,t} = Y_{h,t}/Y_t$ and $\left\{\mu_{h,k}\right\}_{h,k}$ is a deterministic sequence of weights (see online Appendix B for formulas) that add up to one for all t and depend on both relative productivities and Pareto weights. It is easy to check that this measure is increasing in the dispersion of incomes. Next, define $\ln(M_{\mathbb{T},t+k})$ and $\ln(M_{\mathbb{N},t+k})$ as an average of the Lagrange multipliers on budget constraints of traders and non traders, respectively, e.g., $\ln(M_{\mathbb{T},t+k}) \equiv \sum_{h\in\mathbb{T}} \mu_{h,k} \ln(M_{h,t+k}) \Big/ \sum_{h\in\mathbb{T}} \mu_{h,k}$. The imperfect risk sharing force is captured by $\ln(M_{\mathbb{T},t+k}) - \ln(M_{\mathbb{N},t+k})$.[12]

Following steps resembling our derivation of equation (19), we can define covariances $\Sigma_t^{ineq}[j,k] = cov_t\left(\sum_h \mu_{h,k} \ln\left(s_{h,t+k}^{-1}\right), r_{t+1}^j\right)$, $\Sigma_t^M[k,j] = cov_t\left(\mu_{\mathbb{N},k}\left[\ln(M_{\mathbb{T},t+k}) - \ln(M_{\mathbb{N},t+k})\right], r_{t+1}^j\right)$ and weights $s_t^{ineq}[k] = \frac{Q_t^k \mathbb{E}_t Y_{t+k} \xi^2(\tau_{t+k})}{-Q_t^1 B_t \xi'(\tau_{t+k})}$. The optimal portfolio with heterogeneity satisfies

$$\Sigma_t \omega_t \simeq \Sigma_t \omega_t^* + \left(\Sigma_t^{ineq} s_t^{ineq} + \Sigma_t^M s_t^{ineq}\right). \tag{31}$$

The concerns for inequality fluctuations manifest in the sign and the magnitude of Σ_t^{ineq}. If excess returns and inequality are countercyclical, then we would expect Σ_t^{ineq} to be positive and larger in magnitude for longer bonds. Equation (31) then implies that concerns for fluctuating income shares should push the government to issue additional debts at longer maturities.

Besides fluctuations in income inequality, equation (31) shows that heterogeneity adds a term that depends on how ratios of the average Lagrange multipliers across agents covary with returns. When non-traders have more volatile consumption (presumably because they have fewer avenues to smooth) than the traders, the government can use its debt portfolio to shift some risk from non-traders to traders and improve average welfare. A strategy in which the government borrows more in risky securities (security whose returns are low when marginal values of wealth are high) and invests more in (or lowers issuance of) the risk-free asset makes the overall public portfolio less risky. On the margin, this generates a welfare gain because it allows the government to lower the volatility of the non-traders after-tax incomes. When such risky securities are of longer duration (which is generally the case with long duration bonds), such a strategy would also increase the duration of the optimal portfolio.

[12] The formulation of government optimality using aggregated Lagrange multipliers of various groups is closely related to "multiplier approach" of Chien et al. (2011) who show that equilibria of a large class of heterogeneous agent, incomplete markets environments can be characterized and efficiently computed using a multipliers representation.

4.5 Responses of prices to government policies

We now relax the assumption that government policies have no effect on asset prices. There are two broad classes of models of price determination that are commonly used in the literature: closed economy models in which all asset prices are determined by the demand of the representative household (e.g., Lucas and Stokey (1983), Angeletos (2002), Debortoli et al. (2017), Faraglia et al. (2018)) and various models of segmented markets or preferred habitat in which prices of government bonds of different maturities are determined by demands of specific investor clienteles (e.g., Greenwood and Vayanos (2014), Koijen and Yogo (2019), Vayanos and Vila (2021), Bigio et al. (2023)).

In this section, we examine the simplest version of the preferred habitat model. This model is transparent enough to highlight the main effects of price adjustments on optimal portfolios, while also flexible enough to align its predictions with empirical evidence of price responses to government portfolio rebalancing. In online Appendix E, we consider a closed economy and show that it has the channels emphasized by our preferred habitat model, but implies signs of price responses that are inconsistent with the available empirical evidence.

In the spirit of the preferred habitat investors in Vayanos and Vila (2021), we assume that the government faces downward-sloping demand curves for issuances its debt maturities in period t as

$$\ln Q_t^k = \alpha_{k,t} - \varphi_{k,t}(\widetilde{B}_t^k), \tag{32}$$

where $\{\alpha_{k,t}\}_k$ are random variables, and $\{\varphi_{k,t}(\cdot)\}_k$ are differentiable functions, and $\widetilde{B}_t^k$ denotes the quantity of bonds of maturity k issued by the government. This specification of price processes includes, as a special case, our benchmark economy, in which $\varphi_{k,t}(\cdot) = \mathbf{0}$ and $\{\alpha_{k,t}\}_k$ are as implied by equation (5). To abstract from liquidity premium, we assume that parameters are such that domestic households purchase positive quantities of government debt. To streamline exposition, we assume that demand for the one-period bond is perfectly elastic, $\varphi_{1,t}(\cdot) = 0$, but derive all the results in online Appendix B without this restriction.

As in previous section, we restrict our attention to the problem of allocating debts of different maturities. Since debt prices respond to quantity issuances, it will be more convenient to write the government budget constraint in terms of quantities as

$$T_t - G_t + \sum_k Q_t^k \widetilde{B}_t^k = \sum_k \left(Q_t^k + D_t^k\right) \widetilde{B}_{t-1}^{k+1}. \tag{33}$$

We define the debt level and portfolio perturbations in this economy as follows. The debt level perturbation increases quantity of each maturity k by $\omega_t^k \varepsilon / Q_t^k$, the portfolio perturbation changes quantities of a k period bond and a risk-free bond by ε/Q_t^k and $-\varepsilon/Q_t^1$, respectively. Taxes are adjusted in all periods to satisfy government budget constraints. Note that these perturbations coincide with the ones we considered in Section 3 in the absence of price responses.

Price responses introduce two changes in our welfare analysis that will be convenient to describe using inverse price semi-elasticities $\varphi'_{k,t}(\widetilde{B}_t^k) = -\partial \ln Q_t^k / \partial \widetilde{B}^k$. First, when a government issues a price-inelastic bond, its price falls and additional taxes need to be raised to satisfy its budget constraint. If a government issues ε/Q_t^k more units of debt of maturity k taxes need to be adjusted in period t by $\varepsilon\left[-1+\varphi'_{k,t}(\widetilde{B}_t^k)(\widetilde{B}_t^k-\widetilde{B}_{t-1}^{k+1})\right]$ to satisfy the budget constraint. The second term reflects the fact that issuing more debt affects bond prices and revalues both the incoming portfolio inherited from date $t-1$ and the outgoing portfolio chosen at date t. Thus, the direct fiscal impact of price responses depends both on the strength of the price response $\varphi'_{k,t}(\widetilde{B}_t^k)$ and on the quantity of re-balancing of bond k, $\Delta_t^k = \widetilde{B}_t^k - \widetilde{B}_{t-1}^{k+1}$.

The second change is that price responses affect welfare of households directly. This effect is captured, up to $\beta^t \Pr(s^t)$, by $M_t \varphi'_{k,t}(\widetilde{B}_t^k)\delta_t^k$, where $\delta_t^k = \widetilde{b}_t^k - \widetilde{b}_{t-1}^{k+1}$ are portfolio re-balancing of households. The intuition for this term is similar to that of the government, but it has the opposite sign. While $\Delta_t^k > 0$ implies that price response "hurts" the government by requiring it raise taxes to compensate the revenue shortfall, $\delta_t^k > 0$ implies that households benefit from lower bond prices. If households buy a bond from the government, lower bond prices transfer resources from the government to households.

We now derive optimality conditions implied by the debt level and portfolio perturbations. For simplicity, we assume that δ_t^k is small relative to Δ_t^k. Using the envelope theorem, we obtain generalizations of equations (10) and (11),

$$\frac{1}{\xi_t} - \frac{1}{\xi_t}\sum_k \omega_t^k \varphi'_{k,t}(\widetilde{B}_t^k)\Delta_t^k = \mathbb{E}_t \frac{\beta M_{t+1}}{M_t}\frac{1}{\xi_{t+1}}\mathcal{R}_{t+1} \text{ for all } t, \tag{34}$$

$$-\frac{1}{\xi_t}\varphi'_{k,t}(\widetilde{B}_t^k)\Delta_t^k = \mathbb{E}_t \frac{\beta M_{t+1}}{M_t}\frac{1}{\xi_{t+1}} r_{t+1}^k \text{ for all } k, t. \tag{35}$$

If we compare these equations to their analogues in the small open economy, (10) and (11), we see that price responses affect both the debt level dynamics and portfolio composition in proportion to the costs of portfolio re-balancing, $\{\varphi'_{k,t}(\widetilde{B}_t^k)\Delta_t^k\}_k$.

To derive portfolio implications of these equations we take the small noise expansion as in Section 3. Similarly to our study of the liquidity premium in Section 4.3, we consider small noise expansions that scale price effect functions $\{\varphi_{k,t}(\cdot)\}_{k,t}$ with σ^2. The motivation for this approximation is two-fold. Empirically, price responses appear to be comparable to risk premia.[13] Moreover, many microfounded models of price responses, such such as Vayanos and Vila (2021) or the closed economy as in online Appendix E, have the property that price responses are pinned down by the second order moments because they represent compensation for holding duration risk. Proceeding as in Section 4.3, one can show that part (a) of Lemma 2 still holds in this economy but part (b) can now be stated as

$$cov_t\left(r^k_{t+1}, \partial\tau_{t+1}\right) \simeq -\frac{\xi\left(\tau_t\right)}{-\xi'\left(\tau_t\right)Q^1_t}\Delta^k_t\varphi'_{k,t}(\widetilde{B}^k_t). \tag{36}$$

This equation shows the optimal portfolio equalizes covariance of taxes and excess returns of a bond of maturity k to the costs of rebalancing the bond.

Using this optimality condition it is easy to adapt arguments of Section 3 to derive the optimal portfolio. To state the result using intuitive terms, define ω^+_{t-1} be the vector of $\{Q^k_t\widetilde{B}^{k+1}_{t-1}/B_t\}_{k\neq 1}$. Vector ω^+_{t-1} has a simple economic interpretation. These are shares computed by evaluating holdings of securities purchased in $t-1$ at period t prices and normalized by the market value of total debt in period t. The difference $\omega_t - \omega^+_{t-1}$ captures portfolio re-balancing.

Let D_t be a diagonal matrix with elements $\frac{\sum_{k=1}^{\infty}Q^k_t}{\left(Q^1_t\right)^2}\frac{\xi^2(\tau_t)}{-\xi'(\tau_t)}\frac{Y_t}{Q^k_t}$ and Λ_t be a diagonal matrix with of semi-elasticities $\{\varphi'_{k,t}(\widetilde{B}^k_t)\}_{k\neq 1}$. The optimal portfolio satisfies, to the second order,

$$\Sigma_t\omega_t \simeq \Sigma^Q_t s^Q_t + \Sigma^G_t s^G_t - D_t\Lambda_t\left(\omega_t - \omega^+_{t-1}\right), \tag{37}$$

which is a generalization of equation (18) to the case when prices of government bonds respond to their supplies. When matrix Σ_t is invertible, it can be written as

$$\omega_t \approx \omega^*_t - \Sigma^{-1}_t D_t\Lambda_t\left(\omega_t - \omega^+_{t-1}\right), \tag{38}$$

where ω^*_t is given by (19). This equation has a simple interpretation. In the absence of price responses, the planner ought to choose portfolio ω^*_t characterized in previous sections. When prices respond to portfolio re-balancing, the optimal portfolio depends both on ω^*_t and the portfolio chosen in the previous period, ω^+_{t-1}. The relative importance of these two portfolios on ω_t is determined by the covariance-adjusted matrix of price responses, $\Sigma^{-1}_t D_t\Lambda_t$.

Similarly to our discussion in Section 3.1, as long as debt levels are chosen optimally to the first order, the dependence of matrices Σ_t, Σ^Q_t, Σ^G_t, D_t and weights s^Q_t, s^G_t on portfolio choice ω_t and future government policies are all of the third order in equation (37) and thus

[13] For instance, Greenwood and Vayanos (2014) document that a one standard deviation change in their preferred measure of bond supply affects bond yields by a 10 basis points at the short end and up to 40 basis points for longer maturities. These responses are similar in order to risk premia on those bonds that range typically between 50 to 150 basis points at the annual frequency.

drop out from equation (37). Matrix Λ_t depends on ω_t to the second order and so, in order to explicitly obtain optimal portfolio using (38), one needs to estimate this dependency and solve for a fixed point. One tractable case is when price semi-elasticities are constant, i.e., $\{\varphi_{k,t}(\cdot)\}_k$ can be written as $\varphi_{k,t}(\widetilde{B}_t^k) = \lambda_k \widetilde{B}_t^k$. In this case, Λ_t is simply a diagonal matrix of semi-elasticities $\{\lambda_k\}_k$ and (38) is a linear equation that is easy to invert.

In the discussion above, we assumed that the demand for the one period bond is perfectly elastic. This assumption is easy to relax. In online Appendix B, we derive the counterpart of expression (38) when demand for one-period bonds is imperfectly elastic. The optimal portfolio formula now features extra terms that account for adjustments of the risk-free bond holding but preserve all the insights of equation (38).

5 Quantitative application

In Sections 3 and 4, we obtained formulas for optimal public portfolios in various settings in which key objects have empirical counterparts. In this section, we use U.S. data to quantify those objects, derive implied optimal public portfolios, and compare them with observed U.S. debt portfolios.

Since bonds are the securities governments trade most often in order to respond to business cycle shocks, we focus on optimal portfolio of bonds.[14] We start with terms in the expression for ω_t^* from equation (24) as they are more straightforward to take to data and those terms continue to show up in more general settings. We call ω_t^* as the *target portfolio*. After quantifying the target portfolio, we discuss additional terms that arise from incorporating price impacts and household heterogeneity.

5.1 Data

We use the U.S. national income and product accounts to measure output, tax revenues, and government spending. We use data on average marginal tax rates from Barro and Redlick (2011) extended to 2017. To measure returns on government debts of different maturities, we use the Fama Maturity Portfolios published by CRSP. There are 11 such portfolios, of which ten portfolios correspond to maturities of 6 to 60 months in 6 months intervals, and a final portfolio for maturities between 60 and 120 months. We add a twelfth portfolio that consists of the nominal 3-Month Treasury Bill, published by the Federal Reserve Board of Governors. We use data from Gurkaynak et al. (2007) to estimate the yield curves. All data are quarterly,

[14] Since most U.S. public debt is in the form nominal bonds, we use the nominal versions of the optimal portfolios. See online Appendix C for details.

nominal, and extend from 1952 to 2017. More details about data sources and data construction are in online Appendix C.

In Table 1, we present summary statistics of contemporaneous covariances, means, and autocorrelations. For convenience, all variables are multiplied by 100 and reported in quarterly percentage points. Several patterns that emerge from this table will play an important role in shaping an optimal portfolio. Covariances of excess returns of government bonds of different maturities are several orders of magnitude larger than covariances of excess returns with primary surpluses and tax rates. Furthermore, covariances of excess returns with primary surpluses have a negative sign. This reflects that the primary government surplus is procyclical, but that bond excess returns are countercyclical.

Table 1: COVARIANCE MATRIX

	Excess returns r_t^j for various maturities j											Surplus to GDP	Tax rate
	6m	12m	18m	24m	30m	36m	42m	48m	54m	60m	120m	X_t/Y_t	τ_t
6m	0.092	0.2	0.29	0.36	0.43	0.48	0.5	0.53	0.56	0.61	0.69	-0.01	0.017
12m		0.49	0.73	0.91	1.1	1.2	1.3	1.4	1.5	1.6	1.8	-0.10	-0.021
18m			1.1	1.4	1.7	1.9	2.1	2.2	2.4	2.6	3	-0.17	-0.027
24m				1.8	2.2	2.5	2.7	3	3.1	3.5	3.9	-0.26	-0.068
30m					2.8	3.2	3.5	3.7	3.9	4.4	5	-0.31	-0.091
36m						3.6	4	4.3	4.5	5.1	5.8	-0.40	-0.081
42m							4.4	4.8	5.1	5.6	6.5	-0.45	-0.140
48m								5.4	5.6	6.2	7.2	-0.50	-0.180
54m									6.1	6.7	7.7	-0.56	-0.190
60m										7.8	8.6	-0.62	-0.170
120m											10	-0.75	-0.290
X_t/Y_t												4.30	0.940
τ_t													1.900
Mean	0.076	0.14	0.2	0.23	0.26	0.3	0.33	0.33	0.36	0.29	0.44	2.5	30
Autocorr	-0.11	-0.08	-0.09	-0.08	-0.09	-0.07	-0.05	-0.03	-0.04	-0.07	-0.03	0.96	0.92

Notes: Excess returns 6m, 12m, ... are the nominal excess returns in Fama maturity portfolios corresponding to 6-12 months, 12-18 months, ... maturity bins, respectively. Surplus is measured as federal tax receipts (including contributions to social insurance) less federal government consumption expenditure (including transfer payments to persons) from the BEA. The tax rates series is detrended average marginal tax rate on income computed by Barro and Redlick (2011) and extended to 2017. All data are quarterly and in percentage points. All series are for 1952-2017.

5.2 Target portfolio

Equation (24) indicates that to compute the target portfolio ω_t^*, one needs to specify a market structure, a labor supply function $v(\cdot)$, and measure two sets of objects: (i) covariances Σ_t, Σ_t^Q, Σ_t^G, Σ_t^Θ and (ii) weights s_t^Q, s_t^G, s_t^Θ. For market structure, we assume that the government can invest in pure discount bonds from maturity 1 quarter to N quarters. For labor supply, we assume $v_t(Y_t) = \Theta_t^{-1/\gamma} \frac{Y_t^{1+1/\gamma}}{1+1/\gamma}$ as in Section 4.2 and use the expression $\Theta_t = \frac{Y_t}{(1-\tau_t)^\gamma}$, to back out $\ln \Theta_t$ from data on output and tax rates.

We set the labor supply elasticity parameter $\gamma = \frac{1}{2}$. We set Θ_t to normalize $Y_t = 1$ and $B_t = 4$ to get a debt to annual GDP of 100%. We set $\left(G_t, \left\{Q_t^k\right\}_k\right)$ at their sample averages. In particular, $G_t = 15\%$ and the values for $\{Q_t^k\}_k$ are reported in panel (b) of Figure 1. We compute T_{t+k}^{tax} using the optimal tax rates that satisfy the zeroth order budget constraint given B_t and G_t. For our calibrated parameters and functional forms, these tax rates are constant across periods and equal 18.7%.

Measuring the weights and covariances is more challenging. While in Table 1 we reported sample counterparts of ergodic covariances, our theory requires us to measure the covariances and weights conditional on date t information set. Second, our formulas require an inverse of the covariance matrix of returns, Σ_t^{-1}. It is known that simply calculating an in-sample covariance matrix and then taking its inverse can lead to large sampling errors.[15] Third, we need to measure not only covariances of returns with contemporary realizations of various macroeconomic variables but also their realizations at all future horizons. Finally, the weights $\left\{s_t^Q, s_t^G, s_t^\Theta\right\}_t$ require estimating conditional means of future spending and TFP.

We overcome these challenges by adopting a parsimonious dynamic factor structure representation.[16] We start with a particularly simple representation. This simple representation transparently maps the estimated coefficients to the theoretical objects in the optimal portfolio formulas. Additionally, it enables us to emphasize the key quantitative insights that remain relevant in the more advanced factor models discussed later in this section.

Let z_t be a stacked vector that consists of excess returns $\left\{r_t^j\right\}_j$ for the 11 portfolios of different maturities j, a measure of $\ln \Theta_t$ and expenditures $\ln G_t$. We use z_t^ι to denote the ι^{th} element of this vector, with $\iota \in \{G, \Theta\}$ corresponding to series for $\ln G_t$ and $\ln \Theta_t$, and $\iota = j$

[15] See, for example, early work by Jobson and Korkie (1980), Merton (1980), Michaud (1989) and later work by Jagannathan and Ma (2003) and DeMiguel et al. (2007).

[16] Factor representations are popular in finance for estimating Σ_T^{-1} (see, e.g., MacKinlay and Pastor (2000), Chan et al. (1999), Senneret et al. (2016)). We superimpose a VAR structure on the factor model to obtain covariance estimates at all leads and lags. This extension is similar in spirit to the Factor Augmented Vector Auto Regressions (FAVAR) literature (see, e.g. Bernanke et al. (2005) and Bai et al. (2016)).

corresponding to the returns on the j^{th} maturity. We posit the following stochastic process

$$z_t^\iota = \alpha + z_{t-1}^\iota + \kappa_\iota f_t + \varepsilon_t^\iota \text{ for } \iota \in \{G, \Theta\},$$
$$z_t^j = \alpha_j + \rho_j z_{t-1}^j + \kappa_j f_t + \varepsilon_t^j \text{ for all } j, \quad (39)$$

where f_t is a factor, which we take to be here the first principal component extracted from observed excess returns, primary surplus, output, and the risk-free rate, and ε_t^ι are residuals with variances σ_ι^2. For our baseline specification we assume that $f_t = \alpha_f + \varepsilon_t^f$, where ε_t^f are homoskedastic innovations with variance σ_f^2.

This specification implies a very simple and transparent structure on conditional covariances and means. The expected growth rate of aggregate variables $\ln G_t$ and $\ln \Theta_t$ is constant and equal to α. Each variable ι moves because of common shocks, captured here parsimoniously by the factor f_t, and idiosyncratic disturbances. Factor loading κ_ι captures how much each variable responds to the common shock. The date t conditional variance of any variable z_{t+1}^ι is $var_t(z_{t+1}^\iota) = \kappa_\iota^2 \sigma_f^2 + \sigma_\iota^2$, and equal to the sum of a common and the idiosyncratic component. The conditional covariances satisfy $cov_t(z_{t+1}^\iota, z_{t+1}^j) = \kappa_\iota \kappa_j \sigma_f^2$ for all ι, $j \neq \iota$, and $cov_t(z_{t+k+1}^\iota, z_{t+1}^j) = \kappa_\iota \kappa_j \sigma_f^2$ for $\iota \in \{G, \Theta\}$, and all j, and k.[17]

Table 2 reports the estimates of the simple factor model. The factor captures about 90% of the variation in the returns and panel (a) of Figure 1 reports the time-series for the common factor. The return loadings are all statistically significant and are monotonically increasing in maturities. The factor loadings of $\ln G$ and $\ln \Theta$ are statistically significant and have the same signs. This means that spending and tax revenues co-move with returns and partly offset each other when we consider movements in primary surpluses.

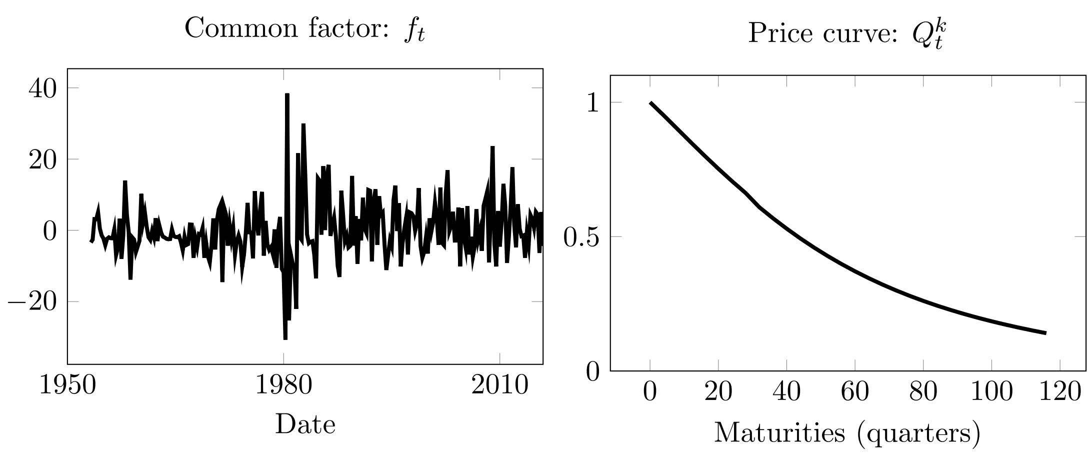
[17] In online Appendix C, we show how the covariances implied by the factor model compare to the raw covariances measured in the data for all versions of the factor model that we use in our analysis.

Table 2: FACTOR MODEL ESTIMATION (BASELINE)

	Excess returns r_t^j for various maturities j													
	6m	12m	18m	24m	30m	36m	42m	48m	54m	60m	120m	$\ln G_t$	$\ln \Theta_t$	f_t
α_k	0.086	0.155	0.220	0.245	0.284	0.315	0.346	0.344	0.372	0.304	0.444	0.005	0.005	0.024
	(0.014)	(0.025)	(0.033)	(0.035)	(0.039)	(0.039)	(0.038)	(0.037)	(0.037)	(0.043)	(0.030)	(nan)	(nan)	(0.501)
ρ_k	-0.107	-0.057	-0.041	-0.043	-0.042	-0.025	-0.022	-0.008	-0.022	-0.027	0.003	1.000	1.000	0.000
	(0.043)	(0.035)	(0.030)	(0.025)	(0.023)	(0.020)	(0.018)	(0.016)	(0.015)	(0.015)	(0.009)	(nan)	(nan)	(nan)
κ_k	0.028	0.074	0.118	0.157	0.199	0.230	0.257	0.285	0.306	0.345	0.404	-0.032	-0.047	0.000
	(0.002)	(0.003)	(0.004)	(0.004)	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)	(0.004)	(0.016)	(0.008)	(nan)
σ_k^2	0.044	0.154	0.267	0.300	0.378	0.384	0.356	0.345	0.341	0.460	0.222	4.231	1.147	63.753
	(0.004)	(0.014)	(0.024)	(0.027)	(0.034)	(0.034)	(0.031)	(0.031)	(0.030)	(0.041)	(0.020)	(0.375)	(0.102)	(5.637)
R2	0.536	0.698	0.771	0.840	0.870	0.898	0.922	0.938	0.946	0.943	0.979	0.015	0.109	0.000

Notes: This table records the OLS estimates of the factor model (39). Standards errors are in parenthesis. The sample for excess returns and primary surpluses normalized by outputs is 1952-2017. The time period is a quarter.

Figure 1: INPUTS FOR TARGET PORTFOLIO



(a) (b)

Notes: The left panel plots the time series for the common factor. The common factor is extracted as the first principal component from observed returns, the government surplus, output, and the risk-free rate. The right panel plots the bond price curve. We use data from Gurkaynak et al. for the period 1952-2017 to compute average yields for maturities spaced 4 quarters apart and then interpolate the yields.

We use this factor model to construct the target portfolio using formula (24). Since that formula requires interest rates and returns for horizons beyond the twelve CRSP maturities, we extrapolate factor loadings and volatilities of missing maturities using exponential curves of the form $e^0 - e^0 \exp\left(-e^1 j\right)$, where the coefficient e^1 captures the slope and the coefficient e^0 bounds the range of values between $\left[0, e^0\right]$. We provide additional details about the fit and discussion of robustness in online Appendix C.

To implement formula (24), covariances Σ_t, Σ_t^G, Σ_t^Θ can be directly constructed using our observations above. We construct Σ_t^Q using the relationship $cov_t(\ln Q_{t+1}^k, r_{t+1}^j) \simeq cov_t(r_{t+1}^k, r_{t+1}^j)/Q_t^1$. Weights $s_t^G[k]$, $s_t^\Theta[k]$, and $s_t^Q[k]$ can be constructed from our estimates since to the appropriate order of approximation they satisfy $\frac{Q_t^k \Gamma^k G_t}{Q_t^1 B_t}$, $\frac{Q_t^k \Gamma^k T_t}{Q_t^1 B_t}$, and $\frac{Q_t^{k+1}\Gamma^{k+1}}{Q_t^1 \sum_{k=1}^\infty Q_t^k \Gamma^k}$ where $\Gamma := \exp(\alpha)$ and α is the growth rate of log G as well as $\ln \Theta$. The terms $\{Q_t^k \Gamma^k\}_k$ will play an important role in the expressions for the optimal portfolios. Recall that Q_t^k is the period-t price of a bond with maturity k (see Figure 1 (b)) so $Q_t^k \Gamma^k$ is the price of that bond adjusted by the expected growth rate Γ^k that occurs by the time this bond matures. We use $\hat{\beta}_t := 1 - \frac{1}{\sum_{k=1}^\infty Q_t^k \Gamma^k}$ to denote the "discount factor" implied by this growth-adjusted price curve.

Using these observations, we can construct the target portfolio ω_t^* for any arbitrary set of bond maturities. We denote the set of available maturities as $\mathcal{G}$. For most of our discussion, we take $\mathcal{G}$ to consist of all maturities up to 30 years, which in our quarterly data specification means $\mathcal{G} = \{1, 2, ..., 120\}$. This choice is in line with issuance practices of the U.S. government. We also discuss implications of choosing other sets $\mathcal{G}$.

We present the target portfolio as a sum of two portfolios, $\omega_t^* = \omega_t^X + \omega_t^Q$ where ω_t^X, ω_t^Q have elements

$$\omega_t^X[j] = \left(\frac{1}{1-\hat{\beta}_t}\right)\left(\frac{\kappa_\Theta T_t - \kappa_G G_t}{Q_t^1 B_t}\right)\left(\frac{\kappa_j}{\sigma_j^2}\chi^2\right), \tag{40}$$

$$\omega_t^Q[j] = \left(1-\hat{\beta}_t\right)\left(\sum_{\ell \notin \mathcal{G}} Q_t^{\ell+1}\Gamma^{\ell+1}\kappa_\ell\right)\left(\frac{\kappa_j}{\sigma_j^2}\chi^2\right) + \left(1-\hat{\beta}_t\right) Q_t^{j+1}\Gamma^{j+1}, \tag{41}$$

and the constant $\chi^{-2} := \sigma_f^{-2} + \sum_{i\in\mathcal{G}} \kappa_i^2 \sigma_i^{-2}$. Portfolios ω_t^X, ω_t^Q have natural economic interpretation. Portfolio ω_t^X equals $\Sigma_t^{-1}\Sigma_t^G s_t^G + \Sigma_t^{-1}\Sigma_t^\Theta s_t^\Theta$ and is the portfolio that hedges the primary surplus risk $\{T_{t+k} - G_{t+k}\}_k$. Portfolio $\omega_t^Q = \Sigma_t^{-1}\Sigma_t^Q s_t^Q$ hedges the interest rate risk.

Formulas (40) and (41) contain several observations about how these risks are hedged. Equation (40) shows how shocks to primary surpluses are hedged. This expression shows that maturities with higher values of $\frac{\kappa_j}{\sigma_j^2}$ have a bigger weight in the portfolio that hedges primary surpluses. This ratio has a following interpretation. Loading κ_j captures how much the returns of bonds of maturity j co-move with a common factor, and σ_j^2 captures the volatility of the component of returns that is orthogonal to the common factor. Thus, $\frac{\kappa_j}{\sigma_j^2}$ is a measure of how good maturity j is at hedging macroeconomic shocks. The other parameters in (40) are scaling terms: $\frac{\kappa_\Theta T_t - \kappa_G G_t}{Q_t^1 B_t}$ is the hedgeable part of primary surpluses in period t in the market value of debt, and $(1-\hat{\beta}_t)^{-1}$ converts that statistic into the present value, which under our balanced growth path factor structure takes a particularly simple form.

Equation (41) shows that the portfolio that hedges interest rate risk has a related but distinct structure. This portfolio consists of two terms. The first term on the right hand side of (41) has a similar structure to the right hand side of (40), that is, portfolio weights depends on the ratio $\frac{\kappa_j}{\sigma_j^2}$ and scaling terms. Importantly, the scaling terms depend only on the interest rates of the maturities not included in $\mathcal{G}$, that is, $\{\ell \notin \mathcal{G}\}$. So that if $\mathcal{G}$ has maturities for the first 30 years, this term captures fluctuations in long interest rates beyond the 30 year horizon.

The second term in (41) has a very simple structure in which holdings of maturity j is proportional to $Q_t^{j+1}\Gamma^{j+1}$. To understand the reason for this structure, and why it is different from (40), recall the maturity matching principle from the discussion of equation (20). When all maturities are available, the maturity matching principle said that a good way to hedge interest rate risks was to align the quantity of debt to the path of expected primary surpluses. With our balanced growth specification, expected deficits grow at rate Γ, and this principle will imply portfolio shares that are proportional to $Q_t^{j+1}\Gamma^{j+1}$.

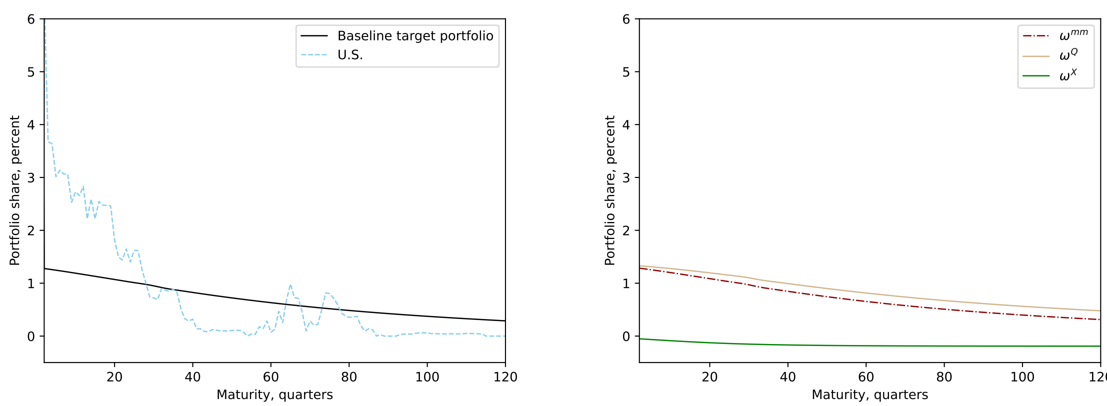
This discussion highlights several take-aways. First, the target portfolio must have a component driven by maturity matching. In the context of a market structure with a cap of the maximum maturity, we denote the component driven by maturity matching as $\omega_t^{mm}[j] \equiv (1-\hat{\beta}_t) \times \{Q_t^{j+1}\Gamma^{j+1}\}_j$ for all available maturities $j \in \mathcal{G}$. The deviations of ω_t^* from ω_t^{mm} are determined by quantitative strength of two forces: the ability of available bonds hedge primary surpluses and very long interest rates. Second, if we increase the number of maturities in $\mathcal{G}$, the relative importance of the second force declines.

We now use our estimation to construct the target portfolio ω_t^*. We choose $\mathcal{G}$ to consist of first 30 year maturities and plot implies ω_t^* in Figure 2 (a). For comparison, we also plot the actual U.S. portfolio of government bonds in 2017. Both graphs show that portfolio shares decline in maturities, roughly geometrically, but the U.S. portfolio overweights short maturities and underweights long maturities. Thus, the duration of the target portfolio is longer. The Macaulay duration, which measures the weighted average time to maturity of cash flows, is approximately 5 years for the U.S. portfolio and 9.6 years for the target portfolio.

Panel (b) of Figure 2 sheds light on what determines quantitative properties of ω_t^*. Here we plot component portfolios ω_t^Q, ω_t^X, and ω_t^{mm}. This panel shows that ω_t^* is extremely similar to ω_t^{mm}. This result comes from the interaction of two forces. The interest rate risk hedging portfolio ω_t^Q has a bit longer duration than ω_t^{mm}, since longer maturities are better at hedging interest rate risk beyond the 30 year horizon. At the same time ω_t^X has negative weights, reflecting the fact that primary surpluses co-move negatively with returns in the data, and so negative holdings of those bonds hedge risk.[18] These two effects approximately cancel out so that ω_t^* resembles portfolio ω_t^{mm}.

Another observation from Figure 2 is that bonds offer modest ability for the government for hedging of the primary surpluses. The ratio $\frac{\kappa_j}{\sigma_j^2}$ peaks for medium-duration maturities but even then the role of these bonds in hedging primary surpluses is fairly small. One way to quantify the importance of hedging of interest rate risk vs primary surplus risk is to compute shares $\frac{\|\omega_t^Q\|_1}{\|\omega_t^Q\|_1+\|\omega_t^X\|_1}$ and $\frac{\|\omega_t^X\|_1}{\|\omega_t^Q\|_1+\|\omega_t^X\|_1}$, where $\|\cdot\|_1$ denotes the l_1 norm (i.e., the sum of absolute values). Using this metric, the importance of hedging of interest rate risk is approximately 85% in the target portfolio, and the importance of hedging of the primary surplus risk is 15%. The poor ability of bonds to hedge primary surplus risk should not be surprising given the observations in Table 1. As we highlighted in that table, covariances of returns on bonds with macroeconomic variables are fairly low, especially compared to the volatility of those returns.

Figure 2: TARGET PORTFOLIO, COMPONENTS, AND U.S. PORTFOLIO



(a) (b)

Notes: Portfolio shares of securities with maturities from 2 quarters to 120 quarters. In panel (a) we plot the target portfolio and compare it to the 2017 U.S. debt portfolio. In panel (b) we plot the maturity matching portfolio ω_t^{mm} and the two components of the target portfolio that hedge interest rate risk ω_t^Q, and primary surplus risk ω_t^X, respectively.

We next discuss the role of certain assumptions made in the baseline calculations.

Multifactor models Our baseline empirical specification (39) assumes that there is one factor f_t. The bond pricing literature going back to the seminal work of Litterman and

[18]For instance, consider states when tax revenues are low. The negative covariance between long maturity rates and primary surplus means that long interest rates will be high in those states. Thus, issuing fewer long duration bonds is helpful because it offsets the loss in tax revenues with lower debt service costs without raising distortionary taxes. The negative (or long) positions in ω_t^X capture this tradeoff.

Scheinkman (1991) showed that a small number of factors explains vast fraction of volatility of bond returns but typically uses more than one factor.[19] We now discuss a multi-factor extension of our factor model (39).

Our general multifactor specification replaces $\kappa_\iota f_t$ with $\sum_m \kappa_{m,\iota} f_t^m$ where m is the index for factors. In this section, we present the result for a two-factor model where factors correspond to the first two principal components of the observed bond returns, the government surplus, output, and the risk-free rate. This two-factor specification explains over 98% of bond excess returns.[20]

We relegate most of the details about the estimation to online Appendix Cand briefly summarize the main takeaways here. The second factor is less volatile as compared to the first factor. The factor loadings on both factors are statistically significant for all the returns but have different shapes as a function of maturity. The loadings on the first factor are monotonically increasing in maturity, while those on the second factor are hump shaped. This corresponds to the level and curvature factors in the Litterman and Scheinkman (1991) terminology. Finally, only the first factor has statistically significant loadings for $\ln G_t$ and $\ln \Theta_t$. Overall, while the two factors are necessary for a more accurate description of returns, the second factor matters little for spending and tax revenue risk.

The multi-factor specification preserves the tractability of our one-factor model. One can still derive the analogue of expressions (40) and (41) and those expressions capture the same economic forces that we emphasized in our baseline specification. We plot the target portfolio ω_t^* implied by the two-factor model in Figure 3(a).[21] For comparison, we also plot the U.S. portfolio and the growth-adjusted price curve ω_t^{mm} that are the same as in Figure 3(a). The two-factor specification slightly shifts portfolio towards shorter maturities but the overall difference from the baseline specification is small, with second factor reducing the Macaulay duration from 9.6 to 9.5 years.

In our empirical specification, we built on the work of Litterman and Scheinkman (1991) and others (for instance, Campbell et al. (1998), Cochrane and Piazzesi (2005), and Ludvigson and Ng (2009)) who express bond returns as being driven by factors common to all maturities and the residuals that are maturity specific. A particularly convenient feature of this approach is that the covariance matrix is easily invertible. There exists another tradition in finance, the

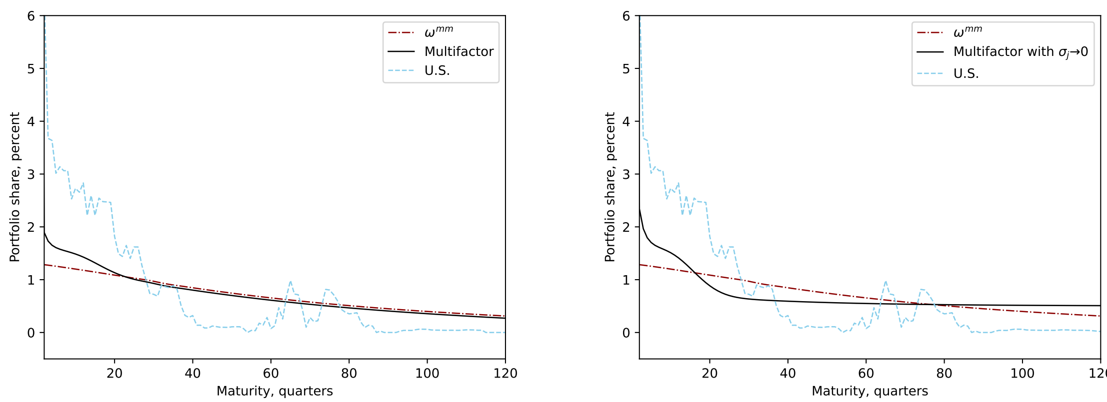
[19] For example, Litterman and Scheinkman used a 3 factor model for bond excess returns. They show that the first three principal components of bond returns explained more than 96% of variation in their sample. Based on the shape of the factor loadings, they interpreted the factors as *level, slope,* and *curvature.*

[20] We experimented with adding more factors and did not find any meaningful changes in the results.

[21] Here and in the rest of the section, we follow a convention of setting statistically insignificant regression coefficients to zero when we implement our formulas.

so-called affine term structure literature, that assumes that it is the stochastic discount factor process that is driven by a small number of factors, and then uses that process to derive prices and returns of bonds of various maturities (see, e.g., Dai and Singleton, 2000, Piazzesi, 2010). This approach uses no maturity specific shock ε_t^j, so the covariance matrix Σ_t is not invertible, which implies that there are multiple optimal portfolios. We assess the role of idiosyncratic shocks ε_t^j in our setting, by sending the estimated $\sigma_j^2 \to 0$ for each j and kept the factor loadings $\{\kappa_{m,\iota}\}_{m,\iota}$ at their estimated values. We report our findings in panel (b) of Figure 3. This figure shows that the limiting target portfolio is quite similar to the target portfolio computed using the estimated $\{\sigma_j^2\}_j$. This is not surprising given that the orthogonal variation captures less than 2% of the variation in returns.

Figure 3: ROLE OF MULTIPLE FACTORS



(a) (b)

Notes: Portfolio shares of securities with maturities from 2 quarters to 120 quarters. In panel (a) we compare the maturity matching portfolio ω_t^{mm} to the target portfolio with multiple factors and the U.S. debt portfolio. In panel (b) we compare the target portfolio with multiple factors to the limiting target portfolio as send the loadings on idiosyncratic components $\sigma_j^2 \to 0$.

Departures from stationarity The baseline factor model assumed stationarity implying that expected deficits and output grow at a constant rate and covariances are constant. In online Appendix C, we discuss several departures from stationarity. Here we summarize the main results.

First, we turn on the autoregressive components in dynamic factor model. The optimal portfolio is largely same with these changes. This is because the excess returns are not very autocorrelated so the estimated ρ_f is close to zero.

Next we investigate predictability in drivers of primary deficits. First, we estimate the top equation of (39) with a more flexible autoregressive structure. We cannot reject that spending and TFP are unit roots. However, economies sometimes experience transitory increases in spending levels. For instance, the public spending during the COVID-19 pandemic represents such a case. Our expressions for the target portfolio provide guidance on how the optimal portfolio should respond to these temporary shocks. In online Appendix C, we consider several experiments with unexpected transitory increases in spending at date t parameterized by the size of the initial impulse and the speed of mean reversion.

The general pattern is that after a transitory increase in spending, the target portfolio "tilts" with lower holdings of short maturity debt and higher holdings of long maturity debt. For a given debt level, high transitory spending leads to lower primary surpluses in the short-run and higher primary surpluses in the long run. The maturity matching motive implicit in the inter-temporal weights on the interest rate hedging component calls for down-weighting maturities when expected surpluses are low and shift the portfolio towards longer maturities when primary surpluses are high.

We also allow for time-varying covariances which we estimate using a Generalized Autoregressive Conditional Heteroskedasticity (GARCH) structure. With heteroskedastic shocks, there is in-sample variation of the covariances and this would generate variation in portfolios even if we kept $\left(G_t, \left\{Q_t^k\right\}_k\right)$ unchanged. However, we find that the portfolios are quite stable. The is because the time-varying volatility of the common factor shows up in both the covariances in returns with spending, tax revenues and the covariances of returns with each other. Since the optimal portfolio depends on the ratio of these two covariances, the effect of time-varying risk is muted in how it affects the target portfolio.

5.3 Price effects

In our previous discussion, we used equation (24) that implicitly assume that prices of government bonds do not depend on bond supplies. A large empirical finance literature (see Krishnamurthy and Vissing-Jorgensen (2012), Greenwood and Vayanos (2014) and more recently Mian et al. (2022)) has documented that changes in supply of bonds affect their prices. In this section, we use empirical estimates of price responses to evaluate optimal portfolio formation with price effects. For quantitative evaluation, we will use the generalization of the setup from Section 4.5 that allows for price effects for bonds of all maturities.[22]

To compute the optimal portfolio with price effects, we need to estimate equation (32)

[22] See online Appendix B for the exact formulas and more details.

which includes nonlinear functions $\left\{\varphi_{k,t}\left(\tilde{B}_t^k\right)\right\}_k$. Following the discussion from Section 4.5, we assume that $\left\{\varphi_{k,t}\right\}_k$ are affine functions of the face value of outstanding debts and thus semi-elasticities $\left\{\varphi'_{k,t}\right\}_k$ are constants. We then leverage estimates from existing literature to recover these constants.

There is a growing empirical finance literature that estimates semi-elasticities $\left\{\frac{\partial \ln \text{yields}_t^k}{\partial \text{Bond supply}_t}\right\}_k$, which capture the effect of changes in bond supply on bond yields of different maturities. These elasticities are obtained using various instrumental variable designs that capture exogenous supply-shifters. A common finding is that these elasticities are positive and increasing in maturities. To incorporate price effects, in principle, we need estimates of semi-elasticities $\left\{\frac{\partial \ln \text{yields}_t^k}{\partial \tilde{B}_t^k}\right\}_k$ for all k, but typically the literature provides such estimates only measures of total supply such as in total debt in Krishnamurthy and Vissing-Jorgensen (2012) or maturity-weighted debt in Greenwood and Vayanos (2014). Under the assumption that the portfolio shares are constant when total debt changes, we can back out semi-elasticities we need to construct price effects. In online Appendix C, we use this assumption and estimates from Greenwood and Vayanos (2014) to construct the matrix Λ_t.

We can now describe the optimal portfolio of public debts with price effects. We use a version of formula (38) that relaxes the assumption that demand for risk-free bond is perfectly elastic.[23] Formula (38) and its generalizations prescribes a non-trivial dependence of portfolio ω_t on portfolio ω_{t-1}. To facilitate comparisons with the target portfolio ω_t^* we focus on $\omega^{ss} = \lim_{t\to\infty} \omega_t$, or the the long run portfolio when the transition dynamics have settled down.[24]

Optimal Portfolio Figure 4 panel (a) reports the optimal steady state portfolio ω^{ss} in our preferred habitat model and compares it to the target portfolio in the Section 5. We see that the optimal portfolio with price effects sits in between the optimal portfolio without price effects and the observed U.S. portfolio.

With price effects, the government faces an additional trade-off when issuing longer maturities relative to hedging motives. As discussed in Section 5, issuing long maturities helps hedge interest rate risk, but it requires constant rebalancing due to a cap on the maximum maturity. For instance, the optimal portfolio without price effects that we computed in Section 5 uses all available maturities. Since the maximum maturity is 30 years, to maintain that portfolio, every period the government has to issue new 30 year bonds. Our estimates of price effects

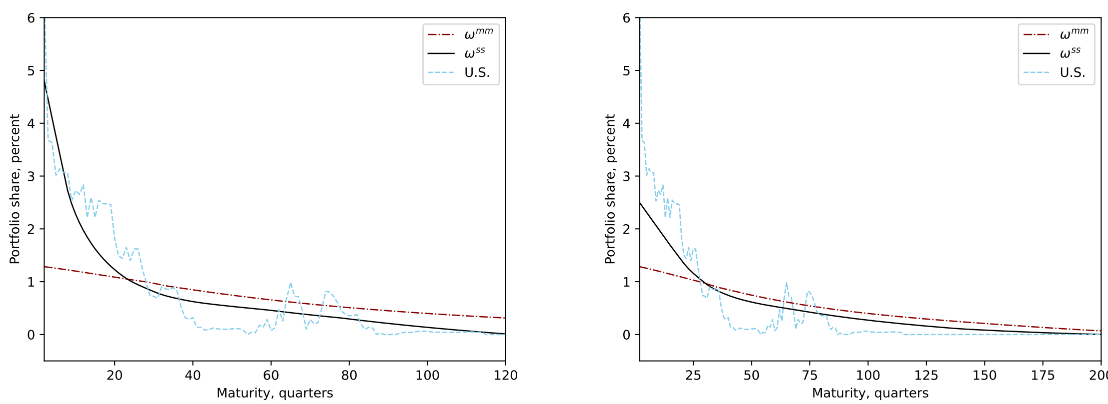
[23]See equation (60) in online Appendix B.

[24]For computing the long run portfolio, we need specify how matrices D_t and Λ_t evolve with t. Under the assumption that $\left\{Q_t^k\right\}$ are set to sample averages and implication of government optimality that $\mathbb{E}_t \tau_{t+\ell} = \tau_t$, we have $D_{t+\ell} = \Gamma^\ell D_t$ where Γ is the growth rate of output along the balanced growth path. Under our assumption that the semi-elasticities $\varphi'_{k,t}$ are constants, and we have $\Lambda_{t+\ell} = \Lambda_t$.

suggest that long bonds are expensive to reissue. Thus, to economize the cost of issuances, the optimal portfolio with price effects tilts towards shorter maturities and away from longer maturities. For our calibration, the Macaulay duration of the optimal portfolio is 6.15 years. This duration is still higher than the U.S. debt portfolio which is about 5 years but lower than the duration of the optimal portfolio ignoring the price effects which was 9.6 years.

This discussion also suggests that the tradeoff between costly reissuances and hedging depends on the cap on the maximum maturity. The ability to issue longer maturities mechanically reduces the amount of debt that needs to be rebalanced, and has an additional benefit of better hedging of interest rate risk as highlighted by the discussion of equation (41) in the previous section. Thus, one should expect the optimal portfolio with and without price effects to come closer as we expand the set of available maturities. We verify this in panel (b) of Figure 4 where we plot the optimal portfolios assuming $N = 200$ or 50 years. The difference in Macaulay durations of the optimal portfolio with and without price effects with $N = 200$ is only 6 months.

Figure 4: OPTIMAL PORTFOLIO WITH PRICE EFFECTS



(a) (b)

Notes: Portfolio shares of securities with maturities from 2 quarters to N quarters. In panel (a) we set $N = 120$ and plot the optimal portfolio with price effects and compare it to the maturity matching portfolio ω_t^{mm} and the U.S. debt portfolio. In panel (b) we repeat the exercise with $N = 200$ quarters. The price effects are calibrated using Greenwood and Vayanos (2014).

5.4 Household heterogeneity

To get a sense of the magnitude of the inequality-hedging portfolio in the optimal portfolio (31), we use the following back-of-the-envelope calculation. Assume that a household type

$h = L$ represents a group of individuals who are in the left-tail (or bottom L percentile) of the income distribution, and that the planner sets $\mu_{L,t} = 1$. Then $\Sigma_t^{ineq}[j,k]$ depends on how the income share of the bottom L percentile covaries with returns. We can use our factor model in equation (39) with an additional equation

$$\ln \frac{Y_t}{y_{L,t}} = \alpha_{ineq} + \rho_{ineq} \ln \frac{Y_{t-1}}{y_{L,t-1}} + \kappa_{ineq} f_t + \sigma_{ineq} \epsilon_t^{ineq},$$

to parameterize $\Sigma_t^{-1} \Sigma_t^{ineq}$ with two new objects: κ_{ineq}, a loading of inequality on the common factor, and ρ_{ineq}, the first-order autocorrelation in a measure of inequality.

We set $L = 25\%$ and use income share data from Guvenen et al. (2014) to obtain $\kappa_{ineq} = 0.002$ and $\rho_{ineq} = 0.92$.[25] Our estimates suggest that the adjustment to the target portfolio is very small, and this comes from a weak correlation of bond returns with movements in income inequality.

To get a sense of what heterogeneous trading frictions mean for the duration of an optimal portfolio, we capture the differences in consumption risk using a parsimonious formulation that sets $\ln(M_{\mathbb{N},t+k}) = (1+\psi)\ln(M_{\mathbb{T},t+k})$; the scalar parameter ψ is intended to measure strength of trading frictions. When non-traders face more risk, so that multiplier $\ln(M_{\mathbb{N},T+t})$ is more volatile than $\ln(M_{\mathbb{T},t+k})$, the parameter $\psi > 0$. Substituting into the definition of Σ_t^M and using the counterpart of the traders' Euler equation we get

$$\Sigma_t^M[k,j] = \psi \mu_{\mathbb{N},k} \left[\mathbb{E}_t r_{t+1}^j - \text{cov}_t \left(\ln Q_{t+1}^{k-1}, r_{t+1}^j \right) \right],$$

where all the terms in the square bracket on the right-hand side can be measured from return data that we used in Section 5. In online Appendix C, we use estimates from factor model (39) to quantify those terms for a special case in which the government trades a risk-free and a growth-adjusted consol and verify that imperfect risk sharing lengthens the optimal maturity.

6 Debt portfolios in neoclassical models

Several papers including Lucas and Stokey (1983), Zhu (1992) and Chari et al. (1994), study optimal public portfolios in "neoclassical" models with complete markets and a representative agent who has time separable expected utility preferences over consumption and leisure. Angeletos (2002) showed that it is both feasible and optimal for a government with access to a sufficiently big set of zero coupon bonds to implement a complete market allocation. He

[25] Guvenen et al. (2014) use SSA data and provide means as well as quantiles of labor earnings at an annual frequency from 1978-2011. We first detrend the raw measure of inequality and then project it onto the unemployment rate to obtain a quarterly inequality series. We estimated κ_{ineq} and ρ_{ineq} by using OLS.

derived explicit expressions for the required portfolio. Buera and Nicolini (2004) and Farhi (2010)) found that plausible calibrations of the neoclassical model requires an optimal portfolio with huge long and short positions.[26] Those portfolios differ markedly from the simple portfolio that we obtained in Section 5.

In this section, we want to understand sources of those differences. We also want to see how well our simple statistical rules for forming an optimal portfolio perform in environments where some of the assumptions used to derive our rules are violated, e.g., absence of income and price effects.[27] We follow Buera and Nicolini (2004) and assume that households are identical, and that they maximize $\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-1/IES}}{1-1/IES} - \frac{Y_t^{1+1/\gamma}}{1+1/\gamma} \right]$. The economy is closed, households and the government trade securities that are in in zero net supply, and the government chooses issuances and taxes, bond sales and purchases to finance an exogenous stochastic government expenditure process. This economy satisfies all of the conditions that underly our benchmark economy except that it is closed and that income effects are present.

We first construct an optimal bond portfolio using standard numerical methods. We call this the *theoretical* optimal portfolio. We follow Buera and Nicolini (2004) and set $IES = 1/2$ and $\gamma = 1$. We assume that $\ln G_t$ follows an AR(1) process and calibrate the mean, variance, and first-order autocorrelation of this process to match the primary surplus to GDP ratio in the U.S. data. We discretize this AR(1) process by confining possible realizations to be on a grid with 20 points. We set the initial level of debt to be four times (quarterly) output in a corresponding complete market economy.[28]

Since the Markov state s^t can take 20 possible values, results of Angeletos (2002) imply that an optimal allocation can be achieved using only the bonds with the first 20 maturities. We use formulas that Angeletos derived in Corollary to his Theorem 1 to compute that optimal portfolio and report it in the green line in Figure 5.[29] By construction, the ratio of the total market value debt to annual GDP is close to 1, but this conceals large variations in market values of positions at specific maturities. Consistent with findings of Buera and Nicolini, our optimal portfolio exhibits huge long-short positions and variations in them across Markov states. Market values of bonds of a given maturity can range *several thousand times* annual

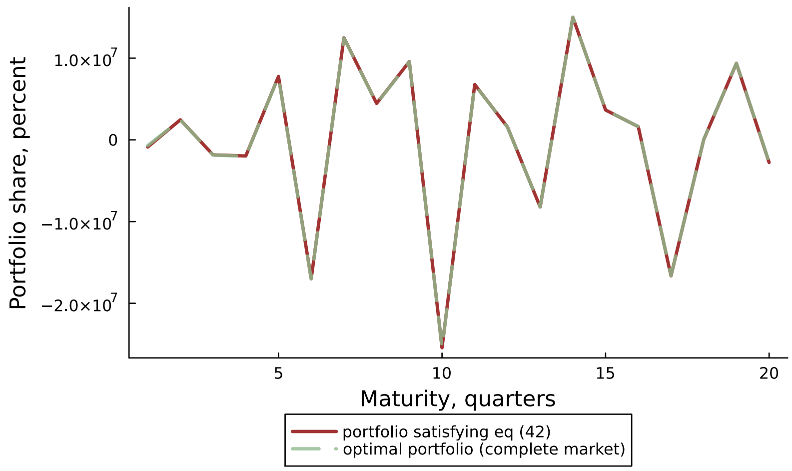
[26] Lustig et al. (2008) study a nominal version of the neoclassical model and impose short-selling as well as maximum maturity restrictions on the government portfolio. They find that these restrictions are binding and that an optimal portfolio issues debt almost exclusively in the maximal maturity bond.

[27] In online Appendix E, we extend our methods to study the target portfolio in a closed economy.

[28] See online Appendix D for detail on how we compute the complete market allocation and the theoretical portfolio.

[29] Actually, there are 20 different portfolios, one for each possible value of G. Here we plot portfolio for one of the middle values ($s = 10$) of realizations of G for concreteness, but it is representative of the portfolio shapes in all other states.

Figure 5: COMPARISON TO NEOCLASSICAL PORTFOLIOS



Notes: Government portfolio shares ω^i of 20 pure discount bonds of maturities $i \in \{1, \ldots, 20\}$ quarters. The green line is the portfolio implementing the complete market allocation following Angeletos (2002). The dark line is the portfolio defined by problem (42) taken for the average state, that is $s = 10$ in the ergodic distribution of G. The tolerance is $\epsilon = 10^{-8}$.

GDP.

What would our statistical summary approach to approximating an optimal portfolio tell us for this economy? Returns on different bonds are highly correlated in the neoclassical economy, which makes the matrix of returns Σ nearly singular. For that reason, we focus on formula (18), which does not require inverting Σ. To make formula (24) operational, we fix a tolerance level $\epsilon > 0$ and let $s_t = s_{10}$, study portfolios ω_t^* that satisfy

$$\left\| \Sigma_t \omega_t^* - \left[\Sigma_t^Q s_t^Q + \Sigma_t^G s_t^G + \Sigma_t^\Theta s_t^\Theta \right] \right\| \leq \epsilon, \tag{42}$$

where $\|\cdot\|$ is the L^1 norm. For sufficiently small tolerance levels that we have studied, we found that a portfolio that satisfies (42) is very close to the theoretical optimal portfolio computed above. The red line in Figure 5 presents this portfolio for $\epsilon = 10^{-8}$. Thus, in the Angeletos environment, ignoring income and price effects in deriving equation (24) does not impair its ability to approximate an optimal portfolio well.

Since our statistical formulas are reliable guides for constructing an optimal portfolios in the neoclassical model, we can use them to understand what drives differences between our prescribed optimal government portfolio and the one that emerges from the standard growth model. In Table 3, we produce version of Table 1 but now estimated from simulations of a neoclassical growth model instead of U.S. data. We scale the moments simulated from the neoclassical model by 100 for ease of comparison. We find that simulations of the neoclassical model generate counterfactual statistics for volatilities of bond prices and also for their co-movements with macroeconomic aggregates. For instance, for long maturities the variance of returns is 300 times smaller than their counterparts in U.S. data. The covariances of returns with primary government surpluses are only 10-20 times smaller, indicating much higher correlations. Furthermore, returns and surpluses are positively correlated and of opposite sign from those in U.S. data.

Table 3: DATA vs NEOCLASSICAL MODEL

	Neoclassical Model		Data	
Mat	100 x Var(r)	100 x Cov(r, X/Y)	Var(r)	Cov(r, X/Y)
6m	0.02	0.29	0.09	-0.01
12m	0.12	0.83	0.49	-0.10
18m	0.30	1.30	1.10	-0.17
24m	0.54	1.80	1.80	-0.26
30m	0.82	2.20	2.80	-0.31
36m	1.10	2.50	3.60	-0.40
42m	1.40	2.90	4.40	-0.45
48m	1.70	3.20	5.40	-0.50
54m	2.00	3.40	6.10	-0.56
60m	2.30	3.70	7.80	-0.62
120m	3.60	4.60	10.00	-0.75

Notes: We simulate the neoclassical model for 265 quarters that correspond to the sample period 1952-2017. The values in the columns for the Neoclassical model are multiplied by 100. Excess returns 6m, 12m, ... are the nominal excess returns in Fama maturity portfolios corresponding to 6-12 months, 12-18 months, ... maturity bins, respectively. The values in the data column are quarterly and in percentage points.

6.1 Reconciling the neoclassical portfolio

Since the matrix Σ_t in the neoclassical setup is nearly singular, other portfolios also approximately satisfy equation (24) and attain levels of welfare that are close to welfare attainable by trading a complete set of Arrow securities. To ensure that our results are not driven by lack of invertibility of Σ_t, we consider a special case in which the underlying Markov state s_t takes two values and the exogenous spending process G_t is calibrated to the same moments as above. The advantage of the two state setup is that we can implement the complete market allocation with a one-period bond and a consol that pays one unit of consumption in perpetuity. In this case ω^* is a scalar and represents the share in the consol.

First, we use the formula from Angeletos (2002), and then we implement formula (19) using objects constructed from the model-simulated economy. In this portfolio, it is optimal for the government to issue debt of 7 to 8 times annual GDP in the consol. This finding is consistent with findings from a similar exercise in Angeletos (2002) and confirms that in the neoclassical growth model, long maturity debt is an excellent hedge against primary surplus risk. As before, from the lens of our formula, we can trace the source of this large position to the values of the covariance of returns and spending, and the volatility of the returns on the consol. In the calibrated model, the ergodic averages of $cov_t\left(X_{t+1}, r_{t+1}^{consol}\right)/var_t(r_{t+1}^{consol}) = 1.48$ and $var_t\left(r_{t+1}^{consol}\right) = (0.18\%)^2$ as compared to -0.061 and $(3.5\%)^2$ in the data[30] implying an hedging spending risk component that is positive, large and equals 7.82 times GDP in line with shares obtain from using the formula from Angeletos (2002).

Our analysis calls for sources of variation in bond returns that are orthogonal to fiscal risks. Bhandari et al. (2017b) described extensions of a neoclassical growth model with discount factor shocks in the spirit of Albuquerque et al. (2016) and shows that the model can produce less extreme portfolios. We can modify the two state risk-free bond and consol setup in a similar fashion to illustrate the main insight of how introducing discount factor shocks can help realign theoretical results with statistics summarized in Tables 1 and thereby imply an optimal public portfolio closer to those prescribed in Section 5.

To that end, we introduce a state-dependent discount factor, $\delta(s^t)\beta^t$ and calibrate $\delta(s^t)$ so that its mean is one and we additionally match the sign and the magnitude of the ratio $cov_t\left(X_{t+1}, r_{t+1}^{consol}\right)/var_t(r_{t+1}^{consol})$ in the ergodic distribution to its data counterpart. The calibrated model produces volatile returns with variance of quarterly returns equal $(4\%)^2$ which is roughly in line with variance of long bonds in the U.S. Applying either Angeletos (2002) formula or our expression (24), we find that matching these asset pricing moments lowers the consol share of total debt by an order of magnitude to 70% of GDP and the rest 30% in the risk-free bond. These holdings are much more similar to ones we found in Section 5. We conclude that neoclassical settings that misrepresents the asset return movements are an inappropriate tool for studying optimal public portfolios whose composition depend critically on the properties of co-movements between returns and macroeconomic variables.

7 Concluding remarks

We have studied determinants of optimal public portfolios in a broad class of dynamic stochastic equilibrium models that encompass various specifications of attitudes towards risk, heterogeneities among households, limits on market participations, and sources of liquidity. We use small noise expansions to summarize determinants of optimal public portfolios in terms of a

[30]We approximate the return on the consol by constructing an weighted average of the 11 CRSP portfolios.

small number of statistics that are functions only of asset prices and macroeconomic variables. For recent U.S. data, we find that an optimal portfolio is simple, stable over time, and has bond shares that decay approximately exponentially with bond maturity. We show that differences between our paper's optimal public portfolio and those prescribed by earlier neoclassical models come from features of those earlier models that lead them to misrepresent observed covariances of asset returns with macroeconomic aggregates.

This paper focuses exclusively on timing protocols in which a government commits to a fiscal plan and cannot default. Natural next steps would explore alternative timing protocols by proceeding along lines advocated by Arellano and Ramanarayanan (2012), Aguiar et al. (2019), Bocola and Dovis (2019), and others.

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Online Appendix

A Additional details for Section 3

A.1 Proof of Lemma 2(a)

$\overline{\mathcal{R}}_{t+k} = \overline{R}^{rf}_{t+k}$ by Lemma 1, so that the zeroth order approximation of equation (10) implies $\overline{\xi}^{-1}_{t+k} = \overline{\frac{\beta M_{t+k+1}}{M_{t+k}}}\overline{\xi}^{-1}_{t+k+1}\overline{R}^{rf}_{t+k+1}$ for all $k \geq 0$. The zeroth order approximation of the households' optimality condition (7) gives us $1 = \overline{\frac{\beta M_{t+k+1}}{M_{t+k}}}\overline{R}^{rf}_{t+k+1}$ for all $k \geq 0$. Combine these two equations to show that $\overline{\xi}_t = \overline{\xi}_{t+k}$ and, therefore, $\overline{\tau}_t = \overline{\tau}_{t+k}$ for all $k \geq 1$.

Multiply equation (10) for period $t+k$ by r^j_{t+1} and take the expectation in period t to get

$$\mathbb{E}_t \frac{1}{\xi_{t+k}} r^j_{t+1} = \mathbb{E}_t \frac{\beta M_{t+k+1}}{M_{t+k}} \frac{1}{\xi_{t+k+1}} \mathcal{R}_{t+k+1} r^j_{t+1}. \tag{43}$$

Lemma 1 implies that $\mathbb{E}_t \partial_\sigma \mathcal{R}_{t+k+1} \partial_\sigma r^i_{t+1} = \mathbb{E}_t \partial_\sigma R^{rf}_{t+k+1} \partial_\sigma r^i_{t+1}$, so that the second order approximation of (43) is

$$\begin{aligned}
&\mathbb{E}_t \partial_\sigma \frac{1}{\xi_{t+k}} \partial_\sigma r^j_{t+1} + \frac{1}{2}\frac{1}{\overline{\xi}_{t+k}} \mathbb{E}_t \partial_{\sigma\sigma} r^j_{t+1} \\
=& \overline{\frac{\beta M_{t+k+1}}{M_{t+k}}} \frac{1}{\overline{\xi}_{t+k+1}} \mathbb{E}_t \partial_\sigma R^{rf}_{t+k+1} \partial_\sigma r^j_{t+1} + \overline{\frac{\beta M_{t+k+1}}{M_{t+k}}}\overline{R}^{rf}_{t+k+1} \mathbb{E}_t \partial_\sigma \frac{1}{\xi_{t+k+1}} \partial_\sigma r^j_{t+1} \\
&+ \frac{\overline{R}^{rf}_{t+k+1}}{\overline{\xi}_{t+k+1}} \mathbb{E}_t \partial_\sigma \frac{\beta M_{t+k+1}}{M_{t+k}} \partial_\sigma r^j_{t+1} + \frac{1}{2} \overline{\frac{\beta M_{t+k+1}}{M_{t+k}}} \frac{\overline{R}^{rf}_{t+k+1}}{\overline{\xi}_{t+k+1}} \mathbb{E}_t \partial_{\sigma\sigma} r^j_{t+1}.
\end{aligned}$$

From the results obtained in the previous paragraph, we know that $\overline{\xi}_{t+k} = \overline{\xi}_t$ and $\partial_\sigma \frac{1}{\xi_{t+k}} = -\frac{\partial_\sigma \tau_{t+k}}{\xi'(\overline{\tau}_t)}$ and that $\overline{\frac{\beta M_{t+k+1}}{M_{t+k}}}\overline{R}^{rf}_{t+k+1} = 1$, so that this equation further simplifies to

$$\begin{aligned}
-\frac{\xi'(\overline{\tau}_t)}{\xi(\overline{\tau}_t)} \mathbb{E}_t \partial_\sigma \tau_{t+k} \partial_\sigma r^j_{t+1} + \frac{1}{2}\mathbb{E}_t \partial_{\sigma\sigma} r^j_{t+1} =& \mathbb{E}_t \partial_\sigma \ln R^{rf}_{t+k+1} \partial_\sigma r^j_{t+1} - \mathbb{E}_t \partial_\sigma \tau_{t+k+1} \partial_\sigma r^j_{t+1} \\
&+ \mathbb{E}_t \partial_\sigma \ln \frac{\beta M_{t+k+1}}{M_{t+k}} \partial_\sigma r^j_{t+1} + \frac{1}{2}\mathbb{E}_t \partial_{\sigma\sigma} r^j_{t+1}.
\end{aligned}$$

Similarly, the household optimality condition (7) implies that $\mathbb{E}_t r^j_{t+1} = \mathbb{E}_t \frac{\beta M_{t+k+1}}{M_{t+k}} R^{rf}_{t+k+1} r^j_{t+1}$, which to the second order of approximation gives

$$\frac{1}{2}\mathbb{E}_t \partial_{\sigma\sigma} r^j_{t+1} = \mathbb{E}_t \partial_\sigma \ln R^{rf}_{t+k+1} \partial_\sigma r^j_{t+1} + \mathbb{E}_t \partial_\sigma \ln \frac{\beta M_{t+k+1}}{M_{t+k}} \partial_\sigma r^j_{t+1} + \frac{1}{2}\mathbb{E}_t \partial_{\sigma\sigma} r^j_{t+1}.$$

Combine these two equations to show that $\mathbb{E}_t \partial_\sigma \tau_{t+k} \partial_\sigma r^j_{t+1} = \mathbb{E}_t \partial_\sigma \tau_{t+k+1} \partial_\sigma r^j_{t+1}$ for all k, j.

A.2 Proof of Theorem 1

We first consider the zeroth order economy. Using the definition of zero coupon bond prices, we have

$$\overline{Q}_t^k = \frac{\overline{S}_{t+1}}{\overline{S}_t} \times \frac{\overline{S}_{t+2}}{\overline{S}_{t+1}} \times ... \times \frac{\overline{S}_{t+k}}{\overline{S}_{t+k-1}} = \overline{Q}_t^1 \times \overline{Q}_{t+1}^1 \times ... \times \overline{Q}_{t+k-1}^1 = \overline{Q}_t^1 \overline{Q}_{t+1}^{k-1}.$$

Furthermore, Lemma 1 implies that excess returns are zero to the zeroth order and $\overline{\mathcal{Q}}_{t+1,t+k} = \overline{Q}_{t+1}^{k-1}$. Thus, the budget constraint (4) gives us

$$\overline{B}_t/\overline{Q}_t^1 = \sum_{k=1}^{\infty} \overline{Q}_{t+1}^{k-1} \overline{X}_{t+k} \quad \Longrightarrow \quad \overline{B}_t = \sum_{k=1}^{\infty} \overline{Q}_t^k \overline{X}_{t+k}, \tag{44}$$

where we used a convention that $\overline{Q}_t^0 = 1$. We have $\overline{X}_{t+k} = \overline{T}_{t+k} - \overline{G}_{t+k}$ and Lemma 2 implies that in the optimum $\overline{T}_{t+k} = \overline{T}_t$ for all k. This allows us to solve for the optimal level of tax revenues, $\overline{T}_t = (\overline{B}_t + \sum_{k=1}^{\infty} \overline{Q}_t^k \overline{G}_{t+k}) / \sum_{k=1}^{\infty} \overline{Q}_t^k$.

Multiply (4) by r_{t+1}^j and take period t expectations to write it as

$$\mathbb{E}_t r_{t+1}^j \sum_{k=1}^{\infty} \mathcal{Q}_{t+1,t+k} X_{t+k} = \mathbb{E}_t r_{t+1}^j \left(R_{t+1}^{rf} + \sum_{i \neq rf} \omega_t^i r_{t+1}^i \right) B_t.$$

The second order approximations of the right hand side and the left hand side of this equation, due to Lemma 1, are

$$RHS \simeq \frac{1}{2} \frac{\overline{B}_t}{\overline{Q}_t^1} \mathbb{E}_t \partial_{\sigma\sigma} r_{t+1}^j + \overline{B}_t \sum_{i \neq rf} \overline{\omega}_t^i \mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma r_{t+1}^i$$

and

$$LHS \simeq \frac{1}{2} \left(\sum_{k=1}^{\infty} \overline{Q}_{t+1}^{k-1} \overline{X}_{t+k} \right) \mathbb{E}_t \partial_{\sigma\sigma} r_{t+1}^j + \sum_{k=1}^{\infty} \overline{Q}_{t+1}^{k-1} \mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma X_{t+k} + \sum_{k=1}^{\infty} \overline{Q}_{t+1}^{k-1} \overline{X}_{t+k} \mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma \ln \mathcal{Q}_{t+1,t+k}.$$

We want to make several observations about these equations. First, the first terms on the right hand sides of these two equations are equal, due to (44). Second, $\mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma \ln \mathcal{Q}_{t+1,t+1} = 0$ since $\mathcal{Q}_{t+1,t+1} = 1$, and $\mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma \ln \mathcal{Q}_{t+1,t+k} = \mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma \ln Q_{t+1}^{k-1}$ for $k > 1$ due to Lemma 1. Therefore, combining these equations and multiplying both sides by $\overline{Q}_t^1$ we have

$$\overline{Q}_t^1 \overline{B}_t \sum_{i \neq rf} \overline{\omega}_t^i \mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma r_{t+1}^i = \sum_{k=1}^{\infty} \overline{Q}_t^{k+1} \overline{X}_{t+k+1} \mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma \ln Q_{t+1}^k \tag{45}$$

$$+ \sum_{k=1}^{\infty} \overline{Q}_t^k \overline{G}_{t+k} \mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma \ln G_{t+k} + \sum_{k=1}^{\infty} \overline{Q}_t^k \mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma T_{t+k}.$$

Note that T_{t+k} is only a function of tax rate τ_{t+k}, so we can write $T_{t+k} = T(\tau_{t+k})$ and $\partial_\sigma T_{t+k} = T'(\overline{\tau}_{t+k})\,\partial_\sigma \tau_{t+k}$. But then we have

$$\mathbb{E}_t \partial_\sigma r^j_{t+1} \partial_\sigma T_{t+k} = T'(\overline{\tau}_{t+k})\,\mathbb{E}_t \partial_\sigma r^j_{t+1} \partial_\sigma \tau_{t+k} = T'(\overline{\tau}_t)\,\mathbb{E}_t \partial_\sigma r^j_{t+1} \partial_\sigma \tau_{t+1} \tag{46}$$

where the second equality follows from Lemma 2(a). Finally, observe that the last equation in (46) must be equal to zero by Lemma 2(b), which establishes (16).

A.3 Proof of Corollary 1

Observe that in the proof of Theorem 1 upto equation (46) we only used first order properties of the debt level optimality condition[31] (10), and did not use portfolio optimality condition (11) at all. Without invoking this optimality condition, the government budget constraint is

$$\begin{aligned}\overline{Q}^1_t \overline{B}_t \sum_{i \neq rf} \overline{\omega}^i_t \mathbb{E}_t \partial_\sigma r^j_{t+1} \partial_\sigma r^i_{t+1} &= \sum_{k=1}^{\infty} \overline{Q}^{k+1}_t \overline{X}_{t+k+1} \mathbb{E}_t \partial_\sigma r^j_{t+1} \partial_\sigma \ln Q^k_{t+1} \\ &+ \sum_{k=1}^{\infty} \overline{Q}^k_t \overline{G}_{t+k} \mathbb{E}_t \partial_\sigma r^j_{t+1} \partial_\sigma \ln G_{t+k} + T'(\overline{\tau}_t) \left(\sum_{k=1}^{\infty} \overline{Q}^k_t \right) \mathbb{E}_t \partial_\sigma r^j_{t+1} \partial_\sigma \tau_{t+1}.\end{aligned}$$

Using definition of $\overline{\omega}^*_t$, this implies that

$$\sum_{i \neq rf} \left(\overline{\omega}^i_t - \overline{\omega}^{*,i}_t \right) \mathbb{E}_t \partial_\sigma r^j_{t+1} \partial_\sigma r^i_{t+1} = \frac{T'(\overline{\tau}_t) \left(\sum_{k=1}^{\infty} \overline{Q}^k_t \right)}{\overline{Q}^1_t \overline{B}_t} \mathbb{E}_t \partial_\sigma r^j_{t+1} \partial_\sigma \tau_{t+1}.$$

Taking the second order approximation of (9), we obtain

$$\partial_{\sigma\sigma} \partial_{prfl,j} V = \beta^t \Pr\left(s^t\right) \overline{M}_{t+1} \frac{\xi'(\overline{\tau}_{t+1})}{\xi(\overline{\tau}_{t+1})^2} \mathbb{E}_t \partial_\sigma \tau_{t+1} \partial_\sigma r^j_{t+1}.$$

Combining these two expressions, and using zeroth order tax smoothing $\overline{\tau}_t = \overline{\tau}_{t+1}$, we get

$$\sum_{i \neq rf} \left(\overline{\omega}^{*,i}_t - \overline{\omega}^i_t \right) \mathbb{E}_t \partial_\sigma r^j_{t+1} \partial_\sigma r^i_{t+1} = \underbrace{\frac{T'(\overline{\tau}_t)\,\xi(\overline{\tau}_t)^2}{-\xi'(\overline{\tau}_t)\,\overline{B}_t} \frac{\sum_{k=1}^{\infty} \overline{Q}^k_t / \overline{Q}^1_t}{\beta^t \Pr(s^t)\,\overline{M}_{t+1}}}_{const_t} \partial_{\sigma\sigma} \partial_{prfl,j} V,$$

which is the expression stated in Corollary 1. Note that $const_t > 0$ if $\overline{B}_t > 0$, $-\xi'(\overline{\tau}_t) > 0$, and $T'(\overline{\tau}_t) = \xi(\overline{\tau}_t)\,\overline{Y}_t > 0$. If v is constant elasticity γ then $\xi(\tau) = 1 - \gamma \frac{\tau}{1-\tau}$ and the peak of the Laffer curve τ^* satisfies $\frac{\tau^*}{1-\tau^*} = \frac{1}{\gamma}$. This implies that if $\tau < \tau^*$ then $\xi(\tau), -\xi'(\tau) > 0$.

[31] This follows since we always pre-multiplied it by r^j_{t+1} prior to taking second order expansions, and $\overline{r}^j_{t+1} = 0$ by Lemma 1.

B Additional details for Section 4

B.1 Additional details for Section 4.2

First of all, observe that since ξ_t is the transformation of τ_t the proof of Lemma 2 remains unchanged. This implies that $\overline{\tau}_{t+k} = \overline{\tau}_t$ for all t and $\overline{\tau}_t$ is the solution to $\overline{B}_t = \sum_{k=1}^{\infty} \overline{Q}_t^k (\overline{\Theta}_{t+k} (\overline{\tau}_t (1-\overline{\tau}_t)) - \overline{G}_{t+k})$ which is the generalization of equation (44). Let $\overline{T}_{t+k}^{tax} = \overline{\Theta}_{t+k}\overline{\tau}_t (1-\overline{\tau}_t)$. We have

$$\mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma T_{t+k} = \mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma T_{t+k}^{tax} = \overline{T}_{t+k} \mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma \ln \Theta_{t+k} + \overline{\Theta}_{t+k} \underbrace{\mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma \overline{\tau}_t (1-\overline{\tau}_t)}_{=0},$$

where the last term is zero following the same steps as in (46). Substitute this equation into (45) to obtain the expression for the optimal portfolio $\overline{\omega}_t$. If Σ_t matrix is invertible, it can be written as (24).

B.2 Additional details for Section 4.3

In the economy with liquidity premia, the second order approximation of portfolio optimality condition, equation (14), holds but the analogue of equation (15) becomes

$$\frac{1}{2} \overline{\frac{\beta M_{t+1}}{M_t}} \mathbb{E}_t \partial_{\sigma\sigma} r_{t+1}^j + \mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma \frac{\beta M_{t+1}}{M_t} = -\frac{1}{2} \partial_{\sigma\sigma} (w_{t,k} - w_{t,1}).$$

Combining this equation with (14) we obtain

$$\frac{1}{2} \partial_{\sigma\sigma} (w_{t,k} - w_{t,1}) = \overline{\frac{\beta M_{t+1}}{M_t}} \mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma \frac{1}{\xi_{t+1}} = \frac{-\xi'(\overline{\tau}_{t+1})}{\xi(\overline{\tau}_{t+1})} \overline{\frac{\beta M_{t+1}}{M_t}} \mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma \tau_{t+1}.$$

Since to the first order the liquidity premium is zero, conclusions of Lemma 2(a) extend to this economy, which allows us to write the above equation as

$$\mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma \tau_{t+1} = \frac{\xi(\overline{\tau}_t)}{-\xi'(\overline{\tau}_t) \overline{Q}_t^1} \frac{1}{2} \partial_{\sigma\sigma} (w_{t,k} - w_{t,1}). \tag{47}$$

Equation (46) still holds but when we combine it with the portfolio optimality condition (47) we obtain

$$\mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma T_{t+k} = T'(\overline{\tau}_t) \frac{\xi(\overline{\tau}_t)^2}{-\xi'(\overline{\tau}_t) \overline{Q}_t^1} \frac{1}{2} \partial_{\sigma\sigma} (w_{t,k} - w_{t,1}) = \frac{\overline{Y}_t \xi(\overline{\tau}_t)^2}{-\xi'(\overline{\tau}_t) \overline{Q}_t^1} \frac{1}{2} \partial_{\sigma\sigma} (w_{t,k} - w_{t,1}).$$

Substitute this equation into (45) to get

$$\begin{aligned} \overline{Q}_t^1 \overline{B}_t \sum_{i \neq rf} \overline{\omega}_t^i \mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma r_{t+1}^i &= \sum_{k=1}^{\infty} \overline{Q}_t^{k+1} \overline{X}_{t+k+1} \mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma \ln Q_{t+1}^k \\ &+ \sum_{k=1}^{\infty} \overline{Q}_t^k \overline{G}_{t+k} \mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma \ln G_{t+k} + \frac{\overline{Y}_t \xi(\overline{\tau}_t)^2 \sum_{k=1}^{\infty} \overline{Q}_t^k}{-\xi'(\overline{\tau}_t) \overline{Q}_t^1} \frac{1}{2} \partial_{\sigma\sigma} (w_{t,k} - w_{t,1}). \end{aligned} \tag{48}$$

This is equation (28) when Σ_t is invertible.

To derive (27), it will be useful to write household optimality conditions (25) in a slightly different form. Consider a perturbation in which households change the quantity of holding a k period bond by an infinitesimal amount until maturity of that bond. The implied optimality condition for that perturbation is

$$M_t Q_t^k = \mathbb{E}_t \beta^k M_{t+k} + M_t Q_t^k w_{t,k} + \mathbb{E}_t M_{t+1} Q_{t+1}^{k-1} w_{t+1,k-1} + ... + \mathbb{E}_t M_{t+k-1} Q_{t+k-1}^1 w_{t+k-1,1}.$$

The optimality condition for the notional private k period bond is $M_t Q_t^{k,pr} = \mathbb{E}_t \beta^k M_{t+k}$, which implies

$$0 = \left(\frac{1}{Q_t^k} - \frac{1}{Q_t^{k,pr}}\right) \mathbb{E}_t \frac{\beta^k M_{t+k}}{M_t} + w_{t,k} + \mathbb{E}_t \frac{\beta M_{t+1}}{M_t} \frac{Q_{t+1}^{k-1}}{Q_t^k} w_{t+1,k-1} + ... + \mathbb{E}_t \frac{\beta^{k-1} M_{t+k-1}}{M_t} \frac{Q_{t+k-1}^1}{Q_t^k} w_{t+k-1,1}.$$

Take the second order approximation of this equation and use the fact that to the first order liquidity premia is zero to obtain

$$0 = \partial_{\sigma\sigma} \left(\frac{1}{Q_t^k} - \frac{1}{Q_t^{k,pr}}\right) \overline{Q_t^k} + \partial_{\sigma\sigma} w_{t,k} + \mathbb{E}_t \partial_{\sigma\sigma} w_{t+1,k-1} + ... + \mathbb{E}_t \partial_{\sigma\sigma} w_{t+k-1,1}.$$

Finally, observe that $\partial_{\sigma\sigma} \left(\frac{1}{Q_t^k} - \frac{1}{Q_t^{k,pr}}\right) \overline{Q_t^k} = -\partial_{\sigma\sigma} \left(\ln Q_t^k - \ln Q_t^{k,pr}\right)$, which implies equation (27).

B.3 Additional details for Section 4.4

Suppose household h has household specific productivity $\theta_{h,t}$ and we partition the households into two groups: $\mathbb{T}$ represent the set of households who can trade bonds and $\mathbb{N}$ represent the set of households who cannot trade bonds. Other than that, we focus on the baseline economy. Individual budget sets are given by

$$c_{h,t+1} + \iota_{h \in \mathbb{T}} \sum_i b_{t+1}^i = (1 - \tau_{t+1}) Y_{h,t+1} + \iota_{h \in \mathbb{T}} \sum_i R_{t+1}^i b_t^i.$$

and optimality implies

$$Y_{h,t} = \theta_{h,t}^{1+\gamma} (1 - \tau_t)^\gamma$$

We can define total output as $Y_t = \sum_h Y_{h,t}$. Assuming a linear tax function, we have

$$\frac{\partial T_t}{\partial \tau_t} = Y_t + \tau_t \sum_h \frac{\partial Y_{h,t}}{\partial \tau_t} = Y_t - \gamma \frac{\tau_t}{1 - \tau_t} \sum_h Y_{h,t} = Y_t \underbrace{\left(1 - \gamma \frac{\tau_t}{1 - \tau_t}\right)}_{\xi_t},$$

so tax revenue elasticity is the same as before.

To compute the welfare effects of the debt and portfolio perturbations, we apply the envelope theorem to each agent h. Changing tax revenues by ϵ will affect household h's budget by $Y_{h,t+1}\frac{\partial\tau_{t+1}}{\partial T_{t+1}^{tax}}$. Following the same steps as the representative agent, welfare gain for agent h is given by

$$\partial_{debt}V_h = \beta^t \Pr\left(s^t\right)\left[M_{h,t}(s^t)\frac{1}{\xi_t(s^t)}\frac{Y_{h,t}(s^t)}{Y_t(s^t)} - \mathbb{E}_{s^t}\beta M_{t+1}\mathcal{R}_{t+1}\frac{1}{\xi_{t+1}}\frac{Y_{h,t+1}}{Y_{t+1}}\right]$$

$$\partial_{\sigma\sigma}\partial_{prfl,j}V_h = \beta^t \Pr\left(s^t\right)\mathbb{E}_t M_{h,t+1} r_{t+1}^j \frac{1}{\xi_{t+1}}\frac{Y_{h,t+1}}{Y_{t+1}}.$$

Combing these for all agents, an optimality condition for the government at $s^t = s^T$ will be

$$\mathbb{E}_t \sum_h \varpi_h M_{h,t+k}\left(\mathcal{Q}_{t+1}^{t-1}\right)^{-1} r_{t+1}^j \frac{Y_{h,t+k}}{Y_{t+k}}\frac{1}{\xi_{t+k}} = 0, \tag{49}$$

where ϖ_h are Pareto weights for household h and $\mathcal{Q}_{t,k} \equiv \frac{1}{\mathcal{R}_{t+1}} \times ... \times \frac{1}{\mathcal{R}_{t+k}}$ is the inverse cumulative return on the government portfolio between periods t and $t+k$. Take second order expansion of (49) to get

$$\begin{aligned}
0 = \mathbb{E}_t\Bigg\{ & \frac{1}{2}\sum_h \varpi_h \left[\overline{M_{h,t+k}}\right]\left(\overline{Q_{t+1}^{t-1}}\right)^{-1}\partial_{\sigma\sigma} r_{t+1}^j \left[\overline{\frac{Y_{h,t+k}}{Y_{t+k}}\frac{1}{\xi_{t+k}}}\right] \\
& + \sum_h \varpi_h \left[\overline{M_{h,t+k}}\right]\left(\overline{Q_{t+1}^{t-1}}\right)^{-1}\partial_\sigma \ln\left(M_{h,t+k}\right)\partial_\sigma r_{t+1}^j \left[\overline{\frac{Y_{h,t+k}}{Y_{t+k}}\frac{1}{\xi_{t+k}}}\right] \\
& + \sum_h \varpi_h \left[\overline{M_{h,t+k}}\right]\left(\overline{Q_{t+1}^{t-1}}\right)^{-1}\partial_\sigma \ln\left(\frac{Y_{h,t+k}}{Y_{t+k}}\right)\partial_\sigma r_{t+1}^j \left[\overline{\frac{Y_{h,t+k}}{Y_{t+k}}\frac{1}{\xi_{t+k}}}\right] \\
& - \sum_h \varpi_h \left[\overline{M_{h,t+k}}\right]\left(\overline{Q_{t+1}^{t-1}}\right)^{-1}\partial_\sigma \ln\left(\xi_{t+k}\right)\partial_\sigma r_{t+1}^j \left[\overline{\frac{Y_{h,t+k}}{Y_{t+k}}\frac{1}{\xi_{t+k}}}\right] \\
& - \sum_h \varpi_h \left[\overline{M_{h,t+k}}\right]\left(\overline{Q_{t+1}^{t-1}}\right)^{-1}\partial_\sigma \ln\left(Q_{t+1}^{t-1}\right)\partial_\sigma r_{t+1}^j \left[\overline{\frac{Y_{h,t+k}}{Y_{t+k}}\frac{1}{\xi_{t+k}}}\right]\Bigg\}.
\end{aligned}$$

Canceling out the terms that do not depend on h and dividing out by the coefficient on $\mathbb{E}_t\partial_{\sigma\sigma}r_{t+1}^j$ yields an approximation to the optimality condition (49)

$$\begin{aligned}
0 = & \mathbb{E}_t\Bigg[\frac{1}{2}\partial_{\sigma\sigma} r_{t+1}^j + \sum_h \mu_{h,t+k}\partial_\sigma \ln\left(M_{h,t+k}\right)\partial_\sigma r_{t+1}^j + \partial_\sigma \ln\left(\xi_{t+k}\right)\partial_\sigma r_{t+1}^j + \partial_\sigma \ln\left(Q_{t+1}^{t-1}\right)\partial_\sigma r_{t+1}^j \\
& + \sum_h \mu_{h,t+k}\partial_\sigma \ln\left(\frac{Y_{h,t+k}}{Y_{t+k}}\right)\partial_\sigma r_{t+1}^j\Bigg]
\end{aligned} \tag{50}$$

where $\mu_{h,t+k} \equiv \varpi_h\left[\overline{M_{h,t+k}}\right]\overline{s_{h,t+k}} \Big/ \left(\sum_h \varpi_h\left[\overline{M_{h,t+k}}\right]\overline{s_{h,t+k}}\right)$ are a deterministic sequence of weights that sum to one with $s_{h,t+k} \equiv \frac{Y_{h,t+k}}{Y_{t+k}}$.

As government bonds are perfect substitutes, for all $h \in \mathbb{T}$ we must have

$$\mathbb{E}_t M_{h,t+k} \left(Q_{t+1}^{t-1}\right)^{-1} r_{t+1}^{j} = 0.$$

Expanding this equation yields

$$\begin{aligned} 0 = &\frac{1}{2} \left[\overline{M_{h,t+k}}\right] \left(\overline{Q_{t+1}^{t-1}}\right)^{-1} \partial_{\sigma\sigma} r_{t+1}^{j} - \left[\overline{M_{h,t+k}}\right] \left(\overline{Q_{t+1}^{t-1}}\right)^{-1} \partial_{\sigma} \ln\left(Q_{t+1}^{t-1}\right) \partial_{\sigma} r_{t+1}^{j} \\ &+ \left[\overline{M_{h,t+k}}\right] \left(\overline{Q_{t+1}^{t-1}}\right)^{-1} \partial_{\sigma} \ln\left(M_{h,t+k}\right) \partial_{\sigma} r_{t+1}^{j} \end{aligned}$$

for all $h \in \mathbb{T}$. This simplifies to

$$\frac{1}{2}\mathbb{E}_t \partial_{\sigma\sigma} r_{t+1}^{j} = \mathbb{E}_t \left[\partial_{\sigma} \ln\left(Q_{t+1}^{t-1}\right) \partial_{\sigma} r_{t+1}^{j} - \partial_{\sigma} \ln\left(M_{h,t+k}\right) \partial_{\sigma} r_{t+1}^{j}\right] \tag{51}$$

As this holds for all $h \in \mathbb{T}$ we can average over all traders, using weights $\mu_{h,t+k}$, to obtain

$$\frac{1}{2}\mathbb{E}_t \partial_{\sigma\sigma} r_{t+1}^{j} = \mathbb{E}_t \left[\partial_{\sigma} \ln\left(Q_{t+1,t-1}\right) \partial_{\sigma} r_{t+1}^{j} - \partial_{\sigma} \ln\left(M_{\mathcal{T},t+k}\right) \partial_{\sigma} r_{t+1}^{j}\right] \tag{52}$$

where $\ln\left(M_{\mathbb{T},t+k}\right)$ is the average SDF of all traders:

$$\ln\left(M_{\mathbb{T},t+k}\right) \equiv \sum_{h\in\mathbb{T}} \mu_{h,t+k} \ln\left(M_{h,t+k}\right) \Big/ \sum_{h\in\mathbb{T}} \mu_{h,t+k}.$$

The same equation does not hold for the non-traders but we do have that for all $h \in \mathbb{N}$

$$\begin{aligned} \frac{1}{2}\mathbb{E}_t \partial_{\sigma\sigma} r_{t+1}^{j} = &\mathbb{E}_t \Big[\partial_{\sigma} \ln\left(Q_{t+1,t-1}\right) \partial_{\sigma} r_{t+1}^{j} - \partial_{\sigma} \ln\left(M_{h,t+k}\right) \partial_{\sigma} r_{t+1}^{j} \\ &+ \left(\partial_{\sigma} \ln\left(M_{h,t+k}\right) - \partial_{\sigma} \ln\left(M_{\mathcal{T},t+k}\right)\right) \partial_{\sigma} r_{t+1}^{j}\Big]. \end{aligned} \tag{53}$$

We can now use equations (51) and (53) substitute for $\frac{1}{2}\partial_{\sigma\sigma} r_{t+1}^{j}$ in (50) to get

$$\begin{aligned} -\mathbb{E}_t \partial_{\sigma} \ln \xi_{t+k} \partial_{\sigma} r_{t+1}^{j} = &\mathbb{E}_t \Big[\partial_{\sigma} \left\{\sum_{h} \mu_{h,t+k} \ln\left(\frac{1}{s_{h,t+k}}\right)\right\} \partial_{\sigma} r_{t+1}^{j} \\ &+ \partial_{\sigma} \left\{\sum_{h\in\mathbb{N}} \mu_{h,t+k} \left(\ln\left(M_{\mathbb{T},t+k}\right) - \ln\left(M_{h,t+k}\right)\right)\right\} \partial_{\sigma} r_{t+1}^{j}\Big]. \end{aligned}$$

We can further simplify this expression by defining

$$\ln\left(M_{\mathbb{N},t+k}\right) \equiv \sum_{h\in\mathbb{N}} \mu_{h,t+k} \ln\left(M_{h,t+k}\right) \Big/ \left(\sum_{h\in\mathbb{N}} \mu_{h,t+k}\right)$$

as the "average" SDF of the non-traders, then

$$\begin{aligned} -\mathrm{cov}_t \left(\ln \xi_{t+k}, r_{t+1}^{j}\right) \simeq &\mathrm{cov}_t \left(\sum_{h} \mu_{h,t+k} \ln\left(\frac{1}{s_{h,t+k}}\right), r_{t+1}^{j}\right) \\ &+ \mu_{\mathbb{N},t+k} \mathrm{cov}_t \left(\ln\left(M_{\mathbb{T},t+k}\right) - \ln\left(M_{\mathbb{N},t+k}\right), \partial_{\sigma} r_{t+1}^{j}\right) \end{aligned} \tag{54}$$

where $\mu_{\mathbb{N},t+k} \equiv \left(\sum_{h\in\mathbb{N}} \mu_{h,t+k}\right)$ is the "share" of non-traders. Equation (54) adds two additional terms to optimality equation in Lemma (2) in main text that capture the effect of heterogeneity on the planners desire to smooth taxes. The first term, $\text{cov}_t\left(\sum_h \mu_{h,t+k} \ln\left(\frac{1}{s_{h,t+k}}\right), r^j_{t+1}\right)$, captures the planners desire to raise taxes in states of the world where inequality is high. The second term, $\mu_{\mathbb{N},t+k}\text{cov}_t\left(\ln\left(M_{\mathbb{T},t+k}\right) - \ln\left(M_{\mathbb{N},t+k}\right), \partial_\sigma r^j_{t+1}\right)$, captures the fact that the planner is trading on behalf of agents without access to asset markets and therefore will want to raise taxes in states of which the non-traders place less weight on relative to those agents with access to asset markets. This effect is scaled by the relative size of the non-traders. Following the steps of Theorem 1 we get (31) where $\Sigma^{ineq}_t[t,k] = cov_k\left(\sum_h \mu_{h,t+k} \ln\left(\frac{1}{s_{h,t+k}}\right), r^j_{t+1}\right)$ is covariance matrix of returns with inequality and $\Sigma^M_t[t,k] = \mu_{\mathbb{N},t+k}\text{cov}_t\left(\ln\left(M_{\mathbb{T},t+k}\right) - \ln\left(M_{\mathbb{N},t+k}\right), r^k_{t+1}\right)$ is the covariance of returns with the relative stochastic discount factors of traders and non-traders with individual weights $\mu_{\mathbb{N},t+k}$ defined above and temporal weights $s^{ineq}_t = \{\frac{Q^k_t \mathbb{E}_t Y_{t+k}\gamma^{-1}(1-(1+\gamma)\tau_{t+k})^2}{Q^1_t B_t}\}_k$.

B.4 Additional details for Section 4.5

B.4.1 Perturbations

We now derive equations (34) and (35) under the assumption that price responses satisfy (32). As the first step, consider a perturbation that increases issuance of maturity k by ε_k in period t. Using (33), the responses ∂_{ε_k} to such perturbation of taxes are given by

$$\partial_{\varepsilon_k} T_t = -Q^k_t - \Delta^k_t \partial_{\varepsilon_k} Q^k_t, \quad \partial_{\varepsilon_k} T_{t+1} = Q^{k-1}_{t+1} + D^{k-1}_{t+1}, \tag{55}$$

where D^{k-1}_{t+1} is equal to 1 if $k=1$ and zero otherwise. Household budget constraint written in the quantity form is

$$C_t + \sum_k Q^k_t \widetilde{b}^i_t = (1-\tau_t) Y_t + \sum_k \left(Q^k_t + D^k_t\right) \widetilde{b}^k_{t-1}.$$

Using the envelope theorem, the welfare impact of this perturbation, up to $\beta^t \Pr\left(s^t\right)$, is given by

$$\partial_{\varepsilon_k} V \propto -\frac{M_t}{\xi_t}\partial_{\varepsilon_k} T_t - \mathbb{E}_{t+1}\frac{\beta M_{t+1}}{\xi_{t+1}}\partial_{\varepsilon_k} T_{t+1} - M_t \delta^k_t \partial_{\varepsilon_k} Q^k_t. \tag{56}$$

Combine (55) and (56), using the fact that (32) implies that $\frac{\partial_{\varepsilon_k} Q^k_t}{Q^k_t} = \partial_{\varepsilon_k} \ln Q^k_t = -\varphi'_{k,t}(\widetilde{B}^k_t)$ and set $\partial_{\varepsilon_k} V = 0$ to obtain

$$\frac{1}{\xi_t} - \varphi'_{k,t}\left(\frac{1}{\xi_t}\Delta^k_t - \delta^k_t\right) = \mathbb{E}_{t+1}\frac{\beta M_{t+1}}{M_t}\frac{1}{\xi_{t+1}} R^{k-1}_{t+1},$$

where R^0_{t+1} is the risk-free interest rate.

Using this observations we can construct both the debt level and the portfolio perturbation. The debt level perturbation is equivalent to setting $\varepsilon_k = \omega_t^k \varepsilon / Q_t^k$ for all maturities k, which implies, due to the previous equation, that

$$\frac{1}{\xi_t} - \sum_k \omega_t^k \varphi_{k,t}' \left(\frac{1}{\xi_t} \Delta_t^k - \delta_t^k \right) = \mathbb{E}_{t+1} \frac{\beta M_{t+1}}{M_t} \frac{1}{\xi_{t+1}} \mathcal{R}_{t+1}.$$

The portfolio perturbation is setting $\varepsilon_k = \varepsilon / Q_t^k$ for some maturity k and $\varepsilon_1 = -\varepsilon / Q_t^1$ to obtain, using our assumption $\varphi_{1,t}(\cdot) = 0$,

$$-\omega_t^k \varphi_{k,t}' \left(\frac{1}{\xi_t} \Delta_t^k - \delta_t^k \right) = \mathbb{E}_{t+1} \frac{\beta M_{t+1}}{M_t} \frac{1}{\xi_{t+1}} r_{t+1}^k.$$

If $\delta_t^k / \Delta_t^k = 0$ these two equations reduce to (34) and (35).

Since price effects are zero to the first order, the result of Lemma 2(a) is then unchanged. Lemma 2(b) is obtained by twice differentiating (35) to get

$$\mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma \tau_{t+1} = -\frac{\xi(\overline{\tau}_t)}{-\xi'(\overline{\tau}_t)\, \overline{Q}_t^1} \overline{\Delta}_t^k \varphi_{k,t}'(\widetilde{B}_t^k). \tag{57}$$

This is equation (36) written in terms of observables.

B.4.2 Approximation of optimal portfolios

Note that equation (57) has a very similar structure to (47). For this reason, arguments analogous to the proof of equation (48) gives

$$\begin{aligned}\overline{Q}_t^1 \overline{B}_t \sum_{i \neq rf} \overline{\omega}_t^i \mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma r_{t+1}^i = {} & \sum_{k=1}^{\infty} \overline{Q}_t^{k+1} \overline{X}_{t+k+1} \mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma \ln Q_{t+1}^k \\ & + \sum_{k=1}^{\infty} \overline{Q}_t^k \overline{G}_{t+k} \mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma \ln G_{t+k} - \frac{\overline{Y}_t \xi(\overline{\tau}_t)^2 \sum_{k=1}^{\infty} \overline{Q}_t^k}{-\xi'(\overline{\tau}_t)\, \overline{Q}_t^1} \overline{\Delta}_t^k \varphi_{k,t}'(\widetilde{B}_t^k).\end{aligned} \tag{58}$$

Finally, observe that Δ_t^k can be written as

$$\Delta_t^k = \frac{B_t}{Q_t^k} \left(\frac{Q_t^k \widetilde{B}_t^k}{B_t} - \frac{Q_t^k \widetilde{B}_{t-1}^{k+1}}{B_t} \right) = \frac{B_t}{Q_t^k} \left(\omega_t^k - \omega_{t-1}^{k,+} \right),$$

where we applied definition of ω_{t-1}^+ given in text. Substitute this into (58) and to obtain expression for the optimal portfolio. If Σ_t is invertible, it can be stated as (38), where ω_t^* is given in (19).

B.4.3 Optimal portfolio without perfectly elastic demand risk-free bond

Optimal portfolio with price effects on all maturities We now extend the analysis to allow for non zero price effects of all maturities including the risk-free bond. The steps are similar to before. The portfolio perturbation now will yield

$$-\frac{1}{\xi_t}\left(\varphi'_{k,t}\Delta_t^k - \varphi'_{1,t}\Delta_t^1\right) = \mathbb{E}_t \frac{\beta M_{t+1}}{M_t}\frac{1}{\xi_{t+1}} r_{t+1}^k$$

or

$$cov_t\left(r_{t+1}^k, \partial\tau_{t+1}\right) \simeq -\frac{\xi\left(\tau_t\right)}{-\xi'\left(\tau_t\right) Q_t^1}\left(\Delta_t^k \varphi'_{k,t} - \Delta_t^1 \varphi'_{1,t}\right)$$

and then equation (58) will be

$$\sum_{i\neq rf} \overline{\omega}_t^i \mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma r_{t+1}^i = \sum_{k=1}^{\infty} \frac{\overline{Q}_t^{k+1}\overline{X}_{t+k+1}}{\overline{Q}_t^1 \overline{B}_t} \mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma \ln Q_{t+1}^k$$
$$+\sum_{k=1}^{\infty} \frac{\overline{Q}_t^k \overline{G}_{t+k}}{\overline{Q}_t^1 \overline{B}_t} \mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma \ln G_{t+k} - \frac{\overline{Y}_t}{\overline{B}_t}\frac{\xi\left(\overline{\tau}_t\right)^2 \sum_{k=1}^{\infty} \overline{Q}_t^k}{-\xi'\left(\overline{\tau}_t\right)\left(\overline{Q}_t^1\right)^2}\left(\overline{\Delta}_t^k \varphi'_{k,t} - \overline{\Delta}_t^1 \varphi'_{1,t}\right).$$

We still have

$$\Delta_t^k = \frac{B_t}{Q_t^k}\left(\frac{Q_t^k \widetilde{B}_t^k}{B_t} - \frac{Q_t^k \widetilde{B}_{t-1}^{k+1}}{B_t}\right) = \frac{B_t}{Q_t^k}\left(\omega_t^k - \omega_{t-1}^{k,+}\right),$$

so the counterpart of formula (38) is

$$\omega_t = \omega_t^* - \Sigma_t^{-1} D_t \left\{\Lambda_t\left(\omega_t - \omega_{t-1}^+\right) - h_t^1\right\}, \tag{59}$$

where h_t^1 is a vector (same dimension as ω) with elements

$$h_t^1[i] = \frac{Y_t \varphi'_{1,t}}{Q_t^1}\left(\omega_t^1 - \omega_{t-1}^{01+}\right) \text{ for all } i, \omega_t^1 = \mathbf{1} - \mathbf{1}^\intercal \omega_t.$$

Under the assumption that $\varphi'_{k,t}$ are constants denoted by λ_k, we can simplify the expressions further and express the law of motion of ω_t as a linear system.

Define L_t^+ and $L_t^{1,+}$ so that

$$L_t^+ \omega_{t-1} = \omega_{t-1}^+ \quad \omega_{t-1}^{1,+} = L_t^{1,+}\omega_{t-1}.$$

It can be shown that $L_t^+ = \begin{bmatrix} 0 & \frac{B_t Q_t^3}{B_{t+1} Q_t^2} & 0 & ... \\ 0 & 0 & \frac{B_t Q_t^4}{B_{t+1} Q_t^3} & ... \\ .. & ... & ... & ... \end{bmatrix}$ and $L_t^{1,+} = \begin{bmatrix} \frac{B_t Q_t^1}{B_{t+1} Q_t^2} & 0 & 0 & \ldots \end{bmatrix}$.

Using this, we can rewrite equation (59) by substituting out h_t^1 as

$$\begin{aligned}
\omega_t &= \omega_t^* - \Sigma_t^{-1} D_t \left[\Lambda_t \left(\omega_t - L_t^+ \omega_{t-1}\right) - h_t^1\right] \\
&= \omega_t^* - \Sigma^{-1} D_t \Lambda_t \omega_t + \Sigma_t^{-1} D_t \Lambda_t L_t^+ \omega_{t-1} + \Sigma_t^{-1} D_t \lambda_1 \left(\mathbf{1} - \mathbf{1}\mathbf{1}^\intercal \omega_t - \mathbf{1} L_t^{1,+} \omega_{t-1}\right) \\
&= \left[I + \Sigma_t^{-1} D_t \Lambda_t + \lambda_1 \Sigma_t^{-1} D_t \mathbf{1}\mathbf{1}^\intercal\right] \left(\omega_t^* + \Sigma_t^{-1} D_t \lambda_1 \mathbf{1}\right) \\
&+ \left[I + \lambda_1 \Sigma_t^{-1} D_t \Lambda_t D_t \Sigma_t^{-1} \mathbf{1}\mathbf{1}^\intercal\right]^{-1} \left[\Sigma_t^{-1} D_t \left(\Lambda_t L_t^+ - \lambda_1 \mathbf{1} L_t^{1,+}\right)\right] \omega_{t-1}. \qquad (60)
\end{aligned}$$

C Additional details for Section 5

C.1 Nominal Economy

We now describe a nominal version of the benchmark economy. Let P_t be the price level and suppose all securities are nominal. The risk-free bond now refers to a nominal one-period bond that pays one dollar next period. The household and government budget constraint in the nominal economy are

$$P_t C_t + \sum_i b_t^i = (1 - \tau_t) P_t Y_t + \sum_i R_t^i b_{t-1}^i$$

and

$$P_t (T_t - G_t) + \sum_i B_t^i = \sum_i R_t^i B_{t-1}^i,$$

respectively, where $\left\{b_t^i, B_t^i\right\}$ are market values in dollars of private and public sector holdings of security i and $R_t^i = \frac{P_t D_t + Q_t^i}{Q_t^i}$ with Q_t^i being the price of security i in dollars is the nominal holding period return on security i. The definition of competitive equilibrium and optimum competitive equilibrium remain unchanged except in nominal economy they are defined for $\{G_t, P_t, S_t\}$.

It is easy to see that the debt perturbation we considered in Section 3 require tax adjustments $\frac{\varepsilon}{P_t(s^t)}$ in s^t and $\frac{\mathcal{R}_{t+1}(s^{t+1})\varepsilon}{P_{t+1}(s^{t+1})}$ in all $s^{t+1} \succeq s^t$ so that equation (8) remains unchanged with the interpretation that $\mathcal{R}_{t+1}\left(s^{t+1}\right)$ is the nominal return on the government portfolio. A similar argument shows that the portfolio perturbation requires $\frac{r_{t+1}^j(s^{t+1})\varepsilon}{P_{t+1}(s^{t+1})}$ as the tax adjustment and equation (9) remains unchanged too. This means that the proof of Lemma 2 can be extended to the nominal economy with the only change that r_{t+1}^j is the excess nominal return on security i.

We can rewrite equation (4) for the nominal economy as

$$\mathbb{E}_{t+1} \sum_{k=1}^{\infty} \mathcal{Q}_{t+1,t+k} \left(P_{t+k} T_{t+k} - P_{t+k} G_{t+k}\right) = (R_{t+1}^{rf} + \sum_{i \neq rf} \omega_t^i r_{t+1}^i) B_t,$$

and applying the same steps as in the proof of Theorem 1, we get that the optimal portfolio satisfies

$$\sum_{i\neq rf}\overline{\omega}_t^i\mathbb{E}_t\partial_\sigma r_{t+1}^i\partial_\sigma r_{t+1}^j=\sum_{k=1}^{\infty}\frac{\overline{Q}_t^{k+1}\overline{X}_{t+k+1}^{\$}}{\overline{Q}_t^1\overline{B}_t}\mathbb{E}_t\partial_\sigma\ln Q_{t+1}^k\partial_\sigma r_{t+1}^j-\sum_{k=1}^{\infty}\frac{\overline{Q}_t^k\overline{G}_{t+k}^{\$}}{\overline{Q}_t^1\overline{B}_t}\mathbb{E}_t\partial_\sigma\ln G_{t+k}^{\$}\partial_\sigma r_{t+1}^j,\tag{61}$$

where $\overline{G}_{t+k}^{\$}=\overline{P_{t+k}G_{t+k}}$, $\overline{X}_{t+k}^{\$}=\overline{T}_t^{\$}-\overline{G}_{t+k}^{\$}$ and $\overline{T}_t^{\$}=\frac{\overline{B}_t+\sum_{k=1}^{\infty}\overline{Q}_t^k\overline{G}_{t+k}^{\$}}{\sum_{k=1}^{\infty}\overline{Q}_t^k}$ with $\{B_t\}$ is market value of public portfolio in dollars, $Q_{t+1}^k$is the nominal price of a hypothetical k period zero coupon bond in dollars, and $\left\{r_{t+1}^j\right\}$ are nominal excess returns on security j. Thus the formula is same as long as we use the appropriate nominal versions of the objects. We drop the \$ superscripts in the main text.

C.2 Data

Output, expenditures, tax revenues

We use the U.S. national income and product accounts to measure output, tax revenues. For our measure of output Y_t we use U.S. GDP. We measure nominal tax revenues T_t as Federal Total Current Tax Receipts + Federal Contribution To Social Insurance and public expenditures G_t as Federal Consumption Expenditures + Federal Transfer Payments To Persons from BEA. All series are nominal and de-trended with constant time trends.

Tax rates

As a measure of tax rates τ_t we use the measure of the average marginal federal tax rate from Barro and Redlick (2011). Their series end in 2012 but we follow their steps and extrapolate this series for the years 2013-2017 using the Statistics of Income publicly available data from the Taxstats website. The series for the raw tax rates are plotted in Figure 6(e). It is clear from the series that there is a structural break in taxes around 1975. In our analysis we use want to focus on movements in taxes around business cycle frequencies and therefore we want to remove this break. We pursue two ways of doing that. First, we follow the business cycle literature and apply a Hodrick-Prescott (HP) filter with the penalty parameter set to 1,600. The resulting series is shown as the teal-blue line in the right panel Figure 6(f). While this procedure eliminates the low frequency movements in taxes, it also makes the resulting series "too smooth" post 1975. As an alternative, we adjust the penalty parameter until we achieve both goals: remove low frequency movements around 1975 and preserve the volatility of tax rates after and before 1975. The resulting series is show in the red line (at a penalty parameter

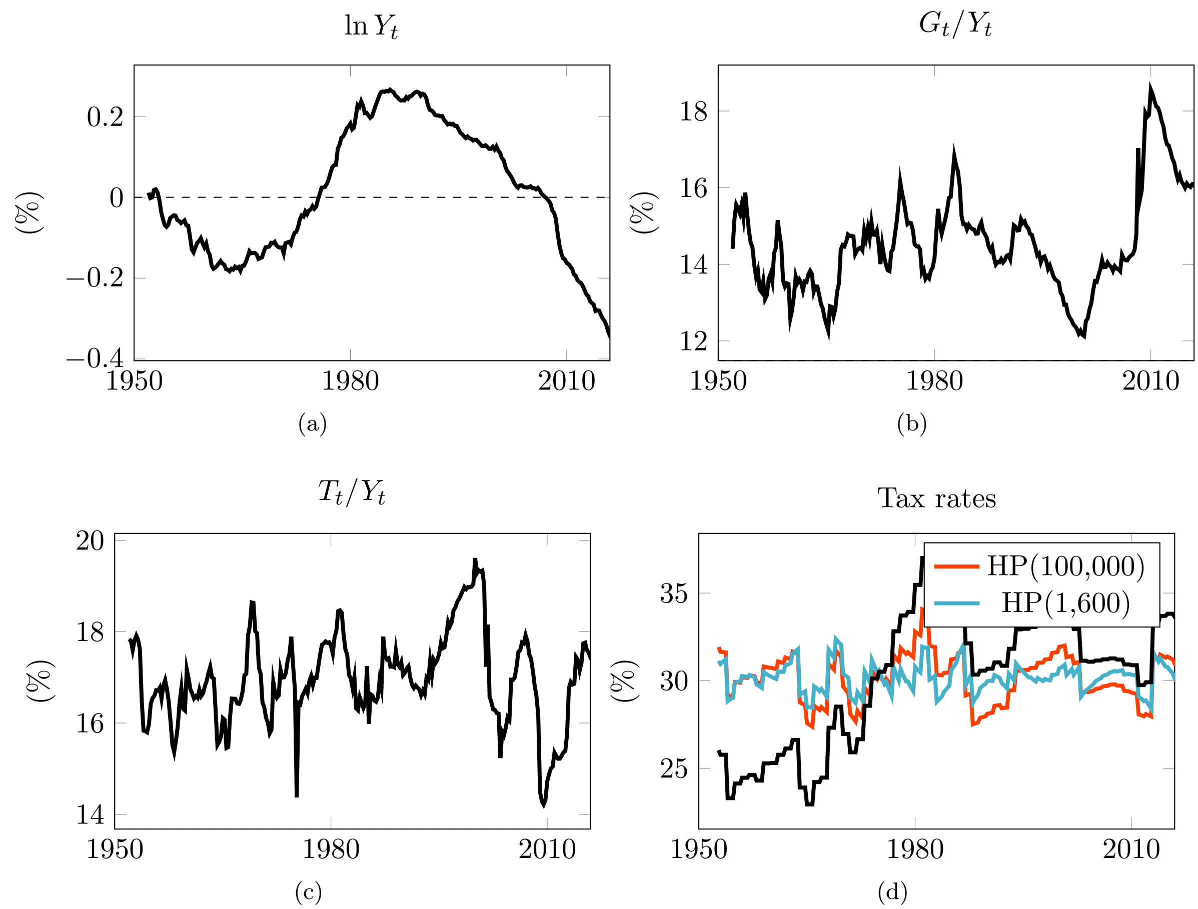

Figure 6: Summary of macroeconomic time series. Panel (a) plots detrended log nominal GDP, panel (b) plots the nominal government expenditure measured as Federal Consumption Expenditures + Federal Transfer Payments To Persons divided by nominal GDP, panel (c) plots nominal revenues divided by nominal GDP, panel (d) plots the average marginal tax rate on income and two ways of detrending the series.

of 100,000) in the right panel. We use the red line as a baseline measure of tax rates, but all our results are virtually unchanged if we use the teal line instead.

Asset returns and government portfolio of bonds

We use the Fama Maturity Portfolios published by CRSP. There are 11 such portfolios, out of which ten portfolios correspond to maturities of 2 to 20 quarter in 2 quarter intervals, and a final portfolio for maturities between 30 and 40 quarters. We use the convention that the upper cut-off for each maturity corresponds to j in the mapping of data to the theory. That is, we use returns on portfolio of bonds of maturities between 2 to 4 quarter to measure r_t^j to $j = 4$, between 4 to 6 quarters to measure r_t^j for $j = 6$, etc. With this convention $j = 40$ is the largest maturity. We aggregate monthly log-returns by summing them across months within each quarter.

To compute the price curve we use yield data from Gurkaynak et al. (2007). The raw data has daily yields for zero coupon bonds of maturities $4, 8, \ldots 120$ quarters. We interpolate the daily yields using a cubic spline to infer yields for all quarters less than 120. The price curve is computed by using the expression $Q^n = \exp\{-n \times y\}$

Maturity structure of the U.S. government debt

We use the CRSP Treasuries Monthly Series to get the amount outstanding B_t^i for all (including TIPS and other inflation-protected bonds) federally issued (marketable) debt between 1952 and 2017, normalized by its face value. Each bond is uniquely identified by its cusips number n. CRSP also supplies us the Macaulay duration i for the outstanding amount, and the nominal market price $Q_t^{n,i}$ of each bond outstanding. For a few bonds where duration is absent, we set the duration equal to maturity date $-$ current date.

We follow Jiang et al. (2019), and construct at each date t, the market value $Q_t^i B_t^i$ held by the US government in bonds of Macaulay duration i, by summing across cusips n, such that $Q_t^i B_t^i = \sum_n Q_t^{n,i} B_t^{n,i}$. We then sum across all Macaulay duration i to get the market value of the government debt portfolio $B_t \equiv \sum_{i \in \mathcal{G}_t} Q_t^i B_t^i$ at each date t. We finally compute the portfolio weight in the US government debt portfolio for each maturity i using that $\omega_t^i = \frac{Q_t^i B_t^i}{B_t}$.

C.3 Derivations for the baseline factor model

In this section we derive expressions (40) and (41).

From Theorem 1, and discussion of equation (24), the covariances Σ_t, Σ_t^Q and Σ_t^G have elements $\{cov_t(r_{t+1}^i, r_{t+1}^j)\}_{i,j}$, $\{cov_t(\ln Q_{t+1}^k, r_{t+1}^j)\}_{j,k}$, $\{cov_t(\ln G_{t+k}, r_{t+1}^j)\}_{j,k}$, $\{cov_t(\ln \Theta_{t+k}, r_{t+1}^j)\}_{j,k}$for all j and $k \geq 1$, and weights s_t^Q and s_t^G , s_t^Θ are vectors with elements $\{\frac{Q_t^{k+1}\mathbb{E}_t X_{t+k+1}}{Q_t^1 B_t}\}_k$ and $\{\frac{-Q_t^k \mathbb{E}_t G_{t+k}}{Q_t^1 B_t}\}_k, \{\frac{-Q_t^k \mathbb{E}_t T_{t+k}^{tax}}{Q_t^1 B_t}\}_k$. In the baseline case, our factor structure boils down to

$$r_t^k = \kappa_k f_t + \sigma_k \varepsilon_t^k$$

$$\ln G_t = t\Gamma + +\kappa_G f_t + \sigma_G \varepsilon_t^G$$

$$\ln \Theta_t = t\Gamma + +\kappa_\Theta f_t + \sigma_\Theta \varepsilon_t^\Theta$$

Under this factor structure, we can calculate $\left(\Sigma_t, \Sigma_t^Q, \Sigma_t^G\right)$ and $\left(s_t^Q, s_t^G, s_t^\Theta\right)$ explicitly. The elements of Σ_t satisfy

$$\begin{aligned}\Sigma_t[j,k] &= cov_t(r_{t+1}^i, r_{t+1}^k)\, forj, k \in \mathcal{G} \\ &= \kappa_j \kappa_k \sigma_f^2 + \iota_{\{j=k\}} \sigma_j^2 \text{ for } j, k \in \mathcal{G}\end{aligned}$$

Lemma 1 implies that $Q_t^1 cov_t(r_{t+1}^k, r_{t+1}^j) \simeq cov_t(\ln Q_{t+1}^k, r_{t+1}^j)$ and we get

$$\Sigma_t^Q[j,k] = Q_t^1 \left(\kappa_j \kappa_k \sigma_f^2 + \iota_{\{j=k\}} \sigma_j^2\right) \text{ for} j \in \mathcal{G}, k \in \mathbb{N}$$

The steps to compute the covariances of spending and tax revenues wit returns next period are similar, so we show the derivation for just one of them. The elements of Σ_t^G and Σ_t^Θ for for$j \in \mathcal{G}, k \in \mathbb{N}$

$$\begin{aligned}\Sigma_t^G[j,k] &= cov_t(\ln G_{t+k}, r_{t+1}^j) \\ &= \kappa_G \kappa_j \sigma_f^2\end{aligned}$$

$$\Sigma_t^\Theta[j,k] = \kappa_\Theta \kappa_j \sigma_f^2$$

$$\Sigma_T = \Delta + \kappa \Delta_f \kappa^\intercal$$

Using the Woodbury matrix identity, Σ_t^{-1} can be explicitly computed as

$$\Sigma_t^{-1}[j,k] = \iota_{\{k=j\}} \frac{1}{\sigma_j^2} - \kappa_j \kappa_k \frac{\chi^2}{\sigma_j^2 \sigma_k^2}\, forj, k \in \mathcal{G}$$

where the constant $\chi^{-2} = \sigma_f^{-2} + \sum_{k \in \mathcal{G}} \kappa_k^2 \sigma_k^{-2}$.

We next derive the weights $\left(s_t^Q, s_t^G, s_t^\Theta\right)$. The weights on interest rate risk are given by

$$s_t^Q[k] = \frac{Q_t^{k+1}\mathbb{E}_t X_{t+k+1}}{Q_t^1 B_t} = \frac{Q_t^{k+1}\Gamma^{k+1}}{Q_t^1 \sum_{k=1}^\infty Q_t^k \Gamma^k}$$

and the s_t^G, s_t^Θ are

$$s_t^G[k] = \frac{-Q_t^k \mathbb{E}_t G_{t+k}}{Q_t^1 B_t} = \frac{-Q_t^k G_t \Gamma^k}{Q_t^1 B_t}, \quad s_t^\Theta[k] = \frac{Q_t^k \mathbb{E}_t T_{t+k}^{tax}}{Q_t^1 B_t} = \frac{Q_t^k T_t \Gamma^k}{Q_t^1 B_t}$$

Now let us derive the expressions for each of the terms in the portfolio. We start with $\Sigma_t^{-1}\Sigma_t^G s_t^G$ which is the portfolio that hedges the spending risk.

$$\Sigma_t^G s_t^G[j] = -\sum_{k=1}^{\infty} \kappa_G \kappa_j \sigma_f^2 \left(\frac{Q_t^k G_t \Gamma^k}{Q_t^1 B_t}\right)$$
$$- \kappa_G \kappa_j \sigma_f^2 \frac{G_t}{Q_t^1 B_t} \sum_{k=1}^{\infty} Q_t^k \Gamma^k$$

Now we multiply with Σ_t^{-1} to get

$$\Sigma_t^{-1}\Sigma_t^G s_t^G = -\kappa_G \frac{G_t}{Q_t^1 B_t} \sum_k Q_t^k \Gamma^k \begin{bmatrix} \frac{\chi^2 \kappa_1}{\sigma_1^2} \\ \frac{\chi^2 \kappa_2}{\sigma_2^2} \\ \vdots \\ \\ \\ \end{bmatrix}$$

The same steps apply to the portfolio that hedges the tax revenue risk. So

$$\Sigma_t^{-1}\Sigma_t^\Theta s_t^\Theta = \kappa_\Theta \kappa_j \frac{T_t}{Q_t^1 B_t} \sum_k Q_t^k \Gamma^k \begin{bmatrix} \frac{\chi^2 \kappa_1}{\sigma_1^2} \\ \frac{\chi^2 \kappa_2}{\sigma_2^2} \\ \vdots \\ \\ \\ \end{bmatrix}.$$

The final portfolio we derive is the one that hedges the interest rate risk. The first step to get $\Sigma_t^Q s_t^Q$

$$\begin{aligned} \frac{\Sigma_t^Q s_t^Q[j]}{Q_t^1} &= \sum_{k=1}^{\infty} \left(\kappa_j \kappa_k \sigma_f^2 + \iota_{\{j=k\}} \sigma_j^2\right) s_t^Q[k] \\ &= \sum_{k=1}^{\infty} \iota_{\{j=k\}} \sigma_j^2 s_t^Q[k] + \kappa_j \sigma_f^2 \sum_{k=1}^{\infty} \kappa_k s_t^Q[k] \\ &= \sigma_j^2 s_t^Q[j] + \kappa_j \sigma_f^2 \sum_{k=1}^{\infty} \kappa_k s_t^Q[k] \end{aligned}$$

and then we multiply with Σ_t^{-1} to get

$$\left(\Sigma^{-1}\Sigma^Q s_T^Q\right)[j] = \frac{1}{\sum_{k=1}^{\infty} Q_t^k \Gamma^k} \left(Q_t^{j+1}\Gamma^{j+1} + \chi^2 \kappa_j \sigma_j^{-2} \left[\sum_{k=1}^{\infty} \kappa_k Q_t^{k+1} \Gamma^{k+1} - \sum_{k \in \mathcal{G}} \kappa_k Q_t^{k+1} \Gamma^{k+1}\right]\right).$$

So finally we get

$$\omega_t^X[j] = \Sigma_t^{-1}\Sigma_t^\Theta s_t^\Theta[j] + \Sigma_t^{-1}\Sigma_t^G s_t^G[j] = \left(\frac{1}{1-\hat{\beta}_t}\right) \left(\frac{\kappa_\Theta T_t - \kappa_G G_t}{Q_t^1 B_t}\right) \left(\frac{\kappa_j}{\sigma_j^2}\chi^2\right),$$

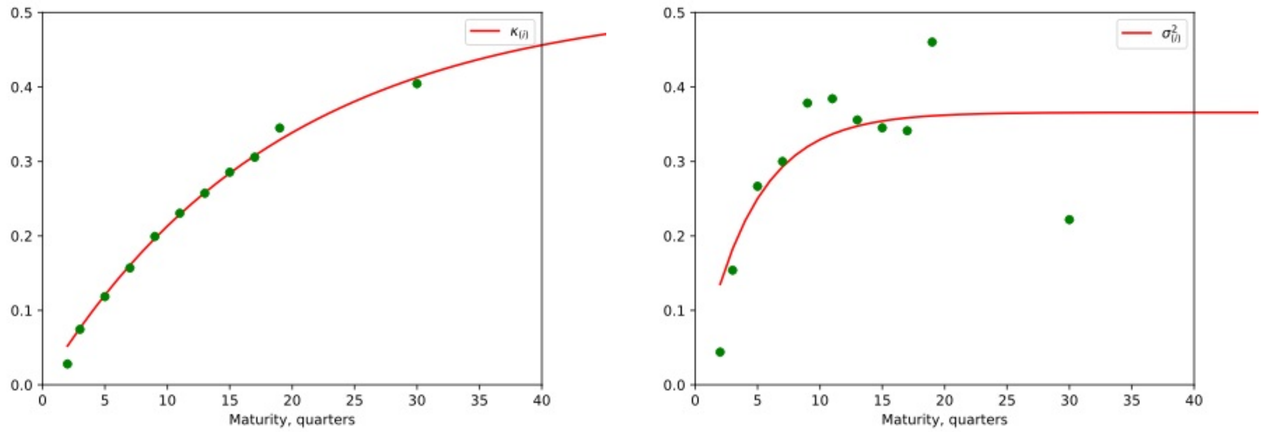

Figure 7: Fit for extrapolation of the factor model estimates of $\left(\kappa_j, \sigma_j^2\right)$ using $f(j) = e^0 - e^0 \exp(-e^1 \times j)$ for factor model (39). The dotted points are the point estimates and the bold line is the interpolation.

Figure 8

$$\omega_t^Q[j] = \left(\Sigma^{-1}\Sigma^Q s_T^Q\right)[j] = \left(1 - \hat{\beta}_t\right)\left[\left(Q_t^{j+1}\Gamma^{j+1}\right) + \left(\sum_{\ell \notin \mathcal{G}} Q_t^{\ell+1}\Gamma^{\ell+1}\kappa_\ell\right)\left(\frac{\kappa_j}{\sigma_j^2}\chi^2\right)\right].$$

C.4 Estimations and extrapolations

We estimate model our factor model (39) using OLS. In the main text (Table 2), we report the estimates for the baseline specification in which we restricted $\rho_G = \rho_\Theta = 1$ and $\rho_f = 0$. This estimation procedure produces estimates of $\left(\alpha_j, \rho_j, \kappa_j, \sigma_j^2\right)$ for eleven j, with the highest being $j = 40$. For constructing our target portfolios, we need to extrapolate $\left(\rho_j, \kappa_j\right)$ for all $j > 1$. In the baseline extrapolation, we estimate δ_j and σ_j^2 by fitting the closest exponential function: $f(j) = e^0 - e^0 \exp(-e^1 \times j)$ for $f(j) \in \left\{\delta_j, \sigma_j^2\right\}$. We fit the parameters e^0 and e^1 to minimize sum of squares between fitted and actual values of δ_j and σ_j^2. The fit is reported in Figure 8

Fit of covariances In the text we mention that a test for the factor model is how well it captures contemporaneous covariances. To implement this test we compute $cov(r^j, G)$ and $cov(r^j, \Theta)$ using the estimated factor model and plot it against the ones computed using the raw data for the 11 portfolios that we use. In Figure 9, we see the fit is good.

Components of the target portfolio In Figure 10, we breakup the target portfolio into the portfolio that hedges government spending ω_t^G, tax revenue risk ω_t^Θ, and interest rate risk ω_t^Q.

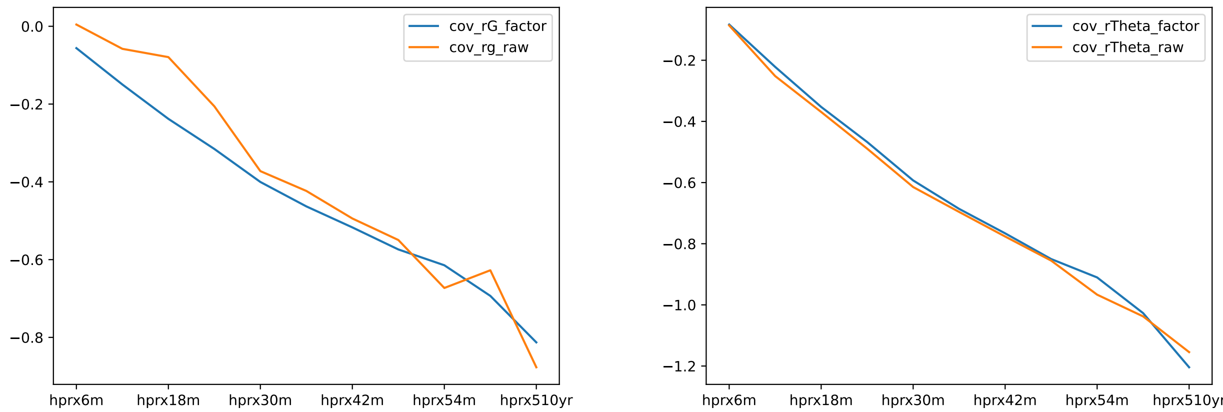

Figure 9: Fit for contemporaneous covariances. The blue lines is computed using the estimates of the factor model and the orange line is computed using the data for the sample period 1952-2017.

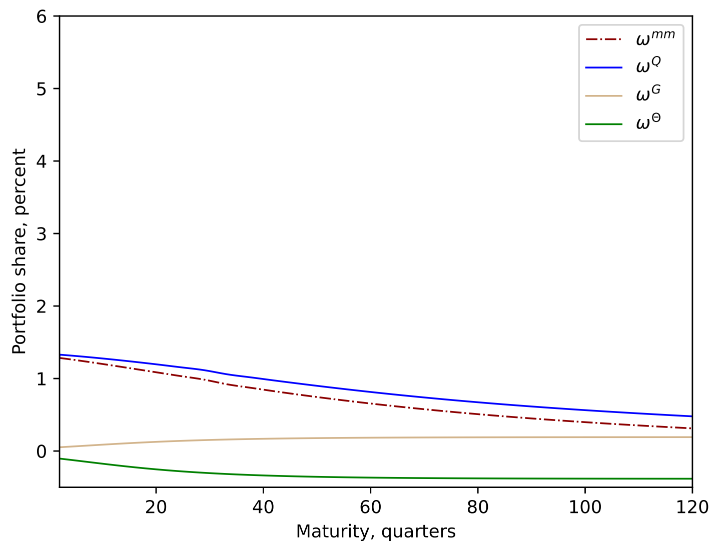

Figure 10: Components of the target portfolio.

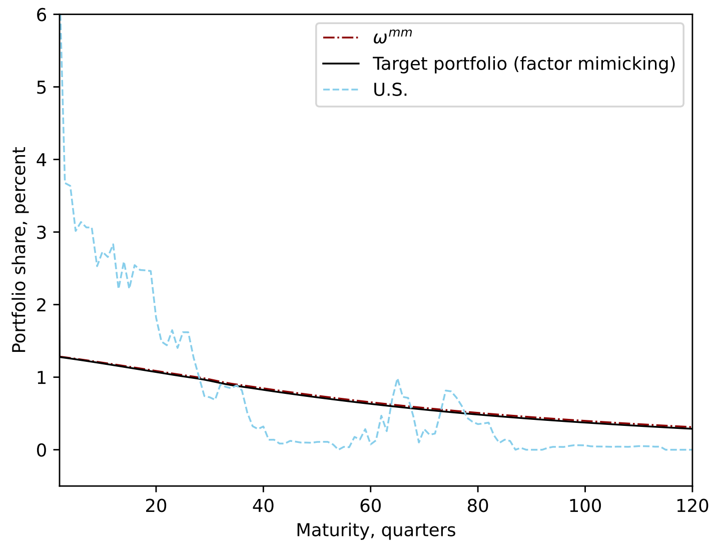

Figure 11: Target portfolio using the common factor extracted from bond excess returns.

C.5 Robustness

Factor-mimicking portfolios In the main text, we extract the common factor using return and macro data. In Figure 11, we report results using the common factor using only bond return data.

Multifactor We extend the factor model to have multiple factors. Let z_t be a stacked vector that consists of excess returns $\left\{r_t^j\right\}_j$ for the 11 portfolios of different maturities j, a measure of $\ln\Theta_t$ and expenditures $\ln G_t$. We use z_t^k to denote the k^{th} element of this vector. We posit the following stochastic process

$$z_t^k = \alpha_k + \rho_k z_{t-1}^k + \sum_m \kappa_{m,k} f_t^m + \varepsilon_t^k \text{ for all } k,$$
$$f_t^n = \alpha_f^n + \sum_m \kappa_{m,f} f_t^m + \varepsilon_t^{f,n} \text{ for all } j = 1 \dots n, \tag{62}$$

where $\{f_t^n\}_n$ are a set of factors and $\left\{\varepsilon_t^k, \varepsilon_t^{f,n}\right\}_{k,t}$ are residuals. We use the subscripts $k \in \{\Theta, G\}$ to denote the variables $\ln\Theta_t$, $\ln G_t$, and $k = j$ to denote returns on bonds of maturity j. In this section, we report the estimates of the factor model and other details skipped in the main text.

With multiple factors the covariance of returns with each other is given by $\Sigma_t = \Delta + \kappa\Delta_f\kappa^\intercal$ where $\Delta = diag\left\{\sigma_j^2\right\}$,$\kappa = [\kappa_1 \quad \kappa_2 \dots]$ is the matrix of factor loadings on returns

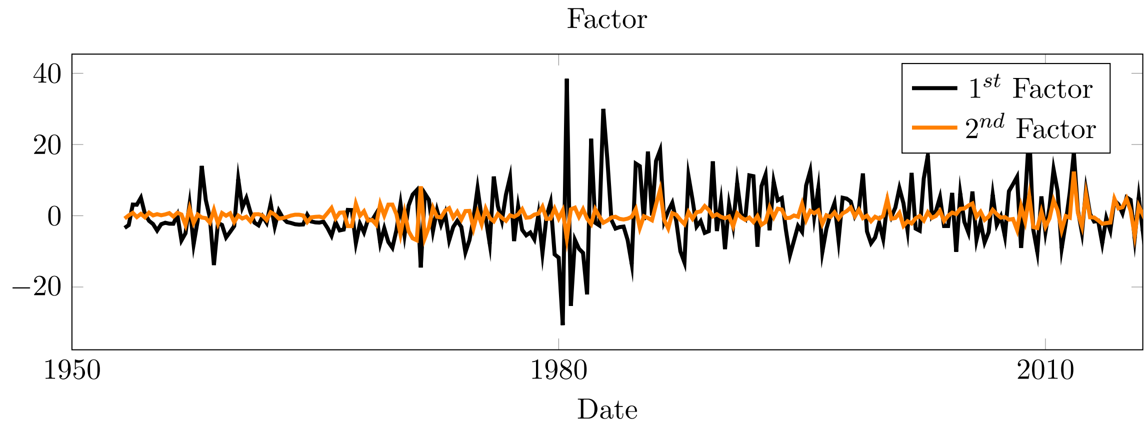

Figure 12: Time series for the first two principal components

Table 4: FACTOR LOADINGS (MULTIFACTOR)

	Excess returns r_t^j for various maturities j											
	6m	12m	18m	24m	30m	36m	42m	48m	54m	60m	120m	$\ln G_t$	$\ln \Theta_t$
$\kappa_{1,k}$	0.028	0.075	0.119	0.157	0.200	0.231	0.257	0.286	0.306	0.345	0.404	-0.032	-0.047
s.e	0.001	0.002	0.002	0.002	0.002	0.002	0.002	0.003	0.003	0.004	0.004	0.016	0.008
$\kappa_{2,k}$	-0.053	-0.126	-0.183	-0.208	-0.244	-0.248	-0.248	-0.218	-0.198	-0.184	-0.066	-0.031	0.024
s.e	0.005	0.008	0.009	0.008	0.008	0.008	0.006	0.009	0.011	0.015	0.013	0.058	0.030
σ_k^2	0.030	0.074	0.098	0.083	0.082	0.077	0.048	0.107	0.145	0.292	0.200	4.227	1.143
s.e	0.003	0.007	0.009	0.007	0.007	0.007	0.004	0.009	0.013	0.026	0.018	0.374	0.101
R^2	0.536	0.698	0.771	0.840	0.870	0.898	0.922	0.938	0.946	0.943	0.979	0.016	0.111

Notes: This table records the OLS factor loadings for the two factor version of the model (39). Standards errors are in parenthesis.The row titled "R2" are values of R-squared for each equation in the system (39). The sample for excess returns and primary surpluses normalized by outputs is 1952-2017, and the sample for the one-period liquidity premium is 1984-2017. The time period is a quarter.

and inverse Σ_t^{-1} can be obtained using Woodbury matrix identity. As before, we use Lemma 1 to compute Σ_t^Q using Σ_t. The covariances with spending and tax revenue risk are given by $\Sigma_t^G[j] = \sum_n \kappa_{n,j}\kappa_{n,G}\sigma_{fn}^2$ and $\Sigma_t^\Theta[j] = \sum_n \kappa_{n,j}\kappa_{n,\Theta}\sigma_{fn}^2$. The weights $\left\{s_t^Q, s_t^G, s_t^\Theta\right\}_t$ are unchanged.

In Figure 12, we plot the first two principal components extracted from excess returns, the risk-free rate, GDP (detrended), deficits/GDP. We see that the second factor is less volatile relative to the first factor. We next report the factor loadings for the two factor model in Table 4.

Fit of covariances Figure 13 plots the counterpart of Figure 9 for the multifactor model. We see that the fit is similar and slightly better than the single factor model.

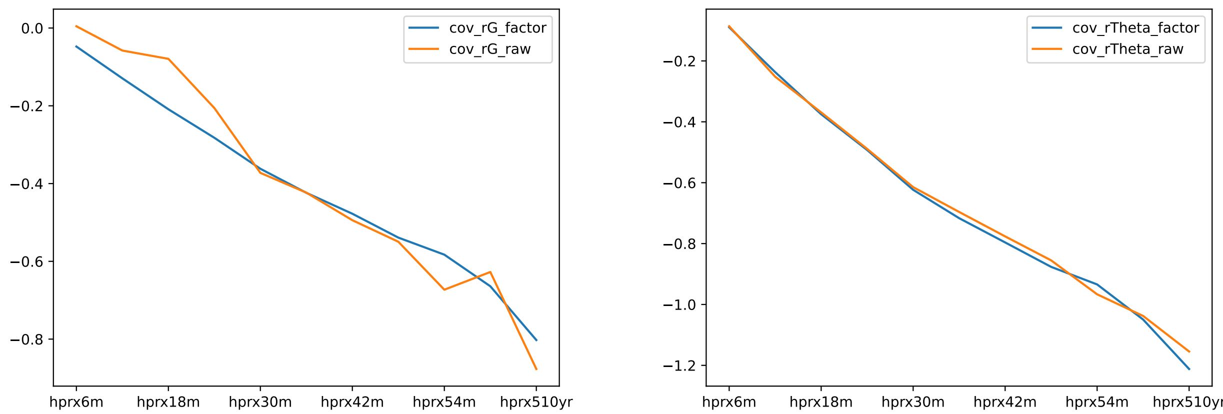

Figure 13: Fit for contemporaneous covariances. The blue lines is computed using the estimates of the multi-factor version of the factor model and the orange line is computed using the data for the sample period 1952-2017.

Extrapolation of loadings In Figure 14, we plot the fit for the factor loadings for the multifactor model. We use the same form for the first factor as the baseline in the text. The second factor is non-monotonic and has a hump shape for intermediate maturities. To capture that hump, we use the functional form $e^0 + e^1 \exp(-(j - e^2)^2/e^3)$. We see in Figure 14 that the interpolated lines fit well the point estimates.

Limiting Portfolio with One Factor In the main text, we compared the limiting portfolio as we send the estimated $\sigma_j^2 \to 0$ for each j and keep the factor loadings $\{\kappa_{n,j}\}_{n,j}$ for the multifactor setting. In Figure (15), we plot the corresponding figure for the baseline target portfolio with one factor.

AR(1) factor structure We consider the general estimation of (39) without any a-priori restrictions on parameters. Table 5 presents estimation results. We see from the table that we cannot reject $\rho_G = \rho_\Theta = 1$ and $\rho_f = 0$.

Transitory Shock We assume that the spending is given by $G_t = G_t^p G_t^{tr}$ with G_t^p following the same structure as the baseline (39) and $G_t^{tr} = \rho_{tr} G_{t-1}^{tr}$. We then simulate the target portfolio for alternative values of the transitory $\left\{G_t^{tr}, \rho_{tr}\right\}$.

Such a shock affects the path of spending, the optimal tax rate, and through the optimal tax rate the expected path of tax revenues. The tax revenue risk weights are unchanged but the weights on spending risk $s_t^G[k] = \frac{-Q_t^k (\Gamma)^k G_t \left(G_t^{tr}\right)^{1-\rho_{tr}^k}}{Q_t^1 B_t}$ and the weights on interest rate risk

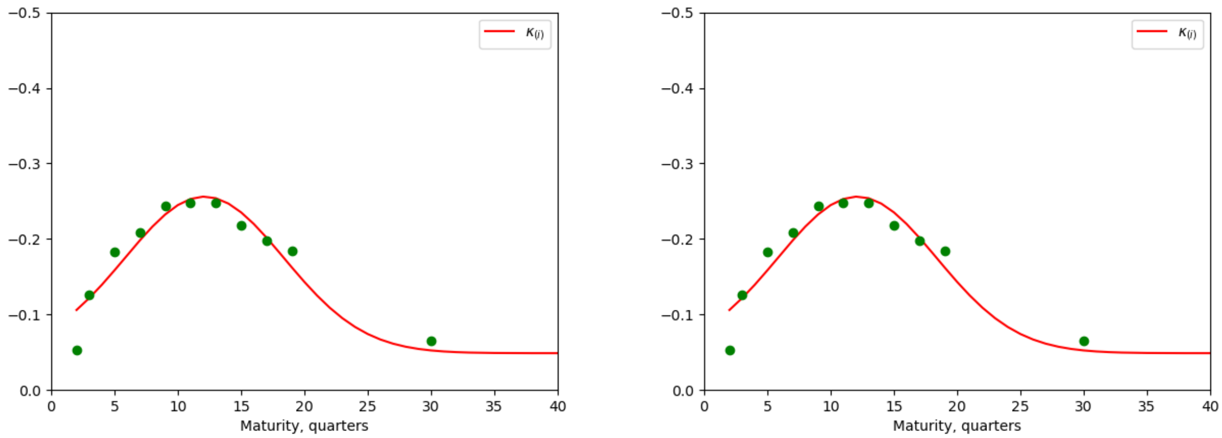

Figure 14: Fit for extrapolation of multifactor model estimates of $(\kappa_{1,j}, \kappa_{2,j})$. For the first factor, we use the functional form $f(j) = e^0 - e^0 \exp(-e^1 \times j)$ and for the second factor we use the functional form $(e^0 + e^1 \exp(-(j - e^2)^2/e^3))$. The dotted points are the point estimates and the bold line is the interpolation.

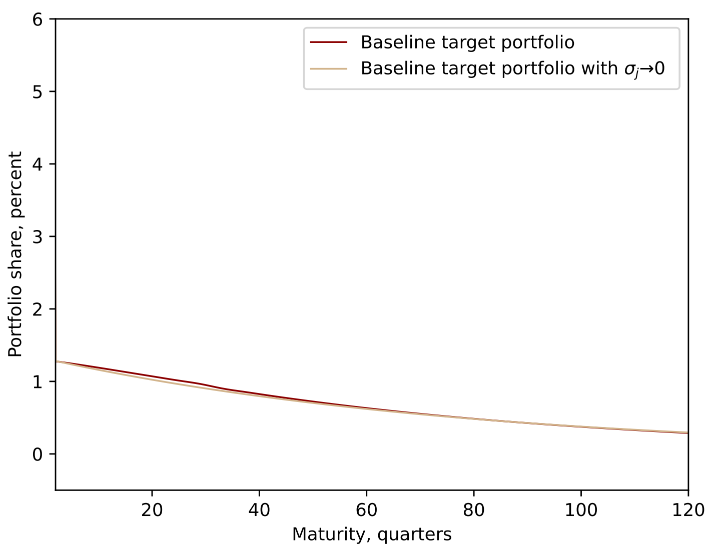

Figure 15: Comparison of the baseline target portfolio to the limiting portfolio in which we set $\sigma_j^2 \to 0$ for each j.

Table 5: FACTOR MODEL ESTIMATION RESULTS (AR(1) FACTOR STRUCTURE)

	Excess returns r_t^j for various maturities j													
	6m	12m	18m	24m	30m	36m	42m	48m	54m	60m	120m	$\ln G_t$	$\ln \Theta_t$	f_t
α_k	0.086 (0.014)	0.155 (0.025)	0.220 (0.033)	0.245 (0.035)	0.284 (0.039)	0.315 (0.039)	0.346 (0.038)	0.344 (0.037)	0.372 (0.037)	0.304 (0.043)	0.444 (0.030)	-0.177 (0.016)	-0.319 (0.008)	0.026 (0.502)
ρ_k	-0.107 (0.043)	-0.057 (0.035)	-0.041 (0.030)	-0.043 (0.025)	-0.042 (0.023)	-0.025 (0.020)	-0.022 (0.018)	-0.008 (0.016)	-0.022 (0.015)	-0.027 (0.015)	0.003 (0.009)	1.001 (0.008)	1.009 (0.004)	-0.035 (0.063)
κ_k	0.028 (0.002)	0.074 (0.003)	0.118 (0.004)	0.157 (0.004)	0.199 (0.005)	0.230 (0.005)	0.257 (0.005)	0.285 (0.005)	0.306 (0.005)	0.345 (0.005)	0.404 (0.004)	-0.032 (0.016)	-0.048 (0.008)	0.000 (nan)
σ_k^2	0.044 (0.004)	0.154 (0.014)	0.267 (0.024)	0.300 (0.027)	0.378 (0.034)	0.384 (0.034)	0.356 (0.031)	0.345 (0.031)	0.341 (0.030)	0.460 (0.041)	0.222 (0.020)	4.231 (0.375)	1.125 (0.100)	63.676 (5.65)
R2	0.536	0.698	0.771	0.840	0.870	0.898	0.922	0.938	0.946	0.943	0.979	0.985	0.996	0.001

Notes: This table records the OLS estimates of the factor model (39) without imposing $\rho_f = 0, \rho_\Theta = \rho_G = 1$. Standards errors are in parenthesis. The sample for excess returns and primary surpluses normalized by outputs is 1952-2017. The time period is a quarter.

$s_t^Q[k] = \frac{Q_t^{k+1}(\Gamma)^{k+1}\left(T_t^{tax} - G_t\left(G_T^{tr}\right)^{1-\rho_{tr}^{k+1}}\right)}{Q_t^1 \sum Q_t^{k+1}(\Gamma)^{k+1}\left(T_t^{tax} - G_t\left(G_t^{tr}\right)^{1-\rho_{tr}^{k+1}}\right)}$ where T_t^{tax} are the tax revenues computed at the optimal tax rate that is constant and balances the inter temporal budget at the zeroth order given the path of spending.

In left panel Figure 16, we show the target portfolio setting for a 10% and 20% increase in the share of spending to GDP with $\rho_{tr} = 0.75$ so that the increase lasts for about 5 years. In right panel, we plot the target portfolios for the same values of G_t^{tr} but a higher value of $\rho_{tr} = .95$.

Heteroskedastic shocks In the main text, we assumed that the shocks $\boldsymbol{\varepsilon}_t$ were homoskedastic, that is, we imposed that $\{\sigma_k\}$ for $k \in \{j, Y, G, A, f\}$ are constant through time. We relax that assumption and augment the baseline factor model 39 with the following univariate GARH processes $\{\sigma_k\}$

$$\sigma_{k,t}^2 = \overline{\sigma}_k^2 + \sum_{j=1}^{p} \rho_{kp}^{GARCH} \varepsilon_{zt-p}^2 + \sum_{j=1}^{q} \varrho_{kq}^{GARCH} \sigma_{\varepsilon z,t-q}^2$$

and impose that all ε are standard Gaussian and independent of each other. We now estimate the system using maximum likelihood and assuming $p = 2$ and $q = 1$.

The consequence of heteroskedastic shocks is that structure of the expressions for Σ_T and Σ_T^{-1} as well as $\Sigma_T^k$for $k \in \{X, A, Q\}$ remains the same but they have time-varying parameters $\sigma_{f,t}$ and $\sigma_{j,t}$ for each return maturities j.[32] We use the same extrapolation scheme as the

[32]The time-variation in $\left\{\sigma_G^2, \sigma_Y^2, \sigma_A^2\right\}$ drops out because the covariances of hedging terms are driven by the common component captured in the factor $\left\{\sigma_{f,t}^2\right\}$.

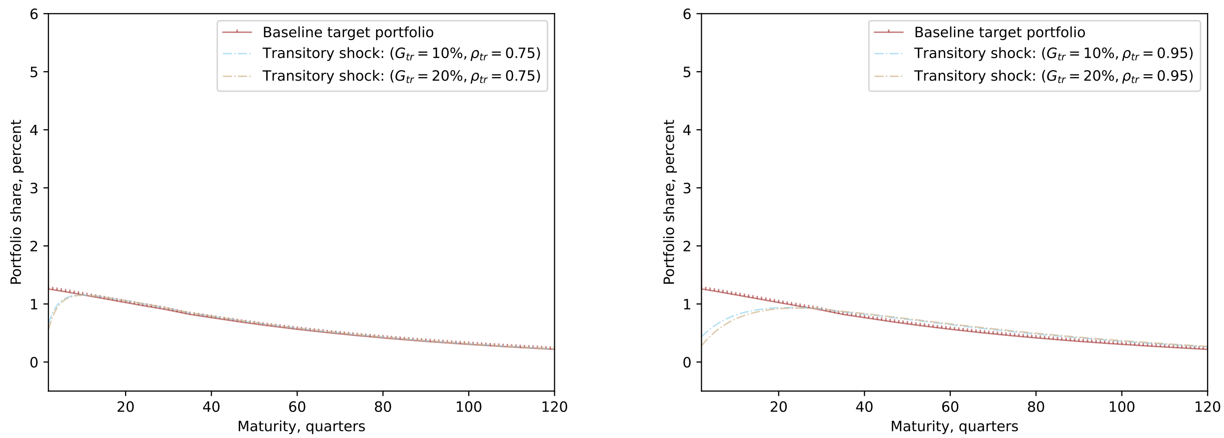

Figure 16: Comparison of optimal portfolios with a transitory increase in spending. In left panel we show the target portfolio for $\ln G_t^{tr} \in \{10\%, 20\%\}$ with $\rho_{tr} = 0.75$ and i n right panel, we plot the target portfolios for for $\ln G_t^{tr} \in \{10\%, 20\%\}$ with $\rho_{tr} = 0.95$.

baseline to obtain (σ_j, κ_j) for other maturities. And finally, as an implication, the optimal target portfolio and its components also inherit that time-variation.[33]

In Figure 17, we plot the time-series for elements in $\{\sigma_{j,t}\}$ and $\sigma_{f,t}$. The volatilities for returns (including the factor) and macro aggregates are high in the early 80s and the great recession of 2008-2010 and quite stable in the intervening periods.

Keeping everything else the same, periods when the factor is more volatile increases the covariance of returns with each other as well as the covariance of returns with surpluses and liquidity risk. Thus, a priori the effect on the optimal portfolio is ambiguous. To gauge how much the portfolio moves overtime, we start by plotting in Figure 18), the 90-10 interval by maturity, that is, for each maturity we construct the 90th and 10th percentile across dates. We see that for lower maturities the portfolio shares varies a few basis points and the fluctuations are much smaller for larger maturities.

Alternative, time aggregation, calculation of returns We also experimented with alternative ways to calculate returns with different time frequencies. In the baseline, we used quarterly measures of returns, surpluses and taxes to ensure the largest sample such that we could measure asset prices and macro data in a consistent way. To verify if our results are driven by our choice of the frequency, we use returns and other macro variables at biannual frequencies. The shortest maturity available is now of 6 months, which we take as our measure

[33] In principle, the fiscal risk and liquidity risk portfolio could vary because quasi-weights π_T^X and π_T^A or $\vec{\beta}$ vary with time. To focus on the impact of heteroskedastic shocks, we keep them constant and equal to the values that we used in the main text and only allow the target portfolio to vary due to time-varying covariances.

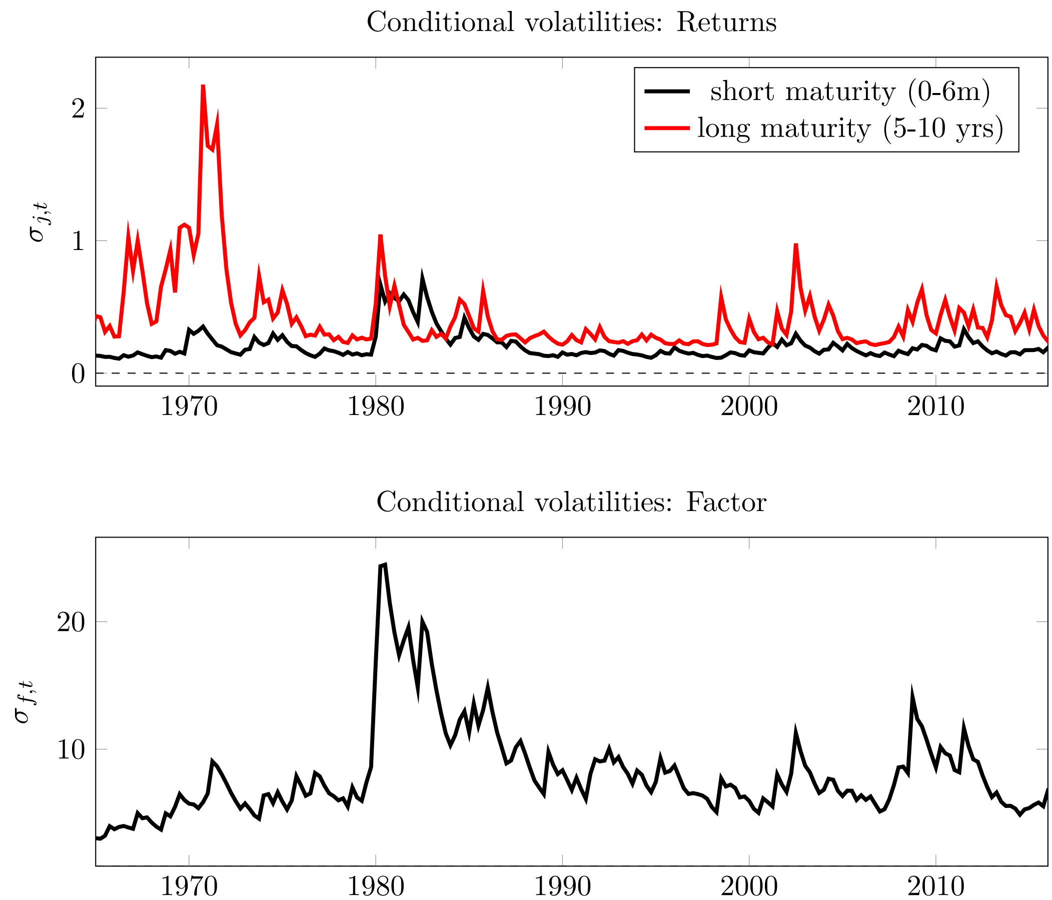

Figure 17: Conditional volatilities of returns, factor, using the estimated GARCH model

Figure 18: 90-10 interval of portfolio shares (maturities from 2 quarters to 120 quarters) with heteroskedastic shocks.

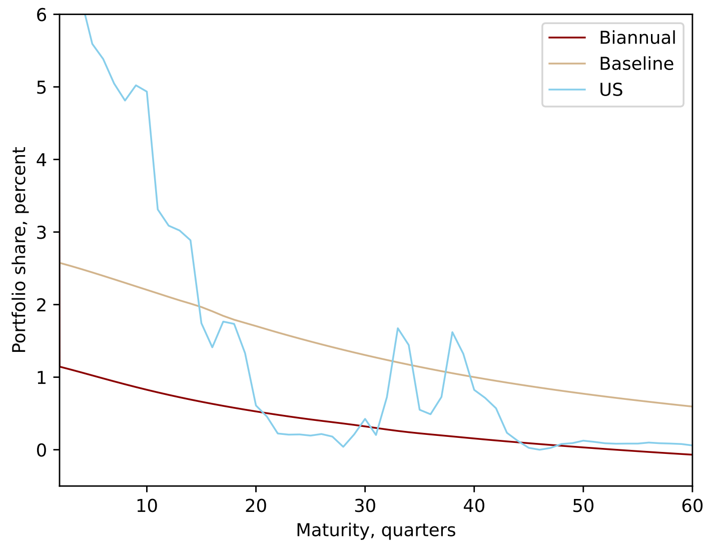

Figure 19: Bi-annual frequency

of the one-period government bond R_t^{rf}. As before, we construct the biannual holding period return by summing monthly returns for each portfolio which are separated by 6 month intervals. For other macro variables, we aggregate two consecutive quarters to obtain the biannual series. Using this data, we apply the same procedure as the baseline (extracting the factor, estimating the factor model, constructing the conditional covariances) and obtain the optimal portfolio. We show the implied unrestricted target portfolios in Figure 19. In order to compare it to our baseline results which have portfolios by quarterly bins, we aggregate the baseline portfolio weights to biannual weights using $\omega_{biannual}[i] \equiv \omega[2i-1] + \omega[2i]$,where i indexes the 6 month intervals and the right hand size is the baseline target portfolio. We find that that the two biannual portfolios are similar.

C.6 Additional details for Section 5.3

In this appendix, we document how we estimate Λ_t using estimates from Greenwood and Vayanos (2014).

Greenwood and Vayanos report estimates of semi-elasticities $\left\{Y_t \frac{\partial \ln \text{yields}_t^k}{\partial \text{Bond Supply}_t}\right\}_k$ by bond maturity for a subset of maturities using maturity-weighted debt $\sum k\tilde{B}_t^k$ as a measure of bond supply. We assume that weights $\tilde{\omega}_t^k = \frac{\tilde{B}_t^k}{\sum_k \tilde{B}_t^k}$ remain constant when $\sum k\tilde{B}_t^k$ changes. Under this assumption, a unit change in $\sum k\tilde{B}_t^k$ leads to a $\frac{1}{\tilde{\omega}_t^k}$ variation in supply of bond of maturity k. Using the definition of Λ_t from the text, and converting the estimate of yields to prices, we

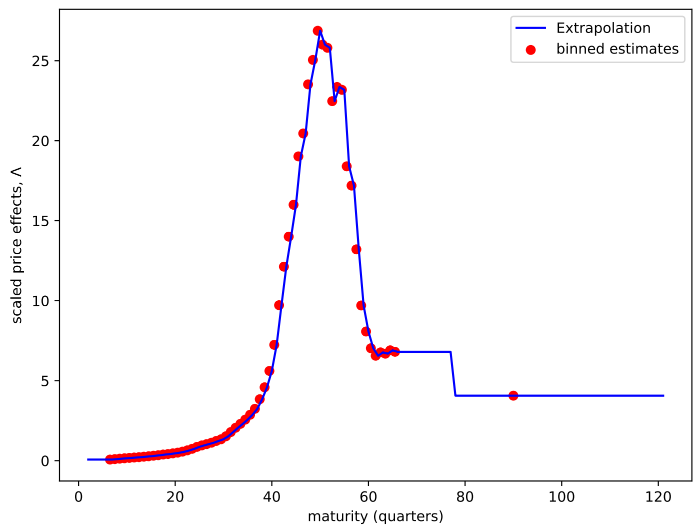

Figure 20: Fit for price effect semi-elaticities. We use 5 year bins with the last bin 15+ years to measure Λ. The red dots indicated estimates of the right hand side of equation 63 for the bins. The blue solid line is interpolated values that we use to measure all elements of Λ_t.

get

$$D_t\Lambda_t[j,k] = \iota_{\{k=j\}}\left(\frac{\sum_{k=1}^{\infty} Q_t^k}{\left(Q_t^1\right)^2}\frac{\xi^2\left(\tau_t\right)}{-\xi'\left(\tau_t\right)}\frac{}{Q_t^k}\right)\left(\frac{k\times\left\{Y_t\frac{\partial\ln\text{yields}_t^k}{\partial\text{Bond Supply}_t}\right\}_k}{\omega_t^{US,k}}\right)\left(\frac{\sum k\tilde{B}_t^{US,k}}{B_t}\right) \quad (63)$$

where all the terms on the right-hand side are measurable.[34]

The first term in (63) is obtained directly from estimates of the price curve $\left\{Q_t^k\right\}$and the tax rate τ_t and we set them to the same values we use in Section 5. To get the second term in (63), we bin maturities in 5 year bins with the last bin for maturities greater than 15 years. The red dots in Figure 20 are the implied values of $\left\{\left(\frac{k\times\left\{Y_t\frac{\partial\ln\text{yields}_t^k}{\partial\text{Bond Supply}_t}\right\}_k}{\omega_t^{US,k}}\right)\left(\frac{\sum k\tilde{B}_t^{US,k}}{B_t}\right)\right\}_k$. For intermediate maturities, we interpolate using the nearest value. The bold line in Figure 20 is the interpolated curve that we use in our implementation.

[34]Greenwood and Vayanos report elasticities $\left\{Y_t\frac{\partial\ln\text{yields}_t^k}{\partial\text{Bond Supply}_t}\right\}_k$ for a subset of maturities. For an arbitrary maturity n, we assume that the elasticity satisfies $a_0 - a_0\exp\left\{-a_1 n^2\right\}$ where parameters a_0 capture the long run level and parameter a_1 captures the speed of convergence. We fit $(a_0, a_1)$to match the long maturity elasticity of 0.003 and the short maturity (1 years) elasticity of 0.001 from Table 3 in Greenwood and Vayanos (2014). Since the Greenwood and Vayanos estimates are relatively flat over the maturities, the exact interpolation scheme is not critical of the results. We set $\left(\frac{\sum k\tilde{B}_t^k}{B_t}\right)$ to 7.3 using the values reported in Panel A of Table 1 in Greenwood and Vayanos (2014).

We then set Λ_t constant across dates at the estimated values and use equation (60) to compute the steady state portfolio.

C.7 Additional details for Section 5.4

We can get a feel for how trading frictions affect the optimal portfolio by studying a special case. We assume stationarity and set $Q_t^k = \beta^k$, and further specialize to a simpler market structure in which the government trades only a risk-free security and a growth-adjusted consol. Let $\overrightarrow{\beta}$ be a geometrically declining series $\beta^k, \beta^{k-1}, \ldots$ and excess return on the consol be denoted by r_t^{∞}. Finally, we impose that the stochastic discount factor of the non-traders is scaled version of the stochastic discount factor of the traders: $\ln(M_{\mathbb{N},t+k}) = (1+\psi)\ln(M_{\mathbb{T},t+k})$. This introduces a new parameter, ψ, that captures the severity of trading frictions as $\psi > 0$ implies that the SDF of the non-traders is more volatile of those of the traders.

Under this last assumption the covariance of the relative stochastic discount factors simplifies to

$$\mathrm{cov}_t\left(\ln(M_{\mathbb{T},t+k}) - \ln(M_{\mathbb{N},t+k}), r_{t+1}^j\right) = -\psi \mathrm{cov}_t\left(\ln(M_{\mathbb{T},t+k}), r_{t+1}^j\right).$$

As the traders trade the consol, we can use the traders' Euler equation, equation (52), to substitute out for this covariance and obtain

$$-\mathrm{cov}_t\left(\ln(M_{\mathbb{T},t+k}), r_{t+1}^j\right) \simeq \mathbb{E}_t r_{t+1}^j - \mathrm{cov}_t\left(\ln Q_{t+1,t-1}, r_{t+1}^j\right).$$

Under our stationarity assumptions we have $\mu_{\mathcal{N},t+k} = \mu_{\mathcal{N},t}$ and can therefore express Σ_t^M as the sum of three terms

$$\Sigma_t^M = \mu_{\mathbb{N},t}\psi\left(\mathcal{R}_t - \overline{Q}_t^1 \Sigma_t^Q\right)$$

where $\mathcal{R}_t[j,k] = Q_t^1 \mathbb{E}_t r_{k+1}^j$.

The effect of non-traders on the optimal portfolio is given by $\Sigma_t^{-1}\Sigma_t^M s_t^{ineq}$. This simplifies under this market structure of a growth adjusted consol and a risk free bond as Σ_t is now a single number representing the conditional covariance of the growth adjusted consol. We can also make progress on the components of $\Sigma_t^M \overrightarrow{\beta}$, starting with $\mathcal{R}_t \overrightarrow{\beta} = \frac{\mathbb{E}_t r_{t+1}^j}{1-\beta}$. Next we note that

$$\Sigma_t^Q \overrightarrow{\beta} = \sum_{k=1}^{\infty} \beta^t \text{cov}_t \left(\frac{1}{Q_t^1} \ln Q_{t+1,k}, r_{t+1}^{\infty} \right) \simeq \frac{\Gamma}{\beta} \sum_{k=1}^{\infty} \beta^t \mathbb{E}_t \partial_\sigma \ln Q_{t+1,k} \partial_\sigma r_{t+1}^{\infty}$$
$$\simeq \frac{\Gamma}{\beta} \mathbb{E}_t \sum_{k=1}^{\infty} \Gamma^t \partial_\sigma Q_{t+1,k} \partial_\sigma r_{t+1}^{\infty}$$
$$\simeq \frac{\Gamma}{\beta} \mathbb{E}_t \partial_\sigma Q_{t+1}^{\infty} \partial_\sigma r_{t+1}^{\infty}$$
$$\simeq \frac{\Gamma}{1-\beta} \text{cov}_t(r_{t+1}^{\infty}, r_{t+1}^{\infty}) = \frac{\Gamma}{1-\beta} \Sigma_t.$$

All put together we have that

$$\Sigma_t^{-1} \Sigma_t^M \overrightarrow{\beta} \approx \frac{\mu_{\mathbb{N},t} \psi}{1-\beta} \left(\frac{\mathbb{E}_t r_{t+1}^{\infty}}{\text{var}_t(r_{t+1}^{\infty})} - \beta \right).$$

So the presence of non-traders will lengthen the maturity as long as $\frac{\mathbb{E}_t r_{t+1}^{\infty}}{\text{var}_t(r_{t+1}^{\infty})} > \beta$. We can construct estimates for both $\mathbb{E}_t r_{t+1}^{\infty}$ and $\text{var}_t(r_{t+1}^{\infty})$ using the fact that the growth adjusted consol is the infinite sum of zero coupon bonds of all maturities weighted by β^j. To check the inequality, we use the estimates of the factor model and find that the $\frac{\mathbb{E}_t r_{t+1}^{\infty}}{\text{var}_t(r_{t+1}^{\infty})}$ is indeed significantly larger that one and hence any reasonable estimate of β.

D Additional details for Section 6

To simulate the neoclassical model, we solve a complete markets Ramsey allocation as in Lucas and Stokey (1983) by posing the following maximization problem. Given some $t = 0$ state $s_0 \in \mathcal{S}$ and household savings $b_0(s^0)$, the Ramsey problem can be expressed as

$$\max_{c_t(s^t), y_t(s^t)} \mathbb{E}_0 \sum_{t=0}^{\infty} u\left(C_t, \frac{Y_t}{\theta_t}\right) \tag{64}$$

subject to

$$Y_t(s^t) = C_t(s^t) + G(s_t), \tag{65}$$

$$b_0(s^0) u_c(s^0) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) \left[u_c(s^t) C_t(s^t) + u_y(s^t) Y_t(s^t) \right], \tag{66}$$

where the *implementability constraints,* equation (66) is derived by taking the time-0 budget constraint and replacing after-tax wages as well as bond prices.

We assume that the state space $\mathcal{S}$ is discrete (described below) and non-linearly solve the optimal allocation using the first-order conditions of the Ramsey planning problem. The resulting optimal allocation is represented using two sets of vectors of dimension $2|\mathcal{S}|$, one set for consumption and labor choices at $t = 0$ and another set for all $s_t \in \mathcal{S}$ for $t \geq 1$. Using the Ramsey allocation $\{c_t, y_t\}$, we can back out other related objects such tax rates $\tau_t = 1 - \frac{\left(\frac{Y_t}{\theta_t}\right)^{\gamma}}{c_t^{-\sigma}}$; primary surplus $X_t = \tau_t Y_t - G_t$; and zero-coupon bond prices $Q_t^n = \mathbb{E}_t \frac{C_{t+n}^{-\sigma}}{C_t^{-\sigma}}$.

We follow Buera and Nicolini (2004) and assume that the preferences of households are isoelastic $U\left(C_t, \frac{Y_t}{\theta_t}\right) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{\left(\frac{Y_t}{\theta_t}\right)^{1+\gamma}}{1+\gamma}$ with parameters $\sigma = 2$ and $\gamma = 1$. The economy is closed, so the demand of assets from foreign investors is zero and there are liquidity services provided by government bonds. The only source of uncertainty comes from the exogenous stochastic process of government expenditures G_t, which follows an AR(1) process

$$\ln G_t = \alpha_G + \rho_G \ln G_{t-1} + \sigma_G \epsilon_t$$

We set $(\alpha_G, \rho_G, \sigma_G)$ to obtain a mean G/Y of 15%, auto correlation of 0.95 and a standard deviation $\frac{1.2}{15}$ which are in line with the U.S. data that we use in Section 5.1. We discretize the $\ln G_t$ process with $|\mathcal{S}| = 20$ grid points. For our calculations, we set the level of initial debt B_0 so that the annualized initial level of government liabilities to GDP is 100%.

We use this parameterization to construct several versions of the optimal portfolio. First, for a given $s \in \mathcal{S}$, we apply Corollary to Theorem 1 in Angeletos (2002) and obtain the optimal portfolio $\omega_T^{CM,n}\left(s^T\right) = \omega^{CM,n}\left(s_T = s\right)$ for $n = 1, \ldots 20$ maturities that implements the complete markets allocation. We use the bond prices and present value of primary surpluses all of which can be backed out given the objects from the Ramsey allocation. In Figure 5, red color line, we plot $\left\{\omega^{CM,n}\right\}_{n=1}^{20}$ for $s_T = s_{10}$ which is the modal state.

Details for Figure 5 To obtain the target portfolio ω_t given some history s^t, we need to solve for a vector of portfolio shares that satisfies

$$\Sigma_t \boldsymbol{\omega}_t = \left[\Sigma_t^Q s_t^Q + \Sigma_t^G s_t^G\right].$$

Before explaining how we get ω_t, we make two observations. First, given the properties of the Ramsey allocations, $\Sigma_t, \Sigma_t^Q s_t^Q, s_t^G, \Sigma_t^G$ only depend on state s_t, which we set to s_{10} and as before can be computed in closed form using the complete market allocation that we have already solved. Second, as mentioned in the main text the returns of different bonds are highly correlated in the neoclassical economy, which makes the matrix of returns Σ_t to be effectively non-invertible and there are a range of portfolios that satisfy inequality (42) for a given ϵ. To obtain the target portfolio that is plotted in Figure 5 blue color, we set $\epsilon = 1e-8$ and pose the following minimization problem

$$\min_{\tilde{\boldsymbol{\omega}}} \left\|\tilde{\boldsymbol{\omega}} - \omega_t^{ABN}\right\|$$

such that

$$\left\|\Sigma_t \tilde{\omega}^n - \left[\Sigma_t^Q s_t^Q + \Sigma_t^G s_t^G\right]\right\| \leq \mathbf{1}^\intercal \epsilon.$$

where $||\cdot\|$ we mean the sup norm . This formulation conveniently delivers an objective that is quadratic while the constraint set is linear and convex; and we use a standard methods (OSQP library) to solve the minimization problem.

E Closed Economy

In this appendix, we study a closed neoclassical version of our *benchmark economy.* Unlike the benchmark open economy specification in Section 2, a change in the governments portfolio will necessarily change the price of assets in economy; and, compared to the segmented markets version of the benchmark economy presented in Section 4.5, a change in the portfolio composition at date t will also affect the price of securities in all other periods.

In what follows, we show how to to adjust our variational approach to incorporate such effects on prices. Our main result is to characterize the price effects and using that we show that the closed economy neoclassical setting implies price responses that are counterfactual relative to the evidence reviewed in Section 4.5. Besides the different structure on price effects, the rest of the analysis of a closed economy including the steps to obtain the expression for the optimal portfolio are similar to Section 3. In Section E.1, we formally describe the neoclassical closed economy environment that we study, then introduce the perturbation and analyze the welfare effects and optimality of the government. The proofs of the main results are in Section E.2.

E.1 Analysis

In addition to the assumptions of the benchmark economy we assume that:

1. Household preferences are time separable

$$V_t = u_t\left(C_t - \frac{(Y_t/\theta_t)^{1+1/\gamma}}{1+1/\gamma}\right) + \beta \mathbb{E}_t V_{t+1}.$$

2. Government expenditures $\{G_t\}$ are exogenous.

3. All assets are in zero net supply.[35]

4. The set of available securities can replicate a consol. We will let Q_t^{∞} denote the price of the consol at date t.

Under these assumptions asset market clearing implies that

$$b_t^i = B_t^i$$

and

$$c_t + G_t = Y_t.$$

Absence of trading frictions and non-pecuniary benefits of government securities the household optimality conditions imply

$$\mathbb{E}_t M_{t+1} R_{t+1}^i = M_t \text{ or } M_t Q_t^i = \mathbb{E}_t \left[M_{t+1} \left(d_{t+1}^i + Q_{t+1}^i \right) \right] \tag{67}$$

Perturbation Following Section 3, we use a perturbational approach to isolate the optimal public portfolio. The perturbations in this section are slightly different and adapted to get tractability in the closed economy environment.

We consider any competitive equilibrium and introduce a perturbation at a particular history s^t by assuming that the government purchases $\frac{\epsilon}{Q_t^j(s^t)}$ units of security j which is funded by selling $\frac{\epsilon}{Q_t^1(s^t)}$ of the risk free bond. This asset swap produces an additional $r_{t+1}^j(s^{t+1})\epsilon$ of excess returns at all histories s^{t+1} following s^t. We assume that the government uses those resources to purchase an additional $\frac{r_{t+1}^j(s^t)\epsilon}{1+Q_{t+1}^{\infty}(s^{t+1})}$ of the consol while keeping its holdings of all other assets constant. Due to its nature of swapping a longer security for a risk-free bond we will refer to this as a Quantitative Easing (or QE) perturbation and formally define it by

$$\partial_{QE} B_t^i(\tilde{s}^\ell) = \begin{cases} \frac{\epsilon}{Q_t^1(s^t)} & \text{if } i = rf \text{ and } \tilde{s}^\ell = s^t, \\ -\frac{\epsilon}{Q_t^j(s^t)} & \text{if } i = j \text{ and } \tilde{s}^\ell = s^t, \\ -\frac{1}{1+Q_{t+1}^{\infty}(s^t)} \left(r_{t+1}^j \left(s^{t+1} \right) \right) \epsilon & \text{if } i = \infty \text{ and } \tilde{s}^\ell \succ s^t, \ell > t, \\ 0 & \text{otherwise.} \end{cases}$$

The change in portfolio composition necessarily requires a change in taxes to balance the governments budget constraint,

$$G_t + \sum_i (Q_t^i + d_t^i) B_{t-1}^i = \tau_t Y_t + \sum_i Q_t^i B_t^i.$$

[35] That all assets are in zero net supply is for notational simplicity. Assuming positive net supply simply adds another term to the resource constraint equivalent to changing exogenous government expenditures.

Differentiating with respect to ϵ in the direction of the QE perturbation yields the following response of tax revenues

$$-\partial_{QE}\left(\tau_\ell Y_\ell\right) = \frac{r^j_{t+1}\left(s^{t+1}\right)}{1+Q^\infty_{t+1}\left(s^{t+1}\right)}\left(I_{\{s^\ell \succ s^t\}}\right) + \sum \partial_\epsilon Q^i_\ell(s^\ell)\left(B^i_\ell(s^{\ell-1}) - B^i_{\ell-1}(s^\ell)\right) \tag{68}$$

where $I_{\{s^\ell \succ s^t\}}$ is an indicator returning 1 if history s^ℓ follows from s^t and zero otherwise. Intuitively the effect of the perturbation on tax revenues is a combination of two effects. The first, $\frac{r^j_{t+1}\left(s^{t+1}\right)}{1+Q^\infty_{t+1}(s^{t+1})}\left(I_{\{s^\ell \succ s^t\}}\right)$, are the direct effects that are a result of the excess returns generated by the asset swap. The second,$\sum \partial_{QE} Q^i_\ell(s^\ell)\left(B^i_\ell(s^{\ell-1}) - B^i_{\ell-1}(s^\ell)\right)$, is the indirect effect that arises because the asset swap in period t changes prices not only in all future periods but also in all past periods starting from the initial date 0.

Assuming that the equilibrium manifold is sufficiently smooth, we can apply the envelope theorem to the household's maximization problem to obtain the welfare impact of this perturbation as $\epsilon \to 0$. The welfare effect of this perturbation comes from its effect on both tax rates and security prices and is given by

$$\begin{aligned}
\partial_{QE} V_0 &= \mathbb{E}_0 \sum_{\ell \geq 0} M_\ell \left(-\frac{\partial_{QE}\left(\tau_\ell Y_\ell\right)}{\xi_\ell} + \sum_{i \geq 0} \partial_{QE} Q^i_\ell \left(b^i_{\ell-1} - b^i_\ell\right)\right) \\
&= \mathbb{E}_0 \sum_{\ell \geq 0} M_\ell \left(-\frac{\partial_{QE}\left(\tau_\ell Y_\ell\right)}{\xi_\ell} + \sum_{i \geq 0} \partial_{QE} Q^i_\ell \left(B^i_{\ell-1} - B^i_\ell\right)\right) \\
&= \mathbb{E}_0 \left[\sum_{\ell \geq 0} M_\ell \sum_{i \geq 0} \partial_{QE} Q^i_\ell \left(\frac{\xi_\ell B^i_{\ell-1} - B^i_{\ell-1}}{\xi_\ell} - \frac{\xi_\ell B^i_\ell - B^i_\ell}{\xi_\ell} \right) + \sum_{\ell \geq T+1} \left(\frac{M_\ell}{\xi_\ell}\right)\left(I_{\{\tilde{s}^\ell \succ s^T\}}\right)\left(\frac{r^j_{T+1}}{1+Q^\infty_{T+1}}\right)\right] \\
&= \mathbb{E}_0 \left[\sum_{\ell \geq 0} M_\ell \left(\frac{\xi_\ell - 1}{\xi_\ell}\right) \sum_{i \geq 0} \partial_{QE} Q^i_\ell \left(B^i_{\ell-1} - B^i_\ell\right) + \sum_{\ell \geq t+1} \left(\frac{M_\ell}{\xi_\ell}\right)\left(I_{\{\tilde{s}^\ell \succ s^t\}}\right)\left(\frac{r^j_{t+1}}{1+Q^\infty_{t+1}}\right)\right] \\
&= \Pr_0\left(s^t\right) M_t\left(s^t\right)\left[PE + \mathbb{E}_t \sum_{k \geq 1} \left(\frac{M_{t+k}}{M_t}\right)\left(\frac{r^j_{t+1}}{1+Q^\infty_{t+1}}\right)\frac{1}{\xi_{t+k}}\right]
\end{aligned} \tag{69}$$

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$$PE = \frac{1}{\Pr_0\left(s^t\right) M_T\left(s^t\right)} \mathbb{E}_0 \left[\sum_{\ell \geq 0} M_\ell \left(\frac{\xi_\ell - 1}{\xi_\ell}\right) \sum_{i \geq 0} \partial_{QE} Q^i_\ell \left(B^i_{\ell-1} - B^i_\ell\right)\right].$$

The term $\mathbb{E}_t \sum_{k \geq 1}\left(\frac{M_{t+k}}{M_t}\right)\left(\frac{r^j_{t+1}}{1+Q^\infty_{t+1}}\right)\frac{1}{\xi_{t+k}}$ parallels the effect of the same perturbation in the open economy benchmark model, and can be analyzed in a similar manner. Now, in addition to that term, we also have PE that captures the effect on asset prices for all histories starting from time 0 onward. In the next section we will show how our second order expansions can allow us express that term using covariances that can be measured in the data.

Characterizing the Price Effects The perturbation affects asset prices through its effect on the stochastic discount factor of the household. This can be seen by differentiating the household Euler equation (67) with respect to ϵ in the direction of the perturbation to get

$$\left(\partial_{QE}M_\ell\right)Q_\ell^i + M_\ell\left(\partial_{QE}Q_\ell^i\right) = \mathbb{E}_\ell\left[\partial_{QE}M_{\ell+1}\left(d_{\ell+1}^i + Q_{\ell+1}^i\right) + M_{\ell+1}\left(\partial_{QE}Q_{\ell+1}^i\right)\right].$$

As the perturbation affects the stochastic discount factor through changes in tax rates we define $\xi_{M,\ell} \equiv \frac{\partial \log M_\ell}{\partial(\tau_\ell y_\ell)}$ as the semi-elasticity of $\log M_t$ with respect to the tax revenues which implies $\partial_{QE}M_\ell = M_\ell \xi_{M,\ell}\partial_{QE}\left(\tau_\ell y_\ell\right)$. Under our assumptions, this semi-elasticity is given by

$$\xi_{M,\ell} = -\psi_\ell \times \frac{1}{Y_\ell - G_\ell - \theta_\ell v\left(Y_\ell\right)} \times \left(\frac{\xi_\ell - 1}{\xi_\ell}\right)$$

where $\psi_\ell \equiv \frac{-[c_\ell - v_\ell(Y_\ell)]U''(c_\ell - v_\ell(Y_\ell))}{U'(c_\ell - v_\ell(Y_\ell))}$ is the coefficient of relative risk aversion.

To get a better understanding of how these terms contribute the price effects in the closed economy we'll focus on a stationary version of the economy

Definition 1. An optimal competitive equilibrium is *stationary from time* t if there exists a constant R_t such that for all $\ell > t$ (i) $\mathbb{E}_t G_\ell \approx G_t$ (iii) $\mathbb{E}_t R_\ell^i \approx R_t$ for all i and (iv) $\mathbb{E}_t c_\ell \approx c_t$.

This definition of stationary differs from the stationarity of the main text in that we assume a growth rate of $\Gamma = 1$. All of our results extend to a positive growth rate assuming that the utility function is CRRA.[36] Our first set of results concern the asset pricing implications of the QE perturbation. We will leave the proof of both propositions to the end of the section.

Proposition 1. *For a neoclassical model which is stationary from time* t

1. The QE perturbation keeps asset prices zero to the first-order

$$\partial_\sigma \partial_{QE} Q_\ell^i = 0 \quad \forall \quad i, \ell \geq 0$$

2. The QE perturbation only affects risk-premia at t

$$\mathbb{E}_\ell \partial_{\sigma\sigma}\partial_{QE} r_{\ell+1}^i = 0 \quad \forall \quad \ell \neq t$$

and at date ℓ

$$\mathbb{E}_\ell \partial_{\sigma\sigma}\partial_{QE} r_{\ell+1}^i = \frac{2\psi_t}{Y_t - G_t - \theta_t v\left(Y_t\right)} \times \left(\frac{1-\xi_t}{\xi_t}\right)\left(\frac{1}{1+\overline{Q}_{t+1}^\infty}\right)\mathbb{E}_\ell \partial_\sigma r_{\ell+1}^j \partial_\sigma r_{\ell+1}^i > 0,$$

where ψ_t is coefficient of relative risk aversion.

[36] The main difference is that we will require that the government smooth excess returns using a growth-adjusted consol rather than a pure consol.

This proposition states that the QE perturbation does not effect prices to zeroth or first order. This is inline with our modeling of price effects in Section 4.5 where we assume that the effect prices is at second order. Intuitively, to zeroth and first-order all assets have the same expected return so the QE perturbation only changes the risk profile of the household's stochastic discount factor which, in turn, will only effect prices to second order. Moreover, the proposition states that the effect on asset prices in the closed economy are counterfactual to what has been documented in the data. Estimates by Greenwood and Vayanos (2014) and others find that find that $\Lambda^{QE}[rf,j] \approx 0$ and $\Lambda^{QE}[i,j] > 0$ for $i > rf$ which implies that expected excess returns should decrease with the QE perturbation rather than increase:

$$\mathbb{E}_t \partial_{\sigma\sigma}\partial_{QE} r^i_{t+1} = -\frac{\overline{Q}^i_{t+1}}{\overline{Q}^i_t}\frac{\partial_{\sigma\sigma}\partial_{QE} Q^i_t}{\overline{Q}^i_t} < 0.$$

When governments buy back long term debt by issuing short term debt, short term rates appear to be unchanged so expected excess returns are driven by the fall in the term premia as the increased demand drives up prices. In contrast, in the closed economy, the government returns the excess returns from the QE swap via taxes which results in making states of the world where excess returns are high (low) better (worse) for the household by lowering (raising) tax rates in those states. As a result, the value of the asset decreases which raises the risk-premia. As noted, this is in inconsistent with the segmented market literature which finds that the excess returns on long maturity debt are lower after QE. Finally, we are able to use our expansions to characterize the price effects

Proposition 2. *For a neoclassical economy which is stationary from time 0, if all initial debt* $\left\{B^i_{-1}\right\}_i$ *was risk-free then* $PE \simeq \left(\frac{\overline{\xi}}{\overline{\xi}-1}\right)^{-1}\Psi_t\left(s^t\right)$ *where*

$$\begin{aligned}
\Psi_t\left(s^t\right) =& \frac{-2\overline{B}\overline{\xi}_M(\overline{Q}^1-1)}{\left(1-\overline{B}(\overline{Q}^1-1)\overline{\xi}_M\right)}\sum_{\ell=t+1}^{\infty}\left(\frac{\left(\overline{Q}^1_t\right)^{\ell-t}}{1+\overline{Q}^{\infty}_{t+1}}\right) cov_t\left(\partial_\sigma \ln M_\ell, \partial_\sigma r^j_{t+1}\right) \\
&- \frac{2\overline{\xi}_M\overline{B}}{\left(1-\overline{B}(\overline{Q}^1-1)\overline{\xi}_M\right)}\sum_{\ell=t+1}\left(\frac{\left(\overline{Q}^1_t\right)^{-t}}{1+\overline{Q}^{\infty}_{t+1}}\right) cov_t\left(\partial_\sigma r^j_{t+1}, \partial_\sigma \ln Q^1_\ell\right) \\
&- \frac{2\overline{\xi}_M}{\left(1-\overline{B}(\overline{Q}^1-1)\overline{\xi}_M\right)}\sum_{j\geq 1}\frac{\overline{Q}^1_t}{1+\overline{Q}^{\infty}_{t+1}\left(\ell^{t+1}\right)} cov_t\left(\partial_\sigma r^j_{t+1}, \partial_\sigma r^j_{t+1}\right) \\
&- \frac{2\overline{B}}{\left(1-\overline{B}(\overline{Q}^1-1)\overline{\xi}_M\right)}\sum_{\ell=t}^{\infty}\left(\frac{\left(\overline{Q}^1_\ell\right)^{\ell-t}}{1+\overline{Q}^{\infty}_{t+1}} cov_t\left(\partial_\sigma \xi_{M,\ell} - \partial_\sigma \xi_{M,\ell+1}, \partial_\sigma r^j_{t+1}\right)\right)
\end{aligned}$$

As we have noted without any assumptions price effects are given by

$$PE = \frac{1}{\Pr_0\left(s^t\right) M_T\left(s^t\right)} \mathbb{E}_0 \left[\sum_{\ell \geq 0} M_\ell \left(\frac{\xi_\ell - 1}{\xi_\ell} \right) \sum_{i \geq 0} \partial_{QE} Q_\ell^i \left(B_{\ell-1}^i - B_\ell^i \right) \right]$$

where a swap of securities at a particular history can affect asset prices at all other histories—past and future—due to general equilibrium effects on the stochastic discount factor that now directly depends on the tax rates. Proposition 2 allows us to characterize these price effects with a closed form expression using entirely time t covariances that are measurable in the data.

E.2 Proofs for Propositions 1 and 2

E.2.1 Proof of Proposition 1

We begin by noting that at the zeroth order, we get $\bar{\xi}_{M,\ell}^j = -\frac{\overline{\psi_\ell}}{\overline{Y}_\ell - \overline{G}_\ell - \bar{\theta}_\ell v\left(\overline{Y}_\ell\right)} \times \left(\frac{\bar{\xi}_\ell - 1}{\bar{\xi}_\ell}\right) = \bar{\xi}_{M,t}$, is independent of time and the details of the perturbation. We proceed by proving a series of lemmas documenting the results of Proposition 1

Lemma 3. *Expected excess returns are zero to the zeroth and the first order*

Proof. The zeroth of (67) gives us

$$\overline{r}_{\ell+1}^i = 0$$

Take first order expansion to get

$$\mathbb{E}_\ell \partial_\sigma r_{\ell+1}^i \overline{M_{\ell+1}} + \mathbb{E}_\ell \overline{r_{\ell+1}^i} \partial_\sigma M_{\ell+1} = 0$$

and thus

$$\mathbb{E}_\ell \partial_\sigma r_{\ell+1}^i = 0.$$

□

Lemma 4. *To the first order, price effects are zero, that is, for all i, ℓ: $\partial_\sigma \partial_{QE} Q_\ell^i = 0$*

Proof. Start from the definition of Q_ℓ^i

$$Q_\ell^i \left(s^\ell\right) = \mathbb{E}_{s^\ell} \sum_{k \geq 1} \frac{M_{\ell+k}}{M_\ell} D_{\ell+k}^i.$$

$$\partial_\sigma \partial_{QE} Q_\ell^i = \mathbb{E}_\ell \sum_{k \geq 1} \left(\partial_{QE} \partial_\sigma \log M_{\ell+k} - \partial_{QE} \partial_\sigma \log M_\ell \right) \left(\frac{\overline{M_{\ell+k}}}{\overline{M}_t`} \right) D_{\ell+k}^i.$$

A necessary and sufficient condition for price effects to be zero at the first order is that $k \geq 1$

$$\mathbb{E}_\ell \left(\partial_{QE} \partial_\sigma \log M_{\ell+k} - \partial_{QE} \partial_\sigma \log M_\ell \right) = 0 \tag{70}$$

Use the definition of $\xi_M\left(s^\ell\right)$ to get $\partial_{QE}\log M_\ell\left(s^\ell\right)=\partial_{QE}\left(\tau_\ell\left(s^\ell\right)Y_\ell\left(s^\ell\right)\right)\times\xi_M\left(s^\ell\right)$. To first order

$$\partial_\sigma\partial_{QE}\log M_\ell=\partial_\sigma\partial_{QE}\left(\tau_\ell Y_\ell\right)\times\overline{\xi}_{M,\ell}$$

Then (70) is equivalently expressed as

$$\mathbb{E}_\ell\left(\partial_\sigma\partial_{QE}\log M_{\ell+k}-\partial_\sigma\partial_{QE}\log M_\ell\right)=\overline{\xi}_{M,\ell}\left(\mathbb{E}_\ell\partial_\sigma\partial_{QE}\left(\tau_{\ell+k}Y_{\ell+k}\right)-\partial_\sigma\partial_{QE}\left(\tau_\ell Y_\ell\right)\right)$$

We check condition (70) by guess and verify.

Suppose $\partial_\sigma\partial_{QE}Q^i_\ell=0$ for $\ell\geq 0$, then for all $\ell\geq 0$ and from equations (68)

$$-\partial_\sigma\partial_{QE}\left(\tau_\ell Y_\ell\right)=\partial_\sigma\left(\frac{r^j_{t+1}\left(s^{t+1}\right)}{1+Q^\infty_{t+1}\left(s^{t+1}\right)}\right)I_{\{s^\ell\succ s^{t+1}\}}=\frac{\partial_\sigma r^j_{t+1}\left(s^{t+1}\right)}{1+\overline{Q}^\infty_{t+1}}I_{\{s^t\succ s^{t+1}\}}$$

When $\ell\geq t+1$

$$\mathbb{E}_\ell\left(\partial_\sigma\partial_{QE}\log M_{t+1+k}-\partial_\sigma\partial_{QE}\log M_{t+1}\right)=\overline{\xi}_{M,t+1}\left(\frac{\partial_\sigma r^j_{t+1}\left(s^{t+1}\right)}{1+\overline{Q}^\infty_{t+1}}-\frac{\partial_\sigma r^j_{t+1}\left(s^{t+1}\right)}{1+\overline{Q}^\infty_{t+1}}\right)I_{\{s^\ell\succ s^{t+1}\}}=0$$

When $\ell\leq t$, we can use the fact that to the first order, expected excess returns are zero from Lemma (4) to establish that (70) holds. □

Lemma 5. *In the closed economy the effect of the perturbation on expected excess returns is*

$$\mathbb{E}_\ell\partial_{\sigma\sigma}\partial_{QE}r^i_{\ell+1}=0\quad\forall\quad\ell\neq t$$

and at date t

$$\mathbb{E}_t\partial_{\sigma\sigma}\partial_{QE}r^i_{t+1}=\frac{2\psi_t}{Y_t-G_t-\theta_t v\left(Y_t\right)}\times\left(\frac{1-\xi_t}{\xi_t}\right)\left(\frac{1}{1+\overline{Q}^\infty_{t+1}}\right)\mathbb{E}_\ell\partial_\sigma r^j_{\ell+1}\partial_\sigma r^i_{\ell+1}>0$$

Proof. The first order expansion $\partial_{QE}M_\ell$ after using Lemma 4 gives us

$$\partial_\sigma\partial_{QE}M_{\ell+1}=-\overline{\xi}_{M,\ell}\overline{M}_{\ell+1}\left\{\partial_\sigma\left(\frac{r^j_{t+1}}{1+Q^\infty_{t+1}}\right)I_{\{\boldsymbol{s}^\ell\succ\boldsymbol{s}^{t+1}\}}\right\}$$

Use this along with the second order expansion of households optimality condition (67) to obtain

$$0=\mathbb{E}_\ell\partial_\sigma r^i_{\ell+1}\left(-\overline{\xi}_{M,\ell}\overline{M}_{\ell+1}\left\{\partial_\sigma\left(\frac{r^j_{t+1}}{1+q^\infty_{t+1}}\right)I_{\{\boldsymbol{s}^\ell\succ\boldsymbol{s}^{t+1}\}}\right\}\right)+\mathbb{E}_\ell\partial_{\sigma\sigma}\partial_{QE}r^i_{\ell+1}\overline{M}_{\ell+1}$$

For $\ell<t$, $I_{\{\boldsymbol{s}^\ell\succ\boldsymbol{s}^{t+1}\}}=0$ and thus $\mathbb{E}_\ell\partial_{\sigma\sigma}\partial_{QE}r^i_{\ell+1}=0$.

For $\boldsymbol{s}^{\ell} \succ \boldsymbol{s}^{t+1}$, use Law of iterated expectations to get

$$0 = \mathbb{E}_{t+1+k} \partial_{\sigma} r^{i}_{t+1+k} \left(-\overline{\xi}_{M,t+1+k} \overline{M}_{t+1+k} \underbrace{\mathbb{E}_{t+1} \left\{ \partial_{\sigma} \left(\frac{r^{j}_{t+1}}{1+Q^{\infty}_{t+1}} \right) \right\}}_{=0} \right) + \mathbb{E}_{t+1+k} \partial_{\sigma\sigma} \partial_{QE} r^{i}_{t+k+2} \overline{M}_{t+2+k}$$

and use Lemma (4) to get $\mathbb{E}_{\ell} \partial_{\sigma\sigma} \partial_{QE} r^{i}_{\ell+1} = 0$ for $\boldsymbol{s}^{\ell} \succ \boldsymbol{s}^{t+1}$.

Finally for $\ell = t$

$$0 = \mathbb{E}_{t} \partial_{\sigma} r^{i}_{t+1} \left(-\overline{\xi}_{M,t} \overline{M}_{t+1} \left\{ \partial_{\sigma} \left(\frac{r^{j}_{t+1}}{1+q^{\infty}_{t+1}} \right) \right\} \right) + \mathbb{E}_{\ell} \partial_{\sigma\sigma} \partial_{QE} r^{i}_{\ell+1} \overline{M}_{t+1}.$$

Substitute for $\overline{\xi}_{M,t}$ and simplify to to get

$$\mathbb{E}_{t} \partial_{\sigma\sigma} \partial_{QE} r^{i}_{t+1} \simeq \frac{2\psi_{t}}{Y_{t} - G_{t} - \theta_{t} v\left(Y_{t}\right)} \times \left(\frac{1-\xi_{t}}{\xi_{t}} \right) \left(\frac{1}{1+\overline{Q}^{\infty}_{t+1}} \right) \mathbb{E}_{\ell} \partial_{\sigma} r^{j}_{t+1} \partial_{\sigma} r^{i}_{\ell+1}.$$

Since $\xi_{t} = 1 - \gamma \frac{\tau_{t}}{1-\tau_{t}} < 1$, $Y_{t} - G_{t} - \theta_{t} v\left(Y_{t}\right) > 0$ from Inada conditions, and $\psi_{t} > 0$, we get that $\mathbb{E}_{t} \partial_{\sigma\sigma} \partial_{QE} r^{j}_{t+1} > 0$. □

E.2.2 Proof of Proposition 2

The second order expansion of the price effects

$$\partial_{\sigma\sigma} \left(\Pr_{0}\left(s^{t}\right) M_{t}\left(s^{t}\right) PE_{j,t,\epsilon} \right) = \mathbb{E}_{0} \left[\sum_{\ell \geq 0} \left(\frac{\overline{\xi}_{\ell} - 1}{\overline{\xi}_{\ell}} \right) M_{\ell} \sum_{i \geq 0} \partial_{\sigma\sigma} \partial_{QE} Q^{i}_{\ell} \left(\overline{B}^{i}_{\ell-1} - \overline{B}^{i}_{\ell} \right) \right] \tag{71}$$

which equals

$$\left(\frac{\overline{\xi}_{0} - 1}{\overline{\xi}_{0}} \right) \overline{M}_{0} \sum_{i \geq 0} \partial_{\sigma\sigma} \partial_{QE} Q^{i}_{0} \overline{B}^{i}_{-1} + \left(\frac{\overline{\xi}_{0} - 1}{\overline{\xi}_{0}} \right) \mathbb{E}_{0} \left[\sum_{t \geq 0} \sum_{i \geq 0} \overline{B}^{i}_{t} \left(\overline{M}_{t+1} \partial_{\sigma\sigma} \partial_{QE} Q^{i}_{t+1} - \overline{M}_{t} \partial_{\sigma\sigma} \partial_{QE} Q^{i}_{t} \right) \right]. \tag{72}$$

Its easy to see that $\left(\frac{\overline{\xi}_{0}-1}{\overline{\xi}_{0}} \right) \overline{M}_{0} \sum_{i \geq 0} \partial_{\sigma\sigma} \partial_{QE} Q^{i}_{0} \overline{B}^{i}_{-1} = \left(\frac{\overline{\xi}_{0}-1}{\overline{\xi}_{0}} \right) \overline{M}_{0} \sum_{i \neq 1} \partial_{\sigma\sigma} \partial_{QE} Q^{i}_{0} \overline{B}^{i}_{-1} = 0$ under the assumption that initial debt was risk-free.

*t*he household pricing equation implies

$$M_{\ell} Q^{i}_{\ell} = \mathbb{E}_{\ell} \left[M_{\ell+1} \left(Q^{i}_{\ell+1} + D^{i}_{\ell+1} \right) \right] \tag{73}$$

Differentiating by ∂_{QE} gives

$$\left(\partial_{QE} M_{\ell} \right) Q^{i}_{\ell} + M_{\ell} \partial_{QE} Q^{i}_{\ell} = \mathbb{E}_{\ell} \left[\left(\partial_{QE} M_{\ell+1} \right) \left(Q^{i}_{\ell+1} + D^{i}_{\ell+1} \right) + M_{\ell+1} \partial_{QE} Q^{i}_{\ell+1} \right]$$

Let's start by looking at $\ell < t$, We know that $\partial_\sigma \partial_{QE} M_{\ell+1} = 0$ so taking the second derivative with respect to σ yields

$$\mathbb{E}_\ell \left[\overline{M}_{\ell+1} \partial_{\sigma\sigma} \partial_{QE} Q^i_{\ell+1} - \overline{M}_\ell \partial_{\sigma\sigma} \partial_{QE} Q^i_\ell\right] = \overline{Q}^i_\ell \mathbb{E}_\ell \left[(\partial_{\sigma\sigma} \partial_{QE} M_\ell) - (\partial_{\sigma\sigma} \partial_{QE} M_{\ell+1}) \overline{R}^{rf}_{\ell+1}\right].$$

For $\ell > t$ and $s^\ell \succ s^t$ we have $\frac{\partial_{QE} M_\ell}{M_\ell} = \xi^j_{M,\ell} \partial_{QE} (\tau_\ell Y_\ell)$ and hence $\partial_\sigma \partial_{QE} M_\ell = \overline{M}_\ell \bar{\xi}_{M,\ell} \frac{\partial_\sigma r^j_{t+1}}{1+\overline{Q}^\infty_{t+1}}$. The second order expansion of equation (73) is

$$\begin{aligned} 2\partial_\sigma \partial_{QE} M_\ell \partial_\sigma Q^i_\ell + (\partial_{\sigma\sigma} \partial_{QE} M_\ell) \overline{Q}^i_\ell + \overline{M}_\ell \partial_{\sigma\sigma} \partial_{QE} Q^i_\ell = \mathbb{E}_\ell &\left[2\partial_\sigma \partial_{QE} M_{\ell+1} \partial_\sigma \left(Q^i_{\ell+1} + D^i_{\ell+1}\right)\right] \\ &+ (\partial_{\sigma\sigma} \partial_{QE} M_{\ell+1}) \left(\overline{Q}^i_{\ell+1} + D^i_{\ell+1}\right) + \overline{M}_{\ell+1} \partial_{\sigma\sigma} \partial_{QE} Q^i_{\ell+1} \end{aligned}$$

We know that$t$

$$\mathbb{E}_\ell \left[\partial_\sigma \partial_{QE} M_{\ell+1} \partial_\sigma \left(Q^i_{\ell+1} + D^i_{\ell+1}\right)\right] = \bar{\xi}_{M,t+1} \frac{\partial_\sigma r^j_{t+1}}{1+\overline{Q}^\infty_{t+1}} \mathbb{E}_\ell \left[\overline{M}_{\ell+1} \partial_\sigma \left(Q^i_{\ell+1} + D^i_{\ell+1}\right)\right]$$

so we get

$$\begin{aligned} &\mathbb{E}_\ell \left[\partial_\sigma \partial_{QE} M_{\ell+1} \partial_\sigma \left(Q^i_{\ell+1} + D^i_{\ell+1}\right)\right] - \partial_\sigma \partial_{QE} M_\ell \partial_\sigma Q^i_\ell \\ &= \frac{\partial_\sigma r^j_{t+1}}{1+\overline{Q}^\infty_{t+1}} \bar{\xi}_{M,t+1} \left(\mathbb{E}_\ell \left[\overline{M}_{\ell+1} \partial_\sigma \left(Q^i_{\ell+1} + D^i_{\ell+1}\right)\right] - \overline{M}_\ell \partial_\sigma Q^i_\ell\right) \\ &= \frac{\partial_\sigma r^j_{t+1}}{1+\overline{Q}^\infty_{t+1}} \bar{\xi}_{M,t+1} \overline{Q}^i_\ell \left(\partial_\sigma M_\ell - \partial_\sigma M_{\ell+1} \overline{R}^{rf}_{\ell+1}\right) \end{aligned}$$

with the last equality coming from

$$\partial_\sigma M_\ell \overline{Q}^i_\ell + \overline{M}_\ell \partial_\sigma Q^i_\ell = \mathbb{E}_\ell \left[\partial_\sigma M_{\ell+1} \left(\overline{Q}^i_{\ell+1} + \overline{D}^i_{\ell+1}\right) + \overline{M}_{\ell+1} \partial_\sigma \left(Q^i_{\ell+1} + D^i_{\ell+1}\right)\right].$$

Noℓe that $t$his only depends on i through $\overline{Q}^i$ thus for $\ell > t$

$$\begin{aligned} &\mathbb{E}_\ell \left[\overline{M}_{\ell+1} \partial_{\sigma\sigma} \partial_{QE} Q^i_{\ell+1} - \overline{M}_\ell \partial_{\sigma\sigma} \partial_{QE} Q^i_\ell\right] \\ &= \overline{Q}^i_\ell \mathbb{E}_\ell \left[(\partial_{\sigma\sigma} \partial_{QE} M_\ell) - (\partial_{\sigma\sigma} \partial_{QE} M_{\ell+1}) \overline{R}^{rf}_{\ell+1} - \bar{\xi}_{M,t+1} \overline{M}_\ell \frac{\partial_\sigma r^j_{t+1}}{1+\overline{Q}^\infty_{t+1}} \frac{\partial_\sigma Q^1_\ell}{\overline{Q}^1_\ell}\right] \end{aligned}$$

where the last term is simplified by noting that $\overline{M}_\ell \frac{\partial_\sigma Q^1_\ell}{\overline{Q}^1_\ell} = \mathbb{E}_\ell \left[\frac{1}{\overline{Q}^1_\ell} \partial_\sigma M_{\ell+1} - \partial_\sigma M_\ell\right]$.

Finally, we have the $\ell = t$ and $s^\ell = s^t$ term which gives

$$\begin{aligned} &\mathbb{E}_t \left[\overline{M}_{t+1} \partial_{\sigma\sigma} \partial_{QE} Q^i_{t+1} - \overline{M}_t \partial_{\sigma\sigma} \partial_{QE} Q^i_t\right] \\ &= \overline{Q}^i_t \mathbb{E}_\ell \left[(\partial_{\sigma\sigma} \partial_{QE} M_t) - (\partial_{\sigma\sigma} \partial_{QE} M_{t+1}) \overline{R}^{rf}_{t+1} - \frac{\bar{\xi}_{M,t+1} \overline{M}_{t+1}}{1+\overline{Q}^\infty_{t+1}(s^{t+1})} \partial_\sigma r^j_{t+1} \partial_\sigma r^i_{t+1}\right]. \end{aligned}$$

Now we note that all the terms $\overline{M}_{\ell+1}\partial_{\sigma\sigma}\partial_{QE}Q^i_{\ell+1} - \overline{M}_\ell\partial_{\sigma\sigma}\partial_{QE}Q^i_\ell$ in the price effect sum have a component $\overline{Q}^i_\ell\left((\partial_{\sigma\sigma}\partial_{QE}M_\ell) - (\partial_{\sigma\sigma}\partial_{QE}M_{\ell+1})\,\overline{R}^{rf}_{\ell+1}\right)$ in them. We gain some tractability by substituting $\partial_{\sigma\sigma}\partial_{QE}M_\ell = \overline{M}_\ell\bar{\xi}_M\partial_{\sigma\sigma}\partial_{QE}\left(\tau_\ell Y_\ell\right) + 2\overline{M}_\ell\partial_\sigma\xi_{M,\ell}\partial_\sigma\partial_{QE}\left(\tau_\ell Y_\ell\right)$ and doing so makes

$$
\begin{aligned}
&\mathbb{E}_0\left[\sum_{\ell\geq 0}\sum_{i\geq 0}\overline{B}^i_\ell\left(\overline{M}_{\ell+1}\partial_{\sigma\sigma}\partial_{QE}Q^i_{\ell+1} - \overline{M}_\ell\partial_{\sigma\sigma}\partial_{QE}Q^i_\ell\right)\right]\\
&= \mathbb{E}_0\left[\sum_{\ell=0}^{\infty}\overline{B}_\ell\overline{M}_\ell\bar{\xi}_{M,\ell}\left(\partial_{\sigma\sigma}\partial_{QE}\left(\tau_\ell Y_\ell\right) - \partial_{\sigma\sigma}\partial_{QE}\left(\tau_{\ell+1}Y_{\ell+1}\right)\right)\right]\\
&- 2\Pr(s^t)\bar{\xi}_{M,t+1}\mathbb{E}_t\left[\sum_{\ell=t+1}\overline{B}_\ell\overline{M}_\ell\frac{\partial_\sigma r^j_{t+1}}{1+\overline{Q}^\infty_{t+1}}\frac{\partial_\sigma Q^1_\ell}{\overline{Q}^1_\ell}\right] \qquad (74)\\
&- 2\Pr(s^t)\bar{\xi}_{M,t+1}\mathbb{E}_t\left[\sum_{j\geq 1}\frac{\overline{M}_{t+1}}{1+\overline{Q}^\infty_{t+1}\left(s^{t+1}\right)}\partial_\sigma r^j_{t+1}\partial_\sigma r^i_{t+1}\right]\\
&+ 2\Pr(s^t)\mathbb{E}_t\left[\sum_{\ell=t}^{\infty}\overline{B}_\ell\overline{M}_\ell\left(\partial_\sigma\xi_{M,\ell}\partial_\sigma\partial_{QE}\left(\tau_\ell Y_\ell\right) - \partial_\sigma\xi_{M,\ell+1}\partial_\sigma\partial_{QE}\left(\tau_{\ell+1}Y_{\ell+1}\right)\right)\right] \qquad (75)
\end{aligned}
$$

Most of these objects we can easily put some structure on except for

$$
\mathbb{E}_0\left[\sum_{\ell=0}^{\infty}\overline{B}_\ell\overline{M}_\ell\bar{\xi}_{M,\ell}\left(\partial_{\sigma\sigma}\partial_{QE}\left(\tau_\ell Y_\ell\right) - \partial_{\sigma\sigma}\partial_{QE}\left(\tau_{\ell+1}Y_{\ell+1}\right)\right)\right],
$$

there we have note that $\overline{B}_\ell = \overline{B}_0 = \overline{B}$, $\overline{M}_\ell = \left(Q^1\right)^\ell\overline{M_0}$ and $\bar{\xi}_{M,\ell} = \bar{\xi}_{M,0} = \bar{\xi}_M$. Put together we have

$$
\begin{aligned}
&\mathbb{E}_0\left[\sum_{\ell=0}^{\infty}\overline{B}_\ell\overline{M}_\ell\bar{\xi}_{M,\ell}\left(\partial_{\sigma\sigma}\partial_{QE}\left(\tau_\ell Y_\ell\right) - \partial_{\sigma\sigma}\partial_{QE}\left(\tau_{\ell+1}Y_{\ell+1}\right)\right)\right]\\
&= \overline{B\xi}_{M,}\mathbb{E}_0\left[\sum_{\ell=0}^{\infty}\left(Q^1\right)^\ell\left(\partial_{\sigma\sigma}\partial_{QE}\left(\tau_\ell Y_\ell\right) - \partial_{\sigma\sigma}\partial_{QE}\left(\tau_{\ell+1}Y_{\ell+1}\right)\right)\right]\overline{M}_0\\
&= \overline{B\xi}_{M,}\mathbb{E}_0\left[\sum_{\ell=0}^{\infty}\left(Q^1\right)^\ell\left(Q^1-1\right)\partial_{\sigma\sigma}\partial_{QE}\left(\tau_\ell Y_\ell\right)\right]\overline{M}_0
\end{aligned}
$$

we can then plug into $\partial_{\sigma\sigma}\partial_{QE}\left(\tau_{\ell}Y_{\ell}\right)$ to get

$$
\begin{aligned}
&\mathbb{E}_0\left[\sum_{\ell=0}^{\infty}\overline{B}_{\ell}\overline{M}_{\ell}\overline{\xi}_{M,\ell}\left(\partial_{\sigma\sigma}\partial_{QE}\left(\tau_{\ell}Y_{\ell}\right)-\partial_{\sigma\sigma}\partial_{QE}\left(\tau_{\ell+1}Y_{\ell+1}\right)\right)\right]\\
&=\overline{B}(Q^1-1)\overline{\xi}_M\overline{M}_0\mathbb{E}_0\left[\sum_{\ell=0}^{\infty}\left(Q^1\right)^{\ell}\sum_{i\geq 0}\partial_{\sigma\sigma}\partial_{QE}Q_{\ell}^{i}\left(\overline{B}_{\ell-1}^{i}-\overline{B}_{\ell}^{i}\right)\right]\\
&+\overline{B\xi}_M\overline{M}_0(Q^1-1)\Pr(s^t)\mathbb{E}_t\left[\sum_{\ell=\ell+1}^{\infty}\left(Q^1\right)^{\ell}\partial_{\sigma\sigma}\left(\frac{r_{t+1}^{j}\left(s^{t+1}\right)}{1+\overline{Q}_{t+1}^{\infty}}\right)\right]\\
=&\overline{B}(Q^1-1)\overline{\xi}_M\frac{\overline{\xi}}{\overline{\xi}-1}\partial_{\sigma\sigma}\left(\Pr_0\left(s^t\right)M_t\left(s^t\right)PE_0^j\left(s^t\right)\right) \qquad (76)\\
&+\overline{B\xi}_M\overline{M}_0(Q^1-1)\Pr(s^t)\mathbb{E}_t\left[\sum_{\ell=\ell+1}^{\infty}\left(Q^1\right)^{\ell}\partial_{\sigma\sigma}\left(\frac{r_{t+1}^{j}\left(s^{t+1}\right)}{1+\overline{Q}_{t+1}^{\infty}}\right)\right]
\end{aligned}
$$

Going back to the HH version of this perturbation we get

$$
\mathbb{E}_t\left[\sum_{\ell=t+1}^{\infty}M_{\ell}\frac{r_{t+1}^{j}}{1+\overline{Q}_{t+1}^{\infty}}\right]=0
$$

As second order expansion of this gives

$$
\overline{M}_0\mathbb{E}_t\left[\sum_{\ell=t+1}^{\infty}\left(Q^1\right)^{\ell}\partial_{\sigma\sigma}\left(\frac{r_{t+1}^{j}\left(s^{t+1}\right)}{1+\overline{Q}_{t+1}^{\infty}}\right)\right]=-2\mathbb{E}_t\left[\sum_{\ell=t+1}^{\infty}\partial_{\sigma}M_{\ell}\frac{\partial_{\sigma}r_{t+1}^{j}\left(s^{t+1}\right)}{1+\overline{Q}_{t+1}^{\infty}}\right] \qquad (77)
$$

Putting all together we get (combining equations (71),(74),(76), and (77))

$$
\begin{aligned}
\left(\frac{\overline{\xi}}{\overline{\xi}-1}\right)\partial_{\sigma\sigma}PE_{j,t,\epsilon}=&\frac{-2\overline{\xi}_M}{\left(1-\overline{B}_0(Q^1-1)\right)\left(Q^1\right)^t}\mathbb{E}_t\left[\sum_{\ell=t+1}\overline{B}_{\ell}\frac{\partial_{\sigma}r_{t+1}^{j}}{1+\overline{Q}_{t+1}^{\infty}}\frac{\partial_{\sigma}Q_t^1}{\overline{Q}_t^1}\right]\\
&-\frac{2\overline{B\xi}_M(Q^1-1)}{\left(1-\overline{B}(Q^1-1)\right)}\mathbb{E}_t\left[\sum_{t=t+1}^{\infty}\left(Q^1\right)^{t-t}\partial_{\sigma}\ln M_t\frac{\partial_{\sigma}r_{t+1}^{j}\left(s^{t+1}\right)}{1+\overline{Q}_{t+1}^{\infty}}\right]\\
&\frac{-2\overline{\xi}_M}{\left(1-\overline{B}(Q^1-1)\right)\left(Q^1\right)^t}\mathbb{E}_t\left[\sum_{j\geq 1}\frac{Q^1}{1+\overline{Q}_{t+1}^{\infty}\left(s^{t+1}\right)}\partial_{\sigma}r_{t+1}^{j}\partial_{\sigma}r_{t+1}^{i}\right]\\
&-\frac{2\overline{B}}{\left(1-\overline{B}(Q^1-1)\right)}\mathbb{E}_t\left[\sum_{t=t}^{\infty}\left(Q^1\right)^{t-t}\frac{\partial_{\sigma}r_{t+1}^{j}\left(s^{t+1}\right)}{1+\overline{Q}_{t+1}^{\infty}}\left(\partial_{\sigma}\xi_{M,t}-\partial_{\sigma}\xi_{M,t+1}\right)\right]
\end{aligned}
$$

as desired.