Macroeconomic Dynamics Near the ZLB: A Tale of Two Equilibria *

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Abstract

This paper studies the dynamics of a New Keynesian DSGE model near the zero lower bound (ZLB) on nominal interest rates. In addition to the standard targeted-inflation equilibrium, we consider a deflation equilibrium as well as a Markov sunspot equilibrium that switches between a targeted-inflation and a deflation regime. We use the particle filter to estimate the state of the U.S. economy in 2008:Q4 under the assumptions that the U.S. economy has been in either the targeted-inflation or the sunspot equilibrium. The two equilibria provide an equally plausible description of the observed data but have different policy implications. We consider a combination of fiscal policy (calibrated to the American Recovery and Reinvestment Act) and monetary policy (that tries to keep interest rates near zero) and compute government spending multipliers. Ex-ante multipliers (cumulative over one year) under the targeted-inflation regime are around 0.91. A monetary policy that keeps interest rates at zero can raise the multiplier to 1.71. The ex-post (conditioning on the realized shocks in 2009-2011) multiplier is estimated to be 1.35. Conditional on the sunspot equilibrium the multipliers are generally smaller and the scope for conventional expansionary monetary policy is severely limited. JEL CLASSIFICATION: C5, E4, E5

KEY WORDS: DSGE Models, Government Spending Multiplier, Multiple Equilibria, Nonlinear Filtering, Nonlinear Solution Methods, ZLB

1 Introduction

Since the beginning of 2009 the U.S. Federal Funds rate has been effectively zero. Investors' access to money, which is an asset that in addition to providing transaction services yields a zero nominal return, prevents nominal interest rates from falling below zero and thereby creates a zero lower bound (ZLB). If an economy is at the ZLB, its central bank is unable to stimulate the economy using a conventional monetary policy that reduces interest rates. Traditionally, the ZLB has mostly been ignored in the specification of dynamic stochastic general equilibrium (DSGE) models that are tailored toward the analysis of U.S. monetary and fiscal policy. With the exception of a short period from 2003:Q3 to 2004:Q2 in which the Federal Funds rate dropped to about 1%, the ZLB did not appear to be empirically relevant. Moreover, accounting for the ZLB in a DSGE model complicates the quantitative analysis of the model considerably.

During the Great Recession of 2008-9 the ZLB has become empirically relevant for the U.S. and since then the literature on the analysis DSGE models with a ZLB constraint has been growing rapidly. Our paper contributes to this literature. We solve a small-scale nonlinear New Keynesian DSGE model with an explicit ZLB constraint using a global approximation to the agents' decision rules. Once the ZLB is explicitly included in a monetary model, there typically exist multiple equilibria. In addition to the widely-studied equilibrium in the neighborhood of a steady state in which actual inflation coincides with the central bank's inflation target (targeted-inflation equilibrium), this paper is the first to study two additional equilibria in a nonlinear DSGE model with a full set of structural shocks. Specifically we consider a minimal-state-variable equilibrium in which the endogenous variables fluctuate around a steady state with zero interest rates (deflation equilibrium); and an equilibrium in which the economy alternates between a targeted-inflation regime and a deflation regime according to the realization of a non-fundamental Markov-switching process (sunspot equilibrium). We show analytically, in a special case, and numerically in the more general cases, how these two equilibria behave differently than the targeted-inflation equilibrium, especially near the ZLB.

We use our model to study the following quantitative questions: conditional on the state of the U.S. economy in January 2009, what is the effect of an increase in government spending of the size of the federal contracts, grants, and loans portion of the American Recovery and Reinvestment Act (ARRA)? Does this effect get amplified by a monetary policy that keeps interest rates near the ZLB for an extended period of time? To answer these questions, we parameterize the DSGE model and use a particle filter to extract the values for the model's state variables from U.S. data over the period 2000:Q1 to 2008:Q4 conditional on the three equilibria. The deflation equilibrium is empirically not viable because during most of the last decades inflation rates were positive. Thus, conditional on filtered states for 2008:Q4 we conduct policy experiments under the assumption that the economy is either in the targeted-inflation or in the sunspot equilibrium. Both of these equilibria provide equally plausible rationalizations of the observed data.

We consider two types of policy exercises, which we label as ex ante and ex post. In the ex-ante analysis we take the states at the beginning of 2009:Q1 as given and simulate the model economy forward, with and without the policy intervention. In the ex-post analysis we condition on the actual filtered shocks from the years 2009 and 2010. A counterfactual set of shocks is constructed by reducing the exogenous government spending process by the amount of the ARRA stimulus. We find that the ex-ante cumulative fiscal multiplier under the targeted-inflation equilibrium is about 0.9 over a one-year horizon. From an ex-ante perspective the economy is expected to revert back to its steady state and the fiscal expansion accelerates the rise in nominal interest rates moving the economy away from the ZLB. The ex-post multiplier is larger, around 1.35. During 2009 and 2010 the realized shocks pushed the economy closer to the ZLB. Once at the ZLB, the feedback portion of the monetary policy rule is inactive and the fiscal stimulus is not accompanied by a rise in nominal interest rates. The resulting lower real rates stimulate demand and amplify the fiscal policy effect. This mechanism is emphasized, for instance, by Eggertsson and Woodford (2003) and Christiano, Eichenbaum, and Rebelo (2011). However, according to our empirical analysis it is not as strong as these authors claim.

Since in the logic of the interest-rate feedback rule the fiscal expansion has a tendency to trigger a rise in interest rates, we combine the fiscal stimulus with a monetary policy that keeps interest rates at or near the ZLB. From an ex ante perspective this leads to an increase of the multiplier from 0.9 to 1.71. From an ex-post perspective, this leaves the multiplier unchanged over one quarter, raises it from 1.35 to 1.52 over four quarters, and from 1.28 to 1.96 over eight quarters. In other words, at the beginning of 2009, in the logic of the targeted-inflation equilibrium, the U.S. central bank had not leverage to stimulate the economy with conventional monetary policy. By the second half of 2010 the actual monetary policy was expansionary in the sense that the model-implied feedback rule would have predicted a positive interest rate. This expansionary monetary policy amplified the effect of the fiscal stimulus.

Viewing the years of 2009 and 2010 through the lens of the sunspot equilibrium delivers a more pessimistic view. The fiscal multipliers are generally smaller both from an ex-ante and from an ex-post perspective. Moreover, the scope for an additional monetary stimulus is severely limited because, at least, conditional on the deflation regime, which was active in early 2009, the probability of staying at the zero lower bound even after an expansionary fiscal policy is high. In turn, the ex post multipliers were between 0.7 and 1.0 over a one-year horizon.

It has been well-known that monetary DSGE models with an explicit ZLB constraint deliver multiple equilibria. This issue was discussed, for instance, by Benhabib, Schmitt-Grohé, and Uribe (2001a,b). In a nutshell, the relationship between nominal interest rates and inflation in a DSGE model are characterized by a consumption Euler equation which embodies a version of the Fisher equation, and a monetary policy rule. The kink in the monetary policy rule induced by the ZLB tends to generate two pairs of steady-state interest and inflation rates that solve both equations. Moreover, near the deflation steady state the so-called *Taylor principle* is violated because nominal interest rates cannot aggressively respond to inflation. In turn, the model exhibits local indeterminacy. We discuss the properties of stochastic equilibria that have the targeted-inflation and deflation steady states as their steady state, respectively. Our policy analysis focuses on the targeted-inflation and a sunspot equilibrium. A sunspot equilibrium is also discussed in Mertens and Ravn (2013). However, unlike in our paper, the sunspot shock is the only shock driving their model. Moreover, Mertens and Ravn (2013) do not attempt to fit their sunspot model to actual data.

In terms of solution method, our work is most closely related to the papers by Judd, Maliar, and Maliar (2011), Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2012), and Gust, Lopez-Salido, and Smith (2012). All three of these papers use global projection methods to approximate agents' decision rules. We study a small-scale New Keynesian DSGE model similar to theirs with two endogenous and three exogenous state variables. The solution is based on a variant of the ergodic-set method proposed by Judd, Maliar, and Maliar (2011). However, we consider several important modifications. First, we use a piece-wise smooth approximation with two separate functions characterizing the decisions when the ZLB is binding and when it is not, while the previous papers use smooth approximations with a single function covering the whole state space. This means all our decision rules allow for kinks at points in the state space where the ZLB becomes binding. The location of these points is determined endogenously. This difference in methodology proves to be very important in obtaining accurate approximations, especially for the deflation and sunspot equilibria. Second, since we are interested in fitting U.S. data from 2000 to 2010 and since some of the observations during this period lie far in the tails of the ergodic distribution of our model, we apply the ergodic-set method to state realizations that are in part obtained from simulating the model and in part from applying the particle filter to U.S. data.[1] Finally, Judd, Maliar, and Maliar (2011), Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2012), and Gust, Lopez-Salido, and Smith (2012) solely compute what we call the targeted-inflation equilibrium, whereas we use the numerical methods to approximate two alternative equilibria.

Most of the other papers that study DSGE models with ZLB constraints take various shortcuts in their solution methods. Braun and Körber (2011) use a variant of extended shooting to solve a set of nonlinear equilibrium conditions. This method assumes that the system reaches its steady state after a fixed number of periods and at any point in time determines the agents' decision under the assumption of perfect foresight, setting future shocks to zero. Adam and Billi (2007) solve a linear-quadratic optimal policy problem

[1]This procedure is iterative: the simulated data as well as the filtered states are obtained from an initial approximation of the model.

with a linearized Euler equation and Phillips curve subject to a ZLB constraint. While the model is solved nonlinearly, it only contains two exogenous state variables. Eggertsson and Woodford (2003) consider a version of the New Keynesian DSGE model in which both the Euler equation and the Phillips curve are log-linearized and the natural rate of interest follows a two-state Markov process. The economy hits the ZLB when the natural rate turns negative. The subsequent exit from the ZLB is exogenous and occurs with a pre-specified probability. A similar approach is used by Christiano, Eichenbaum, and Rebelo (2011). Unfortunately, some of the DSGE model properties are very sensitive to the approximation technique and to implicit or explicit assumptions about the probability of leaving the ZLB. Detailed analyses are provided in Braun and Körber (2011) and Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2012).

The effect of an increase in government spending when the economy is at the ZLB is studied by Braun and Körber (2011), Christiano, Eichenbaum, and Rebelo (2011), Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2012), and Mertens and Ravn (2013). Christiano, Eichenbaum, and Rebelo (2011) argue that the fiscal multiplier at the ZLB can be substantially larger than one. A rise in government spending increases output, marginal costs, and expected inflation. If the economy operates according to the Taylor rule, then the central bank will react to the rise in output and inflation by increasing the nominal interest rate. If the economy is at the ZLB and the systematic part of the Taylor rule is not operative, then the expected inflation translates into a fall in expected real rates, which in turn triggers additional consumption in the current period and raises output further. Thus, the government spending multiplier crucially depends on whether the expansionary fiscal policy triggers an exit from the ZLB. In the target-inflation equilibrium of our model, spells of zero nominal interest rates are short and an expansionary fiscal policy triggers an immediate exit. In turn, the (short-run) fiscal multiplier is substantially less than one. In the sunspot-equilibrium, the economy is in the deflation regime in 2009:Q1 and expected to stay there for several quarters. Although the probability of the fiscal intervention triggering an exit from the ZLB is low the agents' decision rules in the deflation regime imply a government-spending multiplier that is only around 0.8.

The remainder of the paper is organized as follows. Section 2 presents a simple two-equation model that we use to illustrate the multiplicity of equilibria in monetary models with ZLB constraints. We also highlight the types of equilibria studied in this paper. The small-scale New Keynesian model that is used for the quantitative analysis is presented in Section 3. The solution of the model is discussed in Section 4. To fix ideas we first solve a version of the model without endogenous and exogenous persistence in which all equilibrium conditions except for the ZLB constraint are log-linearized. We then proceed with a description of the numerical solution algorithm for the full nonlinear model. Section 5 contains the quantitative analysis. We first illustrate the ergodic distribution of inflation and interest rates under the three equilibria considered in this paper, present some impulse response dynamics of the nonlinear model, and, at last, study the effects of fiscal interventions. Section 6 concludes. Detailed derivations, descriptions of algorithms, and additional quantitative results are summarized in an Online Appendix.

2 A Two-Equation Example

We begin with a simple two-equation example to illustrate the types of equilibria that arise if a ZLB constraint is imposed in a monetary DSGE model. The example is adapted from Benhabib, Schmitt-Grohé, and Uribe (2001a) and Hursey and Wolman (2010). Suppose that the economy can be described by the Fisher relationship

$$R_t = r\mathbb{E}_t[\pi_{t+1}] \tag{1}$$

and the monetary policy rule

$$R_t = \max\left\{1,\ r\pi_*\left(\frac{\pi_t}{\pi_*}\right)^{\psi}\exp[\sigma\epsilon_t]\right\},\quad \epsilon_t \sim iidN(0,1),\quad \psi > 1. \tag{2}$$

Here R_t denotes the gross nominal interest rate, π_t is the gross inflation rate, and ϵ_t is a monetary policy shock. The gross nominal interest rate is bounded from below by one. Throughout this paper we refer to this bound as ZLB because it bounds the net interest rate from below by zero. Combining (1) and (2) yields a nonlinear expectational difference equation for inflation

$$\mathbb{E}_t[\pi_{t+1}] = \max \left\{ \frac{1}{r},\ \pi_* \left(\frac{\pi_t}{\pi_*} \right)^{\psi} \exp[\sigma \epsilon_t] \right\}. \tag{3}$$

This simple model has two steady states ($\sigma = 0$), which we call targeted-inflation steady state and deflation steady state, respectively. In the targeted-inflation steady state inflation equals π_* and the nominal interest rate is $R = r\pi_*$. In the deflation steady state inflation equals $\pi_D = 1/r$ and the nominal interest is $R_D = 1$. While this paper focuses on nonlinear solutions of DSGE models with ZLB constraints, it is instructive to first take a look at the equilibria that arise in a linear approximation. Taking a piece-wise log-linear approximation around the targeted-inflation steady state and denoting percentage deviation of inflation from this steady state by $\hat{\pi}_t = \ln(\pi_t/\pi_*)$ we obtain

$$\mathbb{E}_t[\hat{\pi}_{t+1}] = \max \left\{ -\ln(r\pi_*),\ \psi\hat{\pi}_t + \sigma\epsilon_t \right\}.$$

If the shock standard deviation σ is small then the ZLB is essentially non-binding:

$$\mathbb{E}_t[\hat{\pi}_{t+1}] \approx \psi\hat{\pi}_t + \sigma\epsilon_t \tag{4}$$

and the linearized rational expectations system has the unique stable solution

$$\hat{\pi}_t \approx -\frac{1}{\psi}\sigma\epsilon_t. \tag{5}$$

Alternatively, one can approximate the dynamics of the system near the deflation steady state. Let $\tilde{\pi}_t = \ln(\pi_t/\pi_D)$. Then,

$$\mathbb{E}_t[\tilde{\pi}_{t+1}] = \max \left\{ 0,\ -(\psi - 1)\ln(r\pi_*) + \psi\tilde{\pi}_t + \sigma\epsilon_t \right\}.$$

Since we assumed $\psi > 1$, the ZLB is binding with very probability if σ is small. This leads to

$$\mathbb{E}_t[\tilde{\pi}_{t+1}] \approx 0 \tag{6}$$

This linear rational expectation difference equation has many stable solutions. Following Lubik and Schorfheide (2004), we consider the set of solutions

$$\tilde{\pi}_t = -\frac{1}{\psi}(1 + M)\sigma\epsilon_t + \zeta_t, \tag{7}$$

where M is some constant and ζ_t is a sunspot shock. Setting $M = 1$ and $\zeta_t = 0$ highlights that there is an approximate deflation equilibrium that mimics the fluctuations of the targeted-inflation equilibrium.

The multiplicity of solutions to piece-wise linear approximations of the difference equation (3) suggests that the nonlinear difference equation itself also has multiple stable solutions. It is beyond the scope of this paper to consider all of the solutions that arise in DSGE models with ZLB constraints. Instead, we focus on solutions that mimic the fluctuations near the targeted-inflation steady state. Two such solutions are given by

$$\pi_t^{(*)} = \pi_* \gamma_* \exp\left[-\frac{1}{\psi}\sigma\epsilon_t \right], \quad \gamma_* = \exp\left[\frac{\sigma^2}{2(\psi - 1)\psi^2} \right] \tag{8}$$

and

$$\pi_t^{(D)} = \pi_* \gamma_D \exp\left[-\frac{1}{\psi}\sigma\epsilon_t \right], \quad \gamma_D = \frac{1}{\pi_* r} \exp\left[-\frac{\sigma^2}{2\psi^2} \right]. \tag{9}$$

Notice that

$$\ln \frac{\pi_t^{(*)}}{\pi_* \gamma_*} = \ln \frac{\pi_t^{(D)}}{\pi_* \gamma_D} = -\frac{1}{\psi}\sigma\epsilon_t$$

and equals the term on the right-hand-side of (5). Moreover, for small values of σ the constants $\pi_*\gamma_* \approx \pi_*$ and $\pi_*\gamma_D \approx 1/r_*$. Thus, we refer to $\pi_t^{(*)}$ as the targeted-inflation equilibrium and $\pi_t^{(D)}$ as the deflation equilibrium associated with (3).

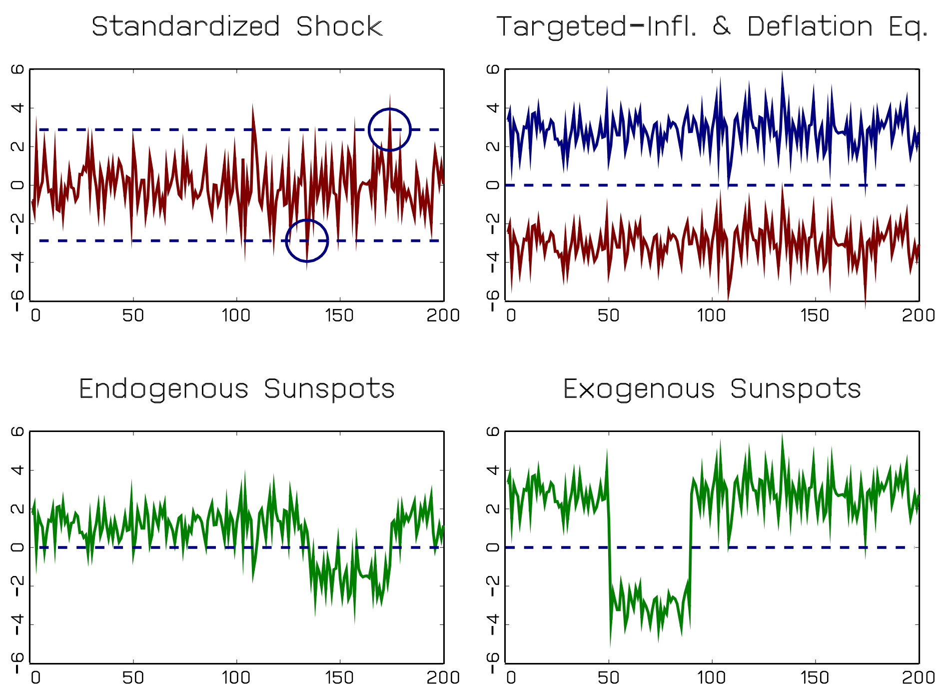
In addition to the equilibria (8) and (9) we also consider a equilibria in which a sunspot s_t triggers moves from targeted-inflation to deflation and vice versa:

$$\pi_t^{(s)} = \pi_* \gamma(s_t) \exp\left[-\frac{1}{\psi}\sigma\epsilon_t \right]. \tag{10}$$

The sunspot shock $s_t \in \{0, 1\}$ evolves according to a two-state discrete Markov switching process. The constants $\gamma(0)$ and $\gamma(1)$ depend on the transition probabilities of the Markov switching process. The fluctuations of $\pi_t^{(s)}$ around $\pi_*\gamma(s_t)$ are identical to the fluctuations in the targeted-inflation and deflation equilibria. The sunspot process could either evolve independently from the fundamental shock or it could be correlated with ϵ_t.[2] For instance, conditional on $s_{t-1} = 1$ (targeted-inflation regime), suppose that $s_t = 0$, i.e., the economy

[2]We thank Mike Woodford for the suggestion to explore equilibria in which the sunspot is triggered by fundamentals.

Figure 1: Inflation Dynamics in the Two-Equation Model



transitions to the deflation regime, if a large negative shock occurs: $\epsilon_t < \underline{\epsilon}_1$. Similarly, the economy exits the deflation regime, if a large positive shock occurs: $\epsilon_t > \underline{\epsilon}_2$.

A numerical illustration is provided in Figure 1. The upper-left panel depicts the evolution of the shock ϵ_t. The upper-right panel compares the paths of net inflation under the targeted-inflation equilibrium and the deflation equilibrium. The difference between the inflation paths is the level shift due to the constants $\ln \gamma_*$ versus $\ln \gamma_D$. The bottom panel shows two sunspot equilibria with visible shifts from the targeted-inflation regime to the deflation regime and back. In the right panel the sunspot evolves exogenously, whereas on the left it is endogenous in the sense that it gets triggered by extreme realizations of ϵ_t, which are marked by circles in the top-left panel.

Before proceeding with a more complicated New Keynesian DSGE model we briefly comment on the treatment of fiscal policy in this paper. We will assume that fiscal policy is passive and that lump-sum taxes are used to balance revenues and outlays. Using the convention that B_t denotes the stock of nominal government bonds at the end of period t, a stylized government budget constraint of the two-equation model is given by

$$R_{t-1}B_{t-1} = B_t + T_t.$$

Dividing both sides by the price level P_t and denoting real government debt by $b_t = B_t/P_t$ and real lump-sum taxes by $\vartheta_t = T_t/P_t$ we obtain

$$b_{t-1}\frac{R_{t-1}}{\pi_t} = b_t + \vartheta_t.$$

A concern might be that in the deflation equilibrium the real value of government debt keeps growing. However, notice that the ex-post real return on government bonds is given by

$$\frac{R_{t-1}}{\pi_t^{(D)}} = r \exp\left[\frac{\sigma^2}{2\psi^2} + \frac{1}{\psi}\sigma\epsilon_t\right]$$

Thus, a stationary lump-sum tax process $\vartheta = b_* R_{t-1}/\pi_t^{(D)}$ will keep the real value of government debt at the level b_*.

There exist many other solutions to (3). For instance, one can use the logic of (7) to construct alternative deflation equilibria. Moreover, Benhabib, Schmitt-Grohé, and Uribe (2001a) studied solutions in which the economy transitions from the targeted-inflation regime to a deflation regime and remains in the deflation regime permanently in continuous-time perfect foresight monetary models. Some of these equilibria are discussed further in the Online Appendix. In the remainder of this paper we will focus on equilibria of a small-scale New Keynesian DSGE model that are akin to $\pi_t^{(*)}$, $\pi_t^{(D)}$, and $\pi_t^{(s)}$ with exogenously evolving sunspot shock.

3 A Prototypical New Keynesian DSGE Model

The DSGE model we consider is the small-scale New Keynesian model studied in An and Schorfheide (2007). The model economy consists of perfectly competitive final-goods-producing firms, a continuum of monopolistically competitive intermediate goods producers, a continuum of identical households, and a government that engages in active monetary and passive fiscal policy. This model has been widely studied in the literature and many of its properties are discussed in the textbook by Woodford (2003). To keep the dimension of the state space manageable we abstract from capital accumulation and wage rigidities. We describe the preferences and technologies of the agents in Section 3.1, summarize the equilibrium conditions in Section 3.2, and characterize the steady states of the model in Section 3.3.

3.1 Preferences and Technologies

Households. Households derive utility from consumption C_t relative to an exogenous habit stock and disutility from hours worked H_t. We assume that the habit stock is given by the level of technology A_t, which ensures that the economy evolves along a balanced growth path despite the quasi-linear preferences. We also assume that the households value transaction services from real money balances, detrended by A_t, and include them in the utility function. The households maximize

$$\mathbb{E}_t\left[\sum_{s=0}^{\infty}\beta^s\left(\frac{(C_{t+s}/A_{t+s})^{1-\tau}-1}{1-\tau}-H_{t+s}+\chi V\left(\frac{M_t}{P_tA_t}\right)\right)\right], \tag{11}$$

subject to budget constraint

$$P_tC_t+T_t+M_t+B_t=P_tW_tH_t+M_{t-1}+R_{t-1}B_{t-1}+P_tD_t+P_tSC_t.$$

Here β is the discount factor, $1/\tau$ is the intertemporal elasticity of substitution, and P_t is the price of the final good. The households supply labor services to the firms, taking the real wage W_t as given. At the end of period t households hold money in the amount of M_t. They have access to a bond market where nominal government bonds B_t that pay gross interest R_t are traded. Furthermore, the households receive profits D_t from the firms and pay lump-sum taxes T_t. SC_t is the net cash inflow from trading a full set of state-contingent securities.

Real money balances enter the utility function in an additively separable fashion. An empirical justification of this assumption is provided by Ireland (2004). As a consequence, the equilibrium has a block diagonal structure under the interest-rate feedback rule that we will specify below: the level of output, inflation, and interest rates can be determined independently of the money stock. We assume that the marginal utility $V'(m)$ is decreasing in real money balances m and reaches zero for $m = \bar{m}$, which is the amount of money held in steady state by households if the net nominal interest rate is zero. Since the return on holding money is zero, it provides the rationale for the ZLB on nominal rates. The usual transversality condition on asset accumulation applies.

Firms. The final-goods producers aggregate intermediate goods, indexed by $j \in [0, 1]$, using the technology:

$$Y_t = \left(\int_0^1 Y_t(j)^{1-\nu} dj \right)^{\frac{1}{1-\nu}}.$$

The firms take input prices $P_t(j)$ and output prices P_t as given. Profit maximization implies that the demand for inputs is given by

$$Y_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-1/\nu} Y_t.$$

Under the assumption of free entry into the final-goods market, profits are zero in equilibrium and the price of the aggregate good is given by

$$P_t = \left(\int_0^1 P_t(j)^{\frac{\nu-1}{\nu}} dj \right)^{\frac{\nu}{\nu-1}}. \tag{12}$$

We define inflation as $\pi_t = P_t / P_{t-1}$.

Intermediate good j is produced by a monopolist who has access to the following production technology:

$$Y_t(j) = A_t H_t(j), \tag{13}$$

where A_t is an exogenous productivity process that is common to all firms and $H_t(j)$ is the firm-specific labor input. Labor is hired in a perfectly competitive factor market at the real wage W_t. Intermediate-goods-producing firms face quadratic price adjustment costs of the form

$$AC_t(j) = \frac{\phi}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - \bar{\pi} \right)^2 Y_t(j),$$

where ϕ governs the price stickiness in the economy and $\bar{\pi}$ is a baseline rate of price change that does not require the payment of any adjustment costs. In our quantitative analysis we set $\bar{\pi} = 1$, that is, it is costless to keep prices constant. Firm j chooses its labor input $N_t(j)$ and the price $P_t(j)$ to maximize the present value of future profits

$$\mathbb{E}_t \left[\sum_{s=0}^{\infty} \beta^s Q_{t+s|t} \left(\frac{P_{t+s}(j)}{P_{t+s}} Y_{t+s}(j) - W_{t+s} H_{t+s} - AC_{t+s} \right) \right]. \tag{14}$$

Here, $Q_{t+s|t}$ is the time t value to the household of a unit of the consumption good in period $t+s$, which is treated as exogenous by the firm.

Government Policies. Monetary policy is described by an interest rate feedback rule of the form

$$R_t = \max \left\{ 1, \left[r\pi_* \left(\frac{\pi_t}{\pi_*} \right)^{\psi_1} \left(\frac{Y_t}{\gamma Y_{t-1}} \right)^{\psi_2} \right]^{1-\rho_R} R_{t-1}^{\rho_R} e^{\sigma_R \epsilon_{R,t}} \right\}. \tag{15}$$

Here r is the steady state real interest rate, π_* is the target-inflation rate, and $\epsilon_{R,t}$ is a monetary policy shock. The key departure from much of the New Keynesian DSGE literature is the use of the max operator to enforce the ZLB. Provided the ZLB is not binding, the central bank reacts to deviations of inflation from the target rate π_* and deviations of output growth from γ.

The government consumes a stochastic fraction of aggregate output and government spending evolves according to

$$G_t = \left(1 - \frac{1}{g_t} \right) Y_t. \tag{16}$$

The government levies a lump-sum tax T_t (or provides a subsidy if T_t is negative) to finance any shortfalls in government revenues (or to rebate any surplus). Its budget constraint is given by

$$P_t G_t + M_{t-1} + R_{t-1} B_{t-1} = T_t + M_t + B_t. \tag{17}$$

Exogenous shocks. The model economy is perturbed by three exogenous processes. Aggregate productivity evolves according to

$$\ln A_t = \ln \gamma + \ln A_{t-1} + \ln z_t, \text{ where } \ln z_t = \rho_z \ln z_{t-1} + \sigma_z \epsilon_{z,t}. \tag{18}$$

Thus, on average the economy grows at the rate γ and z_t generates exogenous fluctuations of the technology growth rate. We assume that the government spending shock follows the AR(1) law of motion

$$\ln g_t = (1 - \rho_g) \ln g_* + \rho_g \ln g_{t-1} + \sigma_g \epsilon_{g,t}. \tag{19}$$

The monetary policy shock $\epsilon_{R,t}$ is assumed to be serially uncorrelated. We stack the three innovations into the vector $\epsilon_t = [\epsilon_{z,t}, \epsilon_{g,t}, \epsilon_{r,t}]'$ and assume that $\epsilon_t \sim iidN(0, I)$.

3.2 Equilibrium Conditions

Since the exogenous productivity process has a stochastic trend, it is convenient to characterize the equilibrium conditions of the model economy in terms of detrended consumption and output: $c_t = C_t/A_t$ and $y_t = Y_t/A_t$. The consumption Euler equation is given by

$$1 = \beta \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\tau} \frac{1}{\gamma z_{t+1}} \frac{R_t}{\pi_{t+1}} \right]. \tag{20}$$

We define

$$\mathcal{E}_t = I\!E_t \left[\frac{c_{t+1}^{-\tau}}{\gamma z_{t+1} \pi_{t+1}} \right], \tag{21}$$

which will be useful in the computational algorithm. In a symmetric equilibrium in which all firms set the same price $P_t(j)$ the price-setting decision of the firms leads to the condition

$$\begin{aligned} 1 \;=\; & \frac{1}{\nu}(1 - c_t^{\tau}) + \phi(\pi_t - \bar{\pi}) \left[\left(1 - \frac{1}{2\nu} \right) \pi_t + \frac{\bar{\pi}}{2\nu} \right] \\ & - \phi \beta \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\tau} \frac{y_{t+1}}{y_t} (\pi_{t+1} - \bar{\pi}) \pi_{t+1} \right] \end{aligned} \tag{22}$$

The aggregate resource constraint can be expressed as

$$c_t = \left[\frac{1}{g_t} - \frac{\phi}{2} (\pi_t - \bar{\pi})^2 \right] y_t. \tag{23}$$

It reflects both government spending as well as the resource cost (in terms of output) caused by price changes. Finally, we reproduce the monetary policy rule

$$R_t = \max \left\{ 1, \left[r \pi_* \left(\frac{\pi_t}{\pi_*} \right)^{\psi_1} \left(\frac{y_t}{y_{t-1}} z_t \right)^{\psi_2} \right]^{1-\rho_R} R_{t-1}^{\rho_R} e^{\sigma_R \epsilon_{R,t}} \right\}. \tag{24}$$

Before exploring the equilibrium dynamics of the stochastic system, it is instructive to consider the steady states of the model.

3.3 Steady States

As the two-equation model in Section 2, the New Keynesian model with ZLB constraint has two steady states, which we refer to as targeted-inflation and deflation steady state. In the targeted-inflation steady state inflation equals π_*. The real interest rate, nominal interest rate output, and consumption are given by

$$\begin{aligned} r &= \gamma/\beta, \quad R_* = r\pi_*, \quad y_* = \frac{c_*}{\left[\frac{1}{g_*} - \frac{\phi}{2}(\pi_* - \bar{\pi})^2\right]} \\ c_* &= \left[1 - \upsilon - \frac{\phi}{2}(1-2\lambda)\left(\pi_* - \frac{1-\lambda}{1-2\lambda}\bar{\pi}\right)^2 + \frac{\phi}{2}\frac{\lambda^2}{1-2\lambda}\bar{\pi}^2\right]^{1/\tau}, \end{aligned} \tag{25}$$

where $\lambda = \nu(1-\beta)$. In the deflation steady state the nominal interest rate is at the ZLB, that is, $R_D = 1$, and provided that $R_* > 1$ and $\psi_1 > 1$:

$$\begin{aligned} r &= \gamma/\beta, \quad \pi_D = \beta/\gamma, \quad y_D = \frac{c_D}{\left[\frac{1}{g_*} - \frac{\phi}{2}(\pi_D - \bar{\pi})^2\right]} \\ c_D &= \left[1 - \upsilon - \frac{\phi}{2}(1-2\lambda)\left(\pi_D - \frac{1-\lambda}{1-2\lambda}\bar{\pi}\right)^2 + \frac{\phi}{2}\frac{\lambda^2}{1-2\lambda}\bar{\pi}^2\right]^{1/\tau}. \end{aligned} \tag{26}$$

If the real rate $r > 1$ then the economy experiences deflation, which is why we label the steady state as deflation steady state.

The relative consumption and welfare in the two steady states depends on the discrepancy between $\bar{\pi}$ and steady state inflation. If $\bar{\pi} = 1$ and the target-inflation rate π_* is substantially greater than one, then consumption and welfare may be higher in the deflation steady state. If, on the other hand, $\bar{\pi} = \pi_*$ then the targeted-inflation steady state is associated with higher welfare. In the next section, we turn to the analysis of equilibrium dynamics. As in the two-equation model of Section 2, we will construct a targeted-inflation equilibrium in which the economy fluctuates around the targeted-inflation steady state; a deflation equilibrium in which the economy fluctuations near the deflation steady state; and an sunspot equilibrium with a targeted-inflation regime and a deflation regime.

4 Solving the Model Subject to the ZLB Constraint

Most of the existing literature focuses on what we call the targeted-inflation equilibrium. In this equilibrium households choose time t consumption as a function of the exogenous state variables z_t, g_t, and $\epsilon_{R,t}$ and the endogenous state variables R_{t-1} and y_{t-1}. Moreover, firms set prices such that time t inflation depends on the same endogenous and exogenous state variables. As illustrated in the example of Section 2, there exist many equilibria in which the economy enters extended periods of deflation. We consider one equilibrium in which consumption and inflation depend on the same set of state variables as in the targeted-inflation equilibrium. Moreover, we consider an equilibrium in which the agents' decision rules also depend on a Markov-switching sunspot s_t. In sum, the goal is to construct functions

$$\begin{aligned} \pi_t &= f_\pi(R_{t-1}, y_{t-1}, z_t, g_t, \epsilon_{R,t}, s_t) \\ c_t &= f_c(R_{t-1}, y_{t-1}, z_t, g_t, \epsilon_{R,t}, s_t) \end{aligned}$$

and so on, such that the equilibrium conditions (20) to (24) as well as the laws of motion for the exogenous processes (and the transversality conditions) are satisfied. In general, there exist other, more complex functions $f_\pi(\cdot)$ and $f_c(\cdot)$ that satisfy the equilibrium conditions. We are focusing on "minimal" state-variable solutions. In Section 4.1 we begin by constructing a solution for a piece-wise log-linear approximation of the equilibrium conditions. The numerical procedure to solve the full model without taking linear approximations is presented in Section 4.2.

4.1 Piece-wise Linear Dynamics Near Steady States

We begin by taking log-linear approximations near the targeted-inflation and the deflation steady state. To simplify the exposition we impose the following restrictions on the DSGE model parameters: $\tau = 1$, $\gamma = 1$, $\bar{\pi} = \pi_*$, $\psi_1 = \psi$, $\psi_2 = 0$, $\rho_R = 0$, $\rho_z = 0$, and $\rho_g = 0$. A general approximation of the equilibrium conditions (20) to (24) is presented in the Online Appendix.

Approximating the Targeted-Inflation Equilibrium. Under the parameter restriction $\bar{\pi} = \pi_*$ the steady states of consumption and output are given by $c_* = 1 - \nu$ and $y_* = c_* g_*$. If we let $\kappa_* = c_*/(\nu\phi\pi_*^2)$, then we obtain the familiar linear system:

$$\begin{aligned}
\hat{R}_t &= \max\left\{ -\ln(r\pi_*),\ \psi\hat{\pi}_t + \sigma_R\epsilon_{R,t} \right\} \\
\hat{c}_t &= \mathbb{E}_t[\hat{c}_{t+1}] - (\hat{R}_t - \mathbb{E}_t[\hat{\pi}_{t+1}]) \\
\hat{\pi}_t &= \beta\mathbb{E}_t[\hat{\pi}_{t+1}] + \kappa_*\hat{c}_t,
\end{aligned} \tag{27}$$

where for each variable $\hat{x}_t = \ln(x_t/x_*)$. The law of motion for output is given by

$$\hat{y}_t = \hat{c}_t + \sigma_g\epsilon_{g,t}. \tag{28}$$

It is well known that if the shocks are small enough such that the ZLB is non-binding, the linearized system has a unique stable solution for $\psi > 1$. Since the exogenous shocks are *iid* and the simplified system has no endogenous propagation mechanism, consumption, output, inflation, and interest rates will also be *iid* and can be expressed as a function of $\epsilon_{R,t}$ and $\epsilon_{g,t}$.

We are interested in the case in which the shocks are large enough such that the ZLB is binding with non-negligible probability. In this case the system (27) exhibits piece-wise linear dynamics:

$$\begin{aligned}
\hat{R}_t(\epsilon_{R,t}) &= \max\left\{ -\ln(r\pi_*),\ \frac{1}{1+\kappa\psi}\Big[\psi(\kappa+\beta)\mu_\pi^* + \kappa\psi\mu_c^* + \sigma_R\epsilon_{R,t}\Big] \right\} \\
\hat{c}_t(\epsilon_{R,t}) &= \begin{cases} \frac{1}{1+\kappa\psi}\Big[(1-\psi\beta)\mu_\pi^* + \mu_c^* - \sigma_R\epsilon_{R,t}\Big] & \text{if } \hat{R}_t \geq -\ln(r\pi_*) \\ \ln(r\pi_*) + \mu_c^* + \mu_\pi^* & \text{otherwise} \end{cases} \\
\hat{\pi}_t(\epsilon_{R,t}) &= \begin{cases} \frac{1}{1+\kappa\psi}\Big[(\kappa+\beta)\mu_\pi^* + \kappa\mu_c^* - \kappa\sigma_R\epsilon_{R,t}\Big] & \text{if } \hat{R}_t \geq -\ln(r\pi_*) \\ \kappa\ln(r\pi_*) + (\kappa+\beta)\mu_\pi^* + \kappa\mu_c^* & \text{otherwise} \end{cases}.
\end{aligned} \tag{29}$$

The constants μ_c^* and μ_π^* are the unconditional expectations of $\hat{c}_t$ and $\hat{\pi}_t$. They can be determined by taking expectations of (29) on both sides of the equalities and solving a nonlinear system of equations.

Approximating a Deflation Equilibrium. In the deflation equilibrium the steady state inflation rate is $\pi_D = \beta$. The log-linearization is more cumbersome because of additional terms that arise from $\pi_D \neq \bar{\pi}$. To ease the expositions assume that $|\pi_D - \bar{\pi}|$ is small and ignore the additional terms from our log-linear approximation. Denote percentage deviations of a variable x_t from its deflation steady state by $\tilde{x}_t = \ln(x_t/x_D)$. If we let $\kappa_D = c_D/(\nu\phi\beta^2)$ and using the steady state relationship $r = 1/\beta$

$$\begin{aligned} \tilde{R}_t &= \max\left\{0,\ -(\psi-1)\ln(r\pi_*) + \psi\tilde{\pi}_t + \sigma_R\epsilon_{R,t}\right\} \\ \tilde{c}_t &= \mathbb{E}_t[\tilde{c}_{t+1}] - (\tilde{R}_t - \mathbb{E}_t[\tilde{\pi}_{t+1}]) \\ \tilde{\pi}_t &= \beta\mathbb{E}_t[\tilde{\pi}_{t+1}] + \kappa_D\tilde{c}_t. \end{aligned} \tag{30}$$

Provided that $\psi > 1$, the ZLB is binding with high probability if the shock standard deviation σ_R is small. In this case $\tilde{R}_t = 0$. It is well-known that if the central bank does not (or is not able to) react to inflation movements the rational expectation system is indeterminate and has many stable solutions. Throughout this paper we focus on the so-called minimum state variable solution, which, in the context of this simple illustrative model, is the solution in which all variables are *iid* in equilibrium. Taking into account that for some shock realizations the ZLB is not binding we can obtain a deflation equilibrium by adjusting the constants in (29):

$$\begin{aligned} \tilde{R}_t(\epsilon_{R,t}) &= \max\left\{0,\ \frac{1}{1+\kappa\psi}\left[\psi(\kappa+\beta)\mu_\pi^D + \kappa\psi\mu_c^D - (\psi-1)\ln(r\pi_*) + \sigma_R\epsilon_{R,t}\right]\right\} \\ \tilde{c}_t(\epsilon_{R,t}) &= \begin{cases} \frac{1}{1+\kappa\psi}\left[(1-\psi\beta)\mu_\pi^D + \mu_c^D + (\psi-1)\ln(r\pi_*) - \sigma_R\epsilon_{R,t}\right] & \text{if } \tilde{R}_t \geq 0 \\ \mu_c^D + \mu_\pi^D & \text{otherwise} \end{cases} \\ \tilde{\pi}_t(\epsilon_{R,t}) &= \begin{cases} \frac{1}{1+\kappa\psi}\left[(\kappa+\beta)\mu_\pi^D + \kappa\mu_c^D + \kappa(\psi-1)\ln(r\pi_*) - \kappa\sigma_R\epsilon_{R,t}\right] & \text{if } \tilde{R}_t \geq 0 \\ (\kappa+\beta)\mu_\pi^D + \kappa\mu_c^D & \text{otherwise} \end{cases}. \end{aligned} \tag{31}$$

Discussion. In the simplified model the government spending shock does not affect interest rates, consumption, and inflation. It simply shifts output according to (28). Moreover, technology growth innovations have no effect on interest rates, inflation, and detrended model variables. A shock $\epsilon_{z,t}$ simply generates a permanent increase in the levels of consumption C_t and Y_t.

The consumption of the household and the pricing decision of the firms depend on the monetary policy shock $\epsilon_{R,t}$. In this simple model, the decision rules have a kink at the point in the state space where the two terms in the max operator of the interest rate equation are equal to each other. In the targeted-inflation equilibrium this point in the state space is given by

$$\bar{\epsilon}_R^* = \frac{1}{\sigma_R}\bigg[-(1+\kappa\psi)\ln(r\pi_*) - (\kappa+\beta)\psi\mu_\pi^* - \kappa\psi\mu_c^*\bigg],$$

whereas in the deflation equilibrium it is

$$\bar{\epsilon}_R^D = \frac{1}{\sigma_R}\bigg[(\psi-1)\ln(r\pi_*) - (\kappa+\beta)\psi\mu_\pi^D - \kappa\psi\mu_c^D\bigg],$$

Once $\epsilon_{R,t}$ falls below the threshold value $\bar{\epsilon}_R^*$ or $\bar{\epsilon}_R^D$, its marginal effect on the endogenous variables is zero. To the extent that $\bar{\epsilon}_R^D > 0 > \bar{\epsilon}_R^*$, it takes a positive shock in the deflation equilibrium to move away from the ZLB, whereas it takes a large negative monetary shock in the targeted-inflation equilibrium to hit the ZLB. The kink in the decision rules implies that the impulse responses of the endogenous variables to the monetary policy shock $\epsilon_{R,t}$ are highly nonlinear. Motivated by the results so far, when we construct a numerical approximation to the decision rules for the more general DSGE model in Section 4.2, we use a piece-wise smooth approximation to separate approximations for the regions of the state space in which $R_t = 1$ and the region in which $R_t > 1$.

4.2 Nonlinear Solution

This section discusses the numerical techniques that we use to solve the model described in Section 3.2. The goal is to characterize the various solutions to the system

$$\xi(c_t, \pi_t, y_t) = \phi\beta I\!\!E_t\left[(c_{t+1})^{-\tau} y_{t+1}(\pi_{t+1}-\bar{\pi})\pi_{t+1}\right] \tag{32}$$

$$c_t^{-\tau} = \beta R_t \mathcal{E}_t \tag{33}$$

$$y_t = \left[\frac{1}{g_t} - \frac{\phi}{2}(\pi_t - \bar{\pi})^2\right]^{-1} c_t \tag{34}$$

$$R_t = \max\left\{1, \left[r_*\pi_*\left(\frac{\pi_t}{\pi_*}\right)^{\psi_1}\left(\frac{y_t}{y_{t-1}}z_t\right)^{\psi_2}\right]^{1-\rho_R} R_{t-1}^{\rho_R} e^{\epsilon_{R,t}}\right\} \tag{35}$$

where $\mathcal{E}_t$ was defined in (21) and $\xi(.)$ is defined as

$$\xi(c,\pi,y) = c^{-\tau} y\left\{\frac{1}{\nu}(1-c^\tau) + \phi(\pi-\bar{\pi})\left[\left(1-\frac{1}{2\nu}\right)\pi + \frac{\bar{\pi}}{2\nu}\right] - 1\right\}. \tag{36}$$

To that end we utilize a global approximation using Chebyshev polynomials following Judd (1992) where all decision rules, $c(.)$, $\pi(.)$, $R(.)$., $y(.)$ and $\mathcal{E}(.)$ are assumed to be functions of the minimum set of state variables $(R_{t-1}, y_{t-1}, g_t, z_t, \epsilon_{R,t})$ and the sunspot variable s_t, where applicable, which we collectively label as $\mathcal{S}_t$. The full solution algorithm is relegated to Section C of the Online Appendix.

The solution algorithm amounts to specifying a grid of points $\mathcal{G} = \{\mathcal{S}_1, \ldots, \mathcal{S}_M\}$ in the model's state space and solving for the vector Θ such that the sum of squared residuals associated with (21) and (32) are minimized for $\mathcal{S}_t \in \mathcal{G}$. By construction, (33)-(35) hold exactly. Since the collocation methods, which require the solution to be accurate on a fixed grid typically obtained by the Kronecker product of grids in each dimension, become exceedingly difficult to implement as the number of state variable go above three, we build on the ergodic-set method discussed in Judd, Maliar, and Maliar (2011). This method requires an iteration between solving and simulating the solved model and the approximation is expected to be accurate only over a set of points that characterizes the model's ergodic distribution of $\mathcal{S}_t$. Since our goal is to fit the model to data from the 2008-09 recession and since explaining these data with our model requires realizations of the states that lie far in the tails of the model-implied ergodic distribution, we combine draws from the ergodic distribution with filtered exogenous state variables[3] based on data on output growth, inflation, and interest rates from 2000 to 2012 to generate the grid $\mathcal{G}$. This ensures that our approximation remains accurate in the area of the state space that is relevant for the empirical analysis.

Unlike Judd, Maliar, and Maliar (2011), Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2012) and Gust, Lopez-Salido, and Smith (2012), we use a piece-wise smooth approximation of the functions $\pi(\mathcal{S}_t)$ and $\mathcal{E}(\mathcal{S}_t)$ by postulating

$$\begin{aligned} \pi_t = \pi(\mathcal{S}_t;\Theta) &= \zeta_t f_\pi^1(\mathcal{S}_t;\Theta) + (1-\zeta_t) f_\pi^2(\mathcal{S}_t;\Theta) \\ \mathcal{E}_t = \mathcal{E}(\mathcal{S}_t;\Theta) &= \zeta_t f_{\mathcal{E}}^1(\mathcal{S}_t;\Theta) + (1-\zeta_t) f_{\mathcal{E}}^1(\mathcal{S}_t;\Theta) \end{aligned}$$

where $\zeta_t = I\{R(\mathcal{S}_t;\Theta) > 1\}$ is an indicator that shows the ZLB is slack. The functions f_j^i are linear combinations of a complete set of Chebyshev polynomials up to fourth order,

[3] See Section 5.3 for a detailed explanation of the filtering procedure

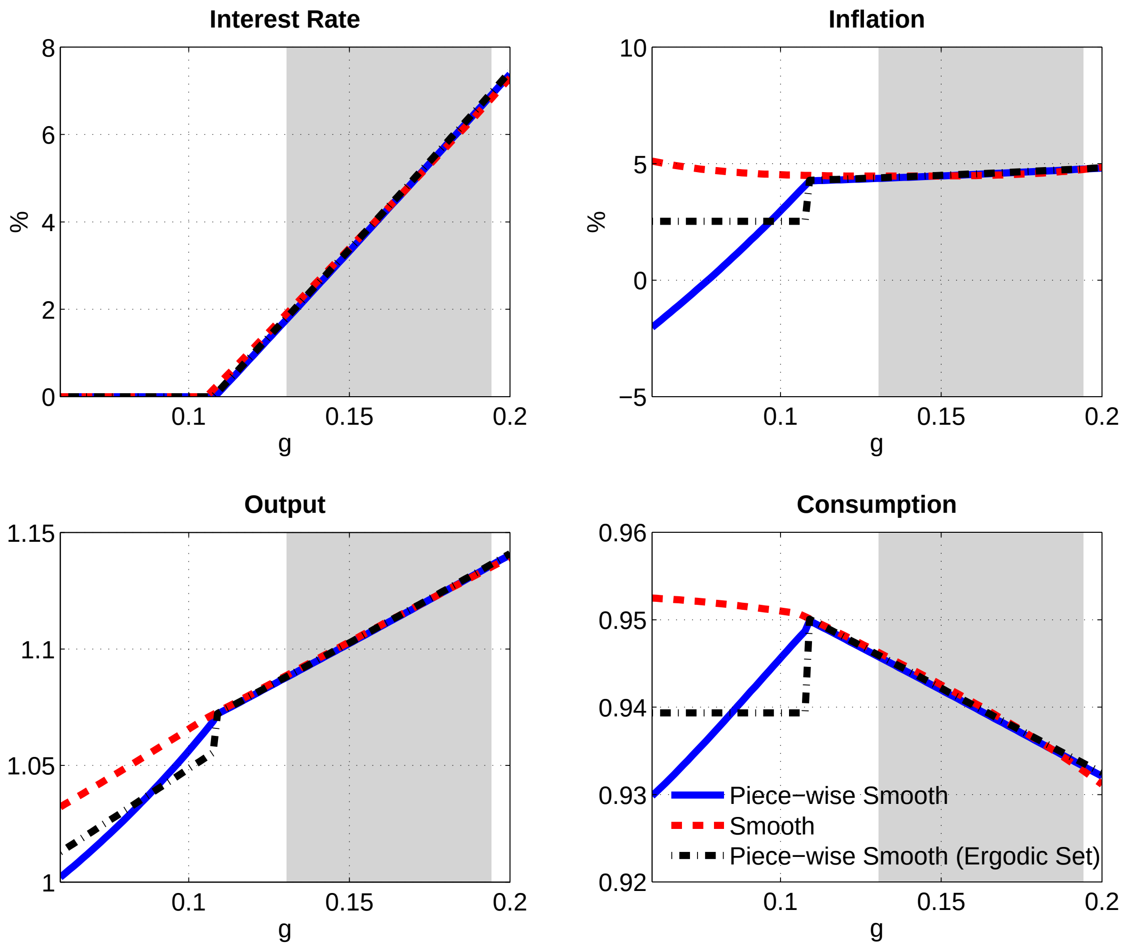
where the weights are given by a vector Θ. Conditional on $\pi(\mathcal{S}_t)$ and $\mathcal{E}(\mathcal{S}_t)$, the decision rules for consumption, output, and interest rates can be obtained recursively from (33), (34), and (35). Our method is flexible enough to allow for a kink in all decision rules and not just R_t which has a kink by its construction. In our experience this flexibility yields much higher accuracy in the approximated decision rules, especially the inflation decision rule. To demonstrate this, Figure 2 shows a slide of the decision rules where we set $R_{t-1} = 1$, $y_{t-1} = y_*$, $z = 0$ and $\epsilon_{R,t} = 0$ and vary g_t in a wide range. We show results from three schemes: our piece-wise smooth benchmark, solved using filtered states as explained in Appendix C, an approximation that assumes π_t and $\mathcal{E}_t$ are smooth functions and a variation of the piece-wise smooth approximation that is only valid on the ergodic distribution. The gray shading shows the 95% coverage of the ergodic distribution. While it may seem like much of the action is significantly away from the ergodic distribution, as we discuss below, this is a region that is visited in 2009 and later, according to our filtered estimates. For value of g smaller than 0.11 the economy hits the ZLB. When approximated smoothly, the decision rules fail to capture the kinks that are apparent in the piece-wise smooth approximation. The decision rule for output illustrates that the (marginal) government-spending multiplier is sensitive to the ZLB - it is noticeably larger in the area of the state space where the ZLB binds, which is not captured by the smooth approximation. This figure shows an example where using filtered states and a piece-wise smooth approximation yields a much superior, and economically different, decision rules.[4]

5 Quantitative Analysis

The quantitative analysis consists of four parts. In Section 5.1 we estimate the parameters of the DSGE model under the assumption that the economy was in the targeted-inflation equilibrium from 1984 to 2007. These parameter estimates are the starting point for the subsequent analysis. In Section 5.2 we compare the ergodic distribution of inflation and

[4]The piece-wise smooth decision rules solved using just the ergodic distribution has no information about what should happen when the ZLB binds and they simply revert to the respective steady states. This is why the dashed line in Figure 2 is constant when the ZLB binds.

Figure 2: Sample Decision Rules



Notes: The gray shading show 95% coverage of the ergodic distribution in the targeted-inflation equilibrium.

interest rates under the three equilibria. Moreover, we examine impulse responses to a monetary policy shock in the targeted-inflation and the deflation equilibrium. In Section 5.3 we use the model to estimate a sequence of historical states for the period 2000:I to 2010:III based on output growth, inflation, and interest rate data. Conditional on the estimated states during the great recession of 2008-09, Section 5.4 assesses the effect of fiscal and monetary policy interventions in the targeted-inflation and the sunspot equilibria.

Table 1: DSGE Model Parameters

$\tau = 1.50$	$r = 1.0070$	$\gamma = 1.0048$	$\nu = 0.1$
$g_* = 1/0.85$	$\phi = 75.75$	$\pi_* = 1.0063$	$\bar{\pi} = 1$
$\psi_1 = 1.36$	$\psi_2 = 0.80$	$\rho_R = 0.65$	$\sigma_R = 0.0021$
$\rho_g = 0.86$	$\sigma_g = 0.0078$	$\rho_Z = 0.11$	$\sigma_z = 0.0103$

5.1 Estimation under Targeted-Inflation Equilibrium

The parameter values for the subsequent analysis are obtained by estimating the DSGE model described in Section 3 under the assumption that the economy is in the targeted-inflation equilibrium. Since the ZLB was not binding during this period we replaced the global approximation discussed in Section 4.2 by a second-order perturbation method. In the area of the state-space that is empirically relevant for our estimation sample, the decision rules obtained under these two solution methods are virtually identical. The data for the estimation was extracted from the FRB St. Louis FRED database (2012-11 vintage). Output growth is defined as real GDP (GDPC96) growth converted into per capita terms. Our measure of population is Civilian Noninstitutional Population (CNP16OV). We compute population growth rates as log differences and apply an eight-quarter backward-looking moving average filter to the growth rates to smooth out abrupt changes in the population growth series. Inflation is defined as the log difference in the GDP deflator (GDPDEF) and the interest rate is the average effective federal funds rate (FEDFUNDS) within each quarter. The estimation period for the DSGE model is 1984:Q1 to 2007:Q4. We used Bayesian techniques described in detail in An and Schorfheide (2007) and report posterior mean estimates in Table 1. These estimates are in line with estimates for small-scale New Keynesian DSGE models that have been reported elsewhere in the literature and will be used throughout the subsequent analysis. A detailed description of the prior distribution underlying this estimation is provided in Appendix D.

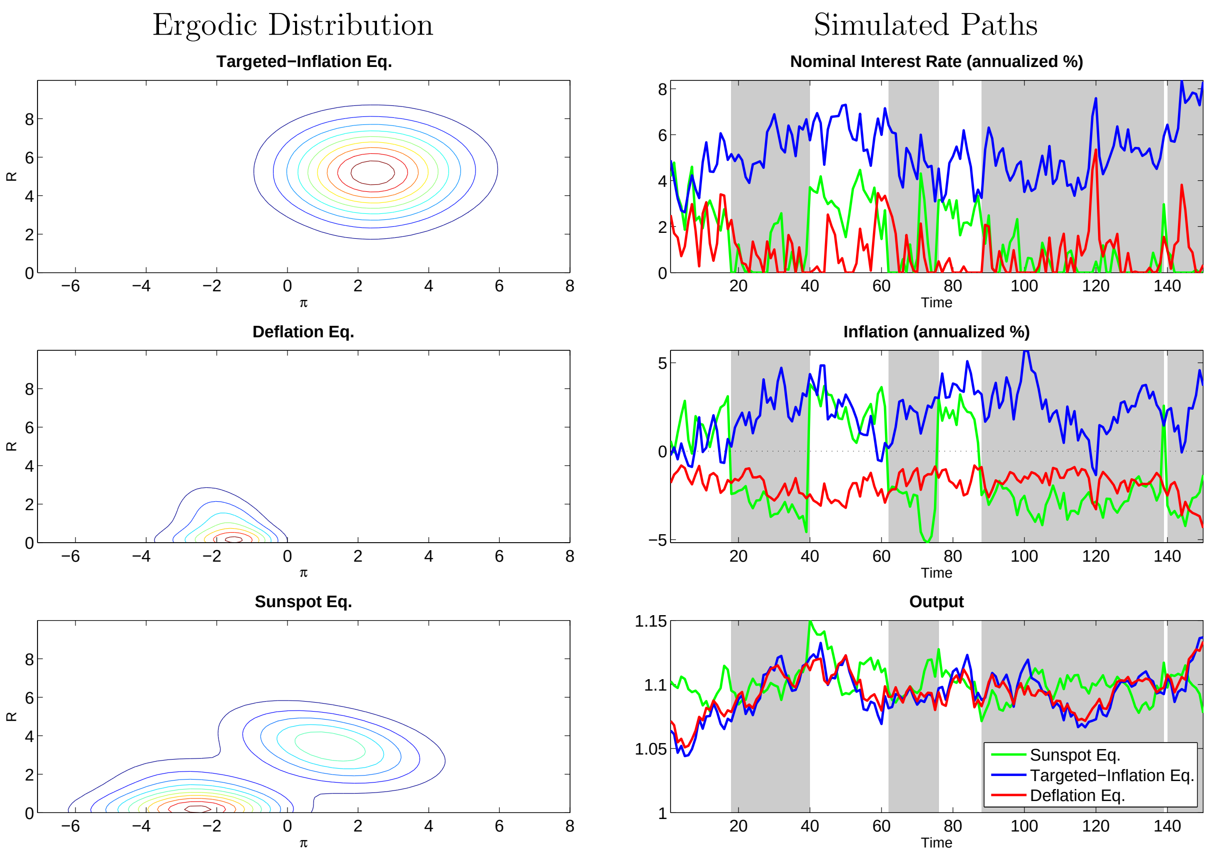
5.2 Equilibrium Dynamics

In order to evaluate the ergodic distributions associated with the targeted-inflation, the deflation, and the sunspot equilibria, we simulate a long sequence of draws from each of the equilibria. The left column of Figure 3 depicts contour plots of the ergodic distributions of the three equilibria. The ergodic distribution of the targeted-inflation equilibrium is approximately centered at the state values, which are 2.5% inflation and an interest rate of 5.3% annually. The contours for the deflation equilibrium are concentrated near the ZLB and peak at an inflation rate of about -1.5%. This inflation rate is larger than the steady state value of -2.8%. By construction, the sunspot equilibrium generates a bimodal ergodic distribution of inflation and interest rates. However, this bimodal distribution is not simply a mixture of the distributions associated with the target-inflation and the deflation equilibria. Since agents expect regime changes to occur in the future the decision rules in the two regimes of the sunspot equilibrium are different from the decision rules in the pure equilibria.

The right column of Figure 3 shows simulated paths of interest rates, inflation rates, and output growth for the three equilibria, using the same shock innovations. The shaded areas correspond to periods in which the deflation regime is active in the sunspot equilibrium. Given our parameter estimates the probability of hitting the ZLB under the targeted-inflation equilibrium is very low. As mentioned above, the estimation sample ranges from 1984 to 2007, which is a period of above-zero interest rates and low macroeconomic volatility. In the deflation regime the interest rate frequently hits the ZLB and may stay at zero for multiple periods. While inflation is mostly positive in the targeted-inflation regime it is always negative in the deflation regime. The simulated paths from the sunspot equilibrium alternate between periods of low interest rates coupled with deflation and periods of high interest and inflation rates. The regime switch induced by the sunspot shock triggers a strong adjustment of the nominal variables. While the time paths of output growth are very similar in the targeted-inflation and deflation equilibrium output appears to be slightly higher in the sunspot equilibrium.

Next, we turn to impulse response functions for monetary policy shocks. While the primary focus of this paper is on government spending shocks we will use monetary policy

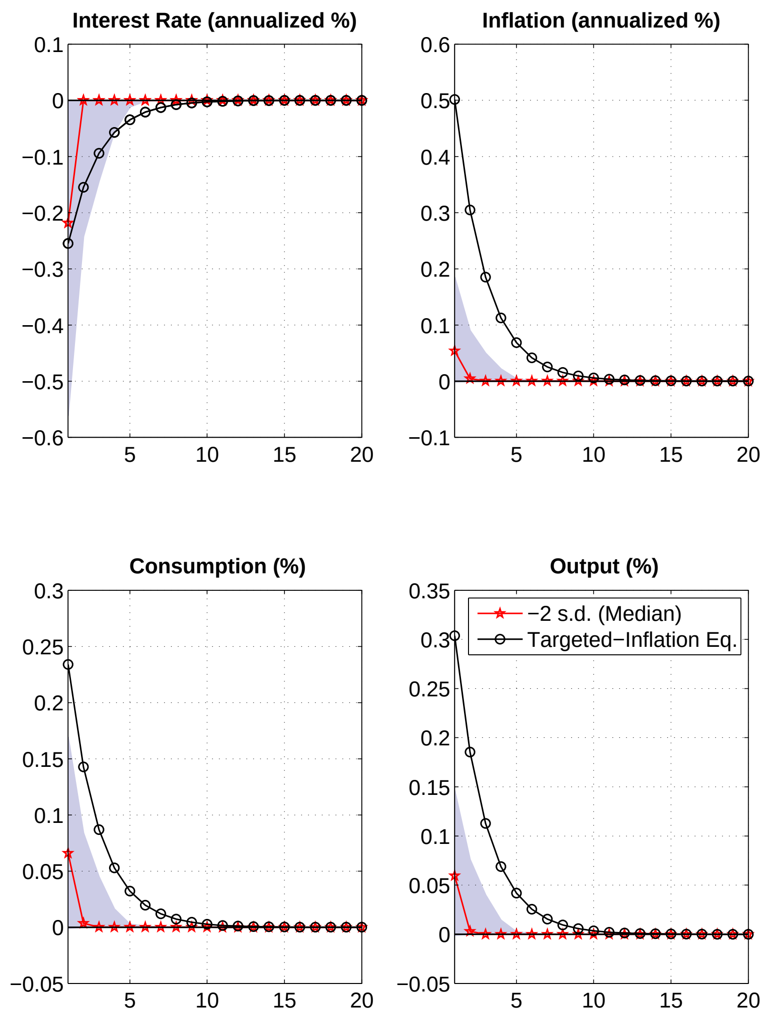
Figure 3: Ergodic Distribution and Simulated Paths



Notes: Figure depicts surface and contour plots of the joint probability density function (kernel density estimate) of interest rates and inflation. Interest rate and inflation are net rates at an annual rate.

shocks in Section 5.4 to assess the effect of an expansionary fiscal policy if it is combined with a monetary policy that keeps interest rates close to zero. Due to the nonlinearity of our model, there are a number of alternative ways of computing impulse response. We compute impulse responses as the difference between a baseline simulation path in which all shocks are drawn from $\epsilon_t \sim N(0, I)$ and an alternative path in which the time $t = 1$ monetary policy shock is shifted by δ and all other shocks are identical to their baseline values. The procedure that is used to compute the impulse responses is summarized in Algorithm 2 in Appendix C. It leads to a distribution of impulse responses that can be summarized by pointwise medians and credible bands. For a linear model, the algorithm produces the standard impulse responses, which are invariant to the size δ of the shock and identical for each simulated path.

Figure 4: Responses to a Negative Monetary Policy Shocks: Targeted-Inflation versus Deflation Equilibrium



The panels of Figure 4 overlay the responses of the economy to an expansionary monetary policy shock under the targeted-inflation equilibrium and the deflation equilibrium. In the former interest rates fall by about 25 basis points (bp) annualized, inflation rises by 50bp and output grows by about 30bp. The impulse responses exhibit no apparent nonlinearities and none of the simulated trajectories hits the ZLB. The impulse response dynamics in the deflation equilibrium, on the other hand, are highly nonlinear and our simulations yield to a distribution of responses summarized by pointwise medians and bands. The median response of the interest rate is slightly smaller than the response in the targeted-inflation regime and reverts back to zero after the initial impact of the shock. More interesting than the median response is the band. Along some trajectories, namely those at which the other shocks push the interest rate to the ZLB, the negative monetary policy shock has no effect on the interest rate because a further reduction is not possible. On other trajectories, the pre-intervention interest rate is positive and the monetary policy shock can lead to a reduction of up to 60bp.

Even if the economy is not at the ZLB in the initial period, an expansionary monetary policy shock increases the probability of hitting the ZLB in the future. The presence of the ZLB leads to a reduction in the expected future nominal interest rate movements compared to an unconstrained environment. Thus, after an expansionary monetary policy intervention agents expect smaller real rate drops in the future. In turn, the increase in consumption in the initial period is lower, because the Euler equation implies that consumption is approximately equal to the discounted sum of expected future real rates. The muted consumption response leads to a muted inflation response via the Phillips curve. Overall, neither inflation nor consumption and output react to the interest rate cut as much as in the targeted-inflation equilibrium. The dampened output and inflation response implies via the interest rate feedback that the interest rate falls more strongly in the deflation equilibrium than in the targeted-inflation equilibrium. Overall, the expansionary monetary policy is less effective in the deflation equilibrium.

5.3 Extracting Historical Shocks

We now use the DSGE model to determine the sequence of shocks that lead to the Great Recession in 2008-09. The filtered state variables for 2009:Q1 will provide the initial conditions for the policy experiments in Section 5.4. We extract two sequences of shocks and states: one sequence is obtained under the assumption that the U.S. economy was in the targeted-inflation equilibrium, whereas the other sequence was obtained assuming that the sunspot equilibrium prevailed since 2000:Q1. Because the U.S. has never experienced a prolonged period of deflation since 1960, the deflation equilibrium is empirically implausible and not considered in the subsequent analysis.

The DSGE model can be represented as a state space model. Let y_t be the 3×1 vector of observables consisting of output growth, inflation, and nominal interest rates. The vector x_t stacks the continuous state variables which are given by $x_t = [R_t, y_t, y_{t-1}, z_t, g_t, A_t]'$ and $s_t \in \{0, 1\}$ is the Markov-switching process.

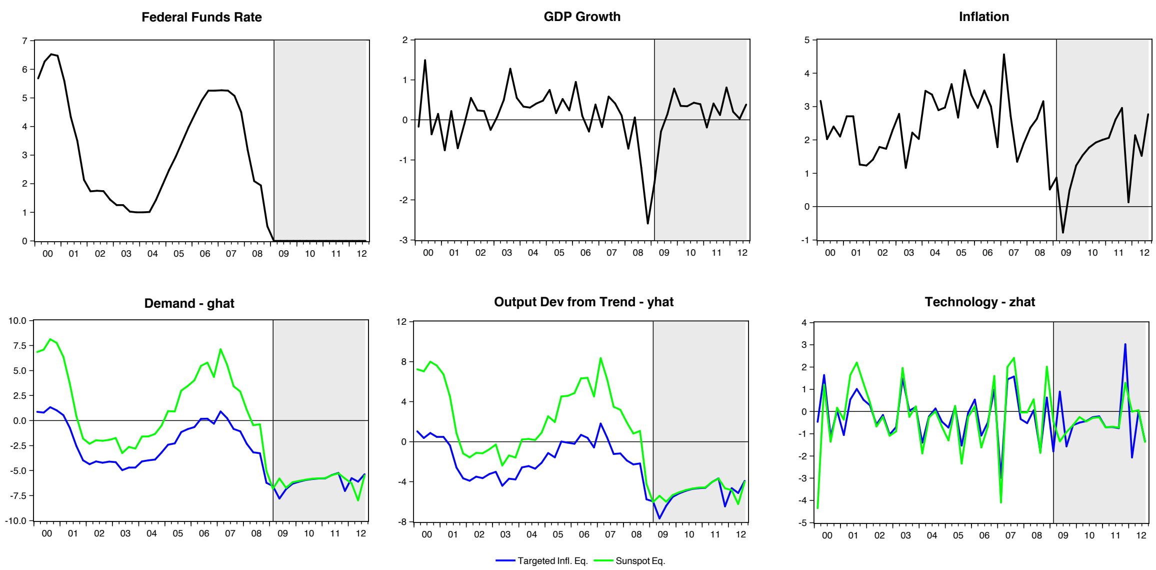
$$\begin{aligned} y_t &= \Psi(x_t) + \nu_t \\ \mathbb{P}\{s_t = 1\} &= \begin{cases} (1 - p_{00}) & \text{if } s_{t-1} = 0 \\ p_{11} & \text{if } s_{t-1} = 1 \end{cases} \\ x_t &= F_{s_t}(x_{t-1}, \epsilon_t) \end{aligned} \tag{37}$$

The first equation in (37) is the measurement equation, where $\nu_t \sim N(0, \Sigma_\nu)$ is a vector of measurement errors. The second equation represents law of motion of the Markov-switching process. The third equation corresponds to the law of motion of the continuous state variables. The vector $\epsilon_t \sim N(0, I)$ stacks the innovations $\epsilon_{z,t}$, $\epsilon_{g,t}$, and $\epsilon_{R,t}$. The functions $F_0(\cdot)$ and $F_1(\cdot)$ are generated by the model solution procedure. We set p_{00} and p_{11} equal to 0.95, which implies that the two regimes are fairly persistent. Under the targeted-inflation equilibrium the state-transition equation $x_t = F(x_{t-1}, \epsilon_t)$ is time-invariant and the Markov switching process s_t does not affect outcomes.

The state vector x_t is extracted from the observables using a particle filter, also known as sequential Monte Carlo filter. Gordon and Salmond (1993) and Kitagawa (1996) made early contributions to the development of particle filters. In the economics literature the particle filter has been applied to analyze stochastic volatility models, e.g., Pitt and Shephard (1999), and nonlinear DSGE models following Fernández-Villaverde and Rubio-Ramírez (2007). Surveys of sequential Monte Carlo filtering are provided, for instance, in the engineering literature by Arulampalam, Maskell, Gordon, and Clapp (2002) and in the econometrics literature by Giordani, Pitt, and Kohn (2011). A detailed description of the particle filter used in the subsequent quantitative analysis is provided in the Online Appendix.

Figure 5 depicts the data described in Section 5.1 (top row) and the time-path of shocks extracted conditional on the two equilibria (bottom row). Interest rates have been essentially

Figure 5: Data and Extracted Shocks



zero since 2009, output growth fell considerably in the second half of 2008 and GDP deflator inflation was below 50bp (annualized) at the beginning of 2009 as well as in late 2011 and early 2012. The filtered states as of 2009:Q1 provide the initial conditions for the subsequent policy analysis and the grey shaded area in the plots indicates the time period for which we analyze the effect of fiscal and monetary policy interventions. Under the sunspot equilibrium the economy was in the targeted-inflation regime until 2008:Q4, switched for a short period in the beginning of 2009 to the deflation regime, reverted back to the targeted-inflation regime until 2011:Q3 when another switch to the deflation regime occurred.

Output deviations from the model-implied stochastic trend mirror the path of the demand shock and are marked by a large drop in the second half of 2008, which generates the fall in output growth in the actual data. While the extracted demand shocks under the targeted-inflation equilibrium and the targeted-inflation regime are qualitatively similar, they are quantitatively different. As evident from Figure 3, holding parameters fixed the mean of the ergodic distribution under the targeted-inflation equilibrium is slightly different from the corresponding regime-conditional mean under the sunspot equilibrium. Likewise, conditioning on the same shocks the time paths of output, inflation, and interest rates in the two equilibria are also different. After 2008, the sunspot equilibrium requires slightly smaller shocks than the targeted-inflation equilibrium to rationalize the observations as the regime switches in the sunspot equilibrium provide additional flexibility to explain this episode of low interest and inflation rates.

5.4 Policy Experiments

Taking the filtered states from 2008:Q4 ($t = T_*$) as given, we now study the effects of policy interventions during the Great Recession. In particular we consider an increase in government spending that is potentially combined with an expansionary monetary policy. To make the experiment more realistic, the fiscal policy intervention is calibrated to a portion of the American Recovery and Reinvestment Act (ARRA) of February 2009. ARRA consisted of a combination of tax cuts and benefits; entitlement programs; and funding for federal contracts, grants, and loans. We focus on the third component because it can be interpreted as an increase in g_t. We will model the ARRA spending as a one-period shock δ^{ARRA} to the demand shock process, where $\delta^{ARRA} = 0.011$. Since $\hat{g}_t$ is serially correlated the effect of the shock in the h'th period on the level of $\hat{g}_t$ is given by $\rho_g^{h-1}\delta^{ARRA}$. In Section F of the Online Appendix we describe how we use data on the disbursement of ARRA funds to determine δ^{ARRA} for our model. While the actual path of the received funds is not perfectly monotone, the calibrated intervention in the DSGE model roughly matches the actual intervention both in terms of magnitude and decay rate.

We consider ex-ante and ex-post policy exercises. For the ex-ante analysis we simulate the model economy forward with and without policy intervention. Along the baseline path the demand shock evolves according to

$$\hat{g}_t = \rho_g \hat{g}_{t-1} + \sigma_g \epsilon_{g,t},$$

whereas along the post-intervention path the demand shock is given by

$$\hat{g}_t^I = \hat{g}_t + \rho_g^{t-T_*-1}\delta^{ARRA}. \tag{38}$$

For the ex-post policy analysis we use the particle filter to obtain estimates of the exogenous shock processes for the years 2009 and 2010. Since the actual path of the demand shock contains the effect of fiscal expansion we define a counterfactual path as

$$\hat{g}^C_{t|t} = \hat{g}_{t|t} - \rho_g^{t-T_*-1}\delta^{ARRA}, \tag{39}$$

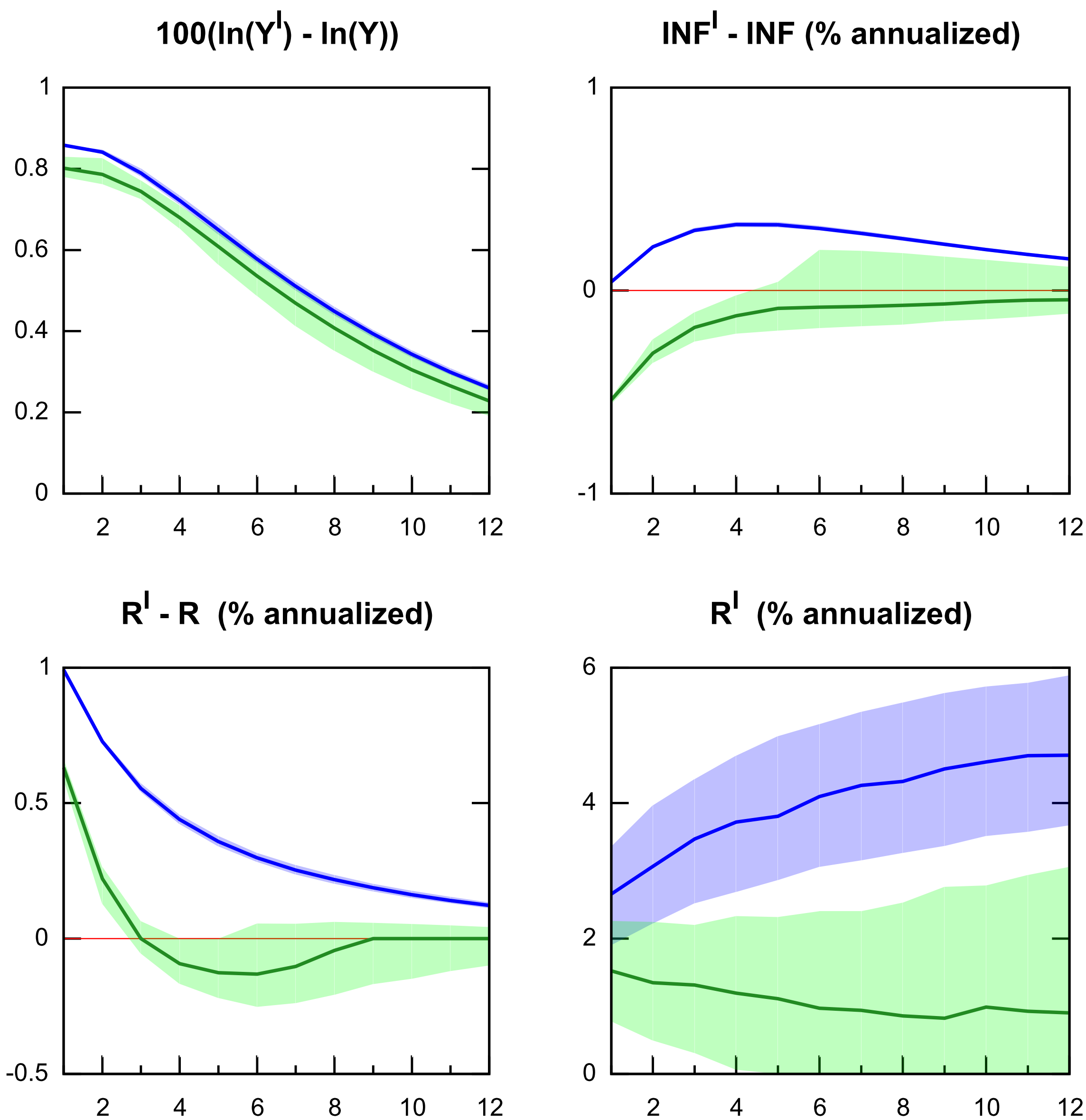
where $\hat{g}_{t|t}$ denotes the filtered demand shock.

Fiscal Policy Intervention (Ex Ante). The effect of the fiscal intervention is computed with an algorithm that is similar to the one used to construct the impulse response functions in Section 5.2. The main difference is that we start from the filtered values of $(R_{T_*}, y_{T_*}, z_{T_*}, g_{T_*})$ and s_{T_*+1} computed with the particle filter.

Figure 6 overlays the effects of the fiscal expansion for the targeted-inflation equilibrium and the sunspot equilibrium. Recall that these effects are computed as the differences of output, inflation, and interest rates along the simulated intervention paths, denoted by an I superscript, and the corresponding baseline paths. The figure shows (pointwise) median responses as well as upper and lower 20% percentiles of the distribution of the intervention effects $X^I - X$. The effects in the targeted-inflation equilibrium mirror a "standard" response to a government spending shock in a New Keynesian DSGE model. Both output and inflation increase and in response the central bank raises interest rates. Output increases by 80bp and monotonically reverts back to the no-intervention level, whereas the response of inflation is hump-shaped and peaks at about 30bp. Since the nonlinearities under the targeted-inflation equilibrium are, weak the bands that characterize the distribution of responses are very narrow.

Since we are conditioning on the filtered $s_{T_*+1} = 1$ for the sunspot equilibrium, agents expect the economy to be in the deflation regime for the subsequent periods. The local dynamics in the sunspot regime differ from the targeted-inflation regime, in part because downward-adjustments of the nominal interest rates are constrained by the ZLB and because the marginal resource costs of price adjustments are larger given the convex adjustment cost schedule. As a result the interest rate response is only 50bp and the fiscal intervention causes inflation to fall. This response is consistent with the simulated paths depicted in Figure 3 which shows that inflation rates in the targeted-inflation and sunspot equilibrium move in opposite directions whenever the sunspot shock signals the deflation regime. The lower right

Figure 6: Fiscal Policy Intervention in Targeted-Inflation and Sunspot Equilibrium



Notes: Figure compares intervention effects from targeted-inflation equilibrium (blue) and sunspot equilibrium (green): pointwise medians (solid); 20%-80% percentiles (shaded area).

panel of Figure 6 shows that due to the intervention and the resulting rise in interest rates the economy moves temporarily away from the ZLB but slowly reverts back to it over the next six quarters as the deflation regime is persisting with high probability.

Based on the impulse response functions we can calculate government spending multipli-

Table 2: Multipliers

Intervention	Targeted-Inflation			Sunspot			
	1Q	4Q	8Q	s_t Path	1Q	4Q	8Q
Ex Ante Policy Analysis – Conditional on 2008:Q4 States							
Fiscal	0.80	0.91	0.97		0.76	0.86	0.91
Fiscal + Monetary	1.22	1.71	2.16		0.99	1.22	1.37
Ex Post Policy Analysis – Conditional on 2009:Q1 - 2011:Q4 Shocks							
Fiscal	1.44	1.35	1.28	01111111	0.54	1.06	1.09
				00000000	0.54	0.72	0.75
Fiscal + Monetary	1.44	1.52	1.96	01111111	0.54	1.20	1.68
				00000000	0.54	0.78	1.29

ers that measure the effect of the fiscal intervention on output relative to the overall increase in government spending. We consider a multiplier defined as

$$\mu = \frac{\sum_{\tau=1}^{H}(Y_\tau^I - Y_\tau)}{\sum_{\tau=1}^{H}(G_\tau^I - G_\tau)}.$$

Our measure is cumulative over the lifetime of the intervention and tabulated for various types of policy interventions in Table 2. The multipliers for the ex-ante policy exercise underlying the impulse response functions in Figure 6 are reported in the fourth row of the table, labeled "Fiscal." Under the targeted-inflation equilibrium the multipliers range from 0.8 ($H = 1$) to 0.97 ($H = 8$). For the sunspot equilibrium the multipliers are slightly smaller, ranging from 0.76 to 0.91.

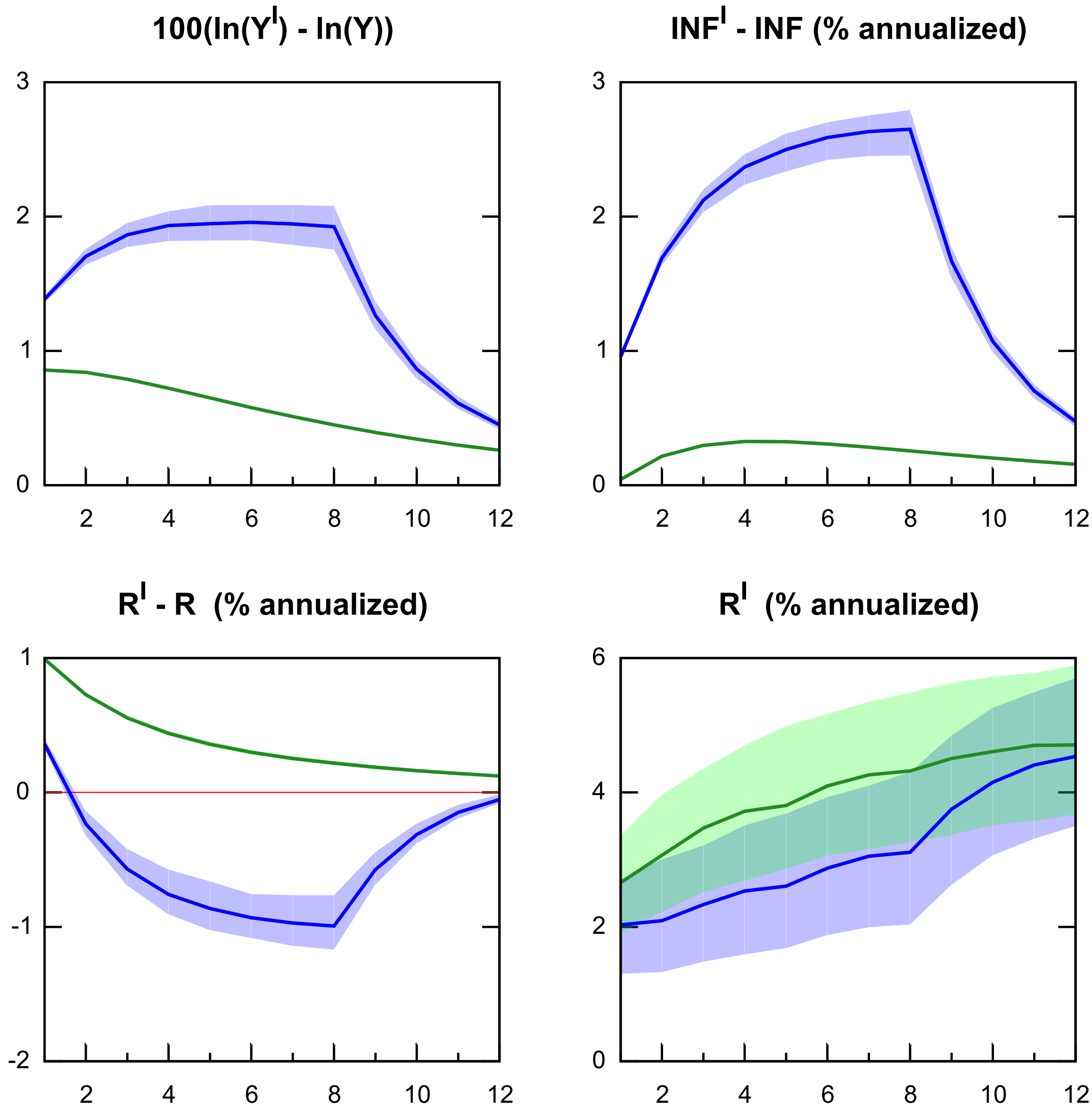
Combined Fiscal and Monetary Policy Intervention (Ex Ante). Since a fiscal expansion creates an upward pressure on the nominal interest rates via the feedback mechanism of the interest rate rule, in principle there is scope for amplifying the effect of the fiscal stimulus by a monetary policy that keeps interest rates near zero. Thus, we now consider a combination of expansionary fiscal and monetary policy. The central bank intervention is assumed to last for eight quarters and is implemented using a sequence of unanticipated monetary policy shocks $\epsilon_{R,t}$. A detailed discussion about the advantages and disadvantages of using unanticipated versus anticipated monetary policy shocks to generate predictions conditional on an interest rate path is provided in Del Negro and Schorfheide (2012). We consider a monetary intervention that is designed to keep interest rates at or near zero for eight quarters.

The DSGE model has the implication that along some of the trajectories interest rates quickly rise. In this case, keeping them at zero through an anticipated monetary policy shock corresponds to an implausibly large intervention. To avoid implausibly large interventions, we constrain the size of the monetary shocks as follows. We choose the shocks such that the difference between the interest rate the obtains with monetary policy intervention does not fall by more than 100bp below the interest rate that would obtain in the absence of a monetary intervention. Thus, we implicitly assume that the FOMC would reneg on a policy to keep interest rates near zero for an extended period of time in states of the world in which output growth and or inflation turn out to be high. The sequence of monetary policy shocks to achieve the ZLB is computed for each simulated trajectory separately. Details on the algorithm to compute the effect of the policy intervention are reported in Appendix C.

Results for the targeted inflation equilibrium are shown in Figure 7. We we overlay the responses to a pure fiscal stimulus, discussed previously. In the targeted-inflation equilibrium interest rates tend to rise fast without the monetary intervention because steady state interest rates are high. Thus, there is a lot of scope for monetary policy interventions because the ZLB poses hardly a constraint. The central bank is able to reduce the interest rate by about 70bp in the first period of the intervention (relative to just the fiscal intervention), reaching about a 150*bp* decrease by the fourth period. This very large monetary intervention leads to an extra 1.5% increase in output and an almost 2% increase in inflation.

Impulse responses for a combined fiscal and monetary stimulus under the sunspot equilibrium are depicted in Figure 8. The median interest rate with just the fiscal intervention is about 1.5% on impact. After two periods, the monetary intervention is able to reduce the interest rate all the way to the ZLB for almost all of the simulated trajectories and keep it there for a total of seven quarters. As a result, the output response increases from about 80bp on impact to about 120bp. This increase persists for the duration of the monetary intervention. After the eight quarter period, however, the additional effect from the

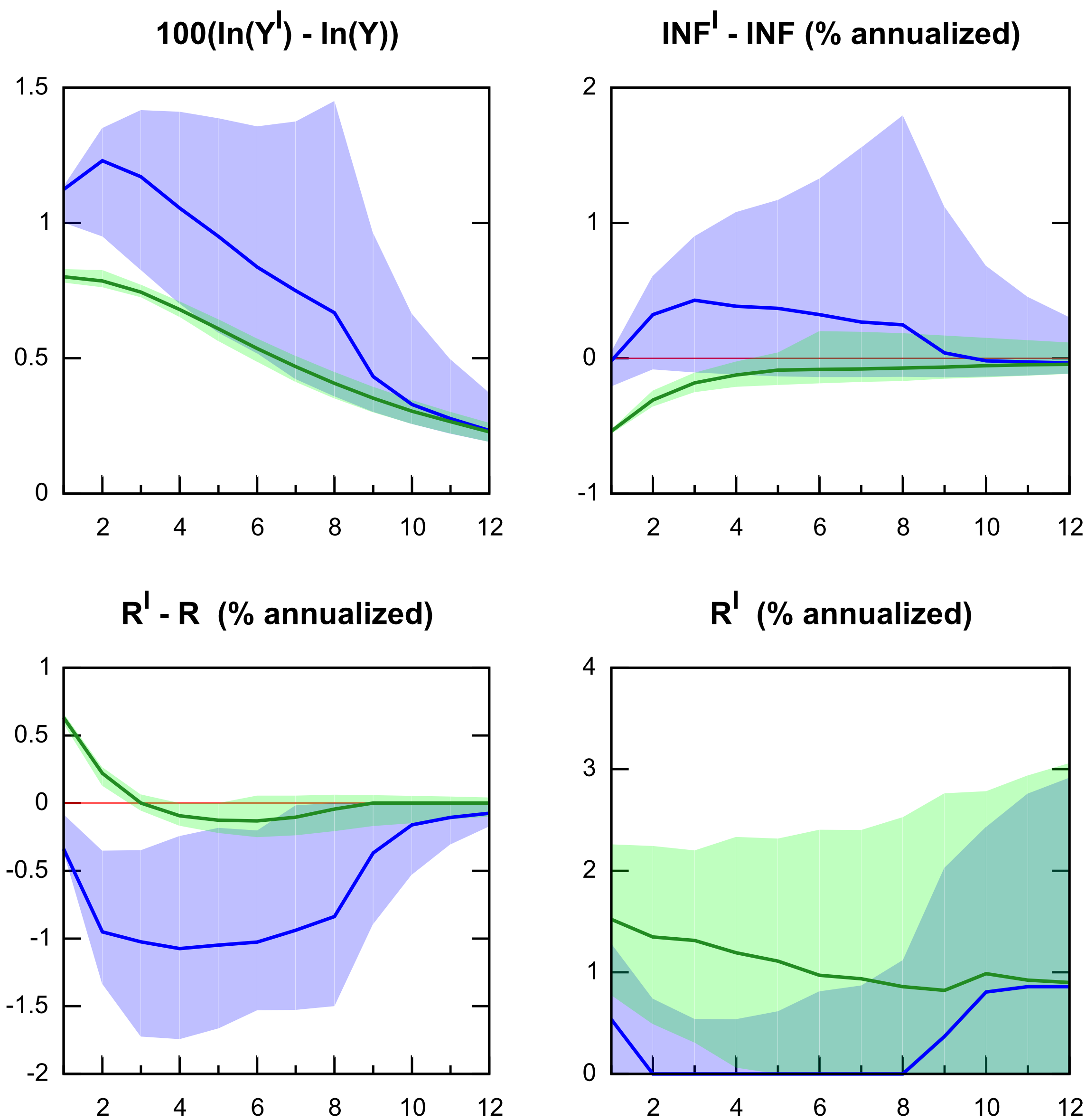
Figure 7: Fiscal and Monetary Policy Intervention in Targeted-Inflation Equilibrium



Notes: Figure depicts pointwise medians (solid); 20%-80% percentiles (shaded area) for fiscal intervention (green) and combined fiscal and monetary intervention (blue).

expansionary monetary intervention dies out quickly. Compared to the targeted-inflation equilibrium the additional boost in output due to the expansionary monetary policy that keeps interest rates near the ZLB is smaller in the sunspot equilibrium because the baseline interest rates are substantially lower. Thus, the scope for stimulating the economy is much

Figure 8: Fiscal and Monetary Policy Intervention in Sunspot Equilibrium



Notes: Figure depicts pointwise medians (solid); 20%-80% percentiles (shaded area) for fiscal intervention (green) and combined fiscal and monetary intervention (blue).

smaller. The spending multipliers, reported in Table 2, for the combined policy intervention range from 0.99 to 1.37 in the sunspot equilibrium, whereas they increase to 1.22 to 2.16 in the targeted-inflation equilibrium.

Ex-Post Policy Analysis. So far, we took an ex-ante perspective in that we simulated the path of the exogenous shocks from 2009:Q1 onwards. Now, we will take an ex-post perspective and condition on the filtered path of the exogenous processes. Since the actual data in 2009-2011 contain the ARRA intervention, we reverse the calculation of the counterfactual path as follows. Rather than adding $(\rho_g^{t-T_*-1}\delta^{ARRA})$ to the filtered $\hat{g}_t$ as in (39), we subtract $(\rho_g^{t-T_*-1}\delta^{ARRA})$ to construct a counterfactual no-intervention path. We can then calculate ex-post government spending multipliers based on the actual and counterfactual level of output.

The resulting multipliers are tabulated in the bottom panel of Table 2. Under the scenario labeled "Fiscal" we simply remove the ARRA intervention from the filtered $\hat{g}_t$ process. Under the targeted-inflation equilibrium, this leads to ex-post multipliers ranging from 1.44 (1 Quarter) to 1.28 (8 Quarters). The ex-post multipliers are larger than the ex-ante multipliers because the filtered shocks pushed the economy toward the ZLB. Once the economy is at the ZLB, the expansionary fiscal policy is less likely to be accompanied by a rise in interested because the feedback portion of the policy rule tends to predict negative interest rates. Without a rising nominal interest rate, real rates tend to be lower and stimulate current-period demand which amplifies the positive effect on output. This mechanism operates very strongly in, for instance, Christiano, Eichenbaum, and Rebelo (2011). While our ex-post analysis does deliver fiscal multipliers that are greater than one, they never exceed 1.5.

Conditional on the sunspot equilibrium, the ex-post fiscal multipliers are considerably smaller. Since there is uncertainty about the realization of the sunspot shock, we consider two extreme scenarios for the s_t process: one in which the economy was in the deflation regime from 2009 to 2011 (denoted by 00000000 in Table 2) and another one in which the economy switched back to the targeted-inflation regime in 2009:Q2. Under the former scenario the cumulative multiplier rises from 0.54 to 0.75, whereas under the latter scenario it rises to 1.09 since in the targeted-inflation regime the stimulating effect of an increase in government spending is larger than in the sunspot regime.

To assess the effect of the combined fiscal and monetary policy interventions we set the filtered monetary policy shocks, which are negative over the period from 2009 to 2011, equal to zero on the counterfactual path. Removing these monetary policy shocks tends to increase output along the counterfactual path and raises the policy multipliers. For instance, in the targeted inflation equilibrium, the multiplier increases from 1.35 to 1.52 after four quarters, whereas it rises from 1.28 to 1.96 after eight quarters. Conditional on the sunspot regime and assuming that the economy was in the targeted-inflation regime after 2009:Q2 the ex-post monetary policy effect is similar, whereas it is lower under the assumption that the economy was in the deflation regime throughout 2009 and 2010.

6 Conclusion

We solve a small-scale New Keynesian DSGE model subject to a ZLB constraint on nominal interest rates, considering three equilibria: the standard targeted-inflation equilibrium, a minimum-state-variable deflation equilibrium, and a sunspot equilibrium. We study the characteristics of these three equilibria. In terms of its ability to fit U.S. output growth, inflation, and interest rate data between 2000 and 2009, the deflation equilibrium is the least compelling, and the targeted-inflation and sunspot equilibria are equally compelling. However, the policy implications are different from both an ex-ante and ex-post perspective. The scope for fiscal stimulus and expansionary monetary policy is much smaller under the sunspot equilibrium view of the world.

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Appendix to "Macroeconomic Dynamics Near the ZLB: A Tale of Two Equilibria"

A Solving the Two-Equation Model

The model is characterized by the nonlinear difference equation

$$\mathbb{E}_t[\pi_{t+1}] = \max\left\{\frac{1}{r},\ \pi_*\left(\frac{\pi_t}{\pi_*}\right)^{\psi}\exp[\epsilon_t]\right\}. \tag{A.1}$$

We assume that $r\pi_* \geq 1$ and $\psi > 1$.

The Targeted-Inflation Equilibrium and Deflation Equilibrium. Consider a solution to (A.1) that takes the following form

$$\pi_t = \pi_*\gamma\exp[\lambda\epsilon_t]. \tag{A.2}$$

We now determine values of γ and λ such that (A.1) is satisfied. We begin by calculating the following expectation

$$\begin{aligned}
\mathbb{E}_t[\pi_{t+1}] &= \pi_*\gamma\frac{1}{\sqrt{2\pi\sigma^2}}\int\exp[\lambda\epsilon]\exp\left[-\frac{1}{2\sigma^2}\epsilon^2\right]d\epsilon \\
&= \pi_*\gamma\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[\frac{1}{2}\lambda^2\sigma^2\right]\int\exp\left[-\frac{1}{2\sigma^2}(\epsilon-\lambda\sigma^2)^2\right]d\epsilon \\
&= \pi_*\gamma\exp\left[\frac{1}{2}\lambda^2\sigma^2\right].
\end{aligned}$$

Combining this expression with (A.1) yields

$$\gamma\exp[\lambda^2\sigma^2/2] = \max\left\{\frac{1}{r\pi_*},\ \gamma^{\psi}\exp[(\psi\lambda+1)\epsilon_t]\right\}. \tag{A.3}$$

By choosing $\lambda = -1/\psi$ we ensure that the right-hand-side of (A.3) is always constant. Thus, (A.3) reduces to

$$\gamma\exp[\sigma^2/(2\psi^2)] = \max\left\{\frac{1}{r\pi_*},\ \gamma^{\psi}\right\} \tag{A.4}$$

Depending on whether the nominal interest rate is at the ZLB ($R_t = 1$) or not, we obtain two solutions for γ by equating the left-hand-side of (A.4) with either the first or the second term in the max operator:

$$\gamma_D = \frac{1}{r\pi_*}\exp\left[-\frac{\sigma^2}{2\psi^2}\right] \quad \text{and} \quad \gamma_* = \exp\left[\frac{\sigma^2}{2(\psi-1)\psi^2}\right]. \tag{A.5}$$

The derivation is completed by noting that

$$\begin{aligned}\gamma_D^\psi &= \frac{1}{r\pi_*}\exp\left[-\frac{\sigma^2}{2\psi}\right] \le \frac{1}{r\pi_*}\\ \gamma_*^\psi &= \exp\left[\frac{\sigma^2}{2(\psi-1)\psi}\right] \ge 1 \ge \frac{1}{r\pi_*}.\end{aligned}$$

A Sunspot Equilibrium. Let $s_t \in \{0,1\}$ denote the Markov-switching sunspot process. Assume that the system is in the targeted-inflation regime if $s_t = 1$ and that it is in the deflation regime if $s_t = 0$ (the 0 is used to indicate that the system is near the ZLB). The probabilities of staying in state 0 and 1, respectively, are denoted by ψ_{00} and ψ_{11}. We conjecture that the inflation dynamics follow the process

$$\pi_t^{(s)} = \pi_*\gamma(s_t)\exp[-\epsilon_t/\psi] \tag{A.6}$$

In this case condition (A.4) turns into

$$\begin{aligned}\mathbb{E}_t[\pi_{t+1}|s_t=0]/\pi_* &= \big(\psi_{00}\gamma(0)+(1-\psi_{00})\gamma(1)\big)\exp[\sigma^2/(2\psi^2)] = \frac{1}{r\pi_*}\\ \mathbb{E}_t[\pi_{t+1}|s_t=1]/\pi_* &= \big(\psi_{11}\gamma(1)+(1-\psi_{11})\gamma(0)\big)\exp[\sigma^2/(2\psi^2)] = [\gamma(1)]^\psi.\end{aligned}$$

This system of two equations can be solved for $\gamma(0)$ and $\gamma(1)$ as a function of the Markov-transition probabilities ψ_{00} and ψ_{11}. Then (A.6) is a stable solution of (A.1) provided that

$$[\gamma(0)]^\psi \le \frac{1}{r\pi_*} \quad \text{and} \quad [\gamma(1)]^\psi \ge \frac{1}{r\pi_*}.$$

Sunspot Shock is Correlated with Fundamentals. As before, let $s_t \in \{0,1\}$ be a Markov-switching sunspot process. However, now assume that a state transition is triggered by certain realizations of the monetary policy shock ϵ_t. In particular, if $s_t = 0$, then suppose $s_{t+1} = 0$ whenever $\epsilon_{t+1} \le \underline{\epsilon}_0$, such that

$$\psi_{00} = \Phi(\underline{\epsilon}_0),$$

where $\Phi(\cdot)$ is the cumulative density function of a $N(0,1)$. Likewise, if $s_t = 1$, then let $s_{t+1} = 0$ whenever $\epsilon_{t+1} > \underline{\epsilon}_0$, such that

$$\psi_{11} = 1 - \Phi(\underline{\epsilon}_1).$$

To find the constants $\gamma(0)$ and $\gamma(1)$, we need to evaluate

$$\begin{aligned}&\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\underline{\epsilon}} \exp\left[-\frac{1}{2\sigma^2}(\epsilon+\sigma^2/\psi)^2\right] d\epsilon \\ &= \mathbb{P}\left\{\frac{\epsilon+\sigma^2/\psi}{\sigma} \leq \frac{\underline{\epsilon}+\sigma^2/\psi}{\sigma}\right\} = \Phi\left(\frac{\underline{\epsilon}+\sigma^2/\psi}{\sigma}\right).\end{aligned}$$

Thus, condition (A.4) turns into

$$\begin{aligned}\frac{1}{r\pi_*} &= \left[\gamma(0)\Phi(\underline{\epsilon}_0)\Phi\left(\frac{\underline{\epsilon}_0+\sigma^2/\psi}{\sigma}\right) + \gamma(1)(1-\Phi(\underline{\epsilon}_0))\left(1-\Phi\left(\frac{\underline{\epsilon}_0+\sigma^2/\psi}{\sigma}\right)\right)\right]\exp[\sigma^2/(2\psi^2)] \\ \gamma^{\psi}(1) &= \left[\gamma(1)(1-\Phi(\underline{\epsilon}_1))\left(1-\Phi\left(\frac{\underline{\epsilon}_1+\sigma^2/\psi}{\sigma}\right)\right) + \gamma(0)\Phi(\underline{\epsilon}_1)\Phi\left(\frac{\underline{\epsilon}_1+\sigma^2/\psi}{\sigma}\right)\right]\exp[\sigma^2/(2\psi^2)].\end{aligned}$$

This system of two equations can be solved for $\gamma(0)$ and $\gamma(1)$ as a function of the thresholds $\underline{\epsilon}_0$ and $\underline{\epsilon}_1$. Then (A.6) is a stable solution of (A.1) provided that

$$[\gamma(0)]^{\psi} \leq \frac{1}{r\pi_*} \quad \text{and} \quad [\gamma(1)]^{\psi} \geq \frac{1}{r\pi_*}.$$

Benhabib, Schmitt-Grohé, and Uribe (2001a) Dynamics. BSGU constructed equilibria in which the economy transitioned from the targeted-inflation equilibrium to the deflation equilibrium. Consider the following law of motion for inflation

$$\pi_t^{(BGSU)} = \pi_*\gamma_* \exp[-\epsilon_t/\psi]\exp\big[-\psi^{t-t_0}\big]. \tag{A.7}$$

Here, γ_* was defined in (A.5) and $-t_0$ can be viewed as the initialization period for the inflation process. We need to verify that $\pi_t^{(BGSU)}$ satisfies (A.1). From the derivations that lead to (A.4) we deduce that

$$\gamma_*\mathbb{E}_{t+1}\big[\exp[-\epsilon_{t+1}/\psi]\big] = \gamma_*^{\psi}.$$

Since

$$\exp\big[-\psi^{t+1-t_0}\big] = \big(\exp\big[-\psi^{t-t_0}\big]\big)^{\psi},$$

we deduce that the law of motion for $\pi_t^{(BGSU)}$ in (A.7) satisfies the relationship

$$\mathbb{E}_t[\pi_{t+1}] = \pi_*\left(\frac{\pi_t}{\pi_*}\right)^{\psi}\exp[\epsilon_t].$$

Moreover, since $\psi > 1$ the term $\exp\big[-\psi^{t-t_0}\psi\big] \longrightarrow 0$ as $t \longrightarrow \infty$. Thus, the economy will move away from the targeted-inflation equilibrium and at some suitably defined t_* reach the deflation equilibrium and remain there permanently. Overall the inflation dynamics take the form

$$\pi_t = \pi_* \begin{cases} \gamma_* \exp[-\epsilon_t/\psi] \exp\big[-\psi^{t-t_0}\big] & \text{if } t \leq t_* \\ \gamma_D \exp[-\epsilon_t/\psi] & \text{otherwise} \end{cases}, \tag{A.8}$$

where γ_* and γ_D were defined in (A.5).

Alternative Deflation Equilibria. Around the deflation steady state the system is locally indeterminate. This suggests that we might be able to construct alternative solutions to (A.1). Consider the following conjecture for inflation

$$\pi_t = \pi_* \gamma \min\big\{ \exp[-c/\psi], \exp[-\epsilon/\psi] \big\}, \tag{A.9}$$

where c is a cutoff value. The intuition for this solution is the following. Large positive shocks ϵ that could push the nominal interest rate above one, are off-set by downward movements in inflation. Negative shocks do not need to be off-set, because they push the desired gross interest rate below one and the max operator in the policy rule keeps the interest rate at one. Formally, we can compute the expected value of inflation as follows:

$$\begin{aligned} \mathbb{E}_t[\pi_{t+1}] &= \pi_* \gamma \left[\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{c} \exp[-c/\psi] \exp\left[-\frac{1}{2\sigma^2}\epsilon^2 \right] d\epsilon \right. \\ & \quad \left. \frac{1}{\sqrt{2\pi\sigma^2}} \int_{c}^{\infty} \exp[-\epsilon/\psi] \exp\left[-\frac{1}{2\sigma^2}\epsilon^2 \right] d\epsilon \right] \\ &= \pi_* \gamma \left[\exp[-c/\psi]\Phi(c/\sigma) + \exp\left[\frac{\sigma^2}{2\psi^2}\right] \int_c^\infty \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(\epsilon + \sigma^2/\psi)^2 \right] d\epsilon \right] \\ &= \pi_* \gamma \left[\exp[-c/\psi]\Phi(c/\sigma) + \exp\left[\frac{\sigma^2}{2\psi^2}\right] \left(1 - \Phi\left(\frac{c}{\sigma} + \frac{\sigma}{\psi} \right) \right) \right] \end{aligned} \tag{A.10}$$

Here $\Phi(\cdot)$ denotes the cdf of a standard Normal random variable. Now define

$$f(c, \psi, \sigma) = \left[\exp[-c/\psi]\Phi(c/\sigma) + \exp\left[\frac{\sigma^2}{2\psi^2}\right] \left(1 - \Phi\left(\frac{c}{\sigma} + \frac{\sigma}{\psi} \right) \right) \right].$$

Then another solution for which interest rates stay at the ZLB is given by

$$\bar{\gamma} = \frac{1}{r_* \pi_* f(c, \psi, \sigma)}$$

It can be verified that for c small enough the condition

$$\frac{1}{r_* \pi_*} \geq \bar{\gamma}^{\psi} \min\left\{ \exp[-c + \epsilon], 1 \right\}$$

is satisfied.

B Model Solution

The equilibrium conditions (in terms of detrended variables, i.e., $c_t = C_t/A_t$ and $y_t = Y_t/A_t$) take the form

$$
\begin{aligned}
1 &= \beta \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\tau} \frac{1}{\gamma z_{t+1}} \frac{R_t}{\pi_{t+1}} \right] && \text{(A.11)} \\
1 &= \frac{1}{\nu} (1 - c_t^\tau) + \phi(\pi_t - \bar{\pi}) \left[\left(1 - \frac{1}{2\nu} \right) \pi_t + \frac{\bar{\pi}}{2\nu} \right] && \text{(A.12)} \\
&\quad - \phi \beta \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\tau} \frac{y_{t+1}}{y_t} (\pi_{t+1} - \bar{\pi}) \pi_{t+1} \right] \\
c_t &= \left[\frac{1}{g_t} - \frac{\phi}{2} (\pi_t - \bar{\pi})^2 \right] y_t && \text{(A.13)} \\
R_t &= \max \left\{ 1, \left[r\pi_* \left(\frac{\pi_t}{\pi_*} \right)^{\psi_1} \left(\frac{y_t}{y_{t-1}} z_t \right)^{\psi_2} \right]^{1-\rho_R} R_{t-1}^{\rho_R} e^{\sigma_R \epsilon_{R,t}} \right\}. && \text{(A.14)}
\end{aligned}
$$

B.1 Approximation Near the Targeted-Inflation Steady State

Steady State. Steady state inflation equals π_*. Let $\lambda = \nu(1-\beta)$, then

$$
\begin{aligned}
r &= \gamma/\beta \\
R_* &= r\pi_* \\
c_* &= \left[1 - v - \frac{\phi}{2}(1-2\lambda) \left(\pi_* - \frac{1-\lambda}{1-2\lambda} \bar{\pi} \right)^2 + \frac{\phi}{2} \frac{\lambda^2}{1-2\lambda} \bar{\pi}^2 \right]^{1/\tau} \\
y_* &= \frac{c_*}{\left[\frac{1}{g_*} - \frac{\phi}{2}(\pi_* - \bar{\pi})^2 \right]}.
\end{aligned}
$$

Log-linearization. We omit the hats from variables that capture deviations from the targeted-inflation steady state. The linearized consumption Euler equation (A.11) is

$$
c_t = \mathbb{E}_t[c_{t+1}] - \frac{1}{\tau}(R_t - \mathbb{E}_t[\pi_{t+1} + z_{t+1}]).
$$

The price setting equation (A.12) takes the form

$$
\begin{aligned}
0 &= -\frac{\tau c_*^\tau}{\nu} c_t + \phi \pi_* \left[\left(1 - \frac{1}{2\nu} \right) \pi_* + \frac{\bar{\pi}}{2\nu} \right] \pi_t + \phi \pi_* (\pi_* - \bar{\pi}) \left(1 - \frac{1}{2\nu} \right) \pi_t \\
&\quad - \phi \beta \pi_* (\pi_* - \bar{\pi}) \Big(\tau c_t - y_t - \mathbb{E}_t[\tau c_{t+1} - y_{t+1}] + \mathbb{E}[\pi_{t+1}] \Big) - \phi \beta \pi_*^2 \mathbb{E}_t[\pi_{t+1}].
\end{aligned}
$$

Log-linearizing the aggregate resource constraint (A.12) yields

$$c_t = y_t - \frac{1/g_*}{1/g_* - \phi(\pi_* - \bar{\pi})^2} g_t - \frac{\phi \pi_*(\pi_* - \bar{\pi})}{1/g_* - \phi(\pi_* - \bar{\pi})^2} \pi_t$$

Finally, the monetary policy rule becomes

$$R_t = \max\left\{ -\ln(r\pi_*),\ (1-\rho_R)\psi_1 \pi_t + (1-\rho_R)\psi_2(y_t - y_{t-1} + z_t) + \rho R_{t-1} + \sigma_R \epsilon_{R,t} \right\}.$$

B.2 Approximation Near the Deflation Steady State

Steady State. As before, let $\lambda = \nu(1-\beta)$. The steady state nominal interest rate is $R_D = 1$ and provided that $\beta/(\gamma\pi_*) < 1$ and $\psi_1 > 1$:

$$\begin{aligned}
r &= \gamma/\beta \\
\pi_D &= \beta/\gamma \\
c_D &= \left[1 - v - \frac{\phi}{2}(1-2\lambda)\left(\pi_D - \frac{1-\lambda}{1-2\lambda}\bar{\pi}\right)^2 + \frac{\phi}{2}\frac{\lambda^2}{1-2\lambda}\bar{\pi}^2\right]^{1/\tau} \\
y_D &= \frac{c_D}{\left[\frac{1}{g_*} - \frac{\phi}{2}(\pi_D - \bar{\pi})^2\right]}.
\end{aligned}$$

Log-linearization. We omit the tildes from variables that capture deviations from the deflation steady state. The linearized consumption Euler equation (A.11) is

$$c_t = \mathbb{E}_t[c_{t+1}] - \frac{1}{\tau}(R_t - \mathbb{E}_t[\pi_{t+1} + z_{t+1}]).$$

The price setting equation (A.12) takes the form

$$\begin{aligned}
0 \;=\; & -\frac{\tau c_D^\tau}{\nu} c_t + \phi\beta\left[\left(1 - \frac{1}{2\nu}\right)\beta + \frac{\bar{\pi}}{2\nu}\right]\pi_t + \phi\beta(\beta - \bar{\pi})\left(1 - \frac{1}{2\nu}\right)\pi_t \\
& -\phi\beta^2(\beta - \bar{\pi})\Big(\tau c_t - y_t - \mathbb{E}_t[\tau c_{t+1} - y_{t+1}] + \mathbb{E}[\pi_{t+1}]\Big) - \phi\beta^3 \mathbb{E}_t[\pi_{t+1}].
\end{aligned}$$

Log-linearizing the aggregate resource constraint (A.12) yields

$$c_t = y_t - \frac{1/g_*}{1/g_* - \phi(\beta - \bar{\pi})^2} g_t - \frac{\phi\beta(\beta - \bar{\pi})}{1/g_* - \phi(\beta - \bar{\pi})^2} \pi_t$$

Finally, the monetary policy rule becomes

$$\begin{aligned}
R_t \;=\; & \max\Big\{0,\ -(1-\rho_R)\ln(r\pi_*) - (1-\rho_R)\psi_1 \ln(\pi_*/\beta) \\
& +(1-\rho_R)\psi_1\pi_t + (1-\rho_R)\psi_2(y_t - y_{t-1} + z_t) + \rho R_{t-1} + \sigma_R\epsilon_{R,t}\Big\}.
\end{aligned}$$

B.3 Targeted-Inflation Equilibrium in Simplified Model

After imposing the parameter restrictions discussed in the main text we obtain the system (omitting hats)

$$\begin{aligned} R_t &= \max\left\{ -\ln(r\pi_*),\ \psi\pi_t + \sigma_R\epsilon_{R,t}\right\} \\ c_t &= \mathbb{E}_t[c_{t+1}] - (R_t - \mathbb{E}_t[\pi_{t+1}]) \\ \pi_t &= \beta\mathbb{E}_t[\pi_{t+1}] + \kappa c_t \end{aligned} \tag{A.15}$$

Since the conjectured law of motion is *iid*, the conditional expectations of inflation and consumption equal their unconditional means which we denote by μ_π and μ_c, respectively. In turn, the Euler equation in (A.15) simplifies to the static relationship

$$c_t = -R_t + \mu_c + \mu_\pi. \tag{A.16}$$

Similarly, the Phillips curve in (A.15) becomes

$$\pi_t = \kappa c_t + \beta\mu_\pi. \tag{A.17}$$

Combining (A.16) and (A.17) yields

$$\pi_t = -\kappa R_t + (\kappa+\beta)\mu_\pi + \kappa\mu_c. \tag{A.18}$$

We now can use (A.18) to eliminate inflation from the monetary policy rule:

$$R_t = \max\left\{ -\ln(r\pi_*),\ -\kappa\psi R_t + (\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t}\right\} \tag{A.19}$$

Define

$$R_t^{(1)} = -\ln(r\pi_*) \quad \text{and} \quad R_t^{(2)} = \frac{1}{1+\kappa\psi}\left[(\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t}\right].$$

Let $\bar{\epsilon}_{R,t}$ be the value of the monetary policy shock for which $R_t = -\ln(r\pi_*)$ and the two terms in the max operator of (A.19) are equal

$$\sigma_R\bar{\epsilon}_{R,t} = -(1+\kappa\psi)\ln(r\pi_*) - (\kappa+\beta)\psi\mu_\pi - \kappa\psi\mu_c.$$

To complete the derivation of the equilibrium interest rate it is useful to distinguish the following two cases. Case (i): suppose that $\epsilon_{R,t} < \bar{\epsilon}_{R,t}$. We will verify that $R_t = R_t^{(1)}$ is consistent with (A.19). If the monetary policy shock is less than the threshold value, then

$$(\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\bar{\epsilon}_{R,t} < -(1+\kappa\psi)\ln(r\pi_*).$$

Thus,

$$-\kappa\psi R_t^{(1)} + (\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t} < -\kappa\psi R_t^{(1)} - (1+\kappa\psi)\ln(r\pi_*) = -\ln(r\pi_*),$$

which confirms that (A.19) is satisfied.

Case (ii): suppose that $\epsilon_{R,t} > \bar{\epsilon}_{R,t}$. We will verify that $R_t = R_t^{(2)}$ is consistent with (A.19). If the monetary policy shock is greater than the threshold value, then

$$(\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\bar{\epsilon}_{R,t} > -(1+\kappa\psi)\ln(r\pi_*).$$

In turn,

$$\begin{aligned}
&-\kappa\psi R_t^{(2)} + (\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t} \\
&= -\frac{\kappa\psi}{1+\kappa\psi}\Big[(\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t}\Big] + (\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t} \\
&= \frac{1}{1+\kappa\psi}\Big[(\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t}\Big] \\
&> -\ln(r\pi_*),
\end{aligned}$$

which confirms that (A.19) is satisfied.

We can now deduce that

$$R_t = \max\left\{-\ln(r\pi_*),\ \frac{1}{1+\kappa\psi}\Big[\psi(\kappa+\beta)\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t}\Big]\right\}. \tag{A.20}$$

Combining (A.16) and (A.20) yields equilibrium consumption

$$c_t = \begin{cases} \frac{1}{1+\kappa\psi}\Big[(1-\psi\beta)\mu_\pi + \mu_c - \sigma_R\epsilon_{R,t}\Big] & \text{if } R_t \geq -\ln(r\pi_*) \\ \ln(r\pi_*) + \mu_c + \mu_\pi & \text{otherwise} \end{cases}. \tag{A.21}$$

Likewise, combining (A.17) and (A.20) delivers equilibrium inflation

$$\pi_t = \begin{cases} \frac{1}{1+\kappa\psi}\Big[(\kappa+\beta)\mu_\pi + \kappa\mu_c - \kappa\sigma_R\epsilon_{R,t}\Big] & \text{if } R_t \geq -\ln(r\pi_*) \\ \kappa\ln(r\pi_*) + (\kappa+\beta)\mu_\pi + \kappa\mu_c & \text{otherwise} \end{cases}. \tag{A.22}$$

Equations (A.20), (A.21), and (A.22) appear in the main text.

If $X \sim N(\mu, \sigma^2)$ and C is a truncation constant, then

$$\mathbb{E}[X|X \geq C] = \mu + \frac{\sigma \phi_N(\alpha)}{1 - \Phi_N(\alpha)},$$

where $\alpha = (C - \mu)/\sigma$, $\phi_N(x)$ and $\Phi_N(\alpha)$ are the probability density function (pdf) and the cumulative density function (cdf) of a $N(0,1)$. Define the cutoff value

$$C = -(1 + \kappa\psi)\ln(r\pi_*) - (\kappa + \beta)\psi\mu_\pi - \kappa\psi\mu_c. \tag{A.23}$$

Using the definition of a cdf and the formula for the mean of a truncated Normal random variable, we obtain that

$$\begin{aligned}
\mathbb{P}[\epsilon_{R,t} \geq C/\sigma_R] &= 1 - \Phi_N(C_y/\sigma_R) \\
\mathbb{E}[\epsilon_{R,t} \,|\, \epsilon_{R,t} \geq C/\sigma_R] &= \frac{\sigma_R \phi_N(C/\sigma_R)}{1 - \Phi_N(C/\sigma_R)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mu_c &= \frac{1 - \Phi_N(C_y/\sigma_R)}{1 + \kappa\psi}\left[(1 - \psi\beta)\mu_\pi + \mu_c\right] - \frac{\sigma_R \phi_N(C_y/\sigma_R)}{(1 + \kappa\psi)(1 - \Phi_N(C_y/\sigma_R))} \qquad \text{(A.24)} \\
&\quad + \Phi_N(C_y/\sigma_R)\left[\ln(r\pi_*) + \mu_c + \mu_\pi\right] \\
\mu_\pi &= \frac{1 - \Phi_N(C_y/\sigma_R)}{1 + \kappa\psi}\left[(\kappa + \beta)\mu_\pi + \kappa\mu_c\right] - \frac{\kappa\sigma_R \phi_N(C_y/\sigma_R)}{(1 + \kappa\psi)(1 - \Phi_N(C_y/\sigma_R))} \qquad \text{(A.25)} \\
&\quad + \Phi_N(C_y/\sigma_R)\left[\kappa\ln(r\pi_*) + (\kappa + \beta)\mu_\pi + \kappa\mu_c\right]
\end{aligned}$$

The constants C, μ_c and μ_π can be obtained by solving the system of nonlinear equations comprised of (A.23) to (A.25).

C Computational Details

C.1 Model Solution Algorithm

Algorithm 1 (Solution Algorithm) *1. Start with a guess for Θ. For the targeted-inflation equilibrium, this guess is obtained from a linear approximation around the* *inflation target. For the deflation equilibrium, it is obtained by assuming constant decision rules at the deflation steady state. For the sunspot equilibrium it is obtained by letting the $s_t = 1$ decision rules come from the targeted-inflation equilibrium and the $s_t = 0$ decision rules come from the deflation equilibrium.*

2. *Given this guess, simulate the model for a large number of periods.*

3. *Given the simulated path, obtain the grid for the state variables over which the approximation needs to be accurate. Label these grid points as $\{\mathcal{S}_1, ..., \mathcal{S}_M\}$. For a fourth-order approximation, we use $M = 130$. For the targeted-inflation equilibrium, 79 of these grid points come from the ergodic distribution, obtained using a cluster-grid algorithm as in Judd, Maliar, and Maliar (2011). The remaining 51 come from the filtered exogenous state variables from 2000Q1 to 2012Q3.* [5]

4. *Solve for the Θ by minimizing the sum of squared residuals obtained following the steps below, using a variant of a Newton algorithm.*

   (a) *For a generic grid point $\mathcal{S}_i$, and the current value for Θ, compute $f_\pi^1(\mathcal{S}_i; \Theta)$, $f_\pi^2(\mathcal{S}_i; \Theta)$, $f_\mathcal{E}^1(\mathcal{S}_i; \Theta)$ and $f_\mathcal{E}^2(\mathcal{S}_i; \Theta)$.*

   (b) *Assume $\zeta_i \equiv I\{R(\mathcal{S}_i, \Theta) > 1\} = 1$ and compute π_i, $\mathcal{E}_i$, as well as y_i and c_i using (33) and (34), substituting in (35).*

   (c) *If R_i that follows from (35) using π_i and y_i obtained in (b) is greater than unity, then ζ_i is indeed equal to one. Otherwise, set $\zeta_i = 0$ (and thus $R_i = 1$) and recompute all other objects.*

   (d) *The final step is to compute the residual functions. There are four residuals, corresponding to the four functions being approximated. For a given set of state variables $\mathcal{S}_i$, only two of them will be relevant since we either need the constrained decision rules or the unconstrained ones. Regardless, the relevant residual func-*

[5] For the deflation equilibrium, we use a time-separated grid algorithm which suits the behavior of this equilibrium better since there are many periods where the economy is on the "edge" of the ergodic distribution at the ZLB. For the sunspot equilibrium, we use the same time-separated grid algorithm.

tions will be given by

$$\mathcal{R}^1(\mathcal{S}_i) = \mathcal{E}_i - \left[\int\int\int \frac{c(\mathcal{S}')^{-\tau}}{\gamma z' \pi(\mathcal{S}')} dF\left(z'\right) dF\left(g'\right) dF\left(\epsilon_R'\right)\right] \tag{A.26}$$

$$\mathcal{R}^2(\mathcal{S}_i) = f\left(c_i, \pi_i, y_i\right) - \phi\beta \int\int\int c(\mathcal{S}')^{-\tau} y(\mathcal{S}') \left[\pi(\mathcal{S}') - \bar{\pi}\right] \pi(\mathcal{S}') dF\left(z'\right) dF\left(g'\right) dF\left(\epsilon_R'\right) \tag{A.27}$$

Note that this step involves computing $\pi(\mathcal{S}')$, $y(\mathcal{S}')$, $c(\mathcal{S}')$ *and* $R(\mathcal{S}')$ *which is done following steps (a)-(c) above for each value of* $\mathcal{S}'$. *We use a non-product monomial integration rule to evaluate these integrals.*

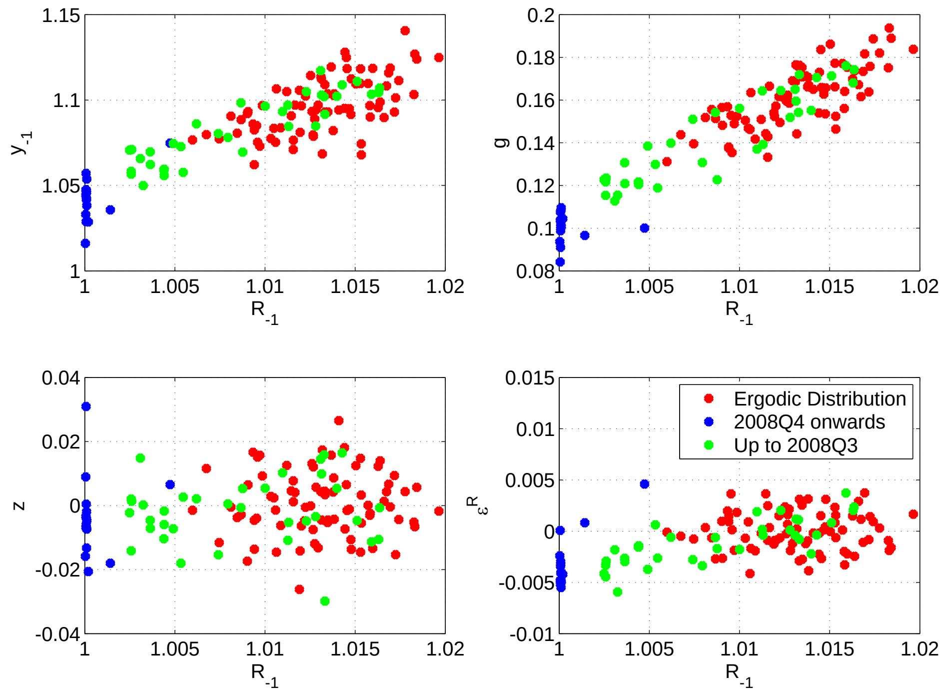
(e) *The objective function to be minimized is the sum of squared residuals obtained in (d).*

5. *Repeat steps 2-4 sufficient number of times so that the ergodic distribution remains unchanged from one iteration to the next. For the targeted-inflation equilibrium, we also iterate between solution and filtering to make sure the filtered states used in the solution grid remain unchanged.*

We start our solution from a second-order approximation and move to a third- and fourth-order approximation by using the previous solution. We use analytical derivatives of the objection function which speeds up the solution by two orders of magnitude. As a measure of accuracy, we compute the approximation errors from A.26 and A.27, converted to consumption units. For the targeted-inflation equilibrium, these are in the order of 10^{-6}. For the deflation and sunspot equilibria they are higher at 10^{-4}, but still very reasonable given the complexity of the model.

Figure A-1 shows the solution grid for the targeted-inflation equilibrium. For each panel we have R_{t-1} on the x axis and the other state variables on the y axis. The red dots are the grid points that represent the ergodic distribution, the green points are the filtered states from 2000:Q1 to 2008:Q3 and the blue points are the filtered state for the period after 2008:Q3. It is evident that the filtered states lie in the tails of the ergodic distribution of the targeted-inflation equilibrium, which assigns negligible probability to zero interest rates and the exogenous states that push interest rates toward the ZLB.

Figure A-1: Solution Grid for the Targeted-Inflation Equilibrium



C.2 Impulse Responses

Algorithm 2 (Scaled Impulse Responses) *For $j = 1$ to $j = n_{sim}$ repeat the following steps:*

1. *Draw initial values $(R_0^{(j)}, y_0^{(j)}, z_0^{(j)}, g_0^{(j)})$ from ergodic distribution.*

2. *Generate baseline trajectories based on the innovation sequence $\{\epsilon_t^{(j)}\}_{t=1}^H$, where $\epsilon_t^{(j)} = [\epsilon_{z,t}^{(j)}, \epsilon_{g,t}^{(j)}, \epsilon_{R,t}^{(j)}]' \sim N(0, I)$.*

3. *Generate counterfactual trajectories based on the innovation sequence*

$$
\begin{aligned}
\epsilon_{z,t}^{I(j)} &= \epsilon_{z,t}^{(j)} \quad \textit{and} \quad \epsilon_{g,t}^{I(j)} = \epsilon_{g,t}^{(j)} \quad \textit{for} \quad t = 1, \ldots, H; \\
\epsilon_{R,1}^{I} &= \delta + \epsilon_{R,1}; \quad \epsilon_{R,t}^{I} = \epsilon_{R,t} \quad \textit{for} \quad t = 2, \ldots, H
\end{aligned}
$$

4. *Conditional on $(R_0^{(j)}, y_0^{(j)}, z_0^{(j)}, g_0^{(j)})$ compute $\{R_t^{(j)}, y_t^{(j)}, \pi_t^{(j)}\}_{t=1}^H$ and $\{R_t^{I(j)}, y_t^{I(j)}, \pi_t^{I(j)}\}_{t=1}^H$ based on $\{\epsilon_t^{(j)}\}$ and $\{\epsilon_t^{I(j)}\}$, respectively, and for a generic variable x, let*

$$
IRF_\delta^{(j)}(x_t|\epsilon_{R,1}) = (\ln x_t^{I(j)} - \ln x_t^{(j)})/|\delta|. \tag{A.28}
$$

Compute the mean of $IRF^{(j)}_\delta(x_t|\epsilon_{R,1})$ across j:

$$IRF_\delta(x_t|\epsilon_{R,1}) = \frac{1}{n_{sim}} \sum_{j=1}^{n_{sim}} IRF^{(j)}_\delta(x_t|\epsilon_{R,1}).\square \tag{A.29}$$

or any percentile.

Algorithm 3 (Effect of Combined Fiscal and Monetary Policy Intervention) *For $j = 1$ to $j = n_{sim}$ repeat the following steps:*

1. *Initialize the simulation by setting $(R^{(j)}_0, y^{(j)}_0, z^{(j)}_0, g^{(j)}_0)$ equal to the mean estimate obtained with the particle filter.*

2. *Generate baseline trajectories based on the innovation sequence $\{\epsilon^{(j)}_t\}^H_{t=1}$ by letting $[\epsilon^{(j)}_{z,t}, \epsilon^{(j)}_{g,t}]' \sim N(0, I)$ and setting $\epsilon_{R,t} = 0$.*

3. *Generate the innovation sequence for the counterfactual trajectories according to*

$$\begin{aligned} \epsilon^{I(j)}_{g,1} &= \delta^{ARRA} + \epsilon^{(j)}_{g,1}; \quad \epsilon^{I(j)}_{g,t} = \epsilon^{(j)}_{g,t} \quad \textit{for} \quad t = 2, \ldots, H; \\ \epsilon^{I(j)}_{z,t} &= \epsilon^{(j)}_{z,t} \quad \textit{for} \quad t = 1, \ldots, H; \\ \epsilon^{I(j)}_{R,t} &= \epsilon^{(j)}_{R,t} = 0 \quad \textit{for} \quad t = 9, \ldots, H; \end{aligned}$$

   In periods $t = 1, \ldots, 8$, conditional on $\{\epsilon^{I(j)}_{g,t}, \epsilon^{I(j)}_{z,t}\}^4_{t=1}$, determine $\epsilon^{I(j)}_{R,t}$ by solving for the smallest $\tilde{\epsilon}_{R,t}$ such that it is less than $2\sigma_R$ that yields either

$$R^{I(j)}_t(\epsilon^{I(j)}_{R,t} = \tilde{\epsilon}_{R,t}) = 1 \quad \textit{or} \quad 400 \ln \left(R^{I(j)}_t(\epsilon^{I(j)}_{R,t} = 0) - R^{I(j)}_t(\epsilon^{I(j)}_{R,t} = \tilde{\epsilon}_{R,t}) \right) = 1.$$

4. *Conditional on $(R^{(j)}_0, y^{(j)}_0, z^{(j)}_0, g^{(j)}_0)$ compute $\{R^{(j)}_t, y^{(j)}_t, \pi^{(j)}_t\}^H_{t=1}$ and $\{R^{I(j)}_t, y^{I(j)}_t, \pi^{I(j)}_t\}^H_{t=1}$ based on $\{\epsilon^{(j)}_t\}$ and $\{\epsilon^{I(j)}_t\}$, respectively, and let*

$$IRF^{(j)}(x_t|\epsilon_{g,1}, \epsilon_{R,1:8}) = (\ln x^{I(j)}_t - \ln x^{(j)}_t). \tag{A.30}$$

Compute medians and percentile bands based on $IRF^{(j)}(x_t|\epsilon_{g,1}, \epsilon_{R,1:8})$, $j = 1, \ldots, n_{sim}$. $\square$

D Estimation of 2nd-Order Approximated DSGE Model

Table A-1 summarizes the prior and posterior distribution from the Bayesian estimation of the 2nd-order approximated version of the DSGE model. The estimation sample is 1984:Q1 to 2007:Q4. The parameter ϕ that is used in the main text is related to the parameter κ (Phillips curve slope of a linearized version of the DSGE model) according to $\phi = \frac{\tau(1-\nu)}{(\nu\pi^2\kappa)}$. The parameters r^*, π^*, and γ are fixed at the sample means of the ex-post real rate, the inflation rate, and output growth. We assume that $\bar{\pi} = 1$, meaning that any price change is costly.

Table A-1: Posterior Estimates for DSGE Model Parameters

		Prior		Posterior	
Parameter	Density	Para 1	Para2	Mean	90% Interval
τ	Gamma	2.00	0.25	1.50	[1.14, 1.89]
κ	Gamma	0.30	0.10	0.17	[0.05, 0.30]
ψ_1	Gamma	1.50	0.10	1.36	[1.27, 1.43]
ρ_r	Beta	0.50	0.20	0.64	[0.55, 0.72]
ρ_g	Beta	0.80	0.10	0.86	[0.82, 0.91]
ρ_z	Beta	0.20	0.10	0.11	[0.03, 0.24]
$100\sigma_r$	Inv Gamma	0.30	4.00	0.21	[0.17, 0.26]
$100\sigma_g$	Inv Gamma	0.40	4.00	0.78	[0.66, 0.93]
$100\sigma_z$	Inv Gamma	0.40	4.00	1.03	[0.83, 1.32]
$400(r^* - 1)$	Fixed	2.78			
$400(\pi^* - 1)$	Fixed	2.52			
$100(\gamma - 1)$	Fixed	0.48			
$\overline{\pi}$	Fixed	1.00			
ψ_2	Fixed	0.80			
ν	Fixed	0.10			
$\frac{1}{g}$	Fixed	0.85			

Notes: Para (1) and Para (2) list the means and the standard deviations for Beta and Gamma; and s and ν for the Inverse Gamma distribution, where $p_{\mathcal{IG}}(\sigma|\nu, s) \propto \sigma^{-\nu-1}e^{-\nu s^2/2\sigma^2}$. The effective prior is truncated at the boundary of the determinacy region. Estimation sample is 1984:Q1 to 2007:Q4. As 90% credible interval we are reporting the 5th and 95th percentile of the posterior distribution.

E Particle Filter

The particle filter is used to extract information about the state variables of the model from data on output growth, inflation, and nominal interest rates over the period 2000:Q1 to 2010:Q4. Throughout this section we focus on the particle filter for the sunspot equilibrium because it involves an additional state variable. The analysis for the targeted-inflation equilibrium and the deflation equilibrium is a special case in which the discrete state s_t is constant.

E.1 State-Space Representation

Let y_t be the 3×1 vector of observables consisting of output growth, inflation, and nominal interest rates. The vector x_t stacks the continuous state variables which are given by $x_t = [R_t, y_t, y_{t-1}, z_t, g_t, A_t]'$ and $s_t \in \{0, 1\}$ is the Markov-switching process.

$$y_t = \Psi(x_t) + \nu_t \tag{A.31}$$

$$\mathbb{P}\{s_t = 1\} = \begin{cases} (1 - p_{00}) & \text{if } s_{t-1} = 0 \\ p_{11} & \text{if } s_{t-1} = 1 \end{cases} \tag{A.32}$$

$$x_t = F_{s_t}(x_{t-1}, \epsilon_t) \tag{A.33}$$

The first equation is the measurement equation, where $\nu_t \sim N(0, \Sigma_\nu)$ is a vector of measurement errors. The second equation represents law of motion of the Markov-switching process. The third equation corresponds to the law of motion of the continuous state variables. The vector $\epsilon_t \sim N(0, I)$ stacks the innovations $\epsilon_{z,t}$, $\epsilon_{g,t}$, and $\epsilon_{R,t}$. The functions $F_0(\cdot)$ and $F_1(\cdot)$ are generated by the model solution procedure. We subsequently use the densities $p(y_t|s_t)$, $p(s_t|s_{t-1})$, and $p(x_t|x_{t-1}, s_t)$ to summarize the measurement and the state transition equations.

Let $z_t = [x_t', s_t]'$ and $Y_{t_0:t_1} = \{y_{t_0}, \dots, y_{t_1}\}$. The distribution $p(z_t|Y_{1:t})$ is approximated by a set of pairs $\{(z_t^{(i)}, \pi_t^{(i)})\}_{i=1}^N$, where $z_t^{(i)}$ is the i'th particle, $\pi_t^{(i)}$ is its weight, and N is the number of particles. The particles $z_t^{(i)}$ are generated from some proposal density and the

$\pi_t^{(i)}$'s correspond to normalized weights in an importance sampling approximation:

$$\begin{aligned}
\mathbb{E}[f(z_t)|Y_{1:t}] &= \int_{z_t} f(z_t) \frac{p(y_t|z_t)p(z_t|Y_{1:t-1})}{p(y_t|Y_{1:t-1})} dz_t \\
&= \int_{z_{t-1:t}} f(z_t) \frac{p(y_t|z_t)p(z_t|z_{t-1})p(z_{t-1}|Y_{1:t-1})}{p(y_t|Y_{1:t-1})} dz_{t-1:t} \\
&\approx \frac{\sum_{i=1}^N f(z_t^{(i)}) \left(\frac{1}{N} \frac{p(y_t|z_t^{(i)})p(z_t^{(i)}|z_{t-1}^{(i)})p(z_{t-1}^{(i)}|Y_{1:t-1})}{g(z_{t-1:t}^{(i)}|Y_{1:t})} \right)}{\sum_{j=1}^N \left(\frac{1}{N} \frac{p(y_t|z_t^{(j)})p(z_t^{(j)}|z_{t-1}^{(j)})p(z_{t-1}^{(j)}|Y_{1:t-1})}{g(z_{t-1:t}^{(j)}|Y_{1:T})} \right)} \\
&= \sum_{i=1}^N f(z_t^{(i)}) \frac{\tilde{\pi}_t^{(i)}}{\sum_{j=1}^N \tilde{\pi}_t^{(j)}} = \sum_{i=1}^N f(z_t^{(i)}) \pi_t^{(i)},
\end{aligned}$$

where the un-normalized and normalized probability weights are given by

$$\tilde{\pi}_t^{(i)} = \frac{1}{N} \frac{p(y_t|z_t^{(i)})p(z_t^{(i)}|z_{t-1}^{(i)})p(z_{t-1}^{(i)}|Y_{1:t-1})}{g(z_{t-1:t}^{(i)}|Y_{1:T})} \quad \text{and} \quad \pi_t^{(i)} = \frac{\tilde{\pi}_t^{(i)}}{\sum_{j=1}^N \tilde{\pi}_t^{(j)}},$$

respectively, and the $z_{t-1:t}^{(i)}$'s are drawn from a probability distribution with a density that is proportional to $g(z_{t-1:t}^{(i)}|Y_{1:t})$. In particular, we adopt an approach known as auxiliary particle filtering, e.g. Pitt and Shephard (1999), and consider proposal densities of the form

$$g(z_{t-1:t}|Y_{1:t}) \propto p(z_{t-1}|Y_{1:t-1})q(z_t|z_{t-1}, y_t)$$

such that

$$\tilde{\pi}_t^{(i)} = \frac{1}{N} \frac{p(y_t|z_t^{(i)})p(z_t^{(i)}|z_{t-1}^{(i)})}{q(z_t^{(i)}|z_{t-1}^{(i)}, y_t)}. \tag{A.34}$$

Since our model has discrete and continuous state variables, we write

$$p(z_t|z_{t-1}) = \begin{cases} p_0(x_t|x_{t-1}, s_t = 0)\mathbb{P}\{s_t = 0|s_{t-1}\} & \text{if } s_t = 0 \\ p_1(x_t|x_{t-1}, s_t = 1)\mathbb{P}\{s_t = 1|s_{t-1}\} & \text{if } s_t = 1 \end{cases}$$

and consider proposal densities of the form

$$q(z_t|z_{t-1}, y_t) = \begin{cases} q_0(x_t|x_{t-1}, y_t, s_t = 0)\lambda(z_{t-1}, y_t) & \text{if } s_t = 0 \\ q_1(x_t|x_{t-1}, y_t, s_t = 1)(1 - \lambda(z_{t-1}, y_t)) & \text{if } s_t = 1 \end{cases},$$

where $\lambda(x_{t-1}, y_t)$ is the probability that $s_t = 0$ under the proposal distribution.

E.2 Filtering

The particle filter generates the importance sampling approximation of $p(z_t|Y_{1:t})$ sequentially for $t = 1, \dots, T$.

Initialization. To generate the initial set of particles $\{(z_0^{(i)}, \pi_0^{(i)}\}_{i=1}^N$, for each i simulate the DSGE model for T_0 periods, starting from the targeted-inflation steady state, and set $\pi_0^{(i)} = 1/N$.

Sequential Importance Sampling. For $t = 1$ to T:

1. $\{z_{t-1}^{(i)}, \pi_{t-1}^{(i)}\}_{i=1}^N$ is the particle approximation of $p(z_{t-1}|Y_{1:t-1})$. For $i = 1$ to N:

   (a) Draw $z_t^{(i)}$ conditional on $z_{t-1}^{(i)}$ from $q(z_t|z_{t-1}^{(i)}, y_t)$.

   (b) Compute the unnormalized particle weights $\tilde{\pi}_t^{(i)}$. Period $t-1$ particles were effectively sampled with probability $1/N$ instead of drawn from the mixture

   $$\hat{p}(z_{t-1}|Y_{1:t-1}) = \sum_{j=1}^N \pi_{t-1}^{(j)} \delta(z_{t-1} - z_{t-1}^{(j)}),$$

   where $\delta(x)$ is the dirac function with the properties $\delta(0) = \infty$, $\delta(x) = 0$ if $x \neq 0$ and $\int \delta(x)dx = 1$. Thus, (A.34)needs to be adjusted by $\pi_{t-1}^{(i)}$:

   $$\tilde{\pi}_t^{(i)} = \frac{1}{N} \frac{p(y_t|z_t^{(i)})p(z_t^{(i)}|z_{t-1}^{(i)})}{q(z_t^{(i)}|z_{t-1}^{(i)}, y_t)} \pi_{t-1}^{(i)}.$$

2. Compute the normalized particle weights $\pi_t^{(i)}$ and the effective sample size $ESS_t = 1/\sum_{i=1}^N (\pi_t^{(i)})^2$.

3. Resample the particles via deterministic resampling (see Kitagawa (1996)). Reset weights to be $\pi_t^{(i)} = 1/N$ and approximate $p(z_t|Y_{1:t})$ by $\{(z_t^{(i)}, \pi_t^{(i)})\}_{i=1}^n$.

E.3 Tuning of the Filter

In the empirical analysis we set $T_0 = 50$ and $N = 1e6$.

Targeted-Inflation Equilibrium. Since the discrete state s_t is irrelevant in this equilibrium, let $z_t = x_t$. Due to the nonlinear state transition it is difficult to evaluate $p(z_t|z_{t-1})$

directly. Recall that $x_t = F(x_{t-1}, \epsilon_t)$. Let $p_\epsilon(\epsilon_t)$ be the DSGE model-implied density of the innovation distribution, which is $\epsilon_t \sim N(0, I)$. Conditional on $x_{t-1}^{(i)}$ and y_t we apply the Kalman-filter updating equations to a log-linearized version of the DSGE model to obtain a preliminary estimate $\hat{\epsilon}_{t|t}$. We then generate a draw from $\epsilon_t^{(i)} \sim N(\hat{\epsilon}_{t|t}, I)$, denoting the density associated with this distribution by $q_\epsilon(\epsilon_t)$ and let $x_t^{(i)} = F(x_{t-1}^{(i)}, \epsilon_t^{(i)})$. In slight abuse of notation (ignoring that the dimension of x_t is larger than the dimension of ϵ_t and that its distribution is singular), we can apply the change of variable formula to obtain a representation of the proposal density

$$q(x_t^{(i)}|x_{t-1}^{(i)}) = q_\epsilon(F^{-1}(x_t^{(i)}|x_{t-1}^{(i)})) \left| \frac{\partial F^{-1}(x_t^{(i)}|x_{t-1}^{(i)})}{\partial x_t} \right|$$

Using the same change-of-variable formula, we can represent

$$p(x_t^{(i)}|x_{t-1}^{(i)}) = p_\epsilon(F^{-1}(x_t^{(i)}|x_{t-1}^{(i)})) \left| \frac{\partial F^{-1}(x_t^{(i)}|x_{t-1}^{(i)})}{\partial x_t} \right|$$

By construction, the Jacobian terms cancel and the ratio that is needed to calculate the unnormalized particle weights for period t in (A.34) simplifies to

$$\frac{p(x_t^{(i)}|x_{t-1}^{(i)})}{q(x_t^{(i)}|x_{t-1}^{(i)})} = \frac{p_\epsilon(\epsilon_t^{(i)})}{q_\epsilon(\epsilon_t^{(i)})}.$$

Sunspot Equilibrium. Discuss choice of $\lambda(z_{t-1}, y_t)$.

F Calibration of the Policy Experiment

Table A-2 summarizes the award and disbursements of funds for federal contracts, grants, and loans. We translate the numbers in the table into a one-period location shift of the distribution of $\epsilon_{g,t}$. In our model total government spending is a fraction ζ_t of aggregate output, where ζ_t evolves according to an exogenous process:

$$G_t = \zeta_t Y_t; \quad \zeta_t = 1 - \frac{1}{g_t}; \quad \ln(g_t/g_*) = \rho_g \ln(g_{t-1}/g_*) + \sigma_g \epsilon_{g,t}$$

For the subsequent calibration of the fiscal intervention it is convenient to define the percentage deviations of g_t and ζ_t from their respective steady states: $\hat{g}_t = \ln(g_t/g_*)$ and

$\hat{\zeta}_t = \ln(\zeta_t/\zeta_*)$. According to the parameterization of the DSGE model in Table 1 $\zeta_* = 0.15$ and $g_* = 1.177$. Thus, government spending is approximately 15% of GDP. We assume that the fiscal expansion approximately shifts $\hat{\zeta}_t$ to $\hat{\zeta}_t^I = \hat{\zeta}_t + \hat{\zeta}_t^{ARRA}$.

We construct $\hat{\zeta}_t^{ARRA}$ as follows. Let G_t^{ARRA} correspond to the additional government spending stipulated by ARRA. Since we focus on received rather than awarded funds, G_t^{ARRA} corresponds to the third column of Table A-2. The size of the fiscal expansion as a fraction of GDP is

$$\zeta_t^{ARRA} = G_t^{ARRA}/Y_t,$$

where Y_t here corresponds to the GDP data reported in the last column of Table A-2. We then divide by ζ_* to convert it into deviations from the steady state level: $\hat{\zeta}_t^{ARRA} = \zeta_t^{ARRA}/\zeta_*$. Taking a log-linear approximation of the relationship between g_t and ζ_t leads to

$$\hat{g}_t^{ARRA} = 0.177 \cdot G_t^{ARRA}/(\zeta_* Y_t).$$

In Figure A-2 we compare $\hat{g}_t^{ARRA}$ constructed from the data in Table A-2 to $(\hat{g}_t^I - \hat{g}_t)$, where $\delta^{ARRA} = 0.011$.[6] While the actual path of the received funds is not perfectly monotone, the calibrated intervention in the DSGE model roughly matches the actual intervention both in terms of magnitude and decay rate.

[6]Recall that $\sigma_g = 0.0078$

Table A-2: ARRA Funds for Contracts, Grant, and Loans

	Awarded	Received	Nom. GDP
2009:3	158	36	3488
2009:4	17	18	3533
2010:1	26	8	3568
2010:2	16	24	3603
2010:3	33	26	3644
2010:4	9	21	3684
2011:1	4	19	3704
2011:2	4	20	3751
2011:3	8	17	3791
2011:4	0	12	3830
2012:1	3	9	3870
2012:2	0	8	3899

Source: www.recovery.org.

Figure A-2: Calibration of Fiscal Policy Intervention