Small Shocks: Equivalence(s) between State and Time Dependent Models

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- First, present a simple general Equilibrium monetary model
- Second, consider abstract setup
- General on Price Setting model (state & time dependent)
- Consider the case of *small aggregate shocks*
- Aggregation of Firms's price setting problem:

  $\rightarrow$ Integrated Phillips Curve
- **Main Result: frequency price changes stays constant.**
- Implication:

  test of state dependent using frequency must use large shocks.

Simple GE model

- Consider frictionless labor market -as in 3 equations model.
- *Study large class (state & time dependent) price setting models*
- Start at steady state w/zero inflation.
- Consider small shock (i.e. linearize model). MIT shocks that
  - Changes nominal cost once and for all
  - Path of nominal interest rates returning to steady state
  - Path of real aggregate productivity returning to steady state
- GE model has features that amplify/dampen shocks.

Example: Simple General Equilibrium Set Up

$$\text{HH Utility}: \int_0^{\infty} e^{-\rho t} \left(\frac{c(t)^{1-\epsilon_c} - 1}{1-\epsilon_c} - \frac{\beta}{1+\epsilon_l} \ell(t)^{1+\epsilon_l} + \log \frac{M(t)}{P(t)} \right) dt$$

$$\text{Kimball Aggregator } C(t) : \quad 1 = \int_0^1 \Upsilon \left(\frac{c_k(t)}{C(t)} A_k(t) dk \right)$$

- Gross output of firm k: Labor $\ell_k(t)$ & Intermediate $I_k(t)$ share α:
  $q_k(t) + c_k(t) = Z_k(t)^{-\alpha} \left[\ell_k(t)^{\alpha} I_k(t)^{1-\alpha} \right]^{\zeta} \quad \Longrightarrow$ CRTS if $\zeta = 1$
- Intermediates: $\int_0^1 I_k(t)\, dk = X(t)$ and $1 = \int_0^1 \Upsilon \left(\frac{q_k(t)}{X(t)} A_k(t) \right) dk$
- Labor: $\ell_p(t) + \int_0^1 \ell_k(t)\, dk = \ell(t)$ where $\ell_p(t)$ used in price-set.
- Random walk quality $\log Z_k(t)$: thus $d \log Z_k(t) = \sigma d\mathcal{W}(t)$ is i.i.d.
  $A_k(t) = Z_k(t)$ gives the quality interpretation –for tractability.

Amplification/dampening of firms problem in GE

- Consider the firm's price setting problem close to steady state.
- A higher average price deviation of others firms:
  1. Increases my ideal price: strategic *complementarity*

     (*amplifies* stickyness, "real rigidities", increasing persistence):
     - Final good used as Intermediate $\alpha > 0$
     - Decreasing returns to scale $\zeta < 1$.
     - Convex log demand in Kimball's aggregator
  2. Decreases my ideal price: strategic *substitutability*

     (*dampens* stickyness, decreasing persistence):
     - Finite Frisch Labor elasticity $\epsilon_L > 0$
     - Concave log demand in Kimball's aggregator

Captured by sign of coefficient on quadratic approximation of firms's profit as function of avg. price deviation of others.

Abstract Set Up

- Three State Dependent Price Setting Problems
- Aggregation and Fixed Point
- Steady State
- Impulse response for small shock, i.e. linearization.
- Statistics with **zero impulse response**
  1. Frequency of price changes
  2. Hazard rate of price changes as function of duration
  3. Cross sectional variance of markups / relative prices.
  4. Even moments of the distribution of price changes.
- Interpretation, Examples, and Implications

Common Features for Price Setting Problem

- Common discount factor ρ
- Period Profits $-F(x, X(t), S(t))$
- Idiosyncratic state $x \in \mathbb{R}$, evolves as $dx(t) = \sigma dW(t)$ if uncontrolled, W is BM independent across firms. No drift.
  - Interpretation$x$ is markup deviation.
- Path of endogenous variables $\{X(t)\}_{t\in[0,T]}$,
  - Interpretation/*main example*: $X(t)$ cross sectional average of $x's$.
  - strategic complementarity (Kimball), or decreasing returns, or final good as intermediate input.
- Path of forcing variables $\{S(t)\}_{t\in[0,T]}$
  - Interpretation/example: interest rates $\to$ nominal wages, rental capital, real shocks, etc.
- If firms can set a new price, markup goes to $x = x^*(t)$.

I. Costly Adjustment Probability

- Value function $u(x, t)$
- Cost minimization problem
- If firm pays $c(\lambda)$ it obtains adjustment probability λ
- cost $c(\cdot)$ assumed convex and increasing

$$\begin{aligned}\rho u(x,t) &= F(x, X(t), S(t)) + u_t(x,t) + \frac{\sigma^2}{2} u_{xx}(x,t) \\ &\quad + \min_{\lambda \geq 0} \left[c(\lambda) - \lambda \left(u(x,t) - u(x^*(t), t) \right) \right] \text{ for all } x \in \mathbb{R}, t \geq 0 \\ x^*(t) &= \arg\min_x u(x,t) \text{ all } t \geq 0\end{aligned}$$

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$$x^*(t) = \arg\min_x u(x,t) \text{ all } t \geq 0$$

- Define $H(v) = \min_{\lambda \geq 0} c(\lambda) - \lambda v$ where $v = u(x,t) - u(x^*(t),t)$
- $H(v)$ is concave & $\lambda^*(v) = \arg\min_{\lambda \geq 0} c(\lambda, x) - \lambda v = -H_v(v)$
- $\Lambda(x,t) = -H_v\left(u(x,t) - u(x^*(t),t)\right)$ adj. prob. depends of (x,t).

II. Random Menu Cost

- Value function $u(x, t)$
- Cost minimization problem
- At Poisson rate κ per unit of time draw random fixed cost ψ
- Random cost has CDF and density $g(\psi) = G'(\psi)$.

$$\rho u(x,t) = F(x, X(t), S(t)) + u_t(x,t) + \frac{\sigma^2}{2} u_{xx}(x,t)$$
$$+\kappa \int_0^\infty \min\{0\,,\ \psi + u(x^*(t), t) - u(x,t)\}\, g(\psi) d\psi \text{ all } x \in \mathbb{R}, t \geq 0$$

$$x^*(t) = \arg\min_x u(x,t) \text{ all } t \geq 0$$

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$$x^*(t) = \arg\min_x u(x,t) \text{ all } t \geq 0$$

- Define $H(v) = \kappa \int_0^\infty \min\{0, -\psi\, v\}\, g(\psi) d\psi$ for $v = u(x,t) - u(x^*(t),t)$
- $H(v)$ is concave & $\lambda^*(v) = \kappa G(v) = -H_v(v)$
- $\Lambda(x,t) = -H_v\left(u(x,t) - u(x^*(t),t)\right)$ adj. prob. depends of (x,t).

Costly Adjustment Prob or Random Menu Cost

- For any $H(\cdot) \leq 0$ and $H(\cdot)$ concave :
  - There is a cost $c(\cdot)$ that rationalizes $H(\cdot)$.
  - There is Poisson rate κ and CDF $G(\cdot)$ so $\kappa G(\cdot)$ rationalize $H(\cdot)$

  (shown in Alvarez, Lippi and Oskolkov, 2022)

- So, we write directly the firms problem in terms of any function H that is non-positive and concave.

Adding fixed (non-random) Menu Cost

- Cost minimization problem, all $x \in \mathbb{R}$ and $t \geq 0$:

$$\rho u(x,t) = \min \Big\{ F(x, X(t), S(t)) + u_t(x,t) + \frac{\sigma^2}{2} u_{xx}(x,t) + H(u(x,t) - u(x^*(t),t)) \, , \; \Psi + u(x^*(t),t) \Big\}$$

$$x^*(t) = \arg\min_x u(x,t) \text{ all } t \geq 0$$

- Thresholds satisfy value matching and smooth pasting:

$$u(\underline{x}(t),t) = u(\bar{x}(t),t) = u(x^*(t),t) + \Psi \text{ all } t \geq 0$$
$$u_x(\underline{x}(t),t) = u_x(\bar{x}(t),t) = u_x(x^*(t),t) = 0 \text{ all } t \geq 0$$

- Adjustment probability $\Lambda(x,t)dt$ in time interval $[t, t+dt]$

$$\Lambda(x,t)dt = \begin{cases} -H_v(u(x,t) - u(x^*(t),t))dt & \text{if } x \in (\underline{x}(t), \bar{x}(t)) \\ 1 = \infty \, dt & \text{if } x \notin (\underline{x}(t), \bar{x}(t)) \end{cases}$$

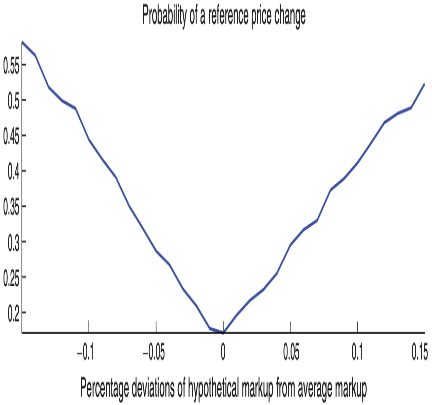
- If $\Psi = \infty$, then $\bar{x}(t) = -\underline{x}(t) = \infty$ all $t \geq 0$.

Examples of Price Setting Models

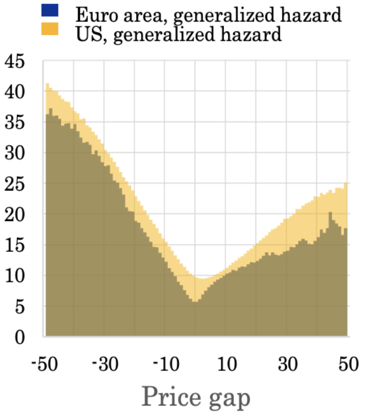
- Golosov & Lucas (2007), set $H = 0$, and $0 < \Psi < \infty$.
- $-H_v(v) > 0$ *constant*: Calvo, either with random menu cost or costly adjustment probabilities
- $-H_v(v) > 0$ *constant, and* $+\infty$ *at boundaries*: Calvo$^+$ of Nakamura & Steinsson's (2010)
- *General concave* $H(\cdot)$: Woodford (2009), and Constain & Nakov (2011), as costly adjustment probabilities.
- *General concave* $H(\cdot)$: Caballero & Engel (1999, 2007), Dotsey, King & Wollman (1999), Berger & Vavra (2018), as random menu cost models.
- Estimates of $\Lambda(x) = -H_v(u(x) - u(x^*))$: Eichenbaum, Jaimovich & Rebelo (2011), Karadi, Amann, Bachiller, & Seiler (2023), Cavallo, Lippi & Myiahara (2023), Gautier, Marx, & Vertier (2023).

Estimates of Hazard rate Adjustment $\Lambda(x)$:

Eichenbaum et al.
AER 2011
Scanner one store

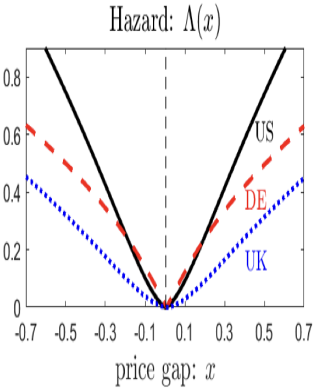


Karadi et al.
JME 2023
CPI micro



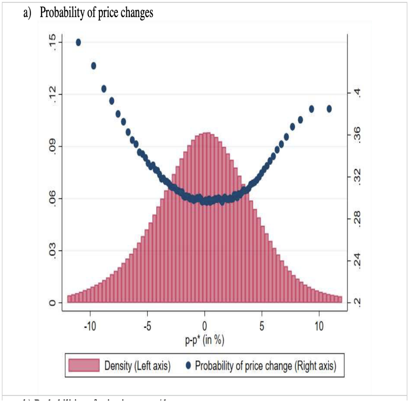
Estimates of Hazard rate Adjustment $\Lambda(x)$:

Cavallo Myiahara Lippi
AER Insights 2024
Online prices



Gautier, Marx & Vertier
JPE Macro 2023
Gasoline stations

Figure 3: Adjustment hazard rates



Law of motion of cross sectional distribution x

- Fix function $\Lambda(x,t) = -H_v\left(u(x,t) - u(x^*(t),t)\right)$, and the path of thresholds $\underline{x}(t), x^*(t), \bar{x}(t)$.
- $m(x,t)$ denotes the density of x at time t
- Law motion of density is:

$$m_t(x,t) = \frac{\sigma^2}{2} m_{xx}(x,t) + H_v\left(u(x,t) - u(x^*(t),t)\right) m(x,t)$$
$$\text{for all } t \geq 0, x \in [\underline{x}(t), \bar{x}(t)], x \neq x^*(t)$$

- Zero density at exit points, and continuity at entry:

$$0 = m(\underline{x}(t),t) = m(\bar{x}(t),t) \text{ for all } t \geq 0$$
$$m(\cdot,t) \text{ is continuous at } x = \bar{x}(t) \text{ for all } t \geq 0$$

- mass preservation and initial condition

$$1 = \int_{\underline{x}(t)}^{\bar{x}(t)} m(x,t)dx \text{ for all } t \geq 0$$
$$m_0(x) = m(x,0) \text{ for all } x \in \mathbb{R}$$

“Endogenous" X variables

- Given density $m(x,t)$, then $X_j(t)$ solves:

$$X_j(t) = \int \mu_j(x)\, m(x,t)\, dx \text{ for } j = 1, 2, \ldots, J \text{ and } t \geq 0$$

  for given function $\mu_j : \mathbb{R} \to \mathbb{R}$

- *Main example*: $X_j(t)$ cross sectional moments of $m(\cdot, t)$, for instance first moment:

$$X(t) = \int x\, m(x,t)\, dx \text{ for } t \geq 0$$

- We can summarize this writing $m \implies X$.

- Firms HJB + This equation: defines a fixed point:

$$X \implies u \implies (\Lambda, \underline{x}, x^*, \bar{x}) \implies m \implies X$$

Stationary State

- We are interested in a impulse response to a “shock", starting at stationary state. So we first state properties of such state.

- At stationary state value function, prob adjustment, and density are constant in time:

$$u(x,t) = \tilde{u}(x),\ \Lambda(x,t) = \tilde{\Lambda}(x),\ \text{and } m(x,t) = \tilde{m}(x)$$

- Three thresholds and optimal return path are constant in time:

$$\underline{x}(t) = \underline{x}_{ss},\ x^*(t) = x^*_{ss},\ \text{and } \bar{x}(t) = \bar{x}_{ss}$$

- Exogenous and Endogenous paths are constant in time

$$X_j(t) = \tilde{X}_j,\ \text{and } S_j(t) = \tilde{S}$$

Assumptions on F

1. $F(\cdot, 0, 0)$ is symmetric and convex in x, i.e.

$$F(x, 0, 0) = F(-x, 0, 0) \text{ and } \frac{\partial^2}{\partial x^2} F(x, 0) > 0.$$

2. $\frac{\partial}{\partial X_j} F(\cdot, 0, 0)$ is antisymmetric, i.e.

$$\frac{\partial}{\partial X_j} F(x, 0, 0) = -\frac{\partial}{\partial X_j} F(-x, 0, 0), \text{ for all } j.$$

3. $\frac{\partial}{\partial S_i} F(\cdot, 0, 0)$ is antisymmetric, i.e.

$$\frac{\partial}{\partial S_i} F(x, 0, 0) = -\frac{\partial}{\partial S_i} F(-x, 0, 0), \text{ for all } i.$$

Assumptions on F and μ: main case of interest I

- Second order Taylor expansion around optimal price

  $$F(x,X,S) = B\left(x + \theta_x X_1 + \theta_s S_1\right)^2 + \Gamma(X_1,S_1) + o\left(\|\,(x,X_1,S_1)\,\|^2\right)$$

  B, θ_x, θ_s are three constants.

- When is $\|\,(x,X_1,S_1)\,\|^2$ small? Small adjustment cost, so x is small, and X, S close to steady state.

- In this expansion the value of X_1 is the cross sectional average of x across firms:

  $$X_1 = \int x\, m(x) dx, \quad \text{corresponds to } \mu_1(x) = x$$

- We consider stationary equilibrium where $X_1 = S_1 = 0$. Note that

  $$F(x,0,0) = B\, x^2 = F(-x,0,0)\,,$$
  $$\frac{\partial}{\partial X} F(x,0,0) = 2\, B\, \theta_x\, x = -\frac{\partial}{\partial X} F(-x,0,0) \text{ and}$$
  $$\frac{\partial}{\partial S} F(x,0,0) = 2\, B\, \theta_s\, x = -\frac{\partial}{\partial S} F(-x,0,0)$$

Assumptions on F: main case of interest II

- Second order Taylor expansion around optimal price

$$F(x, X, S) = B\left(x + \theta_x X_1 + \theta_s S_1\right)^2$$

B, θ_x, θ_s are three constants

- Constant $B > 0$ determines curvature of profit function, value of adjustment.

- In this expansion the value of X_1 is the cross sectional average of x across firms:

$$X_1 = \int x \, m(x) dx, \quad \text{corresponds to } \mu_1(x) = x$$

- Sign of constant $B\,\theta_x$ determines whether prices are **strategic substitutes of complements**.

- Sign constant $B\,\theta_s$ determines effect of forcing variables, say wages, on optimal prices.

Stationary Solution

We use "tildes" or subindex ss for stationary objects.

Under the maintained assumptions:

1. The barriers are symmetric, $\underline{x}_{ss} = -\bar{x}_{ss}$ and $x^*_{ss} = 0$,

2. The stationary value function $\tilde{u}(x)$ and density $\tilde{m}(x)$ are symmetric around $x = 0$,

3. The adjustment hazard $\tilde{\Lambda}(x)$ is symmetric around $x = 0$,

$\Longrightarrow$ It implies distribution of price changes is symmetric.

(observable implication of symmetry)

Main Statistic of Interest

- Number of price changes per unit of time (frequency)

$$N(t) = \int_{\underline{x}(t)}^{\bar{x}(t)} \underbrace{-H_v\left(u(x,t) - u(x^*(t),t)\right)}_{\text{prob a change in p. firm w/}x} \; \underbrace{m(x,t)}_{\text{numb. firms w/}x} \; dx$$

$$+ \underbrace{\frac{\sigma^2}{2}\left(m_x\left(\underline{x}(t),t\right) - m_x\left(\bar{x}(t),t\right)\right)}_{\text{p changes at boundary}}$$

- Cross sectional variance of markups/relative prices

$$V(t) = \int_{\underline{x}(t)}^{\bar{x}(t)} x^2\, m(x,t)dx - \left(\int_{\underline{x}(t)}^{\bar{x}(t)} x\, m(x,t)dx\right)^2$$

- Recall $V(t)$ is the first order term of welfare cost of inflation.

Other Statistic of Interest I: moments of price changes

- A firm with price gaps x that reprice to $x^*(t)$, has a price change $x^*(t) - x$.
- To simplify, set large cost $\Psi = \infty$ so that $\bar{x}(t) = \infty = -\underline{x}(t)$, and thus all price changes are probabilistic.
- j^{th} moment of price changes at time t:

$$M_j(t) = \int_{-\infty}^{\infty} \underbrace{-H_v(u(x,t) - u(x^*(t),t))}_{\text{prob p change firm w}x} \underbrace{m(x,t)}_{\#\text{ firms w.}x} \underbrace{(x^*(t) - x)^j}_{\text{p change to } j} dx$$

- j^{th} moment of price changes at steady state:

$$\tilde{M}_j = \int_{-\infty}^{\infty} \underbrace{-H_v(\tilde{u}(x) - \tilde{u}(x^*_{ss}))}_{\text{prob p change firm w}x} \underbrace{\tilde{m}(x)}_{\#\text{ firms w.}x} \underbrace{(x - x^*_{ss})^j}_{\text{p change to } j} dx$$

- Note $\tilde{M}_j = 0$ for j odd, (implied by symmetry).

IRF as Perturbation

- Solve for equilibrium $\{u, m, \underline{x}, x^*, \bar{x}, X\}$ given:
  1. Initial perturbed distribution: $m_0(x) = \tilde{m}(x) + \epsilon\, \nu(x)$ all $x \in \mathbb{R}$
  2. Path of forcing variable: $S_i(t) = 0 + \epsilon\, s_i(t)$ all $t \geq 0$, with $\lim_{\to\infty} s_i(t) = 0$.
- Fix path $s_i(t)$ and function $\nu(x)$, then IRF(t), denoted by "hats" :

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\left(u(x,t;\epsilon) - \tilde{u}(x)\right) = \frac{\partial}{\partial\epsilon} u(x,t;\epsilon)|_{\epsilon=0} \equiv \hat{u}(x,t)$$

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\left(m(x,t;\epsilon) - \tilde{m}(x)\right) = \frac{\partial}{\partial\epsilon} m(x,t;\epsilon)|_{\epsilon=0} \equiv \hat{m}(x,t),$$

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\left(\underline{x}(t;\epsilon) - \underline{x}_{ss}\right) = \frac{\partial}{\partial\epsilon} \underline{x}(t;\epsilon)|_{\epsilon=0} \equiv \hat{\underline{x}}(t),$$

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\left(x^*(t;\epsilon) - x^*_{ss}\right) = \frac{\partial}{\partial\epsilon} x^*(t;\epsilon)|_{\epsilon=0} \equiv \hat{x}^*(t),$$

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\left(\bar{x}(t;\epsilon) - \bar{x}_{ss}\right) = \frac{\partial}{\partial\epsilon} \bar{x}(t;\epsilon)|_{\epsilon=0} \equiv \hat{\bar{x}}(t),$$

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\left(X(t;\epsilon) - \tilde{X}\right) = \frac{\partial}{\partial\epsilon} X(t;\epsilon)|_{\epsilon=0} \equiv \hat{X}(t).$$

Main result

- We consider an “MIT-shock", i.e. a change in the initial conditions and path of forcing variables starting at the stationary state.
- The “shock" has to be small, since we are using first order approximations – similar to linearization in standard model.
- In particular we assume that
  1. Path of forcing variable: $S(t) = \epsilon s(t)$
  2. The initial condition: $m_0(x) = \tilde{m}(x) + \epsilon \nu(x)$ where $\nu(x) = -\nu(-x)$
- The IRF(t) for Frequency N and Cross sectional Variance of x (markups or rel. prices) V, and even moments of p changes M_j

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(N(t, \epsilon) - N_{ss}\right) = \frac{\partial}{\partial \epsilon} N(t; \epsilon)|_{\epsilon=0} \equiv \hat{N}(t) = 0, \text{ all } t$$

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(V(t, \epsilon) - V_{ss}\right) = \frac{\partial}{\partial \epsilon} V(t; \epsilon)|_{\epsilon=0} \equiv \hat{V}(t) = 0, \text{ all } t$$

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(M_j(t, \epsilon) - \tilde{M}_j\right) = \frac{\partial}{\partial \epsilon} M_j(t, \epsilon)|_{\epsilon=0} \equiv \hat{M}_j(t) = 0, \text{ all } t \text{ for even } j$$

Main example of Assumption on m_0: jump in cost at zero

- Assumption $m_0(x) = \tilde{m}(x) + \epsilon\, \nu(x)$ for $\nu(x) = -\nu(-x)$.
- Recall interpretation of x is markup deviation.
  - Consider an initial jump cost $\mathcal{J}(\epsilon)$, when the “shock" of size ϵ
  - So price gaps of a firm x becomes $x - \mathcal{J}(\epsilon)$.
  - This shifts the distribution from $\tilde{m}(x)$ to $m_0(x) = \tilde{m}(x + \mathcal{J}(\epsilon))$.
  - To first order $m_0(x) = \tilde{m}(x) + \epsilon\, \tilde{m}'(x)\mathcal{J}'(0) + o(\epsilon)$.
  - Since $\tilde{m}(x)$ is symmetric, then $\tilde{m}'(x)$ is antisymmetric, i.e. satisfies the assumption for $\nu(x) = \tilde{m}'(x)\mathcal{J}'(0)$.
- In very simple set ups, e.g. Golosov & Lucas, this is the only effect, since they analyze once-and-for all shock.

Outline of the logic of the proof

- Split, price changes into decreases and increases:

$$N(t) = N^{-}(t) + N^{+}(t)$$

$$= \int_{x^*(t)}^{\bar{x}(t)} \underbrace{-H_v\left(u(x,t) - u(x^*(t),t)\right)}_{\text{prob p. decrease firm w/}x} \quad \underbrace{m(x,t)}_{\text{numb. firms w/}x} \quad dx - \underbrace{\frac{\sigma^2}{2} m_x\left(\bar{x}(t),t\right)}_{\text{p. decrease at boundary}}$$

$$+ \int_{\underline{x}(t)}^{x^*(t)} \underbrace{-H_v\left(u(x,t) - u(x^*(t),t)\right)}_{\text{prob p. increase firm w/}x} \quad \underbrace{m(x,t)}_{\text{numb. firms w/}x} \quad dx + \underbrace{\frac{\sigma^2}{2} m_x\left(\underline{x}(t),t\right)}_{\text{p. increase at boundary}}$$

- Corresponding decomposition for IRF

$$\hat{N}(t) = \hat{N}^{-}(t) + \hat{N}^{+}(t)$$

- We use assumptions to show symmetry-antisymmetry of $\hat{u}(\cdot,t), \hat{m}(\cdot,t)$ and its derivatives, & that $\hat{\underline{x}}(t) = \hat{\bar{x}}(t)$ to show:

$$\hat{N}^{-}(t) = -\hat{N}^{+}(t) \implies \hat{N}(t) = 0$$

- Technical tool: maximum principle for p.d.e.'s.

Outline of the logic of the proof, Symmetry

- Uses that $\tilde{u}(x), \tilde{m}(x)$ are symmetric, and $\bar{x}_{ss} = -\underline{x}_{ss}$ and $x^*_{ss} = 0$.
- Uses that $\frac{\partial u(x,t,\epsilon)}{\partial \epsilon}\Big|_{\epsilon=0} = -\frac{\partial u(-x,t,\epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}$
- Why? differentiate the HJB for all x, t with respect to ϵ, obtain a new p.d.e and boundaries, apply maximum principle for pde's.
- As an illustration of trivial case, consider a simplified version of the period return:

$$\frac{\partial\big(x + s(t)\theta\epsilon\big)^2}{\partial\epsilon}\Big|_{\epsilon=0} = 2\theta s(t)x =$$

$$-\frac{\partial\big(-x + s(t)\theta\epsilon\big)^2}{\partial\epsilon}\Big|_{\epsilon=0} = -2\theta s(t)(-x)$$

- Similar logic applies to $\frac{\partial m(x,t,\epsilon)}{\partial \epsilon}\Big|_{\epsilon=0} = -\frac{\partial m(-x,t,\epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}$

- Special case of only random fixed cost: $\bar{x}(t,\epsilon)=\infty=-\underline{x}(t,\epsilon)$

$$N(t)=N^{-}(t)+N^{+}(t)$$
$$=\int_{x^*(t)}^{\infty}\underbrace{-H_v\left(u(x,t)-u(x^*(t),t)\right)}_{\text{prob p. decrease firm w}/x}\quad\underbrace{m(x,t)}_{\text{numb. firms w}/x}\quad dx$$
$$+\int_{-\infty}^{x^*(t)}\underbrace{-H_v\left(u(x,t)-u(x^*(t),t)\right)}_{\text{prob p. increase firm w}/x}\quad\underbrace{m(x,t)}_{\text{numb. firms w}/x}\quad dx$$

- Price decreases

$$N^{-}(t)$$
$$=-\frac{\partial}{\partial\epsilon}\int_{x^*(t,\epsilon)}^{\infty}H_v\left(u(x,t,\epsilon)-u(x^*(t,\epsilon),t,\epsilon)\right)m(x,t,\epsilon)dx\Big|_{\epsilon=0}$$
$$=-\int_{x^*_{ss}}^{\infty}H_v\left(\tilde{u}(x)-\tilde{u}(x^*_{ss})\right)\frac{\partial m(x,t,0)}{\partial\epsilon}dx$$
$$-\int_{x^*_{ss}}^{\infty}\tilde{m}(x)H_{vv}\left(\tilde{u}(x)-\tilde{u}(x^*_{ss})\right)\left[\frac{\partial u(x,t,0)}{\partial\epsilon}-\underbrace{u_x(\bar{x}^*_{ss})}_{=0}\frac{\partial x^*(t,0)}{\partial\epsilon}\right]dx$$

- Special case of only random fixed cost: $\bar{x}(t) = \infty = -\underline{x}(t)$

$$N(t) = N^-(t) + N^+(t)$$

$$= \int_{x^*(t)}^{\infty} \underbrace{-H_v\left(u(x,t) - u(x^*(t),t)\right)}_{\text{prob p. decrease firm w/}x} \quad \underbrace{m(x,t)}_{\text{numb. firms w/}x} \quad dx$$

$$+ \int_{-\infty}^{x^*(t)} \underbrace{-H_v\left(u(x,t) - u(x^*(t),t)\right)}_{\text{prob p. increase firm w/}x} \quad \underbrace{m(x,t)}_{\text{numb. firms w/}x} \quad dx$$

- Price increases

$$N^+(t)$$

$$= -\frac{\partial}{\partial \epsilon} \int_{-\infty}^{x^*(t,\epsilon)} H_v\left(u(x,t,\epsilon) - u(x^*(t,\epsilon),t,\epsilon)\right) m(x,t,\epsilon) dx \Big|_{\epsilon=0}$$

$$= - \int_{-\infty}^{x^*_{ss}} H_v\left(\tilde{u}(x) - \tilde{u}(x^*_{ss})\right) \frac{\partial m(x,t,0)}{\partial \epsilon} dx$$

$$- \int_{-\infty}^{x^*_{ss}} \tilde{m}(x) H_{vv}\left(\tilde{u}(x) - \tilde{u}(x^*_{ss})\right) \left[\frac{\partial u(x,t,0)}{\partial \epsilon} - \underbrace{u_x(\bar{x}^*_{ss})}_{=0} \frac{\partial x^*(t,0)}{\partial \epsilon} \right] dx$$

Outline of the logic of the proof, finite horizon

- Actual proof uses that at T firms value function $u(x, T) = \tilde{u}(x)$.
- Which is the same as assuming that both X_j and S_j go to steady state, i.e become zero, at $t = T$.
- The result holds by taking $T \to \infty$.
- We have done this for mathematical convenience, i.e., to show that the unique solution has the desired property. We were actually apologetic for that.
- After Mike's plenary, we should reassess its interpretation, and consider its economic content.

Outline of the logic of the proof, cont

- Thus frequency is flat

$$0 = N(t) = N^{-}(t) + N^{+}(t)$$
$$N^{-}(t) = -N^{+}(t)$$

- Different models give different split of increases and increases.

- From examples which can be solved completely (Alvarez & Lippi 2022, Alvarez, Lippi & Souganidis 2023)

  - On a expansionary shocks, where price level will increase in the long run, state dependent models give different answer than time dependent.

  - In this case, state dependent models will have an IRF with higher frequency of price increases.

Interpretation of setup and main result: Phillips curve like

- We are not solving for equilibrium.
- Instead, we solve for firms price setting problem, given other prices (say wages), which we refer to as the path of forcing variables $S(t)$.
- In an equilibrium of a complete model, the path of forcing variables $S(t)$ may have to be determined jointly with prices.
- But recall our result is obtained for an *arbitrary* path of forcing variables $S(t)$.
- **Yet, for small shocks, under the stated assumptions, our result applies for *any equilibrium*, i.e. there is no change in frequency of price changes, the entire hazard rate, or variance of markup after the shock.**
- This is akin to study Phillips curve (in an integrated way), as opposed to the entire equilibrium.
- In special cases, one can solve the entire IRF using these equations, as we return below.

Interpretation of setup: zero steady state inflation, positive σ

- The results apply to a case with **zero steady state inflation**.
  - Price gap evolves, if there are no adjustment as $dx = \sigma dW$, so no drift.
  - Analogous to assumption in linearized NK model of indexation to steady state inflation.
- The positive σ is required to ensure smoothness, and to obtain idiosyncratic price changes.
- Importantly, there are no jumps, or discrete changes on price gaps.

Interpretation of setup and main result: odd moments M_j and X_j

- Even moment of price changes and variance of price gap have zero IRF.
- But, odd moments $M_j(t)$, and average price gap X_j do have a *non-zero* IRF!
- Indeed, in many models, average price gap X_1 is proportional to output deviations.
- Indeed, for special version we can characterize the evolution of the IRF for average price gap, or output, and how this depend on state vs time dependent model.

Interpretation of setup and main result: beyond small shocks

- Changes in total frequency of price changes, i.e. $\hat{N}(t)$, gives evidence of state dependence or lack of thereof.
- "Old" evidence:
  - Mexico: Gagnon (2009)
  - Argentina: Alvarez-Beraja-Neumeyer & Rosada (2019), Alvarez & Neumeyer (2020)
  - Hungary: Karadi and Reif (2019)
  - US: Nakamura-Steinsson-Sun & Villar (2018)
- New evidence, recent inflation:
  - US and various European countrie: Cavallo-Myhara-Lippi (2023)
  - US: Montag & Villar (2024)
  - UK: Blanco, Boar, Jones & Midrigan (2024)

Conclusions

- Analyze the IRF of Frequency of Price Changes to small shocks.
- Consider a relatively large set of Price Setting Problems
- We don't need to solve entire model, i.e. similar to analyzing properties of the Phillips curve in an integrated way.
- Show that for small shocks, **IRF of Frequency is zero**.
- Behavior of IRF to frequency is a good test of state dependence, **only for large shocks**.