Learning about Bond Prices

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Learning and Bond price volatility:

- Many studies on stock price volatility, fewer on bond price volatility
- We learnt a lot about stock price behavior using models where agents learn about prices: *internal rationality*.
- Apply to learning about bond prices

Roadmap

Data

- Nominal US government bonds zero-coupon yields
- Sample 1983:12 to 2019:12.
- Monthly frequency
- Surveys: Reuters' Blue Chip Financial Forecasters survey on expected bond yields.

Flat Term structure of Yield Volatilities

Yearly maturities, leads, lags, forecasting horizons.

Table 1: Yields 1st and 2d moments

	$1y$	$2y$	$3y$	$5y$	$7y$	$10y$
$\widehat{Y}^n$	379	407	431	471	500	532
$\widehat{\sigma}_{Y^n}$	280	284	280	269	260	247
$\widehat{corr}(Y^n)_{-1y}$	0.89	0.90	0.91	0.92	0.92	0.92

Hard to reconcile with RE and reasonable serial correlation of expected inflation.

Failure of Expectations Hypothesis

$$Y_{t+1y}^{n-1} - Y_t^n = \alpha^n + \beta^n \frac{Y_t^n - Y_t^1}{n-1} + U_t^n \tag{1}$$

EH says β's ≈ 1

Table 2: Failure of Expectations Hypothesis

	$2y$	$3y$	$5y$	$7y$	$10y$
$\widehat{\beta}^n$	-0.31	-0.50	-0.99	-1.28	-1.67

horizon= 1y here and in all tables below

Failure of EH: "Modern" version

Let Q_t^n be the price of a zero-coupon bond of maturity n.

Define excess bond returns

$$xr_{t+1y}^n = log\left(\frac{Q_{t+1y}^{n-1}}{Q_t^n}\right) - log\left(\frac{1}{Q_t^1}\right)$$

Or, in terms of log yields $Y_t^n \equiv -\frac{log(Q_t^n)}{n}$

$$xr_{t+1y}^n = -(n-1)Y_{t+1y}^{n-1} + nY_t^n - Y_t^1$$

Failure of EH: ”Modern” version

EH can also be expressed in terms of xr forecasts:

$$xr^n_{t+1y} = \alpha^{xr}_n + b^{xr}_n \frac{Y^n_t - Y^1_t}{n-1} + U^n_t$$

EH implies b^{xr}’s ≈ 0.

Table 3: Failure of EH ”modern” version

	$2y$	$3y$	$6y$	$8y$	$11y$
$\widehat{b}^{xr}_n$	1.37	3.1	10.84	15.53	24.61
$t - stat$	0.52	1.14	2.40	2.87	3.54
R^2	0.32	1.59	6.00	7.88	10.41

High volatility of excess bond returns

$$xr^n_{t+1y} = log\left(\frac{Q^{n-1}_{t+1y}}{Q^n_t}\right) - log\left(\frac{1}{Q^1_t}\right)$$

Table 4: Excess Returns volatility

maturity n	$2y$	$3y$	$5y$	$7y$	$10y$
$\widehat{\sigma}_{xr^n}$	127	239	521	676	895

Hard to explain with RE.

Excess volatility of yields, Intuition for RE failure

$$Q_t^n \quad = \quad Q_t^1 \, E_t^{\mathbb{Q}} \left(Q_{t+1}^{n-1} \right) = E_t^{\mathbb{Q}} \left(\prod_{i=0}^{n-1} Q_{t+i}^1 \right)$$

where $\mathbb{Q}$ refers to "risk-neutral" distribution.
Taking log-linear approximation

$$Y_t^n \quad \approx \quad E_t^{\mathbb{Q}} \left(\frac{\sum_{i=0}^{n-1} Y_{t+i}^1}{n} \right)$$

therefore we expect $\sigma_{Y_t^n} < \sigma_{Y_t^1}$

Excess volatility of excess bond returns, Intuition

Furthermore

$$
\begin{aligned}
xr_{t+1}^{n} &= nY_t^n - (n-1)Y_{t+1}^{n-1} - Y_t^1 \\
&\approx \left(\sum_{i=1}^{n-1} (E_t^{\mathbb{Q}} - E_{t+1}^{\mathbb{Q}})(Y_{t+i}^1) \right)
\end{aligned}
$$

so $\sigma^2_{xr_t^n}$ variance of a change in expectations, so it should be small.

Bond survey expectations

- Much recent work on surveys about asset prices
- Giacoletti, Laursen and Singleton (2020) and Singleton (2021) AFA Presidential Address (GLS) :
  - Describe investor expectations about bond yields
  - Investors (consensus) bond yield expectations are on average similar to those arising from best forecasts of an affine model with principal components

RE Tests

Adam, Marcet, Beutel (2017)
Let $\mathcal{E}$ denote survey expectations.
Consider a regression like EH but with expectations in left hand side

$$\mathcal{E}_t(xr^n_{t+1y}) \quad = \quad \alpha^{\mathcal{E}}_n + b^{\mathcal{E}}_n \frac{Y^n_t - Y^1_t}{n-1} + U^{n,\mathcal{E}}_t$$

Under RE $b^{xr}_n = b^{\mathcal{E}}_n$.

Related to Coibion and Gorodnichenko approach.

For $\mathcal{E}_t(xr^n_{t+1y})$ interpolate BCFF to form monthly expectations 1 year ahead.

RE Regressions

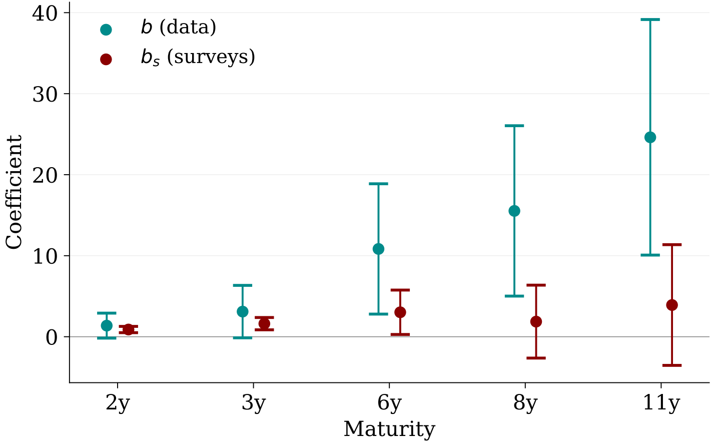
$$
\begin{aligned}
xr^n_{t+1y} &= \alpha^{xr}_n + b^{xr}_n \frac{Y^n_t - Y^1_t}{n-1} + U^{n,xr}_t \\
\mathcal{E}_t(xr^n_{t+1y}) &= \alpha^{\mathcal{E}}_n + b^{\mathcal{E}}_n \frac{Y^n_t - Y^1_t}{n-1} + U^{n,\mathcal{E}}_t
\end{aligned}
$$

Table 5: RE Regressions

	$2y$	$3y$	$6y$	$8y$	$11y$
$\widehat{b}^{xr}_n$	1.37	3.1	10.84	15.53	24.61
$\widehat{b}^{\mathcal{E}}_n$	0.89	1.61	3.02	1.87	3.91
t-stat $b^{xr}_n = b^{\mathcal{E}}_n$	0.37	0.76	1.75	2.47	2.70

$\widehat{b}^{xr}_n > 0$ and $\widehat{b}^{xr}_n > \widehat{b}^{\mathcal{E}}_n$.
Both $\widehat{b}^{xr}_n$ and $\widehat{b}^{xr}_n - \widehat{b}^{\mathcal{E}}_n$ increase with n.
Surveys underestimate the role of the slope in predicting xr^n_{t+1y}
$t - stat$ rejects RE, more strongly for long maturities

RE Regressions



RE Regressions, additional versions

Test is valid for any regressor in the information set at t.

We get a similar picture with alternative regressors that capture the slope:

-PC_t^2: 2d Estimated Principal Component
-$\widehat{\mathbf{s}}_t$: Investor's estimate of underlying slope according to our model

RE Regressions, additional versions

-PC_t^2: 2d Estimated Principal Component
-$\widehat{\mathbf{s}}_t$: Investor's estimate of underlying slope according to our model

Table 6: RE Regressions

regressor		$2y$	$3y$	$6y$	$8y$	$11y$
PC_t^2	$\widehat{b}_n^{xr}$	0.06	0.13	0.43	0.61	0.89
	$\widehat{b}_n^{\mathcal{E}}$	0.02	0.05	0.14	0.13	0.19
	t-stat $b_n^{xr} = b_n^{\mathcal{E}}$	0.85	1.11	1.98	2.57	2.87
$\widehat{\mathbf{s}}_{t-3}$	$\widehat{b}_n^{xr}$	1.04	2.16	9.14	13.90	22.67
	$\widehat{b}_n^{\mathcal{E}}$	0.41	0.57	0.28	−2.28	−2.86
	t-stat $b_n^{xr} = b_n^{\mathcal{E}}$	0.77	0.95	2.12	2.79	3.08
$\widehat{\mathbf{s}}_t, \widehat{\mathbf{s}}_{t-3}$	p-value $b_n^{xr} = b_n^{\mathcal{E}}$	0.70	0.59	0.09	0.01	0.00

Roadmap

Literature

- Standard equilibrium asset pricing models have a hard time matching all previous facts.
- High volatility of xr documented since Shiller (1979) but very rarely addressed.
- Much progress has been made:
  - Under RE homogeneous agents: Time-varying reward for risk:
    Dai and Singleton (2002), Wachter (2006), Gabaix (2008), Bansal and Shaliastovich (2013)
  - Bayesian/RE: heterogeneous beliefs but they know the pricing function:
    Xiong and Yan (2010), Buraschi and Whelan (2022), Buraschi and Jiltsov (2007), Ehling Gallmeyer, Heyerdahl-Larsen and Illeditsch (2018).

Literature

- The old puzzles of Tables 1-4 remain hard to explain jointly.
  Giglio et al (2018) argue standard models can't explain all those observations.
- The RE regressions in Tables 5-6 are a new puzzle.
- It matters whether the data needs RE or learning!!

Literature, Adaptive Learning, partial information and bond yields

- Sinha (2009), Piazzesi and Schneider (2006), Laubach, Tetlow, and Williams (2007) models of adaptive learning about how macro variables or inflation influence bond prices.
- Ellison Tischbirekz (2021) beauty contest.

This Project

Introduce *Internal Rationality*.

- investors have a *consistent system of beliefs about bond prices*
- investors are *fully rational* given their beliefs.

Agents' beliefs are "almost correct"

- close to RE beliefs
- model converges to RE
- "close" to the true distribution, (hard to reject using observed data)
- investors' beliefs match behavior of expectations surveys

C†laim: IR is a "minor" deviation from standard paradigm:

Roadmap

A model of bond prices

N zero-coupon nominal bonds of maturity $n = 1, ..., N$

Bond prices $Q_t = \left[Q_t^1, ..., Q_t^N\right]'$.

Homogeneous investors with $E_0^{\mathcal{P}} \sum_{t=0}^{\infty} \delta^t \; u(c_t)$

Investor beliefs represented by $\mathcal{P}$.

All bonds with $n > 1$ are resold one period after being purchased. Budget constraint:

$$P_t c_t + \sum_{n=1}^{N} Q_t^n B_t^n = w_t P_t + \sum_{n=1}^{N} Q_t^{n-1} B_{t-1}^n$$

$Q_t^0 = 1$

A model of Bond Prices

Assume bond limits:

$$\overline{B} \geq B_t^n \geq \underline{B} \tag{2}$$

Inflation process:

$$\log P_t - \log P_{t-1} \equiv \pi_t \quad = \quad \mu(1-\rho) + \rho\pi_{t-1} + \eta_t^\pi$$

$$\eta_t^\pi \sim \mathcal{N}(0, \sigma^2_{\eta^\pi}), \text{ iid.}$$

The Model

Equilibrium conditions:

$$Q_t^n = \delta E_t^{\mathcal{P}} \left(\frac{u'(c_{t+1})}{u'(c_t)} \frac{P_t}{P_{t+1}} Q_{t+1}^{n-1} \right) \text{ for } n \geq 2$$

The short rate Y_t^1 is exogenous, given by

$$Y_t^1 = -\log(\delta) + \pi_t^e + \xi_t$$

ξ_t a mean-zero exogenous process (policy shock).
So bond limit is binding for $n = 1$ but assumed not to bind for $n > 1$.

Roadmap

Investors' probability space under IR

$$\Omega \equiv \Omega_Q \times \Omega_{\pi,\xi}$$

where Ω_Q contains possible histories of $Q = [Q^n]_{n=1}^N$, and $\Omega_{\pi,\xi}$ contains possible histories of inflation π, ξ.

Investors have a belief system $\mathcal{P}$ with (consistent) probabilities on Ω.

RE is a special case when agents' beliefs about π, ξ are correct and they know the true pricing function determining Q.

Bayesian/RE a special case when agents' don't know everything about π, ξ but they know the true pricing function determining Q.

Model with learning about Q

However if agents don't know the pricing function prices are given by

$$Q_t^n = \delta E_t^{\mathcal{P}} \left(\frac{u'(c_{t+1})}{u'(c_t)} \frac{P_t}{P_{t+1}} Q_{t+1}^{n-1} \right) \text{ for } n \geq 2 \tag{3}$$

Need to state investors' beliefs about **joint** distribution of Q, π.

A common view:

"Investors with beliefs about bond prices Q are irrational."
The reasoning:
"If investors are rational they can apply LIE to their own optimality conditions to obtain:

$$Q_t^n = \delta^n \ E_t^{\mathcal{P}} \left(\frac{u'(c_{t+n})}{u'(c_t)} \frac{P_t}{P_{t+n}} \right)$$

Knowledge of inflation and output process means

$$Q_t^n = \delta^n \ E_t \left(\frac{u'(c_{t+n})}{u'(c_t)} \frac{P_t}{P_{t+n}} \right) = Q_t^{n,RE}$$

This contradicts other price beliefs.
Therefore agents with beliefs about Q are irrational."

But this common view is wrong

1. If some agents are constrained LIE can not be applied with knowledge about inflation and output, it needs market knowledge.
   (Adam, Marcet 2011)
2. Under IR

$$Q_t^n = \delta^n E_t^{\mathcal{P}} \left(\frac{u'(c_{t+n}(Q^{t+n}))}{u'(c_t)} \frac{P_t}{P_{t+n}} \right) \neq Q_t^{n,RE}$$

   so $E_t^{\mathcal{P}}$ depends on beliefs about Q.
   (Adam, Marcet, Beutel, 2017)

Under IR investors' price beliefs are consistent with their information.

Roadmap

Special case

- π unit root: ($\rho = 1$)
- Policy shock ξ iid, independent of π
- Risk neutrality
- Yearly model frequency
- Log linear approximation to Euler equation

A belief system about bond prices

We now specify our assumptions about the system of beliefs.
To build a "good" system of beliefs, we assume

1. Agents know process for (π, ξ).
2. Agents' know yields at nearby maturities are related.
   They express this belief with a perceived yield curve

$$Y_t^n = -\log \delta + \pi_t^e + \mathbf{s}_t n + u_t^n \tag{4}$$

- Investors know the intercept and how inflation affects yields
- but agents believe there is a time-varying, unobserved slope $\mathbf{s}_t$

(in blue we show equations that hold according to investors beliefs)

A Simple Model

Recall perceived yield curve

$$Y_t^n = -\log\delta + \pi_t^e + \mathbf{s}_t n + u_t^n$$

To complete investors' beliefs assume investors believe

$$\mathbf{s}_t = \mathbf{s}_{t-1} + w_t$$

Optimal filtering implies

$$E_t^{\mathcal{P}}(\mathbf{s}_t) \equiv \widehat{\mathbf{s}}_t = \widehat{\mathbf{s}}_{t-1} + g(s_{t-1} - \widehat{\mathbf{s}}_{t-1})$$

g is the "gain" parameter.
s_t is the best estimate of period-t slope given new info at t:

$$s_t = E_t^{\mathcal{P}}(\mathbf{s}_t \mid Y_t^N, ..., Y_t^1)$$

to be specified later. Then

$$E_t^{\mathcal{P}}(Y_{t+1}^n) = -\log\delta + \pi_t^e + \widehat{\mathbf{s}}_t n$$

Beliefs ”near-rational”

Recall perceived yield curve:

$$Y_t^n = -\log\delta + \pi_t^e + \widehat{\mathbf{s}}_t n + \widehat{u}_t^n$$

Simple algebra, actual yield curve:

$$Y_t^n = -\log\delta + \pi_t^e + \widehat{\mathbf{s}}_t \; n \left(\frac{n-1}{n}\right)^2$$

(in brown equations that hold under the true distribution in the learning equilibrium)

Agents’ beliefs are ”quite good”:

- recall agents know process for π, ξ
- when agents believe there is a time-varying slope there is a time-varying slope
- model converges to RE:

A simple estimate s_t (the perceived slope at t)

Investors should use all maturities Y_t^n to estimate $s_t = E_t^{\mathcal{P}}(\mathbf{s}_t \mid Y_t^N, ..., Y_t^1)$.

Additional assumption on beliefs:

Investors think there is a reference bond of maturity m with lower uncertainty

$$var^{\mathcal{P}}(u_t^m) << var^{\mathcal{P}}(u_t^n)$$

Then, GLS says it is optimal to use only $n = m, 1$ to obtain information about $\mathbf{s}_t$:

$$s_t = \frac{Y_t^m - Y_t^1}{m-1}$$

Learning Equilibrium with "simple" estimate s_t

Plugging in equilibrium yields

$$\begin{aligned} Y_t^n &= -\log(\delta) + \widehat{\mathbf{s}}_t \, n \left(\frac{n-1}{n}\right)^2 + \pi_t^e \\ s_t &= \frac{Y_t^m - Y_t^1}{m-1} \\ \widehat{\mathbf{s}}_t &= \widehat{\mathbf{s}}_{t-1} + g\left(s_{t-1} - \widehat{\mathbf{s}}_{t-1}\right) \end{aligned}$$

So, under self-referential learning:

$$\uparrow s_t = \frac{Y_t^m - Y_t^1}{m-1} \Longrightarrow \uparrow \widehat{\mathbf{s}}_{t+1} \Longrightarrow \uparrow Y_{t+1}^m \Longrightarrow \uparrow s_{t+1} = \frac{Y_{t+1}^m - Y_{t+1}^1}{m-1} \ldots$$

Convergence to RE: The T-map

When agents' perceived expectation (PLM) is

$$E_t^{\mathcal{P}} \frac{Y_{t+1}^m - Y_{t+1}^1}{m-1} = \widehat{\mathbf{s}}_t$$

then the true expectation (ALM) is

$$E_t \frac{Y_{t+1}^m - Y_{t+1}^1}{m-1} \equiv \frac{m-1}{m} \widehat{\mathbf{s}}_t \equiv T(\widehat{\mathbf{s}}_t)$$

Convergence to RE

We have $T' < 1$.
Standard theorems in the learning literature imply that model converges to RE:

$$Y_t \longrightarrow Y_t^{RE}$$

if either

- OLS instead of constant gain ($g = 1/t$) or
- $g \longrightarrow 0$

(Marcet and Sargent (1989), Williams (2017), Evans Honkapohja (2002))

Fluctuations around RE

Furthermore, for $g > 0$, plugging in equilibrium value of s_t

$$\widehat{\mathbf{s}}_t = \left(1 - \frac{g}{m}\right)\widehat{\mathbf{s}}_{t-1} - g\frac{\xi_{t-1}}{m-1}$$

So $\widehat{\mathbf{s}}_t$ very persistent and $E(\widehat{\mathbf{s}}_t) = 0$.

The learning model fluctuates around RE.

An Analytic Solution

A linear system of difference equations

$$
\begin{aligned}
\widehat{\mathbf{s}}_t &= \left(1 - \frac{g}{m}\right)\widehat{\mathbf{s}}_{t-1} - g\frac{\xi_{t-1}}{m-1} \\
Y_t^n &= -\log(\delta) + \widehat{\mathbf{s}}_t\ \frac{(n-1)^2}{n} + \pi_t^e \\
xr_{t+1}^n &= \left[2n - 3 + (n-2)^2\frac{g}{m}\right]\widehat{\mathbf{s}}_t \\
&\quad + (n-1)\eta_{t+1}^{\pi} + \left[\frac{(n-2)^2 g}{m-1} - 1\right]\xi_t \\
E_t^{\mathcal{P}}(xr_{t+1}^n) &= -\xi_t
\end{aligned}
$$

Analytic Solution

$var(\widehat{\mathbf{s}})$ contributes to variances, more for higher maturities.

- $var(Y^n)$ has additional variance for long n.
- $var(xr^n)$ increases with n.
- Consistent with RE regression:
  - $\widehat{b}_n^{xr} > 0$
  - $\widehat{b}_n^{xr}$, increases with n
  - $\widehat{b}_n^{xr} > \widehat{b}_n^{\mathcal{E}}$
  - $\widehat{b}_n^{xr} - \widehat{b}_n^{\mathcal{E}}$ increases with n

Intuition for RE regressions

Recall PLM and ALM:

$$E_t^{\mathcal{P}}(Y_{t+1}^n) = -\log\delta + \pi_t^e + \widehat{\mathbf{s}}_t n$$

$$E_t(Y_{t+1}^n) = -\log\delta + \pi_t^e + \widehat{\mathbf{s}}_t n \left(\frac{n-1}{n}\right)^2 \left(1 - \frac{g}{m}\right)$$

So agents *over*predict the influence of $\widehat{\mathbf{s}}_t$ on future yields.

Agents *under*predict the influence of $\widehat{\mathbf{s}}_t$ on future

$$xr_{t+1}^n = -(n-1)Y_{t+1}^{n-1} + nY_t^n - Y_t^1$$

Consistent with $\widehat{b}_n^{xr} > \widehat{b}_n^{\mathcal{E}}$ in the data.

Intuition for RE regressions: general property of IR

Intuition for $b_n^{\mathcal{E}} \approx 0$

- EH holds, approximately, for the agents' belief system

Intuition for $b_n^{xr} > b_n^{\mathcal{E}}$

- Define T as $E_t(Y_{t+1}^n) = T(E_t^{\mathcal{P}}(Y_{t+1}^n))$
- Recall, for stability we need $T' < 1$
- if $T' < 1$ *actual* $E_t(Y_{t+1}^n)$ moves less with slope than $E_t^{\mathcal{P}}(Y_{t+1}^n)$
- Hence *actual* $E_t(xr_{t+1}^n)$ moves more with slope than $E_t^{\mathcal{P}}(xr_{t+1}^n)$
- therefore $b_n^{xr} > b_n^{\mathcal{E}}$ if model is stable.

Therefore $\widehat{b}_n^{xr}, \widehat{b}_n^{\mathcal{E}}$ consistent with IR and standard Euler Eq.

Roadmap

Fitting a simple Calibrated model

- AR(1) inflation with $\rho < 1$.
- Calibrate to sample values σ_{π^e}, ρ, σ_{η^π} ,σ_ξ for $\xi = Y^1 - \pi^e$.
- Fiddled with with the free parameter g, selected $g = 0.3$ (yearly)

Second Moments RE and Learn for the Simpler Model

	3y			10y		
	data	RE	Learn	Data	RE	Learn
σ_{Y^n}	280	90	98	247	73	240
$\rho_{Y^n,-1y}$	0.91	0.94	0.94	0.92	0.94	0.97
σ_{xr^n}	239	214	224	895	295	622
b_n^{xr}	3.1	0	2.02	24.61	0	7.01
$b_n^{\mathcal{E}}$	1.61	0	1.57	3.91	0	3.97

Model III: π, ξ bivariate AR(1)

Now introduce

- Monthly model but yearly horizons and lags
- (π, ξ) bivariate AR(1), estimated to match observed moments (at yearly lags)
- Estimate g by MSM

Model III Moments

Statistic	Maturity	Data	Model	t-stat
Standard deviation yields σ_{Y^n}	1y	280.00	296.11	-0.72
	2y	284.04	275.48	0.36
	3y	280.63	258.46	0.91
	5y	269.98	246.55	0.95
	7y	260.52	267.36	-0.27
	10y	247.59	340.35	-3.85
Serial correlation yields ρ_{Y^n}	1y	0.89	0.91	-0.96
	2y	0.90	0.92	-0.66
	3y	0.91	0.92	-0.36
	5y	0.92	0.92	-0.29
	7y	0.92	0.94	-1.01
	10y	0.92	0.95	-1.79
Standard deviation excess returns σ_{xr^n}	2y	127.11	101.27	2.23
	3y	239.72	201.37	1.91
	5y	521.33	384.68	2.77
	7y	676.19	571.83	1.58
	10y	895.49	917.47	-0.24

$\hat{g}$=0.0536

Model III: RE Regressions

Statistic	Maturity	Data	Model	t-stat
EH coefficient b^n_{xr}	2y	1.37	0.18	1.52
	3y	3.10	1.38	1.04
	6y	10.84	5.06	1.41
	8y	15.53	7.77	1.45
	11y	24.61	12.39	1.65
RE coefficient b^n_s	2y	0.89	0.09	4.05
	3y	1.61	0.12	3.84
	6y	3.02	0.18	2.03
	8y	1.87	0.19	0.73
	11y	3.91	0.19	0.98

Possible improvements

***- risk aversion
- a more elaborate perceived yield curve
- disagreement
- learning about factors

Conclusion

Surveys underpredict the effect of the slope on future yields.

We lay out a simple model of learning about the yield curve.

Improves remarkably fit of moments.

Explains RE regressions

RE prices $\rho < 1$

$$
\begin{aligned}
Q_t^{RE,n} &= \delta^n \, E_t\left(\frac{P_t}{P_{t+n}}\right) \\
Y_t^{RE,n} &= -\log\delta + \mu + \phi^n(\pi_t - \mu) \\
\phi^n &\equiv \frac{\sum_{j=1}^n \rho^j}{n}
\end{aligned}
$$

$$
xr_{t+1}^n = \eta_{t+1}^{\pi} \sum_{j=1}^{n-1} \rho^j - \xi_t
$$

Analytic formulae for simplest case $\rho = 1$

$$\sigma^2_{xr^n} = \left[2n-3+(n-2)^2\frac{g}{m}\right]^2\sigma^2_{\widehat{\mathbf{s}}} \tag{5}$$

$$+(n-1)^2\sigma^2_{\eta^\pi} + \left[\frac{(n-2)^2 g}{m-1} - 1\right]^2\sigma^2_{\xi} \tag{6}$$

$$b^{xr}_n = \frac{cov(xr^n_{t+1}, \frac{Y^n_t - Y^1_t}{n-1})}{\sigma^2_s}$$

$$= \frac{\left[2n-3+(n-2)^2\frac{g}{m}\right]\frac{n-1}{n}\sigma^2_{\widehat{\mathbf{s}}} - \left(\frac{(n-2)^2 g}{m-1} - 1\right)\frac{\sigma^2_\xi}{n-1}}{\left(\frac{n-1}{n}\right)^2\sigma^2_{\widehat{\mathbf{s}}} + \frac{\sigma^2_\xi}{(n-1)^2}} \tag{7}$$

Analytic formulae for simplest case $\rho = 1$

For general π , ξ

$$b_n^{xr} - b_n^{\mathcal{E}} = \frac{(n-1)cov(E_t^{\mathcal{P}}(Y_{t+1}^{n-1}) - Y_{t+1}^{n-1}, s_t)}{\sigma_s^2},$$

$cov(E_t^{\mathcal{P}}(Y_{t+1}^{n}) - Y_{t+1}^{n}, s_t) > 0$, therefore $b_n^{xr} - b_n^{\mathcal{E}} > 0$.
For simplest case

$$b_n^{\mathcal{E}} = \frac{\sigma_\xi^2}{\frac{(n-1)^3}{n^2}\sigma_{\widehat{s}}^2 + \frac{1}{n-1}\sigma_\xi^2}$$

Model II: AR(1) but independent π and ξ

- Monthly model but yearly horizons and lags
- Serially correlated bivariate π and ξ each AR(1) but independent of each other, to match observed moments (at yearly lags)
- Estimate g by MSM

Model II Moments

Statistic	Maturity	Data	Model	t-stat
Standard deviation yields σ_{Y^n}	1y	280.00	273.75	0.28
	2y	284.04	274.94	0.38
	3y	280.63	277.47	0.13
	5y	269.98	285.35	-0.62
	7y	260.52	296.43	-1.42
	10y	247.59	318.00	-2.92
Serial correlation yields ρ_{Y^n}	1y	0.89	0.87	0.65
	2y	0.90	0.87	1.41
	3y	0.91	0.87	2.08
	5y	0.92	0.87	2.79
	7y	0.92	0.88	2.59
	10y	0.92	0.89	2.39
Standard deviation excess returns σ_{xr^n}	2y	127.11	104.86	1.92
	3y	239.72	213.80	1.29
	5y	521.33	433.49	1.78
	7y	676.19	655.58	0.31
	10y	895.49	994.95	-1.08

$\hat{g}$=0.0995

Model II: RE Regressions

Statistic	Maturity	Data	Model	t-stat
EH coefficient b^n_{xr}	2y	1.37	1.80	-0.54
	3y	3.10	4.22	-0.68
	6y	10.84	10.51	0.08
	8y	15.53	14.62	0.17
	11y	24.61	20.82	0.51
RE coefficient b^n_s	2y	0.89	0.04	4.28
	3y	1.61	0.06	4.00
	6y	3.02	0.11	2.08
	8y	1.87	0.14	0.75
	11y	3.91	0.18	0.98

Model III with RE: Moments

Statistic	Maturity	Data	Model	t-stat
Standard deviation yields σ_{Y^n}	1y	280.00	296.11	-0.72
	2y	284.04	275.48	0.36
	3y	280.63	258.46	0.91
	5y	269.98	246.55	0.95
	7y	260.52	267.36	-0.27
	10y	247.59	340.35	-3.85
Serial correlation yields ρ_{Y^n}	1y	0.89	0.91	-0.96
	2y	0.90	0.92	-0.66
	3y	0.91	0.92	-0.36
	5y	0.92	0.92	-0.29
	7y	0.92	0.94	-1.01
	10y	0.92	0.95	-1.79
Standard deviation excess returns σ_{xr^n}	2y	127.11	101.27	2.23
	3y	239.72	201.37	1.91
	5y	521.33	384.68	2.77
	7y	676.19	571.83	1.58
	10y	895.49	917.47	-0.24

$\hat{g}$=0.0536

Model III with RE: RE Regressions

Statistic	Maturity	Data	Model IR	t-stat	Model RE	t-stat
EH coefficient b^n_{xr}	2y	1.37	0.07	1.65	-0.52	2.40
	3y	3.10	1.39	1.03	-0.96	2.46
	6y	10.84	5.95	1.19	-1.98	3.12
	8y	15.53	9.70	1.09	-2.49	3.36
	11y	24.61	16.60	1.08	-3.11	3.74
RE coefficient b^n_s	2y	0.89	0.09	4.05	-0.52	7.11
	3y	1.61	0.11	3.87	-0.96	6.65
	6y	3.02	0.13	2.06	-1.98	3.58
	8y	1.87	0.13	0.76	-2.49	1.90
	11y	3.91	0.11	1.00	-3.11	1.85