Time-Consistent Fiscal Guarantee for Monetary Stability*

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Abstract

This paper provides time-consistent microfoundations to the idea that an authority with fiscal power can credibly sustain the real value of fiat money. We extend a classical OLG monetary model by introducing an authority that maximizes current agents' utility and its own expenditures by imposing taxes and carrying out market operations in the money market. We show that, when the authority has a flexible tax instrument, the optimal policy leads the monetary equilibrium to be unique. Otherwise, the uniqueness of the monetary equilibrium realizes only insofar either the utility of public expenditures is sufficiently low or the authority is sufficiently endowed with real resources. Our analysis points out to the need of microfoundations to the authority behavior in the debate on the price level determination and offers a new perspective on the importance of fiscal backing.

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1 Introduction

The recent evolution of cryptocurrency markets brought into prominence a never settle debate on what makes a fundamentally worthless asset become money and whether a truly private monetary system is possible. Common wisdom has it that monetary stability rests on the credibility of a benevolent authority that guarantees the real value of paper money. The guarantors of major currencies, central banks, typically rely on some form of fiscal backing that in principle could be invoked in case of need.

As a matter of fact, transfers between governments and central banks are negligible if compared to the size of their balance sheets and usually go one way: from the latter to the former. Moreover, the principle of central bank independence has been so remarkably established in the last decades that one may wonder if central banking might constitute an autonomous power devoted to monetary stability.

The issue is of crucial importance. If the ability to enforce transfers, i.e. fiscal power, is not strictly necessary to ensure monetary stability, then a truly private monetary systems must be possible. One, for example, that relies on a sufficiently capitalized private guarantor who is credibly committed to the stability of its paper money. Technological progress on large scale payment systems could make this possibility a concrete policy threat in a near future.

In this paper we take a first pass to this question by providing microfoundations to the idea that fiscal power is needed for monetary stability.

We reconsider the textbook overlapping generation model of Samuelson (1958) extended as in Sims (2013) to include a storage technology with an inefficiently low real return. The presence of an alternative saving asset generates a portfolio allocation problem so that money is acquired only if its real return is not lower than the one on storage. In this model, there exist: i) a unique monetary equilibrium where money is the only store of value in the economy, ii) other suboptimal equilibria where money is used together with storage, and iii) an autarky equilibrium where storage is the only store of value. As money is used together with storage, it gradually loses value up to the autarky point where it becomes worthless. Along these equilibria consumption inequality between the young and the old increases in time as more storage is used instead of money.

To this basic setting we introduce an authority that has the power to tax endowments of the young and carry out money market operations, i.e. it can use tax revenues to buy and sell money. At each period the authority has a one-period objective, which includes the current utility of the agents and its own expenditures.

We derive the optimal time-consistent policy and demonstrates that it selects the monetary equilibrium as the unique equilibrium. Along such equilibrium, the authority's expenditures are financed only through taxes and there are no money market operations. However, the off-equilibrium policy, which is anticipated by the private sector, prevents any savings in storage at the equilibrium. In particular, in response to (aggregate) savings in storage, the authority would buy money to sustain its real value and increase the consumption of the old. However, to maintain the same return between storage and money, young generations should save in storage at increasing rates which make an equilibrium where storage is used unfeasible.

This result holds true even if the authority gives an arbitrarily large weight to its own expenditures relative to agents' ones. In fact, no matter how little the authority cares about agents' utility, she always has the incentive to ensure equal consumption between the old and the young through the manipulation of market prices.

The fiscal power of the authority - i.e. the ability to set taxes in response to aggregate savings in storage - is essential to this result. By regulating money demand the authority maximizes total consumption, whereas by setting taxes she can keep her own expenditures at the optimal level in any possible state of the world. In this sense the availability of a flexible tax instruments aligns private and social preferences: ceteris paribus, both the agents and the authority are better-off in the monetary equilibrium.

We show that the situation is radically different if the authority lacks fiscal power but it can only rely on an endowment in real resources. This situation may entail the case of a public authority that fixes taxes *ex ante*[1] as well as the case of a private guarantor that therefore has no right to force transfers at will.

Without the ability to adjust taxes, there is a mis-alignment of incentives between the authority and the agents: manipulating the price level is now not only a way to equalize consumption levels across generations but also a way to gather real resources for public expenditures.

We show that, in contrast to the previous case, in a monetary equilibrium public expenditures are also financed by seigniorage. Moreover, if the authority is sufficiently endowed and the importance of its expenditures is sufficiently low, the mon-

[1]As, for example, assumed in the fiscal theory of the price level.

etary equilibrium is the only equilibrium; otherwise other equilibria also exist of two types.

First, an autarky equilibrium exists. In response to zero private demand of money, the authority has an incentive to exchange consumption goods for money at a finite ratio; nevertheless, it cannot resist the temptation to implicitly tax future money holders by inflating money away in the following period. As a consequence, zero private money demand is an equilibrium.

Second, there are other equilibria in which agents use storage and the real value of money converges to a finite value. In particular, facing a lack of resources, the authority has an incentive to sell money to increase consumption price and decrease consumption of the old for its own. In this case, as the young saves in storage the authority needs to induce more seigniorage to maintain its own expenditures; an equilibrium obtains when seigniorage alone produces the level of inflation such that the return on money matches the return on storage without for a constant level of storage over time.

To conclude, the ability of the authority to tax, which is proper of a fiscal authority, sustains confidence that policy market operations are indeed carried out to ensure monetary stability and not to raise seigniorage, no matter the degree of benevolence of the authority's objective. On the contrary, authorities that do not have fiscal power (these can also be interpreted as private money issuers), should have a sufficiently large real endowment or a sufficiently strong interest in agents' utility to sustain an exclusive use of money in the system, but also in this case inefficient inflation is produced.

Literature review. In an highly influential paper Obstfeld and Rogoff (1983, 2017) show that preventing hyperinflations in fiat money requires that an authority is committed to an arbitrarily small and probabilistic real redemption value for money. This paper shares the broad idea that an off-equilibrium guarantee may sustain the real value of fiat money at the equilibrium and provides time-consistent microfoundations to it. Microfoundations allow to unveil the conditions under which the commitment may not be credible, as envisaged by Cochrane (2011). More importantly, the argument of Obstfeld and Rogoff (1983, 2017), in contrast to ours, does not require the existence of an authority with fiscal power. In fact, since an arbitrarily small amount of real resources is sufficient to rule out hyperinflations, the power to impose taxes is unnecessary.

This paper puts the implicit guarantee idea vis-á-vis the insights from a popular literature on the interaction between monetary and fiscal policy, as pioneered by Sargent and Wallace (1981) and developed by Bruno and Fischer (1990) among others. In same spirit, we study a framework where the conduct of fiscal policy is crucial for the monetary stability. In contrast to that stream of literature, in our setting the presence of a fiscal authority is not only a source of danger, on the contrary, it does have an active role in preserving monetary stability. Consistently with Wallace (1981)'s irrelevance result, we show that interventions to back money require fiscal backing. Yet, this requirement does not imply fiscal interventions at equilibrium but only out-of-equilibrium. This feature is also distinguishes our theory from the fiscal theory of the price level.

The fiscal theory of the price level – as formulated by Leeper (1991), Sims (1994), Woodford (1995) or Cochrane (2001) and Sims (2013) among others – maintain that government's commitment to future real surplus pins down the real value of circulating nominal debt. By modeling an explicit game between the government and the households, Bassetto (2002) clarify how the fiscal policy can effectively pin down the price level without discussing its optimality. In our analysis instead the authority maximizes in a time-consistent way a well-defined one-period objective and its actions are a best reply to the actions of the private sector. Moreover, in our theory price level determination requires neither fiscal surpluses along the equilibrium nor particular restrictions on long-run behavior. In relation to the fiscal theory of the price level, our analysis clarifies that, in the presence of other storing assets, the microfoundations of the goals of the authority are crucial to ensure a positive demand for money (or government bonds); agents will switch to other saving assets in case the path of public finances does not ensure a sufficiently large return.

More recent works about the determination of the price level includes Benigno (2017), Hagedorn (2016) and Hall and Reis (2016).[2] Benigno (2017) argues that the strategy outlined in the fiscal-theory of the price level can be implemented solely by the central bank and with a 'passive' fiscal policy, when the central bank is appropriately capitalized and can pay interests. Hall and Reis (2016) argue that the price level can be controlled by the central bank by committing to paying interest on reserves. Hagedorn (2016) shows price level determinacy in a model where the government can commit to future nominal deficits.

In contrast with these papers, we do not assume any form of commitment on the

[2]Other references on the topic focusing on monetary rules include Loisel (2009), Atkeson et al. (2010) or Adao et al. (2011).

side of the fiscal/monetary authority, consistently with the discussion by Cochrane (2011) on credibility. Importantly, Cochrane (2011) clarifies that the credibility of the policymaker's actions should not be established only *on equilibrium* to show equilibrium uniqueness, but more importantly *out-of-equilibrium*, when private agents' expectations are not consistent with the central bank's desired policy. Barthlemy and Mengus (2018) investigate a similar approach but they focus on interest rate rules and they assume that currency is traded in any state of the world.

Other approaches to deal with equilibrium multiplicity in monetary models have been considered. In a nutshell, the first one is legal-tender theory of money where, based on the medium of exchange approach of Kiyotaki and Wright (1989), the government can force agents to trade using a given currency Aiyagari and Wallace (1997), Li and Wright (1998). The second one is the tax-theory of money as modeled by Starr (1974) and Goldberg (2012) among others, that postulates that government can impose real value to money. Our model shows that capital controls – in the form of sanctions for not buying a particular asset, in this case money – are not necessary to implement a monetary equilibrium.

Our paper is also related to the literature on bubbles starting with Tirole (1985), although our analysis abstract from money having any transaction value. As Asriyan et al. (2016) we are interested in the interaction of monetary and fiscal policy in economies where money is a rational bubble.[3] The main difference with respect to this paper is that they look at the effects of monetary policy in equilibrium and under commitment, while we consider the off-equilibrium implications of policy decisions under discretion.

Our paper is also related to the literature on bailouts and time-inconsistency as Schneider and Tornell (2004), Farhi and Tirole (2012) or Acharya and Yorulmazer (2007). In our setting, the *ex post* incentive to rescue money holders is *ex ante* desirable as it leads to select the monetary equilibrium. A related paper is Mengus (2017) who shows that government's bailouts may optimally be in the form of asset purchases and, when expected, such bailouts lead even intrinsically worthless to be traded at positive prices. In contrast with his approach, we investigate the impact of public interventions in selecting monetary equilibria so that these interventions are off-equilibrium.

[3] See also the contributions of Gal (2014) and Allen et al. (2017).

2 A Simple Model of Fiat Money

In this section, we introduce a simple model of fiat money along the lines of Sims (2013). Let us consider an economy populated by overlapping generations of homogeneous households of unitary mass and a monetary/fiscal authority. Time is discrete and indexed by $t \in \{1, 2, ...\}$. We assume perfect foresight.

Households At each date, a new generation of homogeneous agents has born. Each agent lives two periods and then disappears. The representative agent born at time t maximizes the following utility function:

$$U_t \equiv u(C_{y,t}) + u(C_{o,t+1}), \tag{1}$$

where $C_{y,t} \geq 0$ and $C_{y,o} \geq 0$ are individual consumption in the first and second period respectively; $u(C)$ is a generic utility function such that $u' > 0, u' < 0$ and $\lim_{C \to 0} u' = \infty$.

Each agent born in period t is endowed with a quantity W_t of consumption good in the first period of his life. The representative young (henceforth "the young") born at time t can exchange one unit of consumption for $P_t \geq 0$ units of money or store consumption goods using a storage technology that yields a gross return $\theta < 1$ in the next period - we denote by $S_t \geq 0$ the amount of goods stored and we assume that the households cannot short-sell money, that is $M_t \geq 0$.

The budget constraint of the first period reads as:

$$C_{y,t} + \frac{M_t}{P_t} + S_t + T_{t,y} = W_t, \tag{2}$$

where M_t denotes the quantity of money purchased by the young at time t; $T_{t,y}$ is a tax imposed by a fiscal authority to the young at time t. Let us call $D_t \equiv S_t + M_t/P_t$ the total stock of savings.

Money has value *only* as long as can be sold in exchange of consumption, in which case money is priced. The consumption the representative old (henceforth just "the old") at time t is:

$$C_{o,t} = \frac{M_{t-1}}{P_t} + \theta S_{t-1} + T_{t,o}, \tag{3}$$

where P_t is the price level of consumption at time t. The first generation is born at date 0, lives just one period, has available a stock of fiat money $M_0 > 0$ does not have storage $S_0 = 0$, and has utility function $U_0 \equiv u(C_{o,1})$.

The monetary/fiscal authority The authority controls transfers to the young $T_{t,y}$ and to the old $T_{t,o}$ and can also buy and store money. Let us denote by $M_{g,t}$ the amount of money held by the authority. In particular, the balance sheet of the authority satisfies:

$$T_{t,y} + \frac{M_{g,t-1}}{P_t} = \frac{M_{g,t}}{P_t} + T_{t,o} + G_t. \tag{4}$$

The left-hand side of (4) represents the resources of the authority – tax revenues and the value of her money holdings; the right-hand side collects emplacements: new money holdings and transfers to the old and government expenditures where $G_t \geq 0$. A policy $\mathcal{P}_t \equiv (T_{y,t}, M_{g,t}, G_t, T_{o,t})$ is a collection of taxes, money purchases and government expenditures that are implemented by the authority at time t.

The policy objective In this section, we introduce an objective for the authority and derive its optimal policy. We assume that the authority, as the agents, is active for one period only. At each date t, we assume that the government selects policies $\{\mathcal{P}_t\}_{t=1}^{\infty}$ so as to maximize the following objective function:

$$\log C_{y,t} + \log C_{o,t} + \lambda \log G_t, \tag{5}$$

that is, the authority cares about the utility of the current young and the old generation, but also the level of public expenditures, G_t weighted by a coefficient $\lambda > 0$. The case $\lambda = 0$ characterizes one in which the authority is fully benevolent; on the opposite, the case $\lambda \to \infty$ is one in which the authority is fully selfish. However, note that G_t does not necessarily entail a "waste". The $\lambda \log G_t$ component can be added to the utility of the agents without that any of our argument is affected. In such a case, G_t would denote a public good whose provision is out of the control of the agents.

Note that, at each date, there is an authority responsible for the current policy and objective; in other words she cannot count on future commitment. This setup leads to a time-consistent policy.[4]

[4] In an equivalent interpretation, the authority has an intra-temporal objective $\sum_{t=0}^{\infty} (U_t + \lambda G_t)$ but lacks of commitment power. The equivalence with a dynamic setup obtains exactly in the case

Policy instruments We first observe that, for any given level of private demand of money $M_t \geq 0$ and a given fiscal stance $G_t - T_{y,t} + T_{o,t}$, the authority can affect the market price by varying its own stock of money. This can be show formally by combining the authority's budget constraint with the market clearing condition for money (8), so to obtain:

$$T_{y,t} - T_{o,t} = \frac{M_{g,t} - M_{g,t-1}}{P_t} + G_t = \frac{M_{t-1} - M_t}{P_t} + G_t. \tag{6}$$

This equality captures formalizes the fact that by controlling $M_{g,t}$ and the fiscal surplus $G_t - T_{y,t} + T_{o,t}$ the authority can effectively choose a market price P_t in response to a private market demand for money M_t.

It is important to stress that the ability of the authority of being price maker derives from the possibility of being a net buyer of money. Even in absence of private demand for money, $M_t = 0$, the authority can act as a buyer of last resort, fixing the rate of exchange between money held by the old and real goods (a variation in fiscal stance). The possibility to determine a *current price* is in the same spirit of Obstfeld and Rogoff (1983). Notice that this is in stark contrast with economies where the authority is always a net supplier, as it is when "money" are one period liabilities of the authority. We will come back on this subtle and important point in due time, just let us clarify here that having the ability to determine the *current* price does not imply inducing private demand M_t being strictly greater than zero.

Equilibrium The definition of a market equilibrium is as follows.

Definition 1. *For a given quantity of fiat money $\bar{M}$ and a given sequence of endowments $\{W_t\}_{t=1}^{\infty}$ and policy $\{\mathcal{P}_t\}_{t=1}^{\infty}$, **a market equilibrium** is profiles of consumption $\{C_{y,t}\}_{t=1}^{\infty}$ and $\{C_{o,t}\}_{t=0}^{\infty}$, money holdings $\{M_t\}_{t=0}^{\infty}$, storage $\{S_t\}_{t=0}^{\infty}$, a sequence of prices $\{P_t\}_{t=1}^{\infty}$, such that, at each period $t \in \{1, 2, ...\}$:*

i) *taking prices as given, the young chooses (M_t, S_t) to maximize* (1) *s.t.* (2)-(3),

ii) *the good market clears:*

$$C_{y,t} + C_{o,t} + S_t + G_t = W_t + \theta S_{t-1}, \tag{7}$$

of logarithmic utility (see later).

iii) *the financial market clears:*

$$M_{g,t} + M_t = \bar{M}. \tag{8}$$

A market equilibrium with optimal policy *is a market equilibrium conditional to a sequence of policies* $\{\mathcal{P}^*{}_t\}_{t=1}^{\infty}$ *such that at each date* $t \geq 1$*, date-*t *policy* $\mathcal{P}_t^*$ *maximizes* (5)*, for given storage decisions* (S_{t-1}, S_t, S_{t+1}) *and given past and future policies* $\left(\mathcal{P}_{t-1}^*, \mathcal{P}_{t+1}^*\right)$ *for any* $t > 1$ *and for given* (S_0, S_1, S_2) *and* $\mathcal{P}_2^*$ *at* $t = 1$*.*

For the moment we assume a fixed stock of money. This assumption is useful to clearly see how the following result do not rely on the ability of the authority to create money. Note that market clearing in the financial market implies that a change in the stock of money held by the authority corresponds to a opposite change in the stock held by the private sector.

Private sector optimization Let us study the problem of the representative agent. This problem is to maximize utility as stated by (1) under the constraints of (2) and (3). In order to get easily computed solutions that give us some insight into how the model works we will assume logarithmic utility $u(C) = \log C$. We also restrict to the case $T_{o,t} = 0$ for each t, which is the relevant case in the following discussions.

We denote by ρ_t the gross per-unit real return on savings D_t. With logarithmic utility the optimal stock of savings at time t is:

$$D_t \equiv S_t + \frac{M_t}{P_t} = \frac{W_t - T_{y,t}}{2}. \tag{9}$$

for any $\rho_t = (\theta S_t + M_t/P_{t+1})/D_t$. Given that M_t and S_t have to be both positive, this optimal stock of savings also implies that:

$$S_t, \frac{M_t}{P_t} \leq \frac{W_t - T_{y,t}}{2}. \tag{10}$$

The portfolio allocation between money and storage depends on the expected return on money. In particular, we have:

$$\frac{M_t}{P_t} = \frac{W_t - T_{y,t}}{2} \quad \text{and } S_t = 0 \qquad \text{if} \qquad \Pi_{t+1} < \frac{1}{\theta}, \tag{11}$$

$$\frac{M_t}{P_t} + S_t = \frac{W_t - T_{y,t}}{2} \qquad \text{if} \qquad \Pi_{t+1} = \frac{1}{\theta}, \tag{12}$$

$$S_t = \frac{W_t - T_{y,t}}{2} \quad \text{and } \frac{M_t}{P_t} = 0 \qquad \text{if} \qquad \Pi_{t+1} > \frac{1}{\theta}, \tag{13}$$

where $\Pi_{t+1} \equiv P_{t+1}/P_t$ is the inflation rate from time t to time $t+1$. The inflation rate is the inverse of the return on money. When the return on money is greater (resp. smaller) than the return on real storage, agents save everything in money (resp. storage). Money and storage may coexist only insofar yield the same return.

Using (9) we can recover the actual law of motion for inflation as:

$$\Pi_{t+1} \equiv \frac{P_{t+1}}{P_t} = \frac{M_{t+1}}{M_t} \left(\frac{W_t - T_{y,t} - 2S_t}{W_{t+1} - T_{y,t+1} - 2S_{t+1}} \right), \tag{14}$$

that, together with (4) and (11)-(13) for any t, describe the equilibrium. Let us study now how the policy affects market equilibrium.

3 Equilibria with constant endowment

In this section we will restrict our attention to the case with a constant endowment $W_t = W$. We will study how the set of equilibria changes with different specification of the policy.

3.1 Absence of Policy

To isolate the role of the authority, it is useful to describe the equilibrium without policy intervention, i.e. with $\mathcal{P}_t = (0, 0, 0, 0)$ at each date t. In this case, the law for inflation (14) becomes

$$\Pi_{t+1} = \frac{W - 2S_t}{W - 2S_{t+1}}, \tag{15}$$

given that $M_t = \bar{M}$ for any $t \geq 1$. We can then easily check that a continuum of market equilibria (absent policy) exists as the following proposition states.

Proposition 1. *For a given $\bar{M}$ and a given sequence of endowments $W_t = W$ for any $t \geq 1$ and policy $\mathcal{P}_t = (0,0,0,0)$ for any $t \geq 1$ and initial conditions $M_0 > 0$ and $S_0 = 0$, multiple market equilibria exist.*

i) *A* ***pure monetary equilibrium*** *exists such that*

$$P_t = P^* \quad \textit{with} \quad P^* \equiv \frac{2M_0}{W} \tag{16}$$

$$\frac{M_t}{P_t} = \frac{W}{2} \tag{17}$$

for any $t \geq 1$,

ii) *An* ***asymptotic autarky equilibrium*** *exists for each $s \geq 1$ such that (16)-(17) holds for $t < s$, and*

$$\begin{aligned} P_t &= \theta^{-t+1} P_s \quad \textit{with} \quad P_{s-1} < P_s < \theta^{-1} P_{s-1} \\ \frac{M_t}{P_t} &= \frac{W}{2} - S_t \quad \textit{with} \quad \lim_{t\to\infty} \frac{M_t}{P_t} = 0, \end{aligned}$$

holds for $t < s$.

iii) *A* ***pure autarky equilibrium*** *exists where $P_t \to \infty, \Pi_t > \theta^{-1}, M_t/P_t = 0$ and $S_t = W/2$ for any $t \geq 1$.*

Proof. Postponed to Appendix A.1 □

A pure monetary equilibrium exists where agents perfectly equalize consumption across periods. This equilibrium, which is denoted with a circle marker in Figure 1, is characterized by a constant real value of money M_t/P_t equal to the real value of savings $W/2$, zero storage and inflation $\Pi_t = 1$, i.e. constant prices.

Asymptotic autarky equilibria also exist. Any initial price level such $P_t > 2\bar{M}/W$ corresponds to an equilibrium with storage $S_t = W/2 - \bar{M}/P_t$, which implies that the real value of the private stock of money is smaller that the desired amount of real saving, which is $W/2$. As long as storage and money are used at the same time, by arbitrage we have $\Pi_t = \theta^{-1}$ which means that, next period, the real value

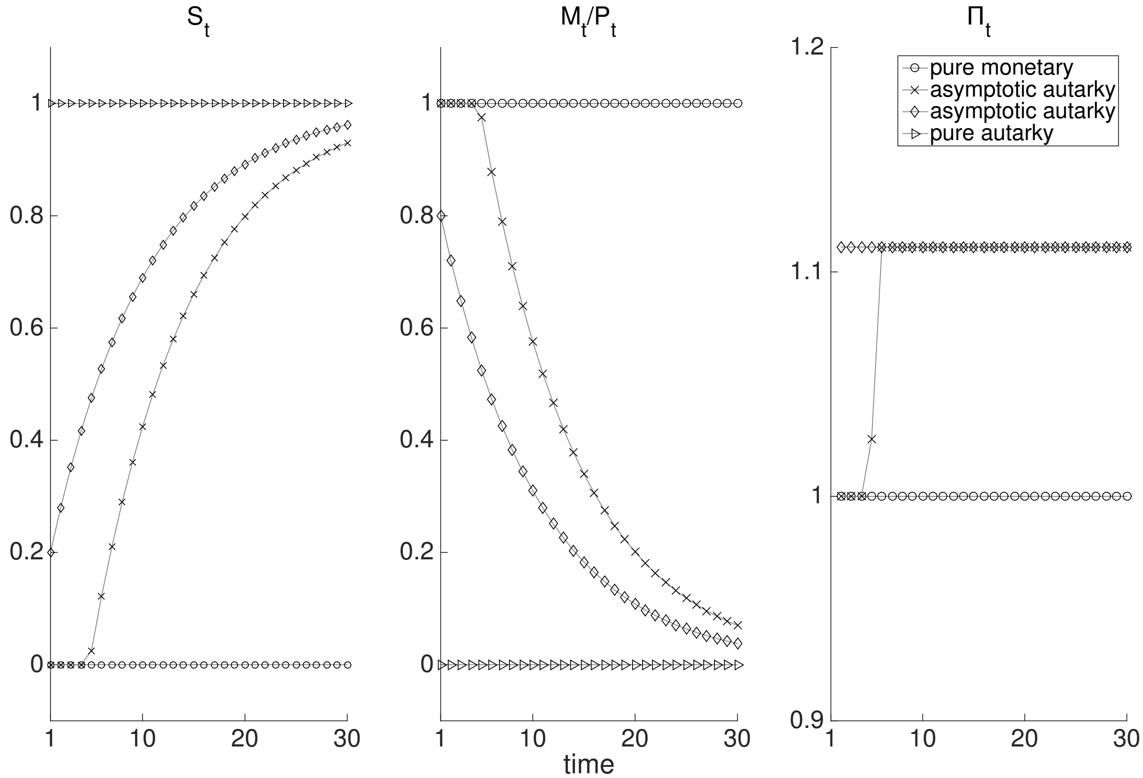

Figure 1: Equilibria without policy intervention for $\theta = 0.9, W = 2, \lambda = 0$ and $\bar{M} = 1$.

of money necessarily reduces and storage expands. Along these paths storage follows the process

$$S_{t+1} = \theta S_t + (1 - \theta)\,\frac{W}{2} \quad \text{with} \quad S_t > 0. \tag{18}$$

In the end, storage converges to $\lim_{t\to\infty} S_t = W/2$ at which money has no real value, i.e. $\lim_{t\to\infty} \bar{M}/P_t = 0$. This equilibrium is denoted with a diamond marker in Figure 1.

It is worth noting that whereas there is a unique initial price associated with the unique pure monetary equilibrium P^*, there exist a continuum of equilibria with storage associated to it. In other words, even nailing down the initial price to P^* is not sufficient for having the unique pure monetary equilibrium and a single path of inflation. In Figure 1 we plot one of these. In period $t = 5$ the price level jumps to a value strictly higher than P^{ast} so that inflation rate at time $t = 5$ reaches a value between 1 and θ^{-1}, which is still compatible with an optimal private choice of having no storage at time $t = 4$. However, as the price departs from P^* necessarily there must be some positive amount of savings in storage, i.e. $S_5 > 0$; this in turns requires $\Pi_6 = \theta^{-1}$ and implies that $S_6 > S_5$. At this point the economy enters in a asymptotic autarky equilibrium.

Finally, **a pure autarky equilibrium** is represented with a triangle marker in Figure 1: in this case, storage is maximal, the real value of monetary savings is nul and with prices being infinitely large and growing at a rate larger than θ^{-1} (not depicted).

3.2 Optimal Time-Consistent Policy

In what follows we restrict our attention to the set of policies without transfers to old, i.e. from here onward we look for optimal policies of the form $\mathcal{P}_t^* = (T_{y,t}^*, M_{g,t}^*, G_t^*, 0)$. Our choice is without loss of generality because, as we will see, direct transfers to the old are not necessary to let the monetary equilibrium being the unique equilibrium outcome.

Let us then state the problem of the authority as follows.

Problem 1 (Flexible taxes). *At any date $t \geq 1$, an optimal policy is a $\mathcal{P}_t^* = (T_{y,t}^*, M_{g,t}^*, G_t^*, 0)$ that solves:*

$$\max_{T_{y,t}, M_{g,t}, G_t} \left\{ \log C_{y,t} + \log C_{o,t} + \lambda \log G_t \right\},$$

subject to

$$T_{y,t} + \frac{M_{g,t-1}}{P_t} = \frac{M_{g,t}}{P_t} + G_t$$

taking into account agents' decision process on consumption:

$$C_{y,t} = \frac{M_t}{P_t} + S_t = \frac{W - T_{y,t}}{2} \tag{19}$$

$$C_{o,t} = \frac{M_{t-1}}{P_t} + \theta S_{t-1} \tag{20}$$

and market clearing conditions (7) *and* (8)*, with $S_0 = 0$ and $M_0 \leq \bar{M}$.*

We can make explicit the power to affect prices by plug (6) into the consumption/saving decision of the young to obtain:

$$\frac{M_t}{P_t} = W - G_t - \frac{M_{t-1}}{P_t} - 2S_t. \tag{21}$$

Such expression for current real saving can be plugged in (19), which together with (20) we use to replace consumption in the objective of the authority. The problem of the authority becomes

$$\max_{P_t,G_t}\left\{\log\underbrace{\left(W - G_t - \frac{M_{t-1}}{P_t} - S_t\right)}_{=C_{y,t}} + \log\underbrace{\left(\frac{M_{t-1}}{P_t} + \theta S_{t-1}\right)}_{=C_{o,t}} + \lambda \log G_t\right\}, \tag{22}$$

as one depending only on current price and government's consumption. Looking into it we note that by increasing the price the authority implicitly dries resources form the young in favor of the old, and that increasing the authority's consumption further reduces consumption of the young.

Intuitively, the optimal amount of public consumption should be such that the marginal utility of consumption of the young is equal to the marginal utility of public consumption weighted by λ. Formally, the solution to this problem is given by:

$$\begin{cases} G_t = \lambda C_{y,t},\ P_t = \frac{(2+\lambda)M_{t-1}}{W-(1+\lambda)\theta S_{t-1}-S_t} & \text{with} \quad \lim_{P_t\to\infty} C_{y,t} \geq \lim_{P_t\to\infty} C_{o,t} \\ G_t = \lambda C_{y,t},\ P_t \to \infty & \text{otherwise.} \end{cases} \tag{23}$$

The optimal price is the price that equalizes consumption of the young with the old. A corner solution emerges when the young consumes less than the young at the autarky limit ($P_t \to \infty$). When this is the case the authority would like to choose a negative price to transfer resources from the latter to the former, which is not feasible; as a second best the authority chooses the price to be infinity; in any case its consumption remains a λ fraction of the consumption of the young.

The following proposition summarizes all these findings:

Proposition 2. *For a given $\lambda \geq 0$, a given $\bar{M}$, a given sequence of endowments $W_t = W$ for any $t \geq 1$ and policy $\mathcal{P}_t = \mathcal{P}_t^*$ for any $t \geq 1$ and initial conditions $M_0 > 0$ and $S_0 = 0$, a unique equilibrium exists. In such equilibrium, $P_1 = P^*$ and* *(i) there is no inflation* $\Pi_t = 1$,

(ii) the real value of money is equal to the desired level of real savings

$$\frac{M_t}{P_t} = \frac{W}{2+\lambda} \quad \textit{and} \quad S_t = 0,$$

(iii) there are no policy market interventions:

$$T_{y,t} = G_t = \frac{\lambda}{2+\lambda} W,$$

for each $t \geq 1$.

Proof. Postponed to Appendix A. □

The pure monetary equilibrium is the only equilibrium outcome when policy is optimally chosen. In contrast to the logic of the fiscal theory of the price level, the authority determines the price although it runs at zero surplus and there are no market interventions along the equilibrium. Out-of-equilibrium instead the authority is active in a way that when anticipated by agents make the monetary equilibrium be the unique outcome.

Asymptotic autarky equilibria are no longer equilibria because policy interventions changes the sequence of equilibrium storage compatible with $\Pi_{t+1} = \theta^{-1}$ from (18) to

$$\theta^2 S_{t-1} - 2\theta S_t + S_{t+1} - (1-\theta)W = 0 \quad \text{with} \quad S_t > 0.. \tag{24}$$

This is a second-order differential equation yielding monotonic paths converging to a steady state storage value of $W/(1-\theta)$, which is an unfeasible value. Thus, such paths illustrated in Figure 2 in light grey cannot be part of an equilibrium. The economic intuition for this dynamic behavior is the following. When young use storage they reduce the real value of money and so the consumption of to the old: inequality increases. The authority has then an incentive to decrease prices, that is to fight inflation. However, since on this equilibrium the sequence of prices is determined by arbitrage conditions, the quantity of storage used has to adjust to make the price path optimal from the point of view of the authority. Such adjustment requires that storage increases faster for the same inflation path; at the same time money shifts from the private sector to the authority's balance sheet.

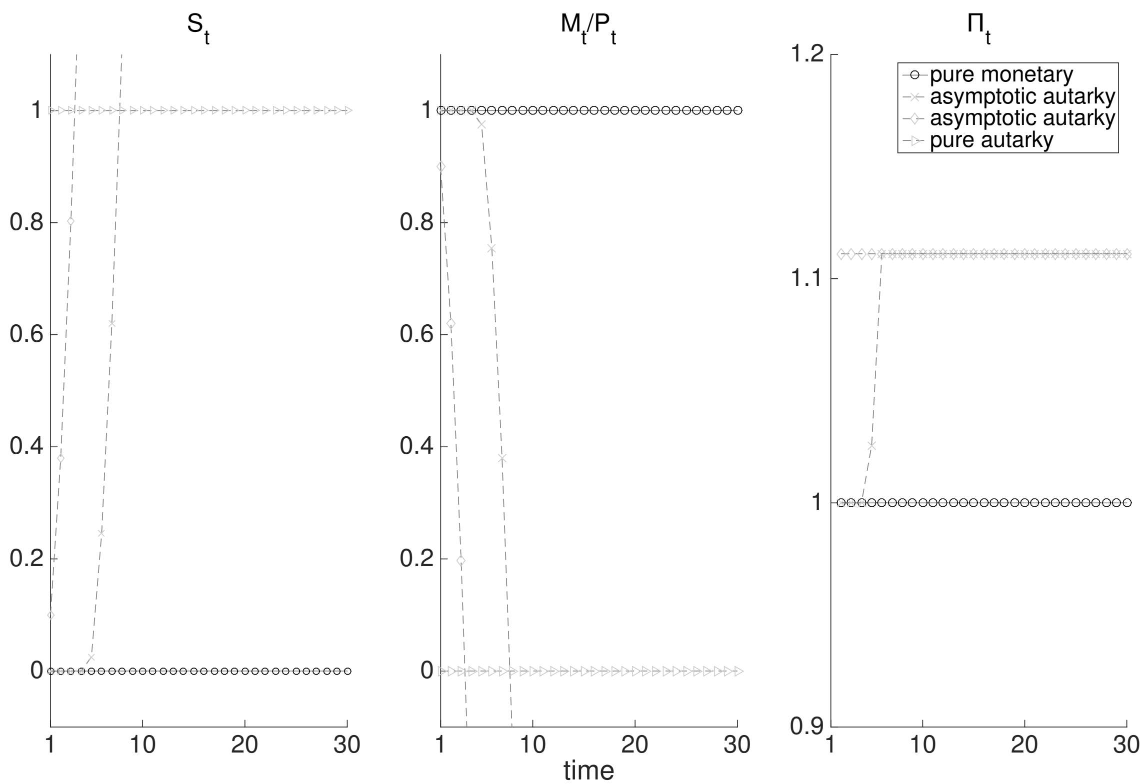

Figure 2: Equilibria without policy intervention (in black) for $\theta = 0.9, W = 2, \lambda = 0$ and $\bar{M} = 1$. Out-of-equilibrium paths in grey.

Pure autarky equilibria are also no longer equilibria. The economic intuition for this result relies on the fact that, in response to a sequence of zero private money demand starting at time t, the authority has always the incentive to create infinite return on money (infinite deflation), i.e. fixing a positive price at t and a zero price level at time $t + 1$. In particular, the authority has an incentive to buy money from the old in exchange of real goods at a positive rate to sustain consumption of old at time t; at the same time it is willing to drive the future price down to zero as money holdings of the next generation of old goes to zero, in the attempt to sustain the real value of their monetary savings. This time-consistent strategy is incompatible with the private choice of not using money and so this cannot be part of an equilibrium.

Finally, note that the proposition is independent of the weight on government's expenditures, λ. The key intuition is that the consumption of the government is a fraction of the consumption of the old (which is equal to one of the young), who is better off in an economy where money has value. As a result, whatever the value of λ, the government always prefers the economy to stay in the monetary equilibrium where everyone is better off.

Remark on the equivalence with unrestricted fiscal policies. The allocation implemented through our restricted policy is equivalent to the optimal one implementable with unrestricted fiscal policies. The difference is that in our original case the implementation occurs through changes in the real value of money. However, in Appendix B we show that the choice of restricting to zero transfers to the old is optimal in at least two cases. First, when there is heterogeneity (even if arbitrarily small) in agents' discount factor but the authority lacks of individual-specific fiscal instruments, the authority would optimally set transfers to the old equal to zero. This happens because, as agents privately acquire the individually optimal stock of money, transfers operated implicitly through changes in the real value of money are more efficient than direct fiscal transfers. Second, when collecting taxes occur with sunk costs, the authority would optimally set transfers to the old equal to zero and raise the minimal amount of taxes needed. This is because operating implicit transfer thought the money market saves on such costs, no matter how small they are.

3.3 Time-consistent Policy with Fixed Baking

Let us now investigate the situation where the authority cannot change taxes in reaction to saving choices. This situation can be interpreted as one in which the authority is committed ex-ante to a given fiscal stance as well as one in which the authority is a private entity lacking therefore any fiscal power. This requires restricting further the policies space to $\hat{\mathcal{P}}_t = (\bar{T}, M^*_{g,t}, G^*_t, 0)$ where, to maintain the analogy with the previous case, taxes on the young $T_{y,t} = \bar{T}$ are taken fixed through time. The authority can still back its interventions in the money market by adjusting its expenditures.[5] The authority solves the following problem:

[5] Our policy specification of no taxes on the old is not a restriction as in this section we do not consider taxes as controls.

Problem 2 (Fixed taxes)**.** *At date t, the authority solves:*

$$\max_{M_{g,t},G_t}\left\{\log C_{y,t}+\log C_{o,t}+\lambda\log G_t\right\},$$

subject to

$$\bar{T}+\frac{M_{g,t-1}}{P_t}=\frac{M_{g,t}}{P_t}+G_t$$

taking into account agents' decision process on consumption:

$$C_{y,t}=\frac{M_t}{P_t}+S_t=\frac{W-\bar{T}}{2}\ \textit{and}\ C_{o,t}=\frac{M_{t-1}}{P_t}+\theta S_{t-1}$$

and market clearing conditions (7) *and* (8)*.*

As before, by combining the authority's budget constraint with the market clearing condition for money, (8), we obtain:

$$\bar{T}=\frac{M_{g,t}-M_{g,t-1}}{P_t}+G_t=\frac{M_{t-1}-M_t}{P_t}+G_t. \tag{25}$$

In this case, controlling $M_{g,t}$ and G_t given $T_{y,t}=\bar{T}$ fixed is equivalent to choosing a market price P_t. It is important to note that the authority can still choose a price level by simply imposing the rate of exchange between the money held by the old and consumption goods.

We can use the budget constraint of the authority to eliminate M_t/P_t from the optimal private saving choice and obtain an expression for G_t. We can then rewrite the problem of the authority as

$$\max_{P_t}\left\{\log\frac{W-\bar{T}}{2}+\log\underbrace{\left(\frac{M_{t-1}}{P_t}+\theta S_{t-1}\right)}_{=C_{o,t}}+\lambda\log\underbrace{\left(\frac{W+\bar{T}}{2}-\frac{M_{t-1}}{P_t}-S_t\right)}_{=G_t}\right\}. \tag{26}$$

The solution to this problem is:

$$\begin{cases} P_t=\frac{2(1+\lambda)M_{t-1}}{W+\bar{T}-2\lambda\theta S_{t-1}-2S_t} & \text{with} \quad \lim_{P_t\to\infty}C_{o,t}\leq\lim_{P_t\to\infty}G_t \\ P_t\to\infty & \text{otherwise.}\end{cases} \tag{27}$$

By loosing the ability to change taxes in response to private saving choices the authority looses the ability to influence the demand of savings and so the consumption of the young. There is now a trade-off in the use of the price for money as an instrument. On the one hand the authority may reduce consumption inequality by lowering the price for money. On the other hand the it can increase public expenditures by decreasing the price for money. Which force prevails depends on the initial level and the importance of public expenditures. Moreover, in contrast to the case studied before, this case leads to a multiplicity of equilibria depending on the level of taxes. The following proposition summarizes the findings.

Proposition 3. *For a given $\bar{M}$ and a given sequence of endowments $W_t = W$ for any $t \geq 1$ and policy $\mathcal{P}_t = \hat{\mathcal{P}}_t$ for any $t \geq 1$ and initial conditions $M_0 > 0$ and $S_0 = 0$, multiple market equilibria with optimal policy exist for any $\lambda \geq 0$.*

(i) *Provided that $1 + \lambda \leq \theta^{-1}$, a **pure monetary equilibrium** exists such that:*

$$\begin{aligned} P_t &= (1+\lambda)^{t-1} P^* \quad \text{with} \quad P^* \equiv \frac{2M_0}{W - \bar{T}} \\ \frac{M_t}{P_t} &= \frac{W - \bar{T}}{2} \end{aligned}$$

for any $t \geq 1$,

(ii) *Provided that $1 + \lambda \leq \theta^{-1}$ and*

$$\bar{T} < \frac{\lambda\theta}{2 - (2+\lambda)\,\theta} W,$$

*an **asymptotic storage equilibrium** exists for each $s \geq 1$ such that (28)-(28) holds for $t < s$, and*

$$\begin{aligned} P_t &= \theta^{-t+1} P_s \quad \text{with} \quad P_{s-1} < P_s < \theta^{-1} P_{s-1} \\ \frac{M_t}{P_t} &= \frac{W - \bar{T}}{2} - S_t \quad \text{with} \quad \lim_{t\to\infty} \frac{M_t}{P_t} = \frac{\theta\lambda}{1-\theta}\frac{W + \bar{T}}{2} - \bar{T} \geq 0, \end{aligned}$$

holds for $t < s$.

(iii) *When*

$$\bar{T} < \frac{\lambda\theta}{2 + \lambda\theta} W,$$

*a **pure autarky equilibrium** exists where $P_t \to \infty$, $\Pi_t > \theta^{-1}$, $M_t/P_t = 0$ and $S_t = (W - \bar{T})/2$ for any $t \geq 1$.*

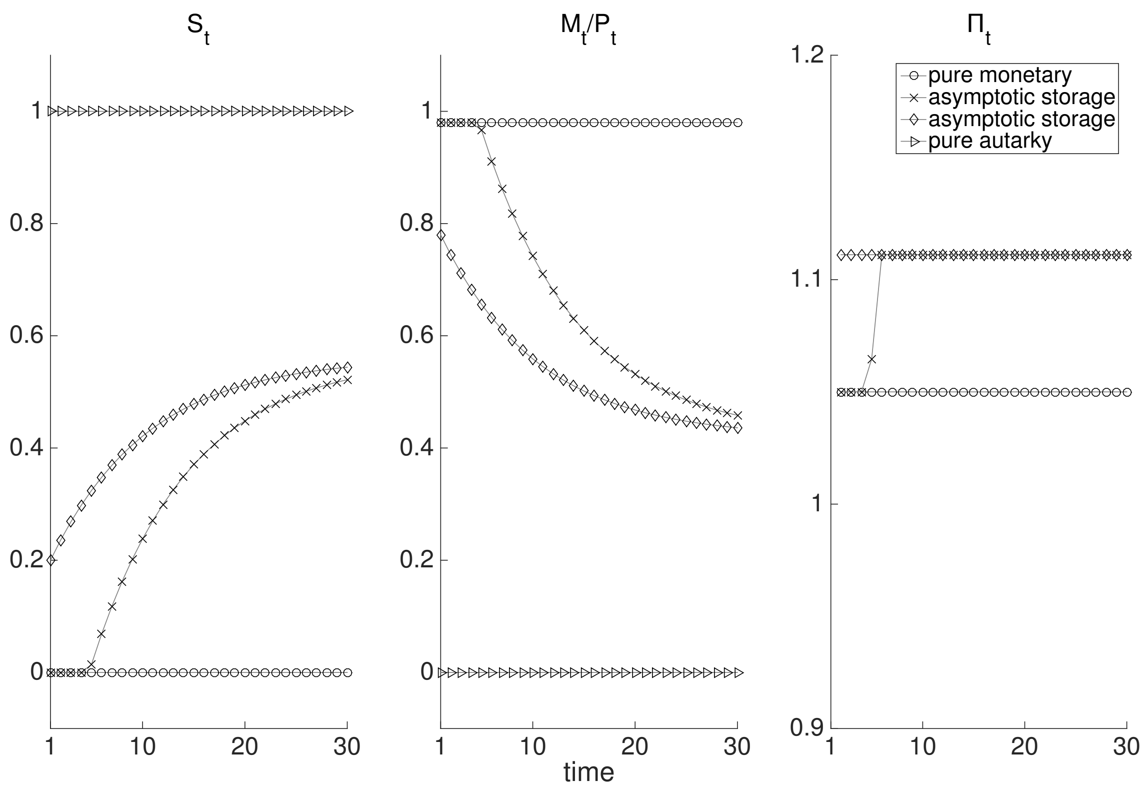

Figure 3: Equilibria with fixed backing for $\theta = 0.9, W = 2$, $\bar{T} = 0.04$, $\bar{M} = 1$ and $\lambda = 0.05$.

The proposition is illustrated by Figure 3. In contrast to the cases before, now the authority will systematically exploits its seigniorage power to raise revenues for public expenditure. This requires that the authority keeps an inflation rate above θ^{-1} by constantly increasing the stock of circulating money. This case requires therefore that the authority may create money in the form of its own liabilities.

The **pure monetary equilibrium** exists provided that the time-consistent optimal rate of inflation does not exceed the inverse of the rate of return on storage. When this condition is violated, money returns are dominated ans so only storage will be used for saving.

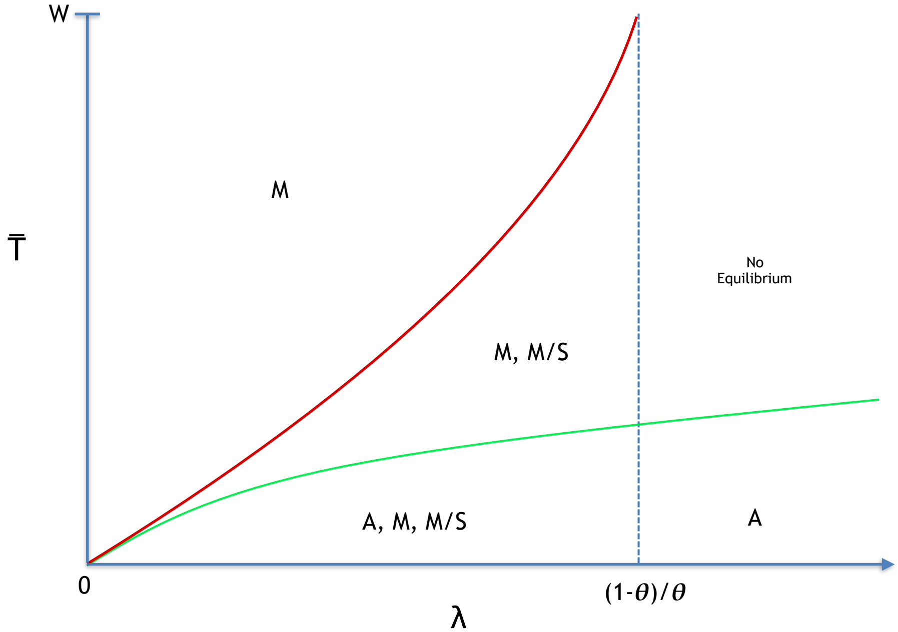

Figure 4: Conditions for multiple equilibria with fixed backing. A, M/S and M denote the regions where the autarky, asymptotic storage and pure monetary equilibria exist, respectively.

Moreover, in contrast to the case plotted in Figure 1, equilibria in which storage is used jointly with money converge to a situation in which the real value of monetary savings and storage reach a steady state level. This equilibria, which we call **asymptotic storage equilibria**, exist when fiscal baking is sufficiently small or the importance of public expenditures is sufficiently large. The law of motion of storage in this case is given by

$$S_{t+1} = \theta S_t + (1 - (1 + \lambda)\theta)\frac{W + \bar{T}}{2} \quad \text{with} \quad S_t > 0,$$

which converges to a feasible value under the conditions uncovered by the proposition. In contrast to (18) and (24), in this case, under certain conditions, the steady state value of storage can reach a positive value strictly between $(W - \bar{T})/2$ and 0. Such equilibrium paths are illustrated in Figure 3. The economic intuition for such a dramatic change in dynamic behavior is the following. In this case the determination of the price level entail a trade-off between the consumption of the old and public expenditures. If expenditures are low, the authority has an incentive to increase prices, that is to boost inflation. As before, since on this equilibrium the sequence of prices is determined by arbitrage conditions, the quantity of storage used has to adjust, but not the adjustment goes in the opposite direction. It requires that storage increases slower for the same inflation path; at the same time money shifts from the authority's balance sheet to the private sector.

With even lower levels of fiscal backing (utility of public expenditures) **the autarky equilibrium** may also exist. When taxes are too low and, thus, government expenditures are low as well, the government may even have the incentive to drive the price level to negative values so as to tax money holdings. Yet, negative price levels are not feasible, but such an incentive prevents credible deflations in absence of private demand of money. As a consequence, autarky can be an equilibrium outcome.

The set of equilibria crucially depends on the level of resources held by the authority and the importance of public expenditures. This is illustrated by Figure 4 showing that for the pure monetary equilibrium to arise the authority must be sufficiently endowed and the importance of expenditures should not exceed a certain threshold.

This trade-off does not arise when taxes can be freely set, as then the government has sufficient tools to adjust government expenditures. In such a case, the authority ensures the value of money to improve total available consumption goods and taxes to ensure the fraction that it needs. In contrast, when taxes are fixed, the government can only adjust expenditures to purchase money and, thus, it trades off the welfare gains of money trading with its cost of cutting expenditures.

Overall, the main reason that committing to taxes does not ensure the uniqueness of the equilibrium as, for example, in Sims (2013) is that agents are not forced to hold money: households can hold no money ($M_t = 0$). In contrast, when money is the only saving asset, as it is generally assumed in the fiscal theory of the price level, it is sufficient to pin down the current real value of the stock of money in circulation to select an equilibrium but this assumes away that the stock of money in circulation is also a private agents' decision.

4 Fluctuations in endowment

In this section we will look at the case of time-varying endowment. To grasp the main forces we will focus on the case a one-time increase in endowment occurring at time τ, i.e. $W_t = W$ for any $t \neq \tau$ and $W_\tau > W$. We will initially look at the dynamics in the absence of policy and then study the optimal policy reaction.

4.1 Absence of policy

Here we look at the case where policy is absent, $\mathcal{P}_t = (0, 0, 0, 0)$ at each t,. In the case of fluctuations in endowment, we have

$$\Pi_{t+1} = \frac{W_t - 2S_t}{W_{t+1} - 2S_{t+1}}. \tag{28}$$

This modification has important consequences on the set of equilibria. As a first result note that the pure monetary equilibrium may not exists any longer. In fact, $S_t = 0$ for any $t \geq 1$ is not an equilibrium when $\Pi_{\tau+1} = W_\tau/W \geq \theta^{-1}$. In this case, there not exist an equilibrium where $S_\tau = 0$ because the return on storage *necessarily* exceeds the one on money. Thus, with $W_\tau > \theta^{-1}W$, S_τ must be strictly positive.

On the other hand, it is possible now an equilibrium where $S_\tau > 0$ and $S_{\tau+1} = 0$. To see this notice that (18) now becomes

$$S_{\tau+1} = \theta S_\tau + \frac{W - \theta W_\tau}{2}, \tag{29}$$

so that, when $\theta W_\tau > W$, there exists a value of S_τ, namely

$$0 < \hat{S}_\tau \equiv \frac{\theta W_\tau - W}{2\theta} < \frac{W_\tau}{2},$$

such that $S_{\tau+1} = 0$.

To conclude that there is an equilibrium where $S_\tau = \hat{S}_\tau > 0$ and $S_{\tau+1} = 0$ we should make sure that $S_t = 0$ for any $t \neq \tau$ are rational choices. The latter is trivially verified for future dates given that from τ onward endowments are constant and (18) is forward looking. However, for $S_{\tau-1} = 0$ being part of the equilibrium we need to have $\Pi_\tau = W/(W_\tau - 2\hat{S}_\tau) < \theta^{-1}$, which is always true given $\theta < 1$. Therefore, $\hat{S}_\tau$ is the amount of storage necessary for the economy to be on a pure monetary equilibrium from $\tau + 1$ onward. Note also that along this equilibrium $\Pi_{\tau-1} = W/(W_\tau - 2\hat{S}_\tau) = \theta$, i.e. a deflation at rate θ occurs in anticipation of the increase in endowment.

Finally, notice that at each $S_\tau > \hat{S}_\tau$ corresponds to an asymptotic autarky equilibrium from τ onward, whereas $S_\tau < \hat{S}_\tau$ cannot be at any equilibrium.

Proposition 4. *For an initial quantity of fiat money $\bar{M}$ and a given sequence of endowments $W_t = W$ for any $t \neq \tau$ and $W_\tau > W$ and policy $\mathcal{P}_t = (0,0,0,0)$ for any $t \geq 1$ and initial conditions $M_0 > 0$ and $S_0 = 0$, multiple market equilibria exist.*

i) *A **almost pure monetary equilibrium** exists such that*

$$P_t = P^* \equiv \frac{2M_0}{W} \quad \text{for} \quad t \neq \tau - 1, \tau \quad \text{and} \quad \Pi_{\tau-1} = \Pi_\tau^{-1} = \theta \tag{30}$$

$$\frac{M_0}{P_t} = \frac{W}{2} \quad \text{for} \quad t \neq \tau \quad \text{and} \quad \frac{M_0}{P_t} = \frac{W}{2} - \hat{S}_t \quad \text{for} \quad t = \tau \tag{31}$$

ii) *An **asymptotic autarky equilibrium** exists for each $s \geq 1$ with $s \neq \tau$ such that (30)-(31) holds for $t < s$, and*

$$\begin{aligned} P_t &= \theta^{-t+1} P_s \quad \text{with} \quad P_{s-1} < P_s < \theta^{-1} P_{s-1} \\ \frac{M_t}{P_t} &= \frac{W}{2} - S_t \quad \text{with} \quad \lim_{t \to \infty} \frac{M_t}{P_t} = 0, \end{aligned}$$

holds for $t < s$.

iii) *A **pure autarky equilibrium** exists where $P_t \to \infty, \Pi_t > \theta^{-1}, M_t/P_t = 0$ and $S_t = W/2$ for any $t \geq 1$.*

Intuitively, an increase in next period endowment pushes the price level down at a rate θ, that is, the return on money increases a because the same amount of money will buy a large amount of consumption in the future: storage at the time just before the increase in endowment is optimally set to zero. The date of the increase the price level shuts up at a rate $theta^{-1}$ and the return on money lowers because now the same quantity of money will buy lower consumption in the future: storage is optimally set positive if the increase in endowment is strong enough. In particular storage at time τ has to be such that money return is equal to the return on storage. However there is only one of such storage values, namely $\hat{S}_\tau$, for which future storage value can be zero. This value is the only optimal positive value of storage consistent with an equilibrium where money does not lose value asymptotically.

4.2 Optimal policy reaction with fluctuations

In this Section, we study the optimal policy reaction to a one-time increase in endowment occurring at time τ, as before, $W_t = W$ for any $t \neq \tau$ and $W_\tau > W$. To get a direct comparison with the case of absence of policy we will focus on the case $\lambda = 0$.

The optimal policy modify the law of motion of storage in two ways. First, whenever $S_t > 0$ we have that

$$\Pi_{t+1} = \frac{W_t + \theta S_{t-1} - 3S_t}{W_{t+1} - \theta S_t - S_{t+1}} = \theta^{-1},$$

or

$$S_t = \frac{S_{t+1} + \theta^2 S_{t-1} - W_{t+1} + \theta W_t}{2\theta}, \tag{32}$$

which is consistent with (24) in case of constant endowments.

Second, whenever $S_t = 0$ instead optimal saving choices require

$$\Pi_{t+1} = \frac{W_t + \theta S_{t-1}}{W_{t+1} - S_{t+1}} < \theta^{-1},$$

or

$$\theta^2 S_{t-1} < W_{t+1} - \theta W_t - S_{t+1}, \tag{33}$$

that is, the return on money is higher than the return on storage.

These two elements nails down the unique equilibrium consistent with an optimal policy response.

Proposition 5. *For an initial quantity of fiat money $\bar{M}$ and a given sequence of endowments $W_t = W$ for any $t \neq \tau$ and $W_\tau = W + \epsilon$ and policy $\mathcal{P}_t = (0, 0, 0, 0)$ for any $t \geq 1$ and initial conditions $M_0 > 0$ and $S_0 = 0$, a unique market equilibrium with optimal policy exists.*

Such equilibrium is characterized by a sequence of $n^* \geq 1$ positive storage values from S_τ to $S_{\tau+n^*-1}$ such that

$$\frac{\sum_{i=1}^{n^*} i\theta^{n^*-i}}{\theta^{n^*-1}}\frac{1-\theta}{\theta}W < \epsilon < \frac{\sum_{i=1}^{n^*+1} i\theta^{n^*+1-i}}{\theta^{n^*}}\frac{1-\theta}{\theta}W$$

and

$$S_\tau = \frac{n^*\theta^{n^*}\epsilon + \left(n^*\theta^{n^*} - \sum_{i=1}^{n^*}\theta^{n^*-i}\right)W}{(1+n^*)\,\theta^{n^*}} \tag{34}$$

$$S_{\tau+n^*-n} = \frac{n\theta^n\theta S_{\tau+n^*-1-n} + \left(n\theta^n - \sum_{i=1}^{n}\theta^{n-i}\right)W}{(1+n)\,\theta^n} \quad \textit{for} \;\; 1 \le n \le n^*-1. \tag{35}$$

Proof. See appendix A.4 □

Figure 5 illustrates the equilibrium with $W = 10$, $\varepsilon = 1$, $M_0 = 1$ and $\theta = 0.99$. Compared with the monetary equilibrium without any intervention, the effect of the policy is in a higher and more persistent use of storage. This is consistent with our intuition that the policy makes optimal higher storage for the same inflation rate. In particular, at the time of the increase in endowment, the authority raises taxes and buy money so that money stock initially decreases. This also leads to a rapid decrease of the price level. From period $\tau = 3$ onward until period $\tau + n * -1 = 5$. i.e. for $n^* = 3$ periods, storage is positive and gradually decreasing. In this period of time inflation is at θ^-1, meaning that the price level increases linearly until reaches the steady state level $W/2$ in period 7. The price increases as the authority now sells money, increasing private monetary holdings and rebating seigniorage revenues in the form of positive transfers to the young. During the whole sequence consumption across agents is equalized by smoothing consumption forward.

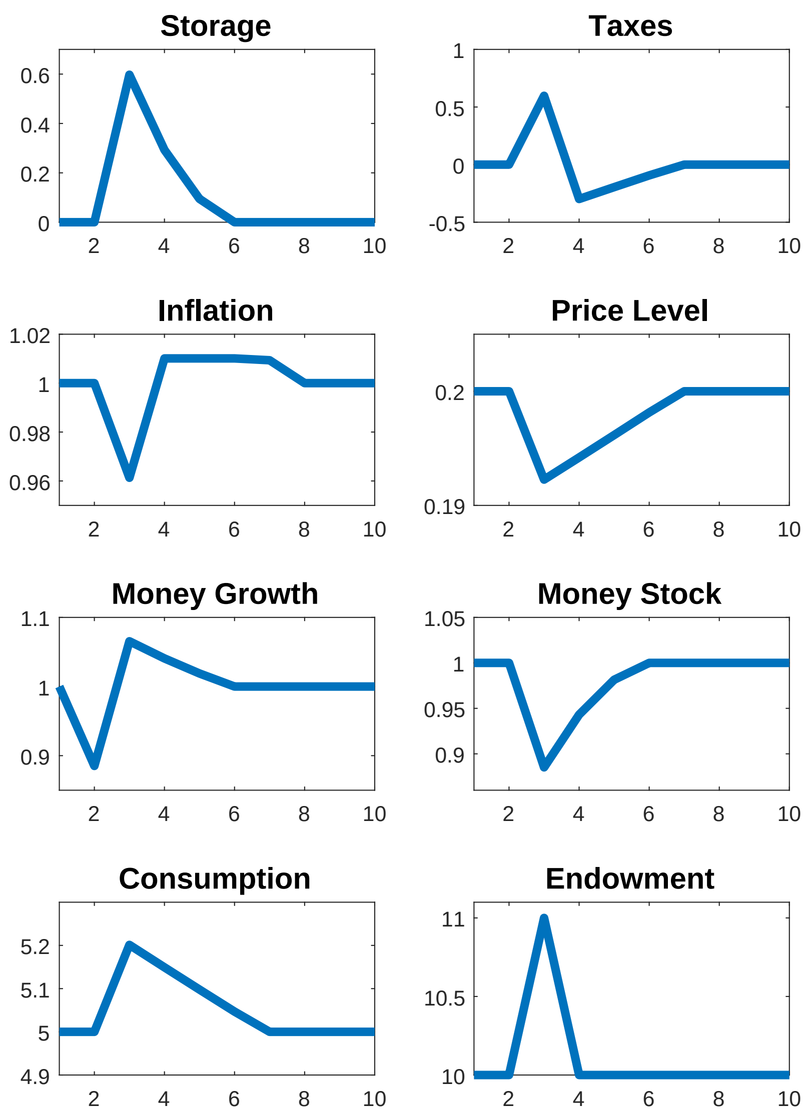

Figure 5: Market equilibrium with optimal policy in the case $W = 10, \tau = 3, \epsilon = 1, M_0 = 1, \lambda = 0$ and $\theta = 0.99$.

A Proofs

A.1 Proposition 1

It is easy to note that $\Pi_{t+1} = 1 < \theta^{-1}$ and $S_t = 0$ for any t is an equilibrium; one in which money is always used and storage never. We refer to this equilibrium as the **pure monetary equilibrium**.

To check if there exist an equilibrium where storage is used jointly with money we should use the arbitrage condition (12). For $S_t > 0$ at time t we must have $\Pi_{t+1} = \theta^{-1}$. In this case, (15) obtains as

$$S_{t+1} = \theta S_t + (1 - \theta) \frac{W}{2}, \tag{36}$$

which implies $S_{t+1} \geq S_t$, given the limit $S_t \leq W/2$ for each date t. Therefore we obtain that, if storage is used in one period, it must necessarily be used on a larger extent next period. In fact, an equilibrium for each initial level of storage $S_1 \in \left[0, \frac{W}{2}\right)$ (S_0 is not an optimal choice, i.e. (18) is not an equilibrium condition for S_0) exists such that storage is always used jointly with money. It is easy to show that in the long run, storage and the real money balance satisfy:

$$\lim_{t\to\infty} S_t = \frac{W}{2} \text{ and } \lim_{t\to\infty} \frac{\bar{M}}{P_t} = 0 \tag{37}$$

for any initial level of storage S_1, where the latter obtains as a consequence of the former because of (9). There are equilibria in which storage is always used, prices grows at a rate $1/\theta$ and money looses value in time until it eventually become worthless; let us call them the **asymptotic autarky equilibria**.

Importantly, all asymptotic autarky equilibria do not necessarily feature storage at date-0 and it is possible to construct asymptotic autarky equilibria where storage is not used until a certain date s after which it is always used. In fact, notice that $S_{s-1} = 0$ only requires that $\Pi_s < \theta^{-1}$, that is

$$0 \leq S_s < (1 - \theta) \frac{W}{2}.$$

Thus, at each date t, after having only used money in past periods, it is possible to start using storage. What is peculiar of the environment with constant endowment is that once storage is used it will used for ever; this is because, for a given S_t, (18) implies a certain S_{t+1} which has the property $S_{t+1} \geq S_t$, given the limit $S_t \leq W/2$ for each t.

Finally, there also exists a **pure autarky equilibrium** defined as one in which $S_t = W/2$ and $\bar{M}/P_t = 0$ for each t in which money is never used and the price level is infinite and grows at a rate larger than $1/\theta$.

A.2 Proposition 2

Using (23) in (21) we get the actual laws of motion of real money and inflation:

$$\frac{M_t}{P_t} = \frac{W - (3+\lambda)\, S_t + \theta S_{t-1}}{2\, +\lambda}, \tag{38}$$

$$\Pi_{t+1} = \frac{W - (3+\lambda)\, S_t + \theta S_{t-1}}{W - (1+\lambda)\theta S_t - S_{t+1}}. \tag{39}$$

provided $W > (1+\lambda)\theta S_t + S_{t+1}$, otherwise we have $M_{t+1}/P_{t+1} \to 0$ and $\Pi_{t+1} \to \infty$ is not defined. We are ready now to investigate investigate how optimally chosen policies affect equilibrium outcomes.

The pure monetary equilibrium. First, the pure monetary equilibrium where $S_t = 0$ at each t is still an equilibrium. This can be easily seen by checking that $S_t = 0$ at any t implies $\Pi_{t+1} = 1$ at any t, which are mutually consistent. Along that equilibrium we also have $M_t = M_0$, $G_t = T_{y,t} = \lambda W/(2+\lambda)$ and $C_{y,t} = C_{o,t} = W/(2+\lambda)$ at any $t \geq 1$.

Non existence of asymptotic autarky equilibria. Here we show that there are no equilibria where both money and storage are used.[6]

Indeed, suppose that one such equilibria exist. This implies that at some date t, storage is positive ($S_t > 0$) and the inflation rate satisfies $\Pi_t = \theta^{-1}$. Combined with the law of motion of inflation (39), this implies:

$$\theta^2 S_{t-1} - 2\theta S_t + S_{t+1} - (1-\theta)W = 0. \tag{40}$$

Let us show first that the law of motion above implies that if $S_t > 0$ then $S_{t+\tau} > 0$ for $\tau \geq 1$. First note that $S_{t-1} = 0$ and $S_t > 0$ implies

$$S_{t+1} = (1-\theta)W + 2\theta S_t > S_t > 0.$$

as $S_t < W$; hence $S_{t-1} = 0$, $S_t > 0$, $S_{t+1} = 0$ cannot be an equilibrium sequence. Then, let us check whether $S_{t-1} > 0$, $S_t > 0$, $S_{t+1} = 0$ is part of a possible solution, that is, if

[6] We should note here that there could be heterogeneity in portfolio allocation between storage and money, i.e. agents may randomize. We check in Appendix C that randomization does not affect our results in no way.

there exists a couple of $S_{t-1} > 0$, $S_t > 0$ such that

$$S_{t-1} = \frac{2\theta S_t + (1-\theta)W}{\theta^2}.$$

in this case, the constraint $S_{t-1} \leq W/2$ requires a negative S_t which is not possible, therefore $S_{t-1} > 0$, $S_t > 0$, $S_{t+1} = 0$ cannot be an equilibrium sequence.

Then, we can rule out the equilibrium where $S_t > 0$ for each $t \geq \tau$. To see that, note that the solution to the difference equation (24) can be rewritten in the homogeneous form:

$$\theta^2 S_{w,t-1} - 2\theta S_{w,t} + S_{w,t+1} = 0$$

with $S_{w,t} \equiv S_t - W/(1-\theta)$. This difference equation has a single real characteristic root $0 < \theta < 1$, so that the sequence $S_{w,t}$ converges monotonically to 0, for any initial conditions. As a consequence, S_t converges to $\bar{S} = W/(1-\theta)$. However, this contradicts that the maximal storage is $S_t = W/2 \leq \bar{S}$ and therefore asymptotic autarky equilibria do not exist.

Non existence of a pure autarky equilibrium. Here we prove that an equilibrium in which real money balance are always valueless ($M_t/P_t = 0$ for any t) do not exist. In such an equilibrium, date-1 real money balance held by private agents also satisfies $M_1/P_1 = 0$. Then the government budget constraint and the financial market clearing condition imply that $T_{y,1} = M_0/P_1 + G_1$. By substituting the latter into the optimal autarky decision $S_1 = (W - T_{y,1})/2$, we get that the storage compatible with $M_1/P_1 = 0$ is

$$S_1 = \bar{S}(S_0) \equiv \frac{W + \theta S_0}{3 + \lambda}, \tag{41}$$

where, note, $\bar{S}(x) \leq W/2$ as $x \leq W/2$. Plugging this into (23), which still holds, for initial conditions $M_0 > 0$ and $S_0 = 0$we get that the price in the first period is

$$\tilde{P}_1 = \frac{3+\lambda}{W} M_0$$

which is finite and positive. Hence $P_1 = \infty$ is not a solution, the authority would always like to exchange money held by the old for consumption good that she collect by raising taxes. Therefore, $M_1/P_1 = 0$ can happen only with $M_1 = 0$, however $M_1 = 0$ cannot be an equilibrium. To see this, note that, in this case, the price at time 2 would be

$$P_2 = \frac{(2+\lambda)M_1}{W - (1+\lambda)\theta \bar{S}(0) - S_2} \geq 0$$

which is finite and positive, even for the maximal storage at period 2, namely $S_2 = \bar{S}\left(\bar{S}\left(0\right)\right)$ (in which case $M_2 = 0$). The important observation is that with $M_1 = 0$ and $S_1 = \bar{S}\left(0\right)$ necessarily $\Pi_2 = P_2/\tilde{P}_1 = 0$ irrespective of M_2. However, $\Pi_2 = 0$ is not compatible with private storage choice $S_1 \neq 0$. The return on money would then be $+\infty$, which obviously exceeds the one on storage θ. The same reasoning applies at any t for $S_{t-1} = 0$. Therefore we conclude that an equilibrium where $M_1/P_1 = 0$ is not possible, and as a consequence, a pure autarky equilibrium does not exist.

A.3 Proof of proposition 3

Using (25) and (27), we get the actual law of motion of inflation of the real value of savings and inflation as:

$$\frac{M_t}{P_t} = \frac{W-\bar{T}}{2} - S_t \tag{42}$$

$$\Pi_{t+1} = \frac{P_{t+1}}{P_t} = \frac{(1+\lambda)\left(W+\bar{T}\right) - 2(1+\lambda)S_t}{W+\bar{T}- \ 2\lambda\theta S_t - 2S_{t+1}} \tag{43}$$

provided $W + \bar{T} \geq 2\lambda\theta S_t + 2S_{t+1}$, otherwise we have $M_{t+1}/P_{t+1} \to 0$ and $\Pi_{t+1} \to \infty$. We are ready now to investigate investigate how optimally chosen policies affect equilibrium outcomes.

The pure monetary equilibrium. The pure monetary equilibrium where $S_t = 0$ at each t is still an equilibrium provided $1 + \lambda < \theta^{-1}$. This can be easily seen by checking that $S_t = 0$ at any t implies $\Pi_{t+1} = 1 + \lambda$ at any t from (43). In turn, $S_t = 0$ requires that $\Pi_{t+1} \leq \theta^{-1}$, thus implying that $1 + \lambda$ does not exceed θ^{-1}. Along that equilibrium, money is growing at a rate $1 + \lambda$:

$$M_t = (1+\lambda)^t M_0. \tag{44}$$

Government expenditures are financed through taxes and seigniorage:

$$G_t = \frac{\lambda}{1+\lambda}\frac{W+\bar{T}}{2}. \tag{45}$$

Finally, private consumption satisfies:

$$C_{y,t} = \frac{W-\bar{T}}{2} \text{ and } C_{o,t} = \frac{W+\bar{T}}{2(1+\lambda)} \text{ at any } t \geq 1. \tag{46}$$

In case $\Pi_{t+1} > \theta^{-1}$ implies $S_t > 0$, so that a pure monetary equilibrium does not exist in that case.

Existence of asymptotic storage equilibria. We investigate now whether there are equilibria where both money and storage are used. $S_t > 0$ implies $\Pi_t = \theta^{-1}$ at t that, is

$$S_{t+1} = \theta S_t + (1 - (1 + \lambda)\theta)\frac{W + \bar{T}}{2}.$$

Let us first consider the case $1 > (1 + \lambda)\theta$. In such a case, $S_t > 0$ implies $S_{t+\tau} > 0$ for $\tau \geq 1$. However, an equilibrium where $S_t > 0$ for each $t \geq \tau$ requires a sequence $\{S_t\}_{t=1}^{\infty}$ converging monotonically to

$$\bar{S} = \frac{1 - (1 + \lambda)\theta}{1 - \theta}\frac{W + \bar{T}}{2}.$$

As previously noted, to be feasible, $\bar{S}$ should satisfy $\bar{S} \leq \left(W - \bar{T}\right)/2$. As a result, a necessary condition to be an equilibrium is:

$$\bar{T} < \frac{\theta\lambda}{2 - (2 + \lambda)\,\theta}W.$$

Otherwise, an equilibrium where money and storage are jointly used does not exist.

Similarly to the case without any policy, all asymptotic storage equilibria do not necessarily feature storage at date-0 and it is possible to construct asymptotic storage equilibria where storage is not used until a certain date s after which it is always used. In fact, notice that $S_{s-1} = 0$ only requires that $\Pi_s \leq \theta^{-1}$, that is

$$0 \leq S_s < (1 - (1 + \lambda)\theta)\frac{W + \bar{T}}{2}$$

Thus, at each date t, after having only used money in past periods, it is possible to start using storage. Also here once storage is used it will used for ever.

Finally, in the case when $1 < (1 + \lambda)\theta$, the sequence of storage S_t converges to a negative value; however this violates the constraint $S_t \geq 0$. Thus, in this case, an equilibrium where storage is used with money does not exist.

Existence of pure autarky equilibria. We study here the conditions for the existence of a pure autarky equilibrium – i.e. one in which $M_t/P_t = 0$ for any t. As before we look at the initial period. Suppose that $M_1/P_1 = 0$. Then $\bar{T} = M_0/P_1 + G_1$ so that (27) still hold at $t = 1$. The storage compatible with $M_t/P_t = 0$ is

$$S_1 = \frac{W - \bar{T}}{2}.$$

Plugging this into (27), for initial conditions $M_0 > 0$ we get that, in this case, the price level in the first period has to satisfy:

$$\underline{P}_1 = \frac{1+\lambda}{\bar{T}} M_0$$

which is positive and finite provided $\bar{T} > 0$. Hence $P_1 = \infty$ is not a solution as long as $\bar{T} > 0$, in this case the authority would always like to exchange money held by the old for consumption good that she collects by raising taxes. Thus, with $\bar{T} > 0$, autarky $M_1/P_1 = 0$ can happen only with $M_1 = 0$, however can be $M_1 = 0$ a solution?

Suppose that $M_1 > 0$. In this case, the price at time 2 is:

$$P_2 = \frac{2(1+\lambda)}{(1-\lambda\theta)W + (1+\lambda\theta)\bar{T} - 2S_2} M_1$$

which is not always positive and finite for maximal storage $S_2 = \left(W - \bar{T}\right)/2$. In particular, $P_t = \infty$ for each t is a possible equilibrium outcome when:

$$\bar{T} < \frac{\lambda\theta}{2+\lambda\theta} W,$$

Only in such a case a pure autarky equilibria exists.

A.4 Proof proposition 5

Let us denote $\epsilon_t = W_t - W$. To start with let us focus on the conditions to have a path where $S_t > 0$, $S_{t+1} = 0$ and $S_{t+2} = 0$ at some t. We apply step (33) to S_{t+1} and we obtain $\theta^2 S_t < W_{t+2} - \theta W_{t+1}$ and then (32) to S_t to get

$$S_t = \frac{\theta^2 S_{t-1} + \theta W_t - W_{t+1}}{2\theta}$$

from which it is obvious that, to get $S_t > 0$ either W_t is sufficiently big or it must be $S_{t-1} > 0$. In particular, consider our endowment process. It should be

$$S_t = \frac{\theta(\theta S_{t-1} + \epsilon_t) - (1-\theta)W}{2\theta} < \frac{(1-\theta)W}{\theta^2} \tag{47}$$

i.e.

$$\frac{1-\theta}{\theta}W < \theta S_{t-1} + \epsilon_t < \frac{\theta+2}{\theta}\frac{1-\theta}{\theta}W.$$

If ϵ_t satisfies the inequality with $S_{t-1} = 0$ then $t = \tau$ is the only time of positive storage is a solution. Otherwise, it must be that also $S_{t-1} > 0$.

Consider then, $S_{t-1} > 0$. Because of (32) (and its implication (47)), this requires in turn that

$$S_{t-1} = \frac{S_t + \theta^2 S_{t-2} - W_t + \theta W_{t-1}}{2\theta} = \frac{2\theta^3 S_{t-2} + 2\theta^2 W_{t-1} - \theta W_t - W_{t+1}}{3\theta^2}$$

from which it is obvious that either W_{t-1} is sufficiently big or it must be $S_{t-2} > 0$. In particular, consider our endowment process. It should be

$$\frac{1-\theta}{\theta}W < \theta S_{t-1} = \frac{2\theta^2(\theta S_{t-2} + \epsilon_{t-1}) + (2\theta^2 - \theta - 1)W}{3\theta} < \frac{\theta+2}{\theta}\frac{1-\theta}{\theta}W \tag{48}$$

i.e.

$$\frac{\theta+2}{\theta}\frac{1-\theta}{\theta}W < \theta S_{t-2} + \epsilon_{t-1} < \frac{\theta^2+2\theta+3}{\theta^2}\frac{1-\theta}{\theta}W$$

then $t-1 = \tau$ and t being the only times of positive storage is a solution. Otherwise, it must be that also $S_{t-2} > 0$.

By iterating we have

$$S_{t+1-n} = \frac{n\theta^n \theta S_{t-n} + n\theta^n W_{t+1-n} - \sum_{i=1}^n \theta^{n-i} W_{t+1-n+i}}{(1+n)\,\theta^n} \geq 0,$$

which requires that either W_{t+1-n} is sufficiently large or S_{t-n} must be positive. In particular, consider our endowment process. It should be

$$S_{t+1-n} = \frac{n\theta^n(\theta S_{t-n} + \epsilon_{t+1-n}) + \left(n\theta^n - \sum_{i=1}^n \theta^{n-i}\right)W}{(1+n)\,\theta^n} < \frac{\sum_{i=1}^n i\theta^{n-i}}{\theta^{n-1}}\frac{1-\theta}{\theta}W \tag{49}$$

we just need to verify that

$$\frac{1}{n\theta^n}\left((1+n)\,\theta^{n-1}\frac{\sum_{i=1}^n i\theta^{n-i}}{\theta^{n-1}}\frac{1-\theta}{\theta}W - \left(n\theta^n - \sum_{i=1}^n \theta^{n-i}\right)W\right) = \frac{\sum_{i=1}^{n+1} i\theta^{n+1-i}}{\theta^n}\frac{1-\theta}{\theta}W$$

$$\frac{1}{n}\left((1+n)\,\theta^{n-1}\frac{\sum_{i=1}^n i\theta^{n-i}}{\theta^{n-1}} - \left(n\theta^n - \sum_{i=1}^n \theta^{n-i}\right)\frac{\theta}{1-\theta}\right) = \sum_{i=1}^{n+1} i\theta^{n+1-i}$$

$$\frac{1}{n}\left((1+n)\sum_{i=1}^{n} i\theta^{n-i}-\left(n\theta^{n+1}-\sum_{i=1}^{n}\theta^{n+1-i}\right)\frac{1}{1-\theta}\right)=\sum_{i=1}^{n+1} i\theta^{n+1-i}$$

$$(1+n)\,(1-\theta)\sum_{i=1}^{n} i\theta^{n-i}-n\theta^{n+1}+\sum_{i=1}^{n}\theta^{n+1-i}=(1-\theta)n\sum_{i=1}^{n+1} i\theta^{n+1-i}$$

which indeed hold for any n.

We are ready now to work out the path of storage under the optimal policy reaction to a one shot increase in endowment, specifically $W_\tau = W + \epsilon$ with $\epsilon > 0$ and $W_t = W$ for $t \neq 1$.

B Micro-foundations for money purchases

In this appendix, we investigate some motives that make money purchases preferred to direct transfers to old households.

Preference heterogeneity To begin with, agents can differ in their preferences. This can translate into heterogeneous savings. Let us elaborate an example of such heterogeneity.

Let us assume that agents' preferences dare as follows: $u(c_O, c_Y) = \log c_Y + \gamma_i \log c_0$ with heterogeneous γ_i. We also assume that a group of mass p of agents are such that $\gamma_i = 1$ – *savers* – and the rest are such that $\gamma_i = 0$ – *consumers*. The former agents save half of their endowment net of taxes to be consumed in the second period of their life – as in the benchmark model – , while the latter do not save at all.

As a result, their consumption while being young are:

$$c^S_{y,t} = \frac{M^S_t}{P_t} + S^S_t = \frac{W - T_{y,t}}{2} \text{ and } c^C_{y,t} = W - T_{y,t}, \tag{50}$$

where $c^S_{y,t}$ is the consumption of savers and $c^S_{y,t}$ the consumption of consumers.

The government's budget constraint is:

$$T_{t,y} = \frac{M_{t-1} - M_t}{P_t} - T_{o,t} \tag{51}$$

and, thus:

$$c^S_{y,t} = \frac{M^S_t}{P_t} + S^S_t = \frac{W - \frac{M_{t-1}-M_t}{P_t} + T_{o,t}}{2} \text{ and } c^C_{y,t} = W - \frac{M_{t-1} - M_t}{P_t} + T_{o,t}. \tag{52}$$

Integrating the first equality across all savers yields:

$$\frac{M_t}{P_t} = \frac{p}{2-p}\left(W - \frac{M_{t-1}}{P_t} + T_{o,t}\right) - \frac{2}{2-p}S_t \tag{53}$$

We can plug this value into the expressions for agents' consumption levels so that the current stock of money M_t disappears:

$$c^S_{y,t} = \frac{M^S_t}{P_t} + S^S_t = \frac{1}{2-p}\left(W - \frac{M_{t-1}}{P_t} + T_{o,t} - S_t\right) \tag{54}$$

$$c^C_{y,t} = \frac{2}{2-p}\left(W - \frac{M_{t-1}}{P_t} + T_{o,t} - S_t\right) = 2c^S_{y,t} \tag{55}$$

The resulting problem for the authority is:

$$\max_{P_t, T_{o,t}} \left\{ \int \log\left(c^i_{y,t}\right) \mathrm{di} + \int \log\left(\frac{M_{i,t-1}}{P_t} + \theta S_{i,t-1} - T_{o,t}\right) \mathrm{di} \right\}. \tag{56}$$

The first order conditions with respect to P_t and $T_{o,t}$ are as follows:

$$M_{t-1}\frac{1}{W - \frac{M_{t-1}}{P_t} + T_{o,t} - S_t} = \int \frac{\gamma_i M_{i,t-1}}{\frac{M_{i,t-1}}{P_t} + \theta S_{i,t-1} - T_{o,t}} \tag{57}$$

$$\frac{1}{W - \frac{M_{t-1}}{P_t} + T_{o,t} - S_t} = \int \frac{\gamma_i}{\frac{M_{i,t-1}}{P_t} + \theta S_{i,t-1} - T_{o,t}} \tag{58}$$

with $c^S_{y,t}$ and $c^C_{y,t}$ defined by equations (54) and (55).

Let us compute the right hand sides of the two conditions:

$$\int \frac{\gamma_i M_{i,t-1}}{\frac{M_{i,t-1}}{P_t} + \theta S_{i,t-1} - T_{o,t}} = \frac{M_{t-1}}{1/p\left(\frac{M_{t-1}}{P_t} + \theta S_{t-1}\right) - T_{o,t}} \tag{59}$$

$$\int \frac{\gamma_i}{\frac{M_{i,t-1}}{P_t} + \theta S_{i,t-1} - T_{o,t}} = \frac{p}{1/p\left(\frac{M_{t-1}}{P_t} + \theta S_{t-1}\right) - T_{o,t}} \tag{60}$$

The first order conditions can then be written:

$$\frac{1}{W - \frac{M_{t-1}}{P_t} + T_{o,t} - S_t} = \frac{1}{1/p\left(\frac{M_{t-1}}{P_t} + \theta S_{t-1}\right) - T_{o,t}} \tag{61}$$

$$\frac{1}{W - \frac{M_{t-1}}{P_t} + T_{o,t} - S_t} = \frac{p}{1/p\left(\frac{M_{t-1}}{P_t} + \theta S_{t-1}\right) - T_{o,t}} \tag{62}$$

Both conditions cannot hold at the same time as soon as $p < 1$, which implies that only:

$$\frac{1}{W - \frac{M_{t-1}}{P_t} - S_t} = \frac{p}{\left(\frac{M_{t-1}}{P_t} + \theta S_{t-1}\right)} \tag{63}$$

may bind in equilibrium. In particular, that means that there exists no interior solution for $T_{o,t}$ that has to equal 0. As a result of these conditions, we obtain the following expression for M_{t-1}/P_t:

$$\frac{M_{t-1}}{P_t} = \frac{pW - \theta S_{t-1} - pS_t}{1+p}, \tag{64}$$

which allows to also rewrite M_t/P_t as follows:

$$\frac{M_t}{P_t} = \frac{1}{(2-p)(1+p)} \left(pW + \theta p S_{t-1} + (p^2 - 2(1+p))S_t\right). \tag{65}$$

The inflation rate at $t+1$ can be expressed as function of storage. Using the no-arbitrage condition between money and storage, we find:

$$\frac{pW + \theta p S_{t-1} + (p^2 - 2(1+p))S_t}{pW - \theta S_t - pS_{t+1}} \frac{1}{2-p} = \theta^{-1} \tag{66}$$

which leads to:

$$(2 - p - \theta)W = (2-p)S_{t+1} + \theta(p-3)S_t + \theta^2 S_{t-1}. \tag{67}$$

As in the benchmark case, the sequences S_t satisfying this equation are of the following form, for $p < 1$:

$$S_t = \lambda_1 \theta^t + \lambda_2 \left(\frac{\theta}{2-p}\right)^t + \frac{2-p-\theta}{2-p-\theta+\theta(p-2)+\theta^2} W \tag{68}$$

As θ and $\theta/(2-p)$ are both below 1, S_t converges to $\frac{2-p-\theta}{2-p-\theta+\theta(p-2)+\theta^2}W$. Given that $\theta(p-2) + \theta^2 = \theta(\theta + p - 2) < 0$, we then obtain that S_t is ultimately above $W/2$. We can then use the same logic as for the proof of Proposition 2.

General conditions for having money purchases Let us investigate more the conditions under which money purchases are preferred to direct transfers. To do so, let us introduce two costs in our benchmark model. First, the cost of transferring $T_{o,t}$ to old agents is $(1+\nu)T_{o,t}$. Second, we assume that the cost of raising $T_{y,t}$ amount of resources cost $(1+\lambda)T_{y,t}$ to young agents.

We plug the government budget constraint $T_{t,y} = \frac{M_{t-1}-M_t}{P_t} + (1+\nu)T_{o,t}$ into individual saving decisions:

$$c^i_{y,t} = \frac{M_{i,t}}{P_t} + S_{i,t} = \frac{W-(1+\lambda)T_{y,t}}{2} \tag{69}$$

to obtain these individual saving decisions as follows:

$$2\frac{M_{i,t}}{P_t} + 2S_{i,t} = W - (1+\lambda)\frac{M_{t-1}}{P_t} + (1+\lambda)\frac{M_t}{P_t} + (1+\lambda)(1+\nu)T_{o,t} \tag{70}$$

Integrated over i, this condition yields:

$$\frac{M_t}{P_t} = \frac{W-(1+\lambda)\frac{M_{t-1}}{P_t} + (1+\lambda)(1+\nu)T_{o,t} - 2S_t}{1-\lambda} \tag{71}$$

and, thus:

$$c^i_{y,t} = \frac{M_{i,t}}{P_t} + S_{i,t} = \frac{W-(1+\lambda)\frac{M_{t-1}}{P_t} + (1+\lambda)(1+\nu)T_{o,t} - (1+\lambda)S_t}{1-\lambda} \tag{72}$$

Except for this, agents are homogeneous. The problem can be rewritten as:

$$\max_{P_t, T_o} \int \log\left(\frac{W-(1+\lambda)\frac{M_{t-1}}{P_t} + (1+\lambda)(1+\nu)T_{o,t} - (1+\lambda)S_t}{1-\lambda}\right) \mathrm{di} + \cdots \tag{73}$$

$$\cdots + \int \log\left(\frac{M_{i,t-1}}{P_t} + \theta S_{i,t-1} - T_{o,t}\right) \mathrm{di}, \tag{74}$$

The first order conditions with respect to P_t and $T_{o,t}$ are:

$$\frac{M_{t-1}(1+\lambda)}{W-(1+\lambda)\frac{M_{t-1}}{P_t} + (1+\lambda)T_{o,t}(1+\nu) - (1+\lambda)S_t} = \frac{M_{t-1}}{\frac{M_{t-1}}{P_t} + \theta S_{t-1} - T_{o,t}} \tag{75}$$

$$\frac{(1+\nu)(1+\lambda)}{W-(1+\lambda)\frac{M_{t-1}}{P_t} + (1+\lambda)T_{o,t}(1+\nu) - (1+\lambda)S_t} = \frac{1}{\frac{M_{t-1}}{P_t} + \theta S_{t-1} - T_{o,t}} \tag{76}$$

These two conditions cannot hold at the same time as soon as $\nu > 0$. In particular, the first constraint always bind while the second binds only when $\nu = 0$, thus implying that $T_{o,t} = 0$. Interestingly, the cost of raising taxes on the young, λ has a symmetric effect on the two conditions, indicating that direct transfers can be ruled out not because of the cost of raising resources but because of the relative cost of transfers over money purchases, as captured by ν. In the end, when the cost of raising resources satisfies $\lambda = 0$ as in the benchmark model, the optimality condition for money purchases leads to the same solution as (23).

In the end, money purchases are preferred to direct transfers only when the cost of transfers to the old (ν) is positive but not when only the cost of transfers to the young (λ) is positive. This then implies that the frictions that lead to money purchases have to increase to cost of transferring resources but not the cost of raising resources, which affects both direct transfers and money purchases.

C Randomization of portfolios

In the case where agents are indifferent between storage and money, they may randomize portfolios so that these portfolios are heterogeneous. In this appendix, we show that such a randomization does not affect our results.

First, let us find the consumption level of a young agent i. To this end, we plug the government budget constraint $T_{t,y} = \frac{M_{t-1}-M_t}{P_t}$ into individual saving decisions:

$$c^i_{y,t} = \frac{M_{i,t}}{P_t} + S_{i,t} = \frac{W - T_{y,t}}{2} \tag{77}$$

to obtain these individual saving decisions as follows:

$$2\frac{M_{i,t}}{P_t} + 2S_{i,t} = W - \frac{M_{t-1}}{P_t} + \frac{M_t}{P_t} \tag{78}$$

Integrated over i, this condition yields:

$$\frac{M_t}{P_t} = W - \frac{M_{t-1}}{P_t} - 2S_t \tag{79}$$

and, thus:

$$c^i_{y,t} = \frac{M_{i,t}}{P_t} + S_{i,t} = W - \frac{M_{t-1}}{P_t} - S_t \tag{80}$$

This leads to the following optimization problem:

$$\max_{P_t} \left\{ \int \log\left(W - \frac{M_{t-1}}{P_t} - S_t\right) \mathrm{di} + \int \log\left(\frac{M_{i,t-1}}{P_t} + \theta S_{i,t-1}\right) \mathrm{di} \right\}, \tag{81}$$

Note that the young generation consume the same, no matter its portfolio choice, consistently with young agents' indifference between portfolios.

The first order conditions with respect to P_t is:

$$\frac{M_{t-1}}{W - \frac{M_{t-1}}{P_t} - S_t} = \int \frac{M_{i,t-1}}{\frac{M_{i,t-1}}{P_t} + \theta S_{i,t-1}} \tag{82}$$

Interestingly, (82) can be rewritten in a more compact way:

$$cov\left(M_{i,t-1}, \frac{1}{\frac{M_{i,t-1}}{P_t} + \theta S_{i,t-1} - T_{o,t}}\right) = 0 \tag{83}$$

In equilibrium, if agents are indifferent between storage and money, a no-arbitrage condition should hold on asset returns: $\theta = P_{t-1}/P_t$. Using (77), this implies that

$$\frac{M_{i,t-1}}{P_t} + \theta S_{i,t-1} = \theta\left(\frac{M_{i,t-1}}{P_{t-1}} + S_{i,t-1}\right) = \theta\frac{W - T_{y,t}}{2}, \tag{84}$$

which implies that $\frac{M_{i,t-1}}{P_t} + \theta S_{i,t-1}$ is constant across individuals. Integrating this condition over households and using the fact that there is a mass 1 of them, we obtain that:

$$\frac{M_{i,t-1}}{P_t} + \theta S_{i,t-1} = \frac{M_{t-1}}{P_t} + \theta S_{t-1}. \tag{85}$$

As a result, in equilibrium, equation (82) simplifies so that we obtain the same first order condition as in the homogeneous case:

$$\frac{M_{t-1}}{W - \frac{M_{t-1}}{P_t} - S_t} = \frac{M_{t-1}}{\frac{M_{t-1}}{P_t} + \theta S_{t-1}} \tag{86}$$

This concludes the proof.

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