Dynamic Factor Models Cointegration and Error Correction Mechanisms

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This talk

- **Non-Stationary Dynamic Factor Models**
- Macroeconomic data
- Impulse-response functions

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- **Non-Stationary Dynamic Factor Models**
- Macroeconomic data
- Impulse-response functions
  1. Representation results
  2. Information criterion
  3. Estimation of non-stationary factors

- **Stationary Dynamic Factor Models (DFM)**
  Forni, Hallin, Lippi & Reichlin (2000); Stock & Watson (2005); Forni, Giannone, Lippi & Reichlin (2009)
- **Cointegration, Error Correction Mechanisms (ECM)**
  Engle & Granger (1987); Johansen (1988, 1991); Stock & Watson (1988)
- **Singular stochastic processes**
  Anderson & Deistler (2008a,b)

Literature

- **Factor models** have become increasingly popular

  Nowadays **commonly used by policy institutions**

  They have proven **successful**

  1. **Forecasting**
     Stock & Watson (2002); Forni, Hallin, Lippi & Reichlin (2005); Giannone, Reichlin & Small (2008); D'Agostino & Giannone (2012)
  2. **Structural analysis**
     Giannone, Reichlin & Sala (2005); Forni, Giannone, Lippi & Reichlin (2009); Forni & Gambetti (2010); Barigozzi, Conti & Luciani (2013); Luciani (2013)

Stationary Dynamic Factor Models

- **Fluctuations** in the economy, $\mathbf{x}_t \sim I(0)$, are due to:
  1. a few **structural shocks**, $\mathbf{u}_t$, orthogonal white noise
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  3. **Generalized DFM**
     Forni, Hallin, Lippi & Reichlin (2000)

$$\underset{n\times 1}{\mathbf{x}_t} = \underset{n\times 1}{\boldsymbol{\chi}_t} + \underset{n\times 1}{\boldsymbol{\xi}_t}$$

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$q < r$

Giannone et al., 2005, Amengual & Watson, 2007, Forni & Gambetti, 2010, and Luciani, 2013 for the **US**, Barigozzi et al., 2013 for the **Euro Area**

$$\boldsymbol{\Phi}(L) = \boldsymbol{\Lambda}\mathbf{C}(L)$$

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  a **finite autoregressive representation**
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▸ Example

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Permanent and Transitory Shocks

- So far, DFM in a **stationary setting**

  Exceptions: Bai (2004); Bai & Ng (2004); Peña & Poncela (2004)

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  all common shocks have a **permanent effect**

  Is this plausible?

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- When $\mathbf{x}_t$ is **non–stationary** but $\boldsymbol{\xi}_t$ is **stationary**
  there are **no idiosyncratic trends**
  strong implications for **cointegration** of $\mathbf{x}_t$

Non-Stationary Dynamic Factor Models

- Let $\mathbf{x}_t, \mathbf{F}_t, \boldsymbol{\xi}_t \sim I(1)$

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1. If all q shocks have **permanent effects**, the **genericity** argument is valid
2. If there are only $q - d$ **common trends**, the **genericity** argument is problematic: $\text{rk}(\mathbf{C}(1)) = q - d$

The research questions

$$\Delta \mathbf{x}_t = \Delta \boldsymbol{\chi}_t + \Delta \boldsymbol{\xi}_t$$
$$\Delta \boldsymbol{\chi}_t = \boldsymbol{\Lambda} \Delta \mathbf{F}_t$$
$$\Delta \mathbf{F}_t = \mathbf{C}(L) \mathbf{u}_t$$

The research questions

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\begin{aligned}
\Delta \mathbf{x}_t &= \Delta \boldsymbol{\chi}_t + \Delta \boldsymbol{\xi}_t \\
\Delta \boldsymbol{\chi}_t &= \boldsymbol{\Lambda} \Delta \mathbf{F}_t \\
\Delta \mathbf{F}_t &= \mathbf{C}(L) \mathbf{u}_t
\end{aligned}
$$

1. What is the correct autoregressive representation for $\Delta \mathbf{F}_t$?
2. How many common trends?
3. How to estimate the model?

Outline

- **Representation Theory**

Main assumptions

$$\underset{r\times 1}{\Delta\mathbf{F}_t} = \underset{r\times q}{\mathbf{C}(L)}\,\underset{q\times 1}{\mathbf{u}_t}$$

- $\Delta\mathbf{F}_t$ is a **rational reduced-rank family**

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- $\Delta \mathbf{F}_t$ is a **rational reduced-rank family**
- with **cointegration rank** c

$$\mathbf{C}(L) = \underset{r\times c}{\boldsymbol{\zeta}}\;\underset{c\times q}{\boldsymbol{\eta}'} + (1-L)\mathbf{D}(L)$$

Granger Representation Th. for Singular Vectors

Theorem

For generic values of the parameters of $\mathbf{C}(L)$*:*

$$\Delta \mathbf{F}_t = \mathbf{C}(L)\mathbf{u}_t$$

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  - d : **transitory shocks**

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  - $r - q$ **singularity**

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- $\boldsymbol{\beta}$ is $r \times c$ with $c = r - q + d$
  - d : **transitory shocks**
  - as if $r - q$ transitory shocks had a **zero loading**

Remarks

- Representation

$$\mathbf{A}(L)\Delta\mathbf{F}_t + \boldsymbol{\alpha}\boldsymbol{\beta}'\mathbf{F}_{t-1} = \mathbf{h} + \mathbf{C}(0)\mathbf{u}_t$$

  is not unique

  1. the number of error terms varies between d and $r-(q-d)$,
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- **empirically this is not a problem**

  1. choose the maximum value for $c = r-(q+d)$
  2. choose the lag of $\mathbf{A}(L)$ in a 'prudent' way

Outline

- **How many common trends?**

Literature

1. In DFM literature
   - Panel tests (PANIC)
     Bai & Ng (2004)
   - Information criterion based on PCA of levels $\mathbf{x}_t$
     Bai (2004)

2. Classical methods
   - Use cointegration tests on estimated factors
     Stock & Watson (1988), Phillips & Ouliaris (1988), Johansen (1991)

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3. Limitations
   - All the DFM procedures assume $q = r$
   - The criterion available assumes $\boldsymbol{\xi}_t$ **stationary**
   - The **estimation error** of factors will affects classical methods

Determining the number of common trends

- $\Delta\boldsymbol{\chi}_t = \mathbf{\Lambda C}(L)\mathbf{u}_t$ imply:

$$\mathbf{\Sigma}_{\Delta\chi}(\theta) = \mathbf{\Lambda C}(e^{-i\theta})\mathbf{C}'(e^{i\theta})\mathbf{\Lambda}'$$

- $\mathbf{rk}(\mathbf{\Sigma}_{\Delta\chi}(0)) = q - d = \tau$
- $\lim_{n\to\infty} \lambda_{j\Delta\chi}(0) = \infty,\ j = 1, \dots, \tau$

$$\widehat{\tau} = \underset{\tau\in[0,\tau_{\max}]}{\operatorname{argmin}} \left[\log\left(\sum_{j=\tau+1}^{n} \widehat{\lambda}_{j\Delta x}(0)\right) + kp(n,T)\right]$$

Hallin and Liška (2007)

Simulation results

				$\xi_i \sim I(1)$		$\xi_i \sim I(1), I(0)$	
T	N	q	d	DGP 1	DGP 2	DGP 1	DGP 2
100	50	2	1	97.3	95.7	93.0	93.9
100	50	3	1	81.9	73.7	87.3	83.2
100	50	3	2	89.6	83.5	92.5	87.8
150	50	2	1	99.2	98.7	85.6	92.1
150	50	3	1	96.7	92.2	97.4	96.8
150	50	3	2	98.3	95.8	96.4	96.7
100	100	2	1	98.1	98.0	96.4	97.8
100	100	3	1	92.2	85.9	94.5	89.7
100	100	3	2	93.9	88.7	95.9	91.6
150	100	2	1	98.9	99.4	91.3	96.5
150	100	3	1	99.3	97.4	98.3	98.9
150	100	3	2	99.2	97.9	98.9	98.9

Percentage of times we estimate τ correctly

DGPs

Outline

- **How to estimate the factors?**

Literature

1. PCA on the levels $\mathbf{x}_t$
   Bai (2004)
   - consistent only under the assumption $\boldsymbol{\xi}_t \sim I(0)$
   - implies that **all variables are cointegrated** ▸ example
   - an assumption that is **not credible in macro-panels**

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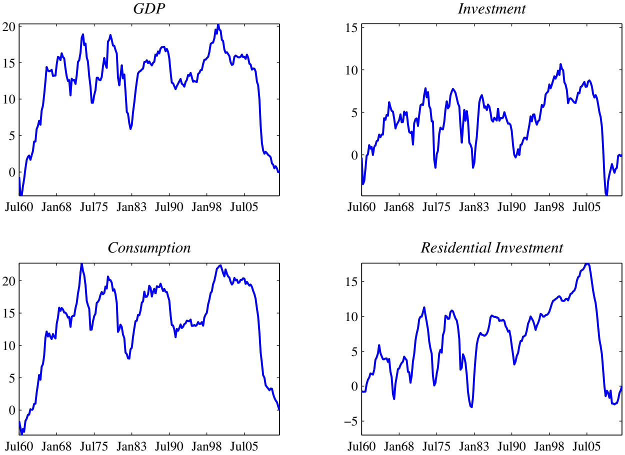
   - an assumption that is **not credible in macro-panels**

2. PCA on first differences $\Delta\mathbf{x}_t$, and cumulating (PANIC)
   Bai & Ng (2004)

$$\widehat{\mathbf{F}}_t = \sum_{t=1}^{T} \widehat{\Delta\mathbf{F}}_t$$

   - $\widehat{\mathbf{F}}_T = \widehat{\boldsymbol{\chi}}_T = 0$ by construction
   - The estimates $\widehat{\boldsymbol{\chi}}_t$ have finite sample problems

Estimation of χ_t with PANIC



Blue: $\mathbf{x}_t$;

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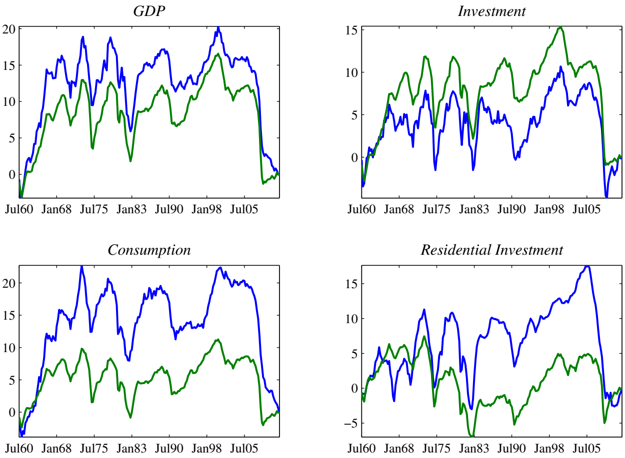


Blue: $\mathbf{x}_t$; Green: $\widehat{\chi}_t$

Estimating the space of common factors

1. Assume $\mathbf{x}_t \sim I(1)$ with no deterministic component
   - Estimate $\mathbf{\Lambda}$ by PCA on $\Delta \mathbf{x}_t$
   - $\widehat{\mathbf{F}}_t = (\widehat{\mathbf{\Lambda}}'\widehat{\mathbf{\Lambda}})^{-1}\widehat{\mathbf{\Lambda}}'\mathbf{x}_t$ and $\widehat{\boldsymbol{\chi}}_t = \widehat{\mathbf{\Lambda}}\widehat{\mathbf{F}}_t$

Estimating the space of common factors

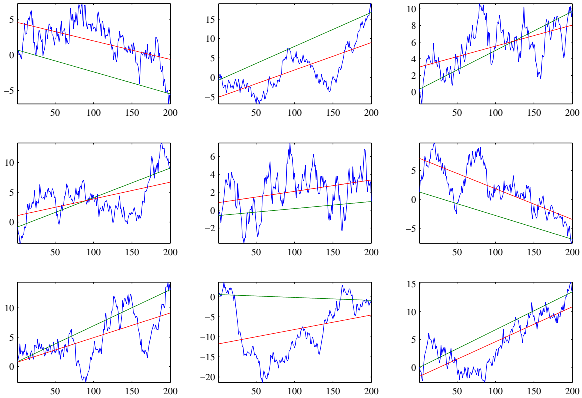
1. Assume $\mathbf{x}_t \sim I(1)$ with $x_{it} = a_i + b_i t + \boldsymbol{\lambda}_i' \mathbf{F}_t + \xi_{it}$
   - Detrend x_{it} and get x_{it}^*
   - Estimate $\boldsymbol{\Lambda}$ by PCA on $\Delta \mathbf{x}_t$
   - $\widehat{\mathbf{F}}_t = (\widehat{\boldsymbol{\Lambda}}'\widehat{\boldsymbol{\Lambda}})^{-1}\widehat{\boldsymbol{\Lambda}}'\mathbf{x}_t^*$ and $\widehat{\boldsymbol{\chi}}_t = \widehat{\boldsymbol{\Lambda}}\widehat{\mathbf{F}}_t$

Estimating the space of common factors

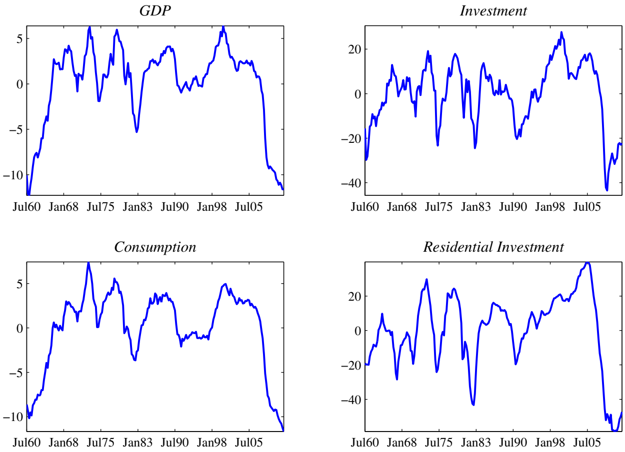
1. Assume $\mathbf{x}_t \sim I(1)$ with $x_{it} = a_i + b_i t + \boldsymbol{\lambda}_i' \mathbf{F}_t + \xi_{it}$
   - Detrend x_{it} and get x_{it}^*
   - Estimate $\boldsymbol{\Lambda}$ by PCA on $\Delta \mathbf{x}_t$
   - $\widehat{\mathbf{F}}_t = (\widehat{\boldsymbol{\Lambda}}'\widehat{\boldsymbol{\Lambda}})^{-1}\widehat{\boldsymbol{\Lambda}}'\mathbf{x}_t^*$ and $\widehat{\boldsymbol{\chi}}_t = \widehat{\boldsymbol{\Lambda}}\widehat{\mathbf{F}}_t$
2. Comparison with PANIC

$$\widehat{\chi}_{it}^P - \widehat{\chi}_{it} = \boldsymbol{\lambda}_i' \boldsymbol{\lambda}_i \left(y_{i0} - a_i + \left(\frac{y_{iT} - y_{i0}}{T} - b_i \right) t \right)$$

Detrending *vs.* Demeaning

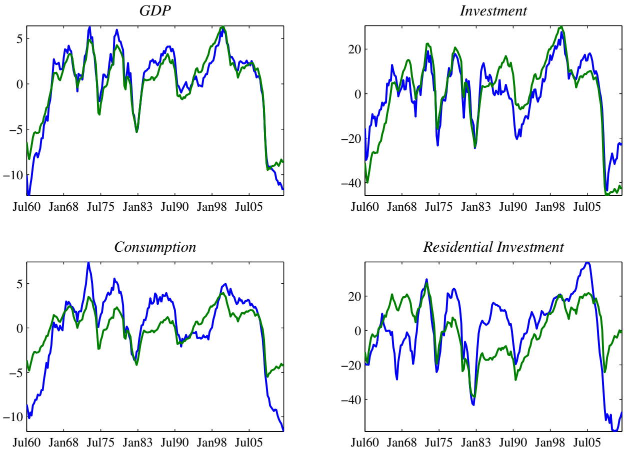


Estimation of χ_t with BLL



Blue: $\mathbf{x}_t$;

Estimation of χ_t with BLL



Blue: $\mathbf{x}_t$; Green: $\widehat{\chi}_t$

Simulation results

				$\xi \sim I(0)$		$\xi_i \sim I(1), I(0)$		$\xi \sim I(1)$	
T	N	q	d	Bai	BLL	Bai	BLL	Bai	BLL
100	50	2	1	0.88	0.93	1.79	0.88	1.88	0.85
100	50	3	1	0.95	0.99	1.36	0.92	1.48	0.88
100	50	3	2	0.88	0.91	1.72	0.84	1.72	0.82
150	50	2	1	0.86	0.93	2.08	0.91	2.27	0.86
150	50	3	1	0.92	0.96	1.52	0.92	1.66	0.89
150	50	3	2	0.86	0.91	2.19	0.88	2.08	0.83
100	100	2	1	0.87	0.92	1.94	0.90	2.17	0.84
100	100	3	1	0.92	0.95	1.43	0.91	1.58	0.87
100	100	3	2	0.85	0.89	1.88	0.85	1.93	0.83
150	100	2	1	0.86	0.93	2.31	0.93	2.65	0.84
150	100	3	1	0.90	0.94	1.62	0.92	1.85	0.91
150	100	3	2	0.87	0.92	2.43	0.85	2.44	0.83

Average $R_i = \frac{\sum_{t=1}^{T}(\hat{\chi}_{it}-\chi_{it})^2}{\sum_{t=1}^{T}\chi_{it}^2}$ relative to PANIC

DGPs

Outline

- **How to estimate the model?**

Estimation: Summary

$$\mathbf{x}_t = \mathbf{\Lambda}\mathbf{F}_t + \boldsymbol{\xi}_t$$

$$\mathbf{A}^*(L)\mathbf{F}_t = \mathbf{A}(L)\Delta\mathbf{F}_t + \boldsymbol{\alpha}\boldsymbol{\beta}'\mathbf{F}_{t-1} = \mathbf{h} + \mathbf{C}(0)\mathbf{u}_t$$

- Impulse responses $\widehat{\mathbf{\Phi}}(L) = \widehat{\mathbf{\Lambda}}(\widehat{\mathbf{A}}^*(L))^{-1}\widehat{\mathbf{C}}(0)$
- $c = r - \tau$
- $\widehat{r}$ Bai & Ng (2002); Alessi, Barigozzi & Capasso (2010), $\widehat{q}$ Hallin & Liška (2007); Onatski (2010), $\widehat{\tau}$ Barigozzi, Lippi & Luciani (2014)
- $\widehat{\mathbf{F}}_t$ Bai & Ng (2004); Barigozzi, Lippi & Luciani (2014)
- VECM on $\widehat{\mathbf{F}}_t \Rightarrow$ residuals $\widehat{\mathbf{v}}_t$
- PCA on $\widehat{\mathbf{v}}_t \Rightarrow \widehat{\mathbf{C}}(0)$

Outline

- **Empirical Analysis**

Data and number of shocks

1. Panel of US 103 quarterly series from 1960:Q3 to 2012:Q4
2. Number of common factors and shocks
   - $r = 7$
   - $q = 3$
   - $d = 2 \Rightarrow \tau = 1$
   - 1 permanent shock, 2 transitory shocks

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   - $r = 7$
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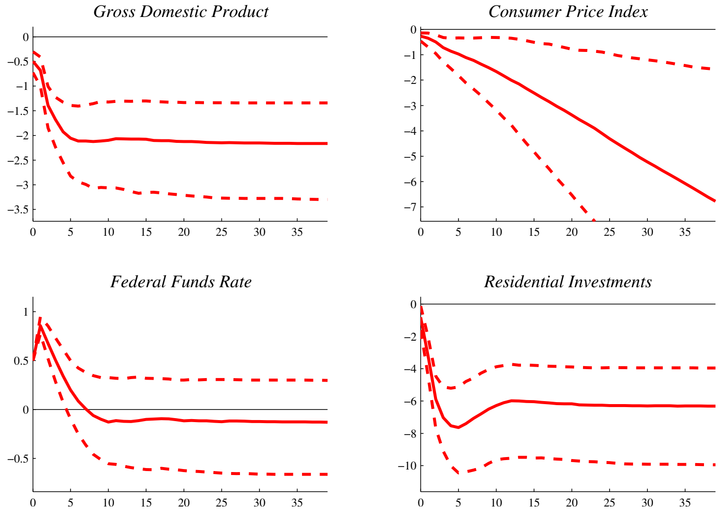
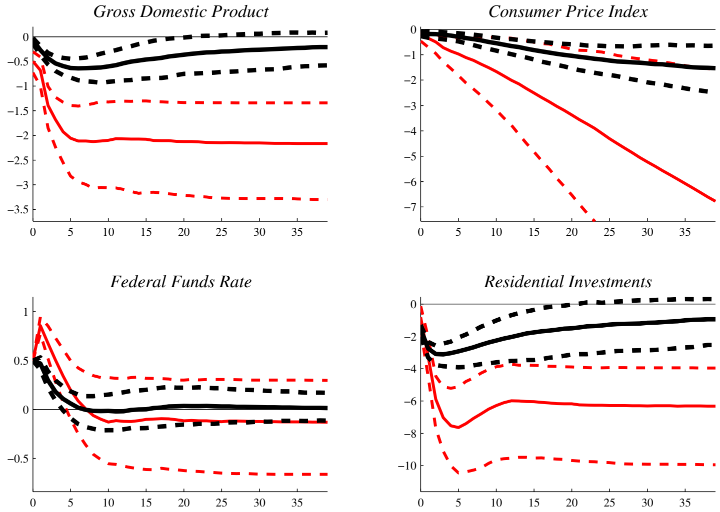
   - 1 permanent shock, 2 transitory shocks
3. Identification
   - Monetary Policy Shock - Sign Restrictions
     Barigozzi, Conti & Luciani (2013)
   - Technology Shock - Long-run Restrictions
     Blanchard & Quah (1992)

Monetary Policy Shock



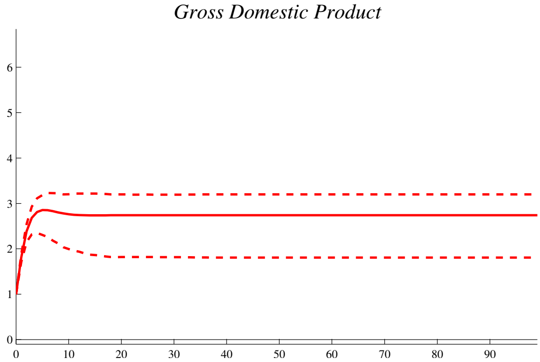
Red: IRF w/o ECM;

Monetary Policy Shock



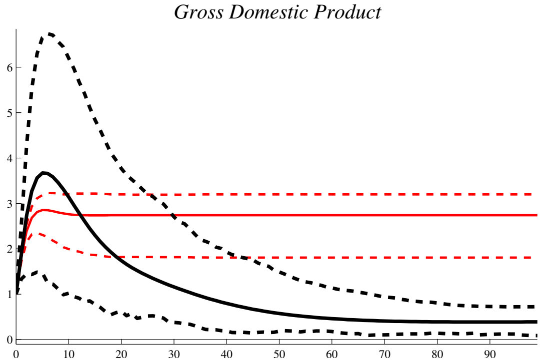
Red: IRF w/o ECM; Black: IRF with ECM

Technology Shock - GDP



Red: IRF w/o ECM;

Technology Shock - GDP



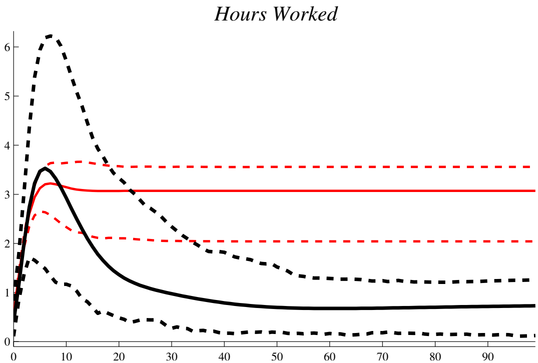
Red: IRF w/o ECM; Black: IRF with ECM

- Detrended data
  Dedola & Neri (2007); Smets & Wouters (2007)

Technology Shock - Hours

- **Hours worked**
- What happens after a positive technology shock?
- Macroeconomic theory:
  - increase – Real Business Cycle model
  - decrease – New Keynesian model
- Empirical evidence:
  - decrease
    Galí (1999); Francis & Ramey (2005)
  - increase
    Christiano, Eichenbaum & Vigfusson (2003); Dedola & Neri (2007)

Technology Shock - Hours



Red: IRF w/o ECM; Black: IRF with ECM

Summary

- **Correct AR specification for $\Delta \mathbf{F}_t$**

1. **VECM** representation

   IRF consistent with macroeconomic theory

2. **VAR** representation

   IRF not necessarily consistent

   - due to cumulation of IRF
   - unrealistically high and persistent

Conclusions

- **Non-stationary Dynamic Factor models**
- **Representation results**

  Granger Representation Theorem for Singular $I(1)$ Vectors
- Criterion for **number of common trends**
- Estimation of **non-stationary common factors**
- Estimation of **impulse response functions**
- **Monetary policy shocks**
- **Technology shocks**

Thank You !

Data Generating Processes

- **DGP 1**:

$$\chi_{it} = \boldsymbol{\lambda}_i' \mathbf{F}_t$$
$$\mathbf{F}_t = \boldsymbol{\Phi}(L)\mathbf{F}_{t-1} + \mathbf{G}\mathbf{u}_t$$

- **DGP 2**:

$$\chi_{it} = \boldsymbol{\lambda}_{i;0}' \mathbf{f}_t + \boldsymbol{\lambda}_{i;1}' \mathbf{f}_{t-1}$$
$$\mathbf{f}_t = \boldsymbol{\Phi}(L)\mathbf{f}_{t-1} + \mathbf{u}_t$$

Data Generating Processes

- **DGP 1**:

$$\chi_{it} = \boldsymbol{\lambda}_i'\mathbf{F}_t$$
$$\mathbf{F}_t = \boldsymbol{\Phi}(L)\mathbf{F}_{t-1} + \mathbf{G}\mathbf{u}_t$$

$$\mathbf{G} = \begin{bmatrix} g_{11} & 0 \\ g_{21} & 0 \\ 0 & g_{32} \\ 0 & g_{42} \end{bmatrix}$$

$$(1-\phi_j L)(1-L)F_{jt} = g_{j1}u_{1t} \quad j = 1,2 \quad \text{permanent}$$
$$(1-\phi_j L)F_{jt} = g_{j2}u_{2t} \quad j = 3,4 \quad \text{transitory}$$

BACK! - Criterion BACK! - Estimation

Data Generating Processes

- **DGP 2**:

$$\chi_{it} = \boldsymbol{\lambda}'_{i;0}\mathbf{f}_t + \boldsymbol{\lambda}'_{i;1}\mathbf{f}_{t-1}$$
$$\mathbf{f}_t = \boldsymbol{\Phi}(L)\mathbf{f}_{t-1} + \mathbf{u}_t$$

$$\begin{aligned}(1-\rho_1 L)(1-L)f_{1t} &= u_{1t} \quad \text{permanent}\\ (1-\rho_2 L)f_{2t} &= u_{2t} \quad \text{transitory}\end{aligned}$$

BACK! - Criterion BACK! - Estimation

Cointegration and Idiosyncratic Component

- Suppose $\mathbf{x}_t = \mathbf{\Lambda} F_t + \boldsymbol{\xi}_t$
- $x_{it} - \beta x_{jt} = z_t$
- $z_t = (\lambda_i - \beta\lambda_j)F_t + (\xi_{it} - \beta\xi_{jt})$
- if $\xi_{it}, \xi_{jt} \sim I(0)$,
- then we can take $\beta = \frac{\lambda_i}{\lambda_j}$
- which implies $z_t = \xi_{it} - \beta\xi_{jt} \Rightarrow z_t \sim I(0)$
- that is x_{it} **and** x_{jt} **are cointegrated**

BACK!

Example for the Genericity argument

$$y_{1t} = u_t + au_{t-1}$$
$$y_{2t} = u_t + bu_{t-1}$$

Example for the Genericity argument

$$\mathbf{y}_t = \begin{pmatrix} 1 + aL \\ 1 + bL \end{pmatrix} u_t$$

Example for the Genericity argument

$$\mathbf{y}_t = \begin{pmatrix} 1 + aL \\ 1 + bL \end{pmatrix} u_t$$

$$\mathbf{y}_t - \mathbf{A}\mathbf{y}_{t-1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} u_t,$$

Example for the Genericity argument

$$\mathbf{y}_t = \begin{pmatrix} 1 + aL \\ 1 + bL \end{pmatrix} u_t$$

$$\mathbf{y}_t - \mathbf{A}\mathbf{y}_{t-1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} u_t,$$

$$\mathbf{A} = \frac{1}{b-a} \begin{pmatrix} ab & a^2 \\ b^2 & -ab \end{pmatrix}$$

BACK!