Bayesian tail risk interdependence using quantile regression

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Introduction

- Recent financial disasters emphasised the need to investigate the consequences associated with the extreme tail co–movements among institutions.
- During the last years particular attention has been devoted to measure and quantify the level of financial risk within a firm or investment portfolio.
- One of the most diffuse risk measurement has become the Value-at-Risk (VaR).
- The VaR is an important capital evaluation tool where different institutions are considered as independent entities.
- Unfortunately, such risk measure fails to consider the institution as part of a system which might itself experience instability and spread new sources of systemic risk.

Systemic risk literature

During financial crises, episodes of contemporaneous distress of many institutions are not rare and thus need to be taken into account in order to analyse the overall health level of a financial system.

For this reason different systemic risk measures have been proposed in literature to analyse the tail-risk interdependence

1. Acharya *et al.* (2010, 2012), Banulescu and Dumitrescu (2012), Adams *et al.* (2010), Brownlees and Engle (2012).
2. Acharya and Richardson (2009), Tarashev *et al.* (2002), Huang *et al.* (2012).

⇒ (**Interesting**): Billio *et al.* (2013) and Ahelegbey and Giudici (2014).

⇒ (**Extensions**): Bernardi, Maruotti and Petrella (2013), Bernardi and Petrella (2014), Bernardi, Durante and Petrella (2014) and Bernardi, Durante, Jaworski and Petrella (2014).

Adrian and Brunnermeier (2011) propose the **CoVaR**, defined as the overall VaR of an institution conditional on another institution being under distress.

Goals

1. We propose to estimate VaR and CoVaR using Bayesian quantile regression approach.
2. We extend the Adrian and Brunnermeier (2011) CoVaR approach to account for the dynamics of the tails behaviour. The idea is to consider time varying quantiles to link the future tail behaviour of a time series to its past movements.
3. We analyse different U.S. companies belonging to several sectors of the Standard and Poor's Composite Index (S&P500) in order to evaluate the marginal contribution to the overall systemic risk of a single institution belonging to it.

CoVaR Definition

Let Y_k and Y_j be two different institutions' returns and $\mathbf{X}$ a set of exogenous covariates, $\text{CoVaR}_{k|j}^{\mathbf{x},\tau}$ is the Value-at-Risk of institution k conditional on institution j being at its τ–VaR level, i.e. $Y_j = \text{VaR}_j^{\mathbf{x},\tau}$

$$\mathbb{P}(Y_k \leq \text{CoVaR}_{k|j}^{\mathbf{x},\tau} \mid \mathbf{X} = \mathbf{x}, Y_j = \text{VaR}_j^{\mathbf{x},\tau}) = \tau. \tag{1}$$

The CoVaR not only capture the systematic risk embedded in each institution, but also reflects individual contribution to the systemic risk, capturing extreme tail risk dependency.

CoVaR estimation

There are many possible ways to infer on VaR and CoVaR. The most common approaches to estimate VaR is the variance–covariance methodology, historical and Monte Carlo simulations. For an overview of alternative parametric and nonparametric methodologies and processes to generate VaR estimates see Jorion (2006) and Lee and Su (2012).

Recently, Chao *et al.* (2012) and Taylor (2008) suggest to estimate VaR using quantile regression.

Since both VaR and CoVaR are distribution quantiles we address the problem of their estimation using a quantile regression approach.

$$\begin{aligned} \mathrm{VaR}_j^{\mathbf{x},\tau} &= \theta_{j,0}^\tau + \theta_{j,1}^\tau x_1 + \theta_{j,2}^\tau x_2 + \ldots + \theta_{j,M}^\tau x_M \\ \mathrm{CoVaR}_{k|j}^{\mathbf{x},\tau} &= \theta_{k,0}^\tau + \theta_{k,1}^\tau x_1 + \theta_{k,2}^\tau x_2 + \ldots + \theta_{k,M}^\tau x_M + \beta^\tau \mathrm{VaR}_j^{\mathbf{x},\tau}. \end{aligned}$$

Time invariant CoVaR model

To estimate $\text{VaR}_j^{\mathbf{x},\tau}$ and $\text{CoVaR}_{k|j}^{\mathbf{x},\tau}$ we follow a quantile regression approach:

$$\begin{aligned} y_{j,t} &= \mathbf{x}_t^\top \boldsymbol{\theta}_j + \epsilon_{j,t} & (2) \\ y_{k,t} &= \mathbf{x}_t^\top \boldsymbol{\theta}_k + \beta y_{j,t} + \epsilon_{k,t}, & (3) \end{aligned}$$

where β, $\boldsymbol{\theta}_j$ and $\boldsymbol{\theta}_k$ are unknown parameters and $\epsilon_{j,t} \sim \mathcal{AL}(\tau, 0, \sigma_j)$ $\epsilon_{j,k} \sim \mathcal{AL}(\tau, 0, \sigma_k)$ are independent Asymmetric Laplace distributions.

Due to the property of AL distributions, the functions $\mathbf{x}^\top \boldsymbol{\theta}_j$ and $\mathbf{x}^\top \boldsymbol{\theta}_k + \beta y_j$ correspond to the τ-th quantiles of $Y_j \mid \mathbf{X} = \mathbf{x}$ and $Y_k \mid \{\mathbf{X} = \mathbf{x},\, Y_j = y_j\}$, respectively.

Prior distributions

In the static regression framework we impose the following structure of conjugate proper priors on the parameters $\boldsymbol{\gamma} = (\boldsymbol{\theta}, \beta, \sigma_j, \sigma_k)$

$$\pi(\boldsymbol{\gamma}) = \pi(\boldsymbol{\theta})\, \pi(\beta)\, \pi(\sigma_j)\, \pi(\sigma_k), \tag{4}$$

where a Gaussian distribution is chosen for the regression parameters $\boldsymbol{\theta}$ and β, and Inverse Gamma distributions are imposed on the nuisance parameters (σ_j, σ_k), (see e.g. Lum and Gelfand, 2012):

$$\begin{aligned} \boldsymbol{\theta} = (\boldsymbol{\theta}_j, \boldsymbol{\theta}_k)^{\mathsf{T}} &\sim \mathcal{N}_{(2M+2)}\left(\boldsymbol{\theta}^0, \Sigma^0\right) & (5)\\ \beta &\sim \mathcal{N}\left(\beta^0, \sigma_\beta^2\right) & (6)\\ \sigma_j &\sim \mathcal{IG}\left(a_j^0, b_j^0\right) & (7)\\ \sigma_k &\sim \mathcal{IG}\left(a_k^0, b_k^0\right), & (8) \end{aligned}$$

and M is the number of covariates.

The Gibbs sampler

To make inference we exploit the representation of Asymmetric Laplace distributions as a location-scale mixture of Normals.

Definition

Let $\epsilon_l \sim \mathcal{ALD}(\tau, 0, \sigma_l)$, for $l \in \{j, k\}$, then

$$\epsilon_l = \lambda W_l + \delta\sqrt{\sigma_l W_l} Z_l, \tag{9}$$

where $W_l \sim \mathcal{E}(\sigma^{-1})$ and $Z_l \sim \mathcal{N}(0, 1)$, for $l \in \{j, k\}$, are independent random variables and $\mathcal{E}(\cdot)$ denotes the Exponential distribution and

$$\lambda = \frac{1 - 2\tau}{\tau(1 - \tau)}, \qquad \delta = \frac{2}{\tau(1 - \tau)}, \tag{10}$$

in order to ensure that the $\tau-$th quantile of ϵ is equal to zero.

The Gibbs Sampler, cont'd

The Partially collapsed Gibbs Sampler (see Liu, 1994; Van Dyk and Park, 2008 and Park and Van Dyk, 2009) consists of the following steps

1. $\pi\left(\sigma_l \mid \mathbf{y}_l, \mathbf{x}, \boldsymbol{\theta}_l\right)$, $l \in \{j,k\}$, **(collapsed step)**
2. $\pi\left(\omega_{j,t}^{-1} \mid y_{j,t}, \mathbf{x}_t, \boldsymbol{\theta}_j, \sigma_j\right)$, $\forall t = 1, \ldots, T$
3. $\pi\left(\omega_{k,t}^{-1} \mid \mathbf{y}_t, \mathbf{x}_t, \boldsymbol{\theta}_k, \beta, \sigma_k\right)$, $\forall t = 1, \ldots, T$,
4. $\pi\left(\boldsymbol{\theta}_j \mid \mathbf{y}_j, \mathbf{x}, \boldsymbol{\omega}_j, \sigma_j\right)$,
5. $\pi\left(\left(\boldsymbol{\theta}_k, \beta\right)^{\mathsf{T}} \mid \mathbf{y}, \mathbf{x}, \boldsymbol{\omega}_k, \sigma_k\right)$.

In this way we generate a Markov chain converging to the desired joint posterior distribution $\pi\left(\boldsymbol{\theta}_j, \right)$ which corresponds to the blocked Gibbs algorithm described in Park and Van Dyk (2009). This is because combining steps **1.**, **2.** and **3.** above produces draws from the conditional posterior distribution of $(\omega_{l,1}, \omega_{l,2}, \ldots, \omega_{l,T}, \sigma_l)$, for $l \in \{j,k\}$.

Dynamic extension

When modelling time varying quantiles, it is important to link future tail behaviours of time series to their past movements.

We model both VaR and CoVaR as a function of latent variables having their own time dynamics.

The observed vector $(y_{j,t}, y_{k,t})$, is a function of independent latent processes $(\mu_{j,t}, \mu_{k,t})$, the regressor terms $(\boldsymbol{\theta}_j, \boldsymbol{\theta}_k)$, and the loading factor β_t:

$$\begin{aligned} y_{j,t} &= \mu_{j,t} + \mathbf{x}_t^{\mathsf{T}} \boldsymbol{\theta}_j + \epsilon_{j,t} \\ y_{k,t} &= \mu_{k,t} + \mathbf{x}_t^{\mathsf{T}} \boldsymbol{\theta}_k + \beta_t y_{j,t} + \epsilon_{k,t}. \end{aligned}$$

Latent factors dynamics

The dynamics of the unobserved factors $\mu_{j,t}$ and $\mu_{k,t}$ are

$$\mu_{l,t+1} = \mu_{l,t} + \mu^*_{l,t} + \eta_{l,t} \quad (11)$$
$$\mu^*_{l,t+1} = \mu^*_{l,t} + \eta^*_{l,t}, \quad (12)$$

for $l = j, k$, where $(\eta_{l,t}, \eta^*_{l,t})^\mathsf{T} \sim \mathcal{N}_2(0, \mathrm{S}_l)$ and $\mathrm{S}_l = s_l^2 \mathrm{V}$.

This dynamic specification allows for a certain degree of smoothness of the quantile process.

Latent factors dynamics, cont'd

Since one of our main focuses is to analyse the dynamic tail co-movement of two institutions, we allow the loading parameter β_t to change over time:

$$\beta_{t+1} = \beta_t + \beta_t^* + \eta_{\beta,t} \quad (13)$$

$$\beta_{t+1}^* = \beta_t^* + \eta_{\beta,t}^*, \quad (14)$$

where $(\eta_{\beta,t}, \eta_{\beta,t}^*)^\mathsf{T} \sim \mathcal{N}_2(0, s_\beta^2 \mathrm{V})$.

Inference

Equations (11)-(14) can be written using a non-Gaussian State Space representation

$$\begin{aligned} \mathbf{y}_t &= \mathrm{Z}_t \boldsymbol{\xi}_t + \mathbf{x}_t^{\mathsf{T}} \boldsymbol{\theta} + \boldsymbol{\epsilon}_t && (15) \\ \boldsymbol{\xi}_{t+1} &= \mathrm{A} \boldsymbol{\xi}_t + \boldsymbol{\eta}_t && (16) \\ \boldsymbol{\xi}_1 &\sim \mathcal{N}\left(0, \kappa \mathbb{1}_6\right) && (17) \end{aligned}$$

where

- $\mathrm{Z}_t = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & y_{j,t} & 0 \end{pmatrix}$ is the matrix of loading factors
- A is a block diagonal matrix
- $\boldsymbol{\xi}_t = \left(\mu_{j,t}, \mu^*_{j,t}, \mu_{k,t}, \mu^*_{k,t}, \beta_t, \beta^*_t\right)^{\mathsf{T}}$ is the vector of latent states
- $\boldsymbol{\theta} = (\boldsymbol{\theta}_j, \boldsymbol{\theta}_k)$ is a $(M \times 2)$ matrix of time invariant coefficients
- $\boldsymbol{\eta}_t = \left(\eta_{j,t}, \eta^*_{j,t}, \eta_{k,t}, \eta^*_{k,t}, \eta_{\beta,t}, \eta^*_{\beta,t}\right)^{\mathsf{T}} \sim \mathcal{N}_6\left(0, \Omega\right)$.

Gaussian State Space representation

Exploiting the location–scale mixture representation of the AL distribution, the linear non-Gaussian State Space model in (15)-(17) admits the following conditionally Gaussian representation:

$$\begin{aligned}\mathbf{y}_t &= \mathbf{c}_t + \mathrm{Z}_t\boldsymbol{\xi}_t + \mathbf{x}_t^{\mathsf{T}}\boldsymbol{\theta} + \mathrm{G}_t\boldsymbol{\nu}_t, \quad \boldsymbol{\nu}_t \sim \mathcal{N}_2\left(0, \mathbb{1}_2\right) & (18)\\ \boldsymbol{\xi}_{t+1} &= \mathrm{A}\boldsymbol{\xi}_t + \boldsymbol{\eta}_t, \quad \boldsymbol{\eta}_t \sim \mathcal{N}\left(0, \Omega\right) & (19)\\ \boldsymbol{\xi}_1 &\sim \mathcal{N}\left(0, \kappa\mathbb{1}_6\right) & (20)\end{aligned}$$

where the time–varying vector $\mathbf{c}_t$, and matrix G_t are respectively

$$\begin{aligned}\mathbf{c}_t &= \lambda\left[\omega_{j,t}, \omega_{k,t}\right]^{\mathsf{T}} & (21)\\ \mathrm{G}_t &= \delta\begin{bmatrix}\sqrt{\sigma_j\omega_{j,t}} & 0\\ 0 & \sqrt{\sigma_k\omega_{k,t}}\end{bmatrix} & (22)\end{aligned}$$

and $\omega_{j,t}$ and $\omega_{k,t}$ are independent with $\omega_{l,t} \sim \mathcal{E}\left(\sigma_l^{-1}\right)$ for $l \in (j, k)$ and $\sigma_l > 0$; λ and δ are defined in equation (10).

Prior distributions

The same prior structures as in the static framework are imposed on regression and nuisance parameters $(\boldsymbol{\theta}, \sigma_j, \sigma_k)$, while the scale of the unobserved components have independent Inverse Gamma disturbutions

$$s_j^2 \sim \mathcal{IG}\left(r_j^0, v_j^0\right) \tag{23}$$

$$s_k^2 \sim \mathcal{IG}\left(r_k^0, v_k^0\right) \tag{24}$$

$$s_\beta^2 \sim \mathcal{IG}\left(r_\beta^0, v_\beta^0\right) \tag{25}$$

where the hyper parameters r_l^0, v_l^0 should be carefully selected because they are related with the prior signal-to-noise ratio.

The Gibbs Sampler

The resulting Gaussian State Space model (18)–(20) combined with conjugate priors on the unknown parameters allow to implement the following Gibbs sampling algorithm:

1. $\pi\left(s_l^2 \mid \mathbf{y}_l, \boldsymbol{\xi}_t^l\right)$, $l \in \{j, k\}$,
2. $\pi\left(\sigma_l \mid \mathbf{y}_l, \boldsymbol{\xi}_t^l, \boldsymbol{\theta}_l\right)$, $l \in \{j, k\}$, **(collapsed step)**
3. $\pi\left(\omega_{j,t}^{-1} \mid y_{j,t}, \mathbf{x}_t, \boldsymbol{\theta}_j, \sigma_j, \mu_{j,t}\right)$, $\forall t = 1, \ldots, T$
4. $\pi\left(\omega_{k,t}^{-1} \mid \mathbf{y}_t, \mathbf{x}_t, \boldsymbol{\theta}_k, \beta_t, \sigma_k, \mu_{k,t}\right)$, $\forall t = 1, \ldots, T$,
5. $\pi\left(\boldsymbol{\theta}_j \mid \mathbf{y}_j, \mathbf{x}, \boldsymbol{\omega}_j, \sigma_j, \boldsymbol{\mu}_j\right)$,
6. $\pi\left(\boldsymbol{\theta}_k \mid \mathbf{y}, \mathbf{x}, \boldsymbol{\omega}_k, \sigma_k, \boldsymbol{\mu}_k\right)$,
7. $\pi\left(\boldsymbol{\xi}_t, \beta_t \mid \mathbf{y}, \mathbf{x}, \boldsymbol{\theta}_j, \boldsymbol{\theta}_k, \sigma_j, \sigma_k, \boldsymbol{\omega}_j, \boldsymbol{\omega}_k, s_j^2, s_k^2\right)$.

Quantile point estimates

In order to make posterior inference we use the Maximum a Posteriori (MaP) summarising criteria.

When dealing with dynamic quantiles it is important to prove that the resulting quantile estimates are properly defined in terms of the conditional or unconditional distributions of observables.

We prove that the estimated sample quantiles (at MaP) have the appropriate number of observations above and below, so that they are unconditional quantiles.

Proposition

For the state space model defined in equations (11)–(14) with the prior distributions specified in equations (5)–(8) and (23)–(25), κ large enough and a diffuse prior on $\boldsymbol{\theta}$, the MaP quantile estimates $\mu_{j,t}^{\text{MaP}} + \mathbf{x}_t^{\mathsf{T}} \boldsymbol{\theta}_j^{\text{MaP}}$ and $\mu_{k,t}^{\text{MaP}} + \mathbf{x}_t^{\mathsf{T}} \boldsymbol{\theta}_k^{\text{MaP}} + y_{j,t}\beta_t^{\text{MaP}}$ satisfy

$$\begin{aligned} \sum_{t \notin C} (x_{m,t} + 1)\, \chi_\tau \left(y_{j,t} - \left(\mu_{j,t}^{\text{MaP}} + \mathbf{x}_t^{\mathsf{T}} \boldsymbol{\theta}_j^{\text{MaP}} \right) \right) &= 0 \\ \sum_{t \notin C} (y_{j,t} + x_{m,t} + 1)\, \chi_\tau \left(y_{k,t} - \left(\mu_{k,t}^{\text{MaP}} + \mathbf{x}_t^{\mathsf{T}} \boldsymbol{\theta}_k^{\text{MaP}} + y_{j,t}\beta_t^{\text{MaP}} \right) \right) &= 0, \end{aligned}$$

$\forall m \in \{1, \ldots, M\}$, where $C \subset \{1, \ldots, T\}$ is the set of all points such that the MaP quantile estimate coincides with observations and

$$\chi_\tau : \; z \to \begin{cases} \tau - 1 & \text{if} \quad z < 0 \\ \tau & \text{if} \quad z > 0. \end{cases}$$

Empirical application

The empirical application analyses tail co-movements between an individual institution j and the whole system k it belongs to (systemic risk).

We consider the S&P500 Composite Index (k) for the U.S market where different sectors (j) are included. The sample period is January 2, 2004 – December 28, 2012.

To control for the general economic conditions we use observations of macroeconomic regressors: *VIX index, weekly change of 3–month Treasury Bill rate, short term liquidity spread, the change in the slope of the yield curve, the change in the credit spread, DJ US Real Estate Index.*

To account for the individual firms' characteristics, we include observations from the following microeconomic regressors: *leverage, the market to book value, size, the maturity mismatch.*

Empirical Application, cont'd

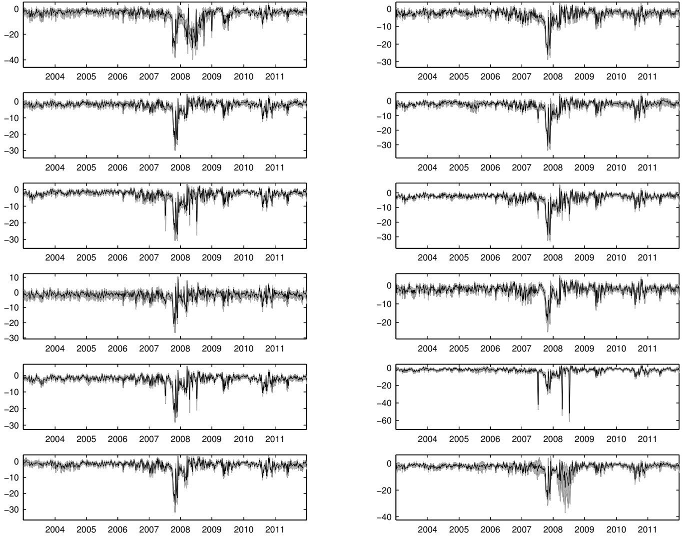
For all the reported institutions the β_t's parameters are positive and significantly different from zero.

By comparing $\text{HPD}_{95\%}$ for the β_t parameters, it is evident that the systemic risk contribution is significantly different across institutions belonging to different sectors.

On average the systemic β_t has lower value for institutions belonging to the financial sector and is higher for institutions belonging to consumer and energy sectors.

This evidence gives the idea of existence of sectors having different sensitivity to the risk exposure.

Comparing the β_ts for two different values of τ, we observe that on average higher values of the parameter tend to be associated with smaller values of the confidence level τ, meaning that the co-movement between asset and market becomes stronger for extreme returns.



Dynamic $\text{CoVaR}_{k|j}^{\mathbf{x},\tau}$, $\tau = 0.01$ and $\text{HPD}_{95\%}$, ***first panel***: **C, GS**; ***second panel***: **MCD, NKE**; ***third panel***: **CVX, XOM**; ***forth panel***: **BA, GE**; ***fifth panel***: **INTC, ORCL**; ***last panel***: **AEE, PEG**.

Conclusion

In this paper we address the problem of estimating the CoVaR in a Bayesian framework using quantile regression.

We first consider a time-invariant model allowing for interactions only among contemporaneous variables.

The model is subsequently extended in a time-varying framework where the constant and the loading parameter β are modelled as unobserved processes having their own dynamics.

The dynamic model shows high flexibility, providing risk measures that promptly react to the economic and financial downturns.

This is the first attempt to implement a Bayesian inference for the CoVaR.

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