THE MISSING CYCLE IN THE HP FILTER AND THE MEASUREMENT OF CYCLICALLY-ADJUSTED BUDGET BALANCES

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The HP filter suffers from a pro-cyclical bias in end-of-sample trend estimates. This paper argues that this feature is related to the "missing cycle" in the stochastic model of the filter. The paper suggest an extensions of the HP filter by including a stochastic cycle component in the underlying model of the filter. As a consequence, the derived trend and cyclical components are more consistent with the underlying filter model, and the end-point behavior improves significantly because the pro-cyclical bias in end-of-sample trend estimates is virtually removed.

1. Introduction

The decomposition of macroeconomic time series into trend and cyclical components is crucial to many macroeconomic concepts such as potential output, p-star, or the natural interest rate, and derived indicators such as cyclically adjusted budget balances. All these concepts imply that short- and long-term movements can be separated. Typically, the components are theoretical concepts and therefore not observable. Rather, they have to be identified on the basis of a theoretical model or plausible *ad hoc* assumptions.

Several tools for trend extraction have been developed in the literature.[1] Some of them allow building multivariate economic models and adjusting the model parameters to the data such as models with unobserved components (UC), others are are purely mechanical transformations of the original data such as the Baxter-King filter (Baxter and King, 1999) and the Hodrick-Prescott filter (Hodrick and Prescott, 1997). From a theoretical perspective, complex unobserved components models are clearly superior to the simpler methods. From a more practical point of view, the estimation of unobserved component models – which is usually carried out using recursive estimation methods such as the Kalman filter – can be difficult: The results depend on well specified initial conditions for unobserved variables and their variances. The final model chosen is usually the outcome of a relatively elaborate

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[1] Comprehensive overviews over trend-cycle decompositions are given in Dupasquier *et al.* (1997) or in Chagny and Döpke (2002).

procedure of model selection.[2] Furthermore, in some cases the Kalman filter approach may not work with annual data.

While simple trend extraction methods are more convenient to use, the economic interpretation of their results may pose problems. This is mainly because it is not possible to adjust the filter to properties of the time series to be filtered. Such mechanical approaches may also give rise to “spurious cycles” (Harvey and Jäger, 1993; Jäger, 1994; Cogley and Nason, 1995) which reflect more the properties of the filter used rather than those of the time series. An additional problem, which all approaches – including UC models – have in common, concerns the instability of trend estimations at the end of the data sample. The trend values of the last sample periods can change significantly when the sample is extended with the arrival of new data.[3]

This paper follows an approach between the two polar methods of trend extraction – UC models on the one hand and mechanical filters on the other. The well known Hodrick-Prescott filter (HP filter) is extended by an explicit stochastic models for both the trend and the cycle. The resulting “trend-cycle filter” (TC filter) allows for the simultaneous extraction of the trend and the cyclical process.

Compared with the HP filter as well as other common univariate filters, the TC filter has several advantages: first, it has better real time properties than other common univariate filters, as for instance the HP filter. Second, as both, trend and cyclical component, are explicitly modelled, it has a better foundation in the time domain than common univariate filters. Third, it can to some extent be adjusted to the data. Fourth, it can be easily extended to incorporate structural breaks. Finally, it is more convenient to use than unobserved components models.

The paper proceeds as follows. Section 2 discusses general properties of the HP filter. In Section 3, the trend-cycle filter is developed by generalising the underlying trend model of the HP filter and by adding an explicit stochastic model for the cycle. Section 4 discusses the instability of trend/cycle estimations at the end of the sample – the so-called “end-point problem” of filters. Second, it assesses the end-point reliability of the TC filter empirically by applying it to real GDP in selected countries and the euro area. Section 5 concludes.

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2 As Planas and Rossi (2004, p. 130) note in an investigation of the real time reliability of UC Phillips curve models:

“...recursive estimation requires a close monitoring of the parameter values, as sudden jumps can strongly increase the revisions. For instance, we found that the proper handling of the Kalman filter starting conditions is critical to the stability of model parameter estimates over time”.

3 The trend also changes if past data are revised *ex post*. Empirically, the instability due to the revision of past data is less problematic than the instability stemming from new data (Döpke, 2004; Rünstler, 2002).

2. The HP Filter

The HP filter is obtained by minimising the objective function:

$$\sum_{t=1}^{N}(x_t - x_t^T)^2 + \lambda\sum_{t=2}^{N}\left[(x_t^T - x_{t-1}^T) - (x_{t-1}^T - x_{t-2}^T)\right]^2 \tag{1}$$

for x_t. It is convenient to express the objective function in matrix form:

$$(X - X^T)'(X - X^T) + \lambda X^{T'}\nabla^{2'}\nabla^2 X^T \tag{2}$$

where X and X^T are $N\times1$ vectors of the original data and the trend and ∇^2 denotes the 2nd difference matrix.[4] The solution[5] of this optimisation problem follows from the first order conditions in matrix form:

$$X^T = (I + \lambda\nabla^{2'}\nabla^2)^{-1}X \tag{3}$$

$$X^C = X - X^T$$

2.1 The stochastic model of the HP Filter

For a more general interpretation of the HP filter one may start with the implicit stochastic trend model, a second order random walk. Let us write the model in matrix notation:

$$\begin{aligned}
&X - X^T - X^C = 0\\
&\nabla^2 X^T = \eta,\ \ E(\eta_t) = 0\ \ E(\eta_t^2) = \sigma_n^2\ \ \forall t = 1...N,\ \ E(\eta\eta') = \sigma_n^2 I_N\\
&X^C = \zeta,\ \ E(\zeta_t) = 0\ \ E(\zeta_t^2) = \sigma_\zeta^2\ \ \forall t = 1...N,\ \ E(\zeta\zeta') = \sigma_\zeta^2 I_N\\
&E(\eta\zeta') = 0_N
\end{aligned} \tag{4}$$

The residuals η and ζ are typically referred to as signal and noise. We assume that these processes have a zero-mean and that their variances exist. Furthermore, they are assumed to be mutually uncorrelated. The signal variable η is a white noise error term, whereas ζ may follow an unspecified stationary ARMA-process.

When inspecting the stochastic model of the filter and the definition of the trend in equation (4), several points are worth mentioning. First, the objective function in equation (2) is a weighted sum of the inner products of the residuals $\zeta'\zeta + \lambda\ \eta'\eta$ with the weight parameter λ.

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4 Lag and difference operators in matrix form are explained in Appendix 1.

5 For a more detailed derivation of the solution see for instance Danthine and Girardin, 1989.

Second, the stochastic model of the trend process as a second order random walk is a prior which may or may not be appropriate, depending on the properties of the series being filtered.[6]

Third, the cyclical component generated with an HP is proportional to the fourth difference of the HP filter trend, shifted forwards by two periods (see Reeves *et al.*, 1996, p. 4) – a highly implausible property.

Fourth, the trend and the cycle add up to the original series, meaning that there is no residual component capturing non-cyclical random impacts. According to the time domain representation of the filter in equation (4), the cycle is not explicitly modelled. Rather, it is defined as a residual process so that an additional residual component cannot be identified.

Finally, under the additional assumptions that the cycle process ζ is white noise and that η and ζ are distributed normally, maximising $\zeta'\zeta + \lambda\,\eta'\eta$ gives an optimal filter for the underlying stochastic process[7] if the parameter λ is set equal to the inverse signal-to-noise variance ratio: $\lambda = \sigma_{\zeta}^{2} / \sigma_{\eta}^{2}$. This interpretation is also consistent with an unobserved components model in which the parameter λ would be estimated as the inverse signal-to-noise variance ratio. These additional assumptions are usually not met in practice. In addition, the choice of the value of λ is based on prior assumptions and not on the concept of an optimal filter. Therefore, the HP filter is in general not an optimal filter in practical applications.[8] Furthermore, the cyclical component obtained from filtering is not a white noise process but follows some auto-correlated process, the properties of which depend on λ.

2.2 *The value of λ*

Since the parameter λ is key for the properties of the HP filter, much has been written about the proper value without, however, providing clear indications as to how to choose the appropriate value of λ. Ideally, the choice of λ should be adjusted so that it reflects prior knowledge on the length of the cycle. However, the smoothing parameter does not only affect the cycle but the volatility of trend growth as well – a consequence of the fact that the HP filter does not contain an explicit model of the cycle. Therefore, many practitioners tend to choose high values for λ when filtering annual data because they feel that lower values – as suggested in the

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6 Many macroeconomic time series are assumed to be $I(1)$ which contradicts the local linear trend model underlying the HP filter.

7 Whittle (1983). A filter is optimal if the sum of squared differences between the true and the estimated cyclical component take a minimum.

8 It follows also that the fixed value of λ is unequal to the observed inverse signal-to-noise variance ratio:

$$\frac{(X - X^T)'(X - X^T)}{N-1} \Big/ \frac{X^{T\prime}\nabla^{2\prime}\nabla^2 X^T}{N-3}$$

econometrics literature – would give rise to implausibly volatile trend growth rates. Thus, the value of λ is often based on a prior assumption of an acceptable trend volatility.

Values of 1600 for quarterly data and of 100 for annual data are commonly used. Ravn and Uhlig (2002) argue on the basis of frequency domain considerations that $\lambda = 1600$ for quarterly data is inconsistent with $\lambda = 100$ but would rather correspond to $\lambda = 6.5$ for annual data. Kaiser and Maravall (1999) propose a value of 8 for annual data, and Pedersen (2001) argues for a value of 1000 for quarterly data and for 3-5 for annual data. In Bouthevillain *et al.* (2001) the filter is applied with $\lambda = 30$ and in Mohr (2001) with $\lambda = 20$ to annual data.

The impact of the value of λ can be best demonstrated in the frequency domain. As the gain functions of the trend and the cyclical component for different λ-values in Figure 1 show, low frequency components are allocated to the trend while high frequency components are allocated to the cycle. Higher values of λ shift the gain function of the trend to the left so that the trend contains less of the higher frequencies, thereby becoming smoother. If $\lambda \rightarrow \infty$, the extracted trend approaches a linear trend. With lower values of the smoothing parameter, the trend becomes more volatile as it contains a larger part of the high-frequency spectrum. In the extreme case of $\lambda = 0$, the trend is equal to the original series.[9]

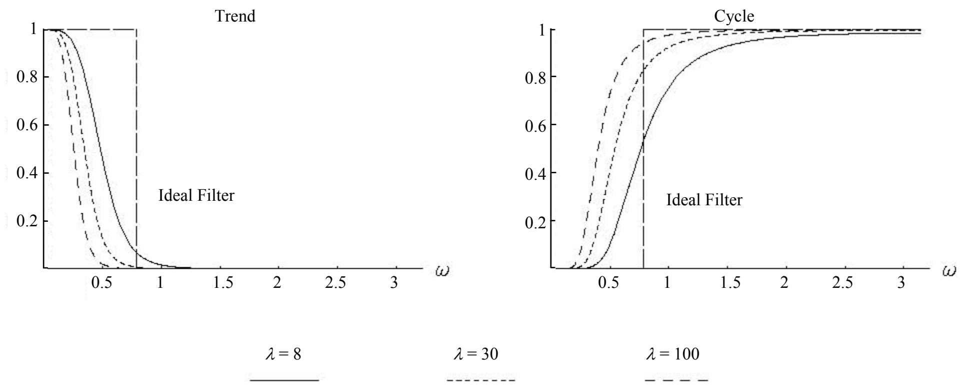
The frequency domain characteristics of the HP filter have well-known implications:

- First, the volatility of the cycle is controlled by the smoothing parameter λ. However, as λ defines the trend-volatility as well, there is no way to model the trend and the cycle independently from each other. Extracting shorter cycles comes automatically at the cost of a more volatile trend.
- Second, the missing model for the cyclical component has important consequences when additional, new data at the end of the sample are processed. There is no other choice than to allocate the information contained in a new data either to the trend or to the cycle, even though it may represent an outlier not generated by the data generating process underlying the HP filter.
- Finally, the HP filter is often used as an approximation to an ideal filter. Suppose, for instance, that the objective is to filter out a cycle of 8 or less periods length implying an ideal filter as shown in Figure 1: all frequencies below the critical frequency of $\frac{2\pi}{8}$ are cut off. By adjusting λ, the HP filter can approximate the desired ideal filter to some extent. However, there is a trade off in the choice of λ: while decreasing λ gives a better approximation to the ideal filter in the low frequency range, it worsens the approximation in the higher range. Therefore, either the trend contains frequencies which ideally should be

9 It is possible to translate the value of λ into a corresponding critical frequency ω_c, determined by $\omega_c / G_{HP}^{-1}(\lambda, .) = 0.5$. In this way, the filter can be characterised by a reference cycle of frequency ω_c.

Figure 1

Gain Function of the Trend and the Cyclical Component of the HP Filter for Different Values for λ



fully captured in the cycle and is therefore overly volatile, or longer waves which – according to the ideal filter – belong to the trend have too much weight in the cycle.

In short, a third component capturing irregular random influences is missing in the HP filter model. This tends to increase the instability of the trend estimate in real time as random influences are partly forced to contribute to the trend variability. This issue will be discussed further in Section 4.1.

3. The TC filter

This section extends the HP filter first by allowing for stochastic trends of arbitrary order and second by adding a stochastic model for the cycle to the filter. The resulting trend-cycle filter provides simultaneous, model-based estimates of the trend and the cyclical component.

3.1 A general stochastic trend model

In the HP filter model, the stochastic trend is restricted to a second order random walk. We generalise the trend model to a stochastic trend of any order. In this way, the order of the stochastic trend can be adjusted to the original series. For instance, many economic time series are $I(1)$ and a first order stochastic trend – possibly with a deterministic drift – would be more appropriate than the second order trend embodied in the HP filter.

The generalised trend model in matrix form can be described as:

$$\nabla^{d-1}(\nabla X^T - U_b) = \eta \tag{5}$$

where U denotes the $(T \times 1)$ vector [0,1,…,1]', b stands for the drift parameter to be determined endogenously, and d denotes the order of the trend. The expression U_b accounts for a deterministic drift if the trend is of first order $(d = 1)$. For a higher order trend $(d > 1)$, the drift term vanishes as $\nabla^{d-1}U_b = 0$.

Replacing the second line in equation (4) by equation (5) leads to the following objective function of the generalised trend filter in matrix form:

$$(X - X^T)'(X - X^T) + \lambda\left[\nabla^{d-1}(\nabla X^T - U_b)\right]'\left[\nabla^{d-1}(\nabla X^T - U_b)\right] \tag{6}$$

In the case of the first order random walk with drift the objective function has to be maximised for both the trend vector X^T and the drift parameter b, yielding:

$$\begin{aligned} X - (I + \lambda\nabla'\nabla)X^T + \nabla' U_b &= 0 \\ b = (U'U)^{-1}\, U'\nabla X^T & \end{aligned} \tag{7}$$

Thus, the drift term is computed as the average change in the trend:

$$b = \frac{x_N^T - x_1^T}{N-1}$$

Note, however, that the drift term b and the trend X^T are determined simultaneously. Another interpretation of the solution for b in equation (7) is that it represents the parameter of a regression of ∇X^T on the vector U. This allows us to define the residual projection matrix $W = I - U(U'U)^{-1}U'$ of this regression and to merge the solutions for b and X^T to yield:

$$X^T = (I + \lambda\nabla' W\nabla)^{-1}X \qquad \text{d} = 1 \tag{8}$$

The solution for $d > 1$ is straightforward, as in this case the trend reduces to a d-th order stochastic trend $\nabla^d X^T = \eta$, and the solution is similar to that of the original HP filter in equation (3):

$$X^T = (I + \lambda\nabla^{d'}\nabla^d)^{-1}X \qquad d \geq 2 \tag{9}$$

The solution in equation (9) can also be applied to a first order random walk with drift if the linear trend is removed from the time series before filtering. The result should not differ too much from the trend as given in equation (8), in which the deterministic and the stochastic trend components are simultaneously determined.

The generalisation of the trend order is well known in the literature. The case of $d = 1$ without simultaneous determination of the deterministic drift is known as "exponential smoothing" and was used by Lucas (1980) in an empirical analysis of the quantity theory of money. The simultaneous determination of the drift was first proposed in Tödter (2002) as the Extended Exponential Smoothing (EES). Furthermore, the Butterworth filter, which is primarily known in the engineering literature, depicts the general case of a stochastic trend of order d (Gomez, 2001).

For macroeconomic time series, stochastic trends of order higher than two do not make much sense. In the following sections, we will therefore concentrate on the EES, the HP filter and on TC filters with first- and second-order stochastic trends.

3.2 A stochastic model for the cycle

In this subsection, the stochastic model for the HP filter is extended by an explicit model for the cycle. The cyclical process is now assumed to follow a stationary ARMA-process, which is not left implicit as in the HP filter. Thus, we amend the stochastic model in equation (7) with the equation $AX^{C} = B\zeta$, in which the elements of the matrices A and B are determined by the parameters of an appropriately specified stationary ARMA process.

A convenient approach to model cyclical movements are stochastic cycles as suggested in Harvey (1989) or in Harvey and Jäger (1993). The original stochastic cycle approach in Harvey (1989) was extended towards stochastic cycles of order c in Harvey and Trimbur (2003). A stochastic cycle of order 2 is a stochastic cycle of order 1 with an error process that itself follows a stochastic cycle. Stochastic cycles of higher order are defined respectively. Stochastic cycles of order c give rise to ARMA($2c$, c) processes as shown in Harvey and Trimbur (2003).

The model for the c-th order stochastic cycle can be specified in state-space form as:

$$\begin{bmatrix} x_{1,t}^{C} \\ x_{1,t}^{C*} \end{bmatrix} = \rho \begin{bmatrix} \cos(\mu) & \sin(\mu) \\ -\sin(\mu) & \cos(\mu) \end{bmatrix} \begin{bmatrix} x_{1,t-1}^{C} \\ x_{1,t-1}^{C*} \end{bmatrix} + \begin{bmatrix} \zeta_t \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} x_{1,t}^{C} \\ x_{1,t}^{C*} \end{bmatrix} = \rho \begin{bmatrix} \cos(\mu) & \sin(\mu) \\ -\sin(\mu) & \cos(\mu) \end{bmatrix} \begin{bmatrix} x_{1,t-1}^{C} \\ x_{1,t-1}^{C*} \end{bmatrix} + \begin{bmatrix} x_{i-1,t}^{C} \\ 0 \end{bmatrix} \qquad (10)$$
$$i = 2 \ldots c$$

where $x_{i,t}^{C^*}$ is an auxiliary variable needed to write the model in state space form. The properties of the cycle are obtained by writing:

$$\begin{bmatrix} x_{1,t}^{C} \\ x_{1,t}^{C*} \end{bmatrix} = \begin{bmatrix} 1-\rho\cos(\mu) & -\rho\sin(\mu) \\ \rho\sin(\mu) & 1-\rho\cos(\mu) \end{bmatrix}^{-1} \begin{bmatrix} x_{i-1,t}^{C} \\ 0 \end{bmatrix}$$
$$i = 2 \ldots c$$

from which one obtains:

$$\begin{gathered} x^C_{i,t} = (\alpha(L)/\beta(L))^{-1} x^C_{i-1,t} \\ \alpha(L) = 1 - 2\rho\cos(\mu)L + \rho^2 L^2 \\ \beta(L) = 1 - \rho\cos(\mu)L^{\;2} \\ i = 2 \dots c \end{gathered} \tag{11}$$

The parameter ρ should be chosen from the open interval]0, 1[. It dampens the cycle, and $\rho < 1$ ensures that the cyclical process is stationary. In practice, ρ will be assigned a value close to 1, for instance $\rho = 0.975$. The parameter μ, which defines the "critical" frequency that dominates the stochastic cycle, is more important. As with the value for ρ, the parameter μ can be determined on the basis of prior knowledge on the length of the cycle.[10]

By iterative substitution, one obtains:

$$(1 - 2\rho\cos(\mu)L + \rho^2 L^2)^c \; x^C_{c,t} = (1 - \rho\cos(\mu)L)^c \,\zeta_t \tag{12}$$

for the c-th order stochastic cycle which we will incorporate in the TC filter: $x^C_t = x^C_{c,t}$.

The stochastic cycle model can be easily transformed to its matrix form $AX^C = B\zeta$ where A and B denote $(N-2c)\times N$ matrices representing the AR and the MA process, respectively:

$$A = \begin{bmatrix} a_{2c} & \dots & a_1 & 1 & 0 & \dots & & 0 \\ 0 & a_{2c} & \dots & a_1 & 1 & 0 & \dots & 0 \\ \vdots & & & a_{2c} & \dots & a_1 & 1 & \vdots \\ 0 & \dots & & & a_{2c} & \dots & a_1 & 1 \end{bmatrix} \quad B = \begin{array}{c} \begin{array}{cccccccccccc} & & & \text{column } c{+}1 & & & & & & & & \\ & & & \downarrow & & & & & & & & \end{array} \\ \begin{bmatrix} 0 & \dots & 0 & b_c & \dots & b_1 & 1 & 0 & \dots & & & 0 \\ \vdots & & \vdots & 0 & b_c & \dots & b_1 & 1 & 0 & \dots & & 0 \\ \vdots & & \vdots & \vdots & & & b_c & \dots & b_1 & 1 & & \vdots \\ 0 & \dots & 0 & \dots & & & & b_c & \dots & b_1 & & 1 \end{bmatrix} \end{array}$$

The first c columns of B are set equal to 0, and the a_i's and b_i's are determined by $\alpha(L)^c$ and $\beta(L)^c$ in equation (12).

3.3 *Putting it all together: The TC filter*

Combining the trend and the cycle model in matrix form gives the model of the TC filter:

[10] Alternatively, the parameters ρ and μ can be estimated from the data in an iterative procedure as shown in (?).

$$X - X^T - X^C = \varepsilon, \qquad E(\varepsilon) = 0, \qquad E(\varepsilon'\varepsilon) = \sigma_\varepsilon^2$$
$$\nabla^{d-1}(\nabla X^T - Eb) = \eta, \qquad E(\eta_t) = 0, \qquad E(\eta_t^2) = \sigma_\eta \quad \forall t = 1...N, \qquad \mathrm{E}(\eta\eta') = \sigma_\eta^2 I_N \tag{13}$$
$$AX^C = B\zeta, \quad E(\zeta_t) = 0, \qquad E(\zeta_t^2) = \sigma_\zeta \quad \forall t = 1...N, \qquad \mathrm{E}(\zeta\zeta') = \sigma_\zeta^2 I_N$$
$$E(\zeta\varepsilon') = 0_N, \quad E(\zeta\eta') = 0_N$$

We assume that ζ and η are white noise error terms. Furthermore, we assume $E(\varepsilon) = 0$, that the variance σ_ε exists and that ε is uncorrelated with the other residuals. ε could follow any stationary ARMA process fulfilling these requirements and is not necessarily a white noise process.

As with the HP filter or the EES, the objective function for this problem is constructed as the sum of the inner products of the residuals $\varepsilon'\varepsilon + \eta'\eta + \zeta'\zeta$. Different from the one-component filters, however, there is no smoothing parameter (such as λ in the HP filter or the EES), and it will be explained below why this is so. This gives the following optimisation problem:[11]

$$\begin{aligned}
&\underset{X^T, X^C, b}{Min} \left(X - X^C - X^T\right)'\left(X - X^C - X^T\right) + \\
&\left[\nabla^{d-1}(\nabla X^T - U_b)\right]'\left[\nabla^{d-1}(\nabla X^T - U_b)\right] + \\
&X^{C'} A'(BB')^{-1} AX^C
\end{aligned} \tag{14}$$

The solutions to this problem for the trend and the cyclical processes are obtained by minimising the objective function for X^T, X^C, and also for b if the trend is assumed to follow a first-order random walk with drift (d = 1). For the sake of simplicity, we define the residual projection matrix W of a regression on U as $W = I - U(U'U)^{-1}U'$ and make use of the following notation:

$$\begin{aligned}
&M_C \equiv \left[(I + A'(BB')^{-1} A\right]^{-1} \\
&M_T \equiv \begin{cases} (I + \nabla' W \nabla)^{-1}, & if\ d = 1 \\ (I + \nabla^{d'} \nabla d)^{-1}, & if\ d > 1 \end{cases}
\end{aligned} \tag{15}$$

We obtain the following system of first order conditions (FOCs):

$$\begin{aligned}
X^T &= M_T(X - X^C) \\
X^C &= M_C(X - X^T)
\end{aligned} \tag{16}$$

[11] The last expression with X^C in equation (14) can be derived as follows: the objective function involves the minimisation of $\zeta'\zeta$. The minimisation can be carried out in two steps: first, minimize $\zeta'\zeta$ for a given X^C under the constraint that the stochastic cycle model $AX^C = B\zeta$ holds. This gives $\zeta = B'\kappa$, with κ as Lagrange multiplier. By replacing ζ in the stochastic cycle model, one obtains $AX^C = BB'\kappa$. From that we derive $\zeta = B'(BB')^{-1} AX^C$ and hence $\zeta\zeta = X^{C'}A'(BB')^{-1} AX^C$.

To explain the intuition behind the system of FOCs, observe that M_T is an one-component trend filter which transforms any series X to a trend series. For instance, assuming $d = 2$, we obtain the HP filter with $\lambda = 1$. Similarily, the matrix M_C transforms any (stationary) series to a cycle series. Indeed, it can be shown (Harvey and Trimbur, 2003) that the matrix M_C defines a band-pass filter with a gain function spreading around the critical frequency μ. If the order of the stochastic cycle c is increased, the cyclical filter approaches a perfect band-pass filter. Thus, the system of FOCs in equation (16), combining the trend and the cyclical band-pass filter, can be interpreted as follows: applying the trend filter to a series from which the cyclical process has been removed (*i.e.*, on $X - X^C$), gives the trend X^T and, similarily, if the band-pass filter is applied to a series from which the trend has been removed (*i.e.*, on $X - X^T$), the cyclical process follows.

From the FOCs we derive the following solutions for the trend and the cyclical process:

$$\begin{aligned} X^T &= (I - M_T M_C)^{-1} M_T (I - M_C) X \Leftrightarrow X^T = M_{TC} X \\ X^C &= (I - M_C M_T)^{-1} M_C (\mathrm{I} - M_T) X \Leftrightarrow X^C = M_{CT} X \end{aligned} \tag{17}$$

Equation (17) defines the $d, c, \frac{2\pi}{\mu}, \rho$ filter with a stochastic trend of order d, a stochastic cycle of order c, a critical cycle length of $\frac{2\pi}{\mu}$ and a dampening parameter of ρ.

As equation (17) shows, the two-components TC filter can be regarded as a combination of the one-component trend and the one-component band-pass filter. For instance, using the trend filter to remove the trend in the first step and applying the band-pass filter on the residual yields:

$$\tilde{X}^C = M_C(I - M_T)X$$

as the cyclical component. However, this stepwise approach would neglect the simultaneity in the computation of the trend and the cycle and is therefore finally corrected by the correction factor $(I - M_C M_T)^{-1}$. In the special case of $M_C M_T = 0$, there is no simultaneity error, so that the stepwise application of the trend and the cyclical filter would not differ from applying the simultaneous TC filter.[12, 13]

As mentioned above, the variance components in the TC filter objective function (14) are not weighted. As the TC filter contains two components which are modelled (the trend and the cycle), two weighting parameters, λ_1 and λ_2, are

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12 Technically, $M_C M_T \rightarrow 0$ means that the intersection of the trend gain with the cycle gain in the frequency domain becomes smaller. This implies that the contribution of the trend to identify the cycle (and vice versa) becomes smaller and that trend and cycle become increasingly independent from each other. *Ceteris paribus*, the intersection of the gain functions decreases when critical cyclical frequency of the cycle μ becomes higher when the order of the stochastic trend, d, or of the stochastic cycle, c, become smaller.

13 Equation (17) gives consistent results if one component is missing. For instance, assume that there is only a trend and no cyclical component, implying $M_C = 0$. It follows that the two-components trend filter collapses to the one-component trend filter: $M_{TC} = M_T$. Respectively, if there is no trend, *i.e.* if $M_{T.} = 0$, it follows that $M_{CT} = M_C$.

necessary to define the objective function with weights as $\varepsilon'\varepsilon + \lambda_1 \eta'\eta + \lambda_2 \zeta'\zeta$. Under certain assumptions in addition to those in equation (13), minimising the weighted objective would provide the optimal filter for the process defined in equation (13).[14] However, deriving an optimal filter is not our objective. Instead, we want to extend the HP filter with a cyclical model in order to improve certain properties of the HP filter and in order to account for prior assumptions on the cyclical process in more straightforward manner.

With the HP filter, prior assumptions about the cyclical process are in principle reflected in the choice of the smoothing parameter λ. However, as discussed above, the relationship between the assumed cyclical process and the value of lambda is unclear. The TC filter trend can be interpreted as an HP filter trend in which the smoothing parameter λ is replaced by a more complex expression reflecting prior assumptions on the length of the cycle. Rewriting the trend in equation (17) as:

$$X^T = (I + (I + (A'(BB')^{-1}A)^{-1})\nabla^{d\prime}\nabla^d)^{-1}X$$

reveals that the trend of the TC filter is similar to the HP filter trend in equation (3) with λ replaced by the matrix expression $I + (A'(BB')^{-1}A)^{-1}$. Since this expression depends on μ, the critical frequency of the cycle, it reflects the prior assumption on the average cycle length.[15] Thus, by amending the HP filter with a model for the cycle, we have replaced the – to a certain extent arbitrary – smoothing parameter λ with a more general model based expression providing a clear-cut relationship between the cycle length and the filter parameter μ.

3.4 *Properties of the TC filter in the time domain*

As equation (17) shows, both the stochastic trend and the stochastic cycle model affect the trend and cycle solutions. This is so because the trend and the cycle are determined simultaneously; prior information on the nature of one component is used to identify the other component.

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14 The additional assumptions are that ε, η and ζ are all normally distributed and that the weights are set equal to the respective inverse signal-to-noise variance ratios:

$$\lambda_1 = \frac{\sigma_\varepsilon^2}{\sigma_\eta^2} \quad \text{and} \quad \lambda_2 = \frac{\sigma_\varepsilon^2}{\sigma_\zeta^2}$$

However, in equation (13) these variance ratios have been implicitly set to 1. This is an important difference to the general Kalman filter approach in (Harvey and Trimbur, 2003), in which signal and noise variances are estimated simultaneously with the trend and cycle. Like the HP filter, the TC filter is in general not an optimal filter.

15 For instance, assuming a relevant cycle length of eight years, μ could be set to $\frac{2\pi}{8} \approx 0.8$ with annual data.

The TC filter reproduces deterministic trends up to order[16] $2d - 1$. This can easily be shown by rewriting the trend in equation (17) as:

$$(I + A'(BB')^{-1}A)\nabla^{d'}\nabla^{d}X^{T} = A'(BB')^{-1}A(X - X^{T})$$

Preserving a deterministic trend implies $X = X^T$, so that the condition $\nabla^{d'}\nabla^{d}X = 0$ follows: the second difference of the trend should vanish. As the $2d$-th difference of any trend of order $2d - 1$ is zero, a trend of order $2d - 1$ fulfills the condition. The TC filter resembles in this respect the HP filter, which preserves deterministic trends of at most third order.

Unlike the HP filter, however, the TC filter preserves deterministic, stationary cycles as well, and its trend is *cyclically neutral* as long as the cycle in the data is consistent with the cyclical model of the filter. This means that applying the TC filter on such a process reproduces the input process completely in the cycle and yields a zero trend. In order to prove this we set $X^C = X$ in equation (17) and derive the condition:

$$(I + \nabla^{d'}\nabla^{d})A'(BB')^{-1}AX = 0$$

For this condition to hold it is sufficient that $AX = 0$. This is the case if X is generated by $\alpha(L)^k X = 0$, for $1 \leq k \leq n$ and with $\alpha(L)$ defined as in equation (11). The cyclical neutrality of the trend follows immediately from equation (17) together with the assumption that $AX = 0$.

The cyclical neutrality of the trend is an important improvement over an HP filter trend, which is not cyclically neutral: depending on the value of the smoothing parameter λ, the HP filter reproduces harmonic oscillations partly in the trend.[17]

The equations of the trend and the cyclical process are symmetrical: the matrices ∇^d and A can be regarded as containers for arbitrary but distinctive stochastic processes. It is even possible to include exogenous variables in order to identify the trend and the cycle as, for instance, the inflation rate, indicators of capacity utilisation or of consumer sentiments. This is similar to the Multivariate HP filter as proposed by Laxton and Tetlow (1992).[18]

3.5 *Properties of the TC filter in the frequency domain*

In this subsection we analyse the properties of the trend-cycle filter in the

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[16] A deterministic trend of order k is defined as $\sum_{i=0}^{k} a_i t^i$ with t denoting the time index.

[17] This is owing to the fact that the HP filter cannot approximate an ideal filter perfectly, as explained in Section 2. The HP filter would give a zero trend only in the limiting case of $\lambda \to \infty$. The other polar case of $\lambda = 0$ just reproduces the input process. The incorporation of cyclical fluctuations in the HP filter trend reflects the leakage effects of the filter explained above.

[18] For a more recent application of the multivariate HP filter, see Gruen *et al.* (2002) and Boone *et al.* (2000).

frequency domain. We derive the polynomial lag forms and subsequently the frequency domain representations – *i.e.* the Power Transfer Functions (PTFs) – of the trend and the cyclical filter in equation (17).

The matrices A and ∇^d in equation (17) are matrix-form translations of the polynomial lags for the stochastic cycle $\gamma(L)^c = [\alpha(L)/\beta(L)]^c$ – with $\alpha(L)$ and $\beta(L)$ defined as in equation (11) – and the stochastic trend, $(1 - L)^d$. The transposes of these matrices represent the respective *lead*-polynomials $\gamma(L^{-1})^c$ and $(1 - L^{-1})^d$ in matrix form. The polynomial lag forms of the trend and the cyclical filter in (17) can therefore easily be derived by replacing ∇ and A with $1 - L$ and $\gamma(L)$ and their transposes with $1 - L^{-1}$ and $\gamma(L^{-1})$, respectively. After simplifying we have:

$$\begin{aligned} x_t^T &= \left\{1 + \left[1 + \left(\gamma(L)\,\gamma(L^{-1})\right)^{-c}\right]\left[(1-L)\,(1-L^{-1})\right]^d\right\}^{-1} x_t = G_{TC}^T(L,\omega)\,x_t \\ x_t^C &= \left\{1 + \left[1 + \left((1-L)\,(1-L^{-1})\right)^{-d}\right]\left[\gamma(L)\,\gamma(L^{-1})\right]^c\right\}^{-1} x_t = G_{TC}^C(L,\omega)\,x_t \end{aligned} \tag{18}$$

The corresponding gain functions, $\mathrm{G}_{TC}^T(\omega)$ and $\mathrm{G}_{TC}^C(\omega)$, can be obtained by replacing the lag operator L in equation (18) with $U^{-i\omega}$ with ω as the frequency in radians.

As the filters are symmetric, the PTFs are equal to the squared gain functions. The impact of the parameters d, c, μ and ρ on the behaviour of the TC filter can be best explained by visual inspection of the PTFs as shown in Figure 2 for different parameter settings.

The order of the stochastic cycle, c, determines the bandwidth of the frequency spectrum contained in the cyclical process. The spectrum expands around the critical frequency μ when c becomes larger. Increasing the order of the stochastic cycle also shifts the trend spectrum to the lower frequency range. This is a consequence of the simultaneous determination of the trend and the cycle. However, the impact of changes in c on the trend-spectrum is minor.

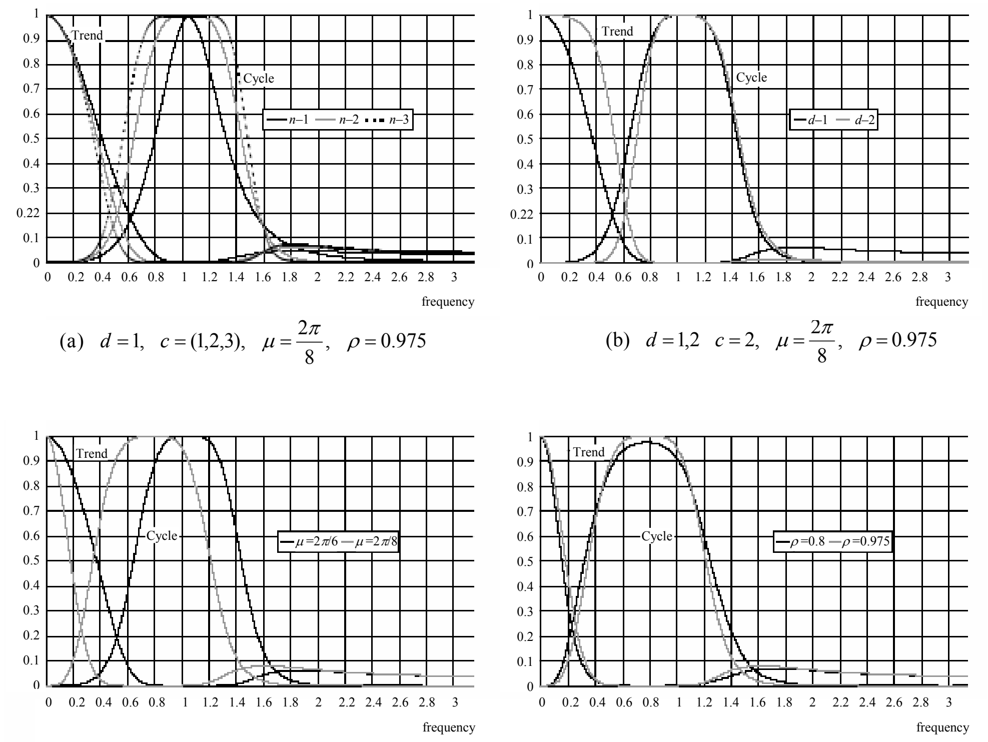
The critical frequency μ determines the center of gravity in the frequency spectrum of the cycle. Changes in μ give also rise to unidirectional shifts in the position of the trend spectrum, implying that μ does not only affect the volatility of the cycle but to some extent the trend volatility as well. Again, this feature follows from the simultaneous determination of the trend and the cycle.

An increase in the order of the stochastic trend d takes higher frequencies in the trend spectrum, implying that the trend becomes more volatile. The impact of changes in d on the cycle-gain are minor. Thus, by setting the order of the stochastic trend, the trend volatility can be manipulated without affecting the cycle too much, whereas the properties of the cycle are mainly determined through μ and c.

The parameter ρ is necessary to ensure the stationarity of the cycle and should be set close to but less than 1. As Figure 2d shows, the power-transfer functions are quite robust against changes in ρ.

Figure 2

Power-transfer Functions of the Trend and the Cycle of the TC filter for Different Parameter Values



(a) $d=1,\ \ c=(1,2,3),\ \ \mu=\dfrac{2\pi}{8},\ \ \rho=0.975$

(b) $d=1{,}2\ \ c=2,\ \ \mu=\dfrac{2\pi}{8},\ \ \rho=0.975$

(c) $d=1,\ \ c=2,\ \ \mu=\left(\dfrac{2\pi}{6},\dfrac{2\pi}{8}\right),\ \ \rho=0.975$

(d) $d=1,\ \ c=2,\ \ \mu=\dfrac{2\pi}{8},\ \ \rho=(0.8,\,0.975)$

4. An application to real GDP in selected countries

Now we apply variants of TC filter to annual real GDP from 1970-2002 in Germany (DE), Spain (ES), France (FR), Italy (IT), the euro area (EURO), and in the US and compare the results to those obtained with the HP filter and the Extended Exponential Smoothing (EES) as suggested by Tödter (2002). The data source is the spring 2004 AMECO database of the European Commission. In order to adjust for the structural jump in the German and the euro area series owing to the German unification, German real GDP was regressed on a constant, a linear trend and a jump dummy which takes a value of 1 from 1991 onwards and of 0 before. The estimated shift parameter value was then added to real GDP before 1991.

We choose a value of 7 for the smoothing parameter for the EES, following Tödter (2002). We fix the λ parameter for the HP filter to 30, as in Bouthevillain *et al.* (2001). We define an 8 years reference cycle for the TC filters, *i.e.* $\mu = \frac{2\pi}{8}$, and set the dampening parameter $\rho = 0.975$.

Figure 3 shows the resulting relative cyclical components for the TC(1,2), the TC(2,2), the HP(30) filter and the EES(7). The cyclical components are very similar to each other in the middle of the sample, with the exception of comparatively large TC(1,2) cycles for Spain and the US. More important, however, are the significant differences we observe at the sample fringes: The procession of end-of-sample information seems to constitute the most distinctive feature.

Furthermore, the patterns of trend growth generated with a TC filter are less smooth than the trend growth pattern derived from the one-component filters (Figure 4). In fact, the HP filter has often been criticised for generating an implausibly cyclical – even pro-cyclical – pattern in trend growth, which is difficult to reconcile with the common assumption that the long run growth path is mainly affected by irregular supply shocks. At the first sight, it seems as if the zig-zag like movements in the TC filter trend growth rates are more in line with this prior assumption than the patterns of the HP filter or the EES trend growth rates.

In the next sections we analyse the properties of trends and cycles computed with the TC filter more thoroughly and compare them with trends and cycles generated with the HP filter and the EES. In the first subsection, the issue of the so-called end-point problem is investigated from a more theoretical perspective. It is argued that the forecasting capability of the stochastic model underlying the filter is the main variable triggering the end-of-sample instability. In the second subsection, we explore the forecasting performance of the filters empirically. We find that the stochastic cycle model improves the forecasting performance of filters considerably. Finally, it is shown that some of the assumptions underlying the TC filter can be tested and that the TC filter can to some extent be adjusted to the data.

4.1 The end-point problem and the predictive capabilities of filters

Many trend-cycle decompositions suffer from the so called end-point problem. The trend in the final period N, x_N^T, is based on information available up to and including period *N*. It can change significantly if new data for period $N + 1$ become available – irrespective of whether the new data point is driven by cyclical or by structural factors. The real-time allocation of the dynamics to structural and cyclical forces is necessarily uncertain as information on the future path of the economy missing. It is only when new data in future periods become available that the trend-cycle decomposition in period N becomes more certain and stabilises.

While the limited amount of real-time information is a general problem for any trend-cycle decomposition that relies on past and future periods, trend extraction tools differ in the significance of the problem. The problem is less significant, the better the model underlying the filter can forecast the original time series. This can be illustrated by taking the example of the HP filter stochastic model.

Figure 3

Cyclical Components of Real GDP
(percent of real GDP)

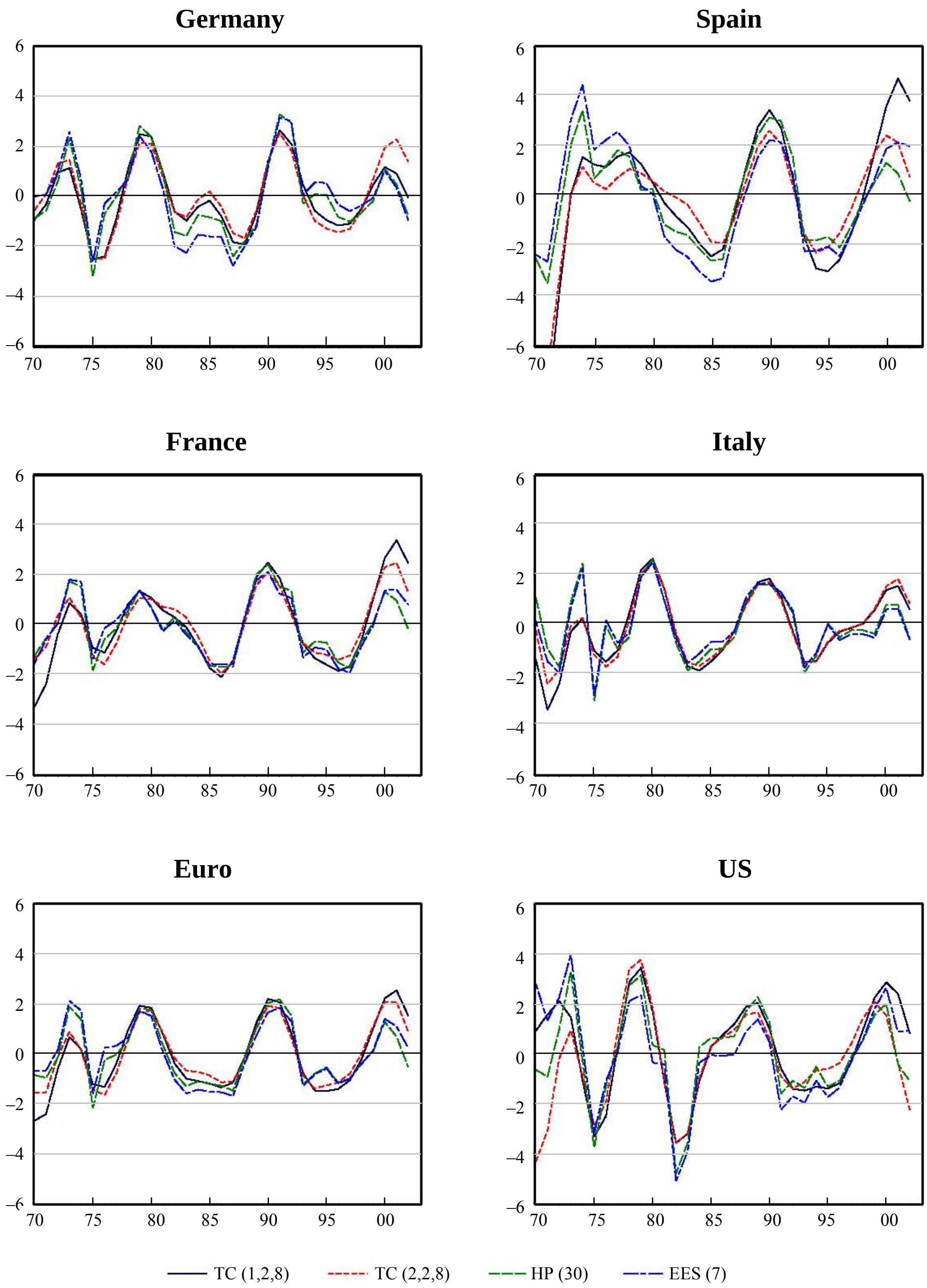
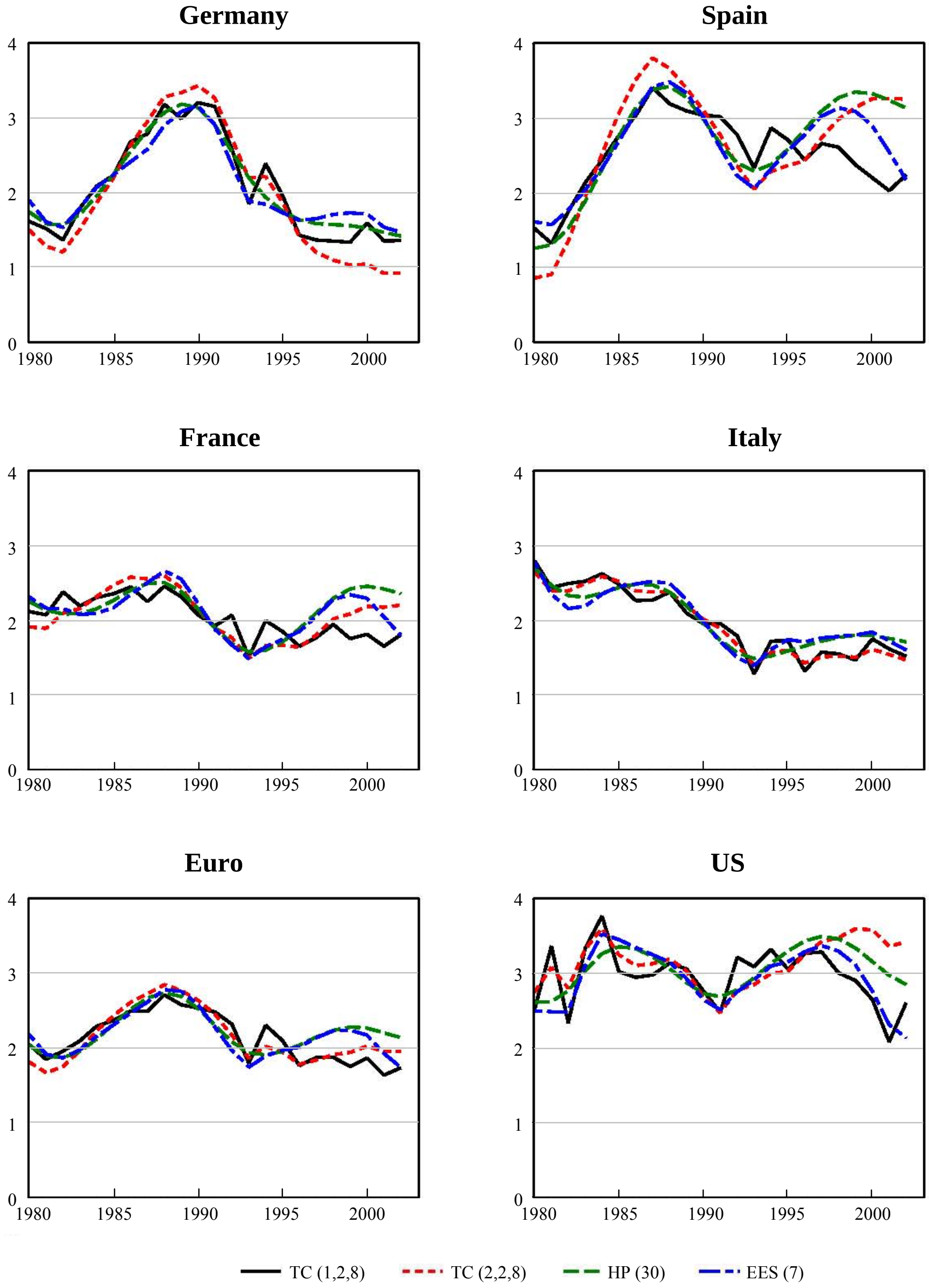


Figure 4

Trend Growth Rates Real GDP
(percent)



The stochastic model of the HP filter can be used to forecast x_{N+1} in period N, once the trend value in N is given. As the trend model is a second order random walk and because the cycle is not modelled, it follows that the optimal forecast for x_{N+1} is equal to:

$$\hat{x}_{N+1} = 2x^T_{N-1} - x^T_{N-2}$$

Now extend the original series by $\hat{x}_{N+1}$ to obtain $[x_1 \ldots x_N, \hat{x}_{N+1}]$ and apply the HP filter to the extended series. As a result, the trend series up to period $N\,[x^T_t \ldots x^T_N]$ is identical to the one obtained from filtering the non-extended series; the HP filter is consistent with its own forecast (Kaiser and Maravall, 1999).

From this we can conclude that there is no end-point problem if new data that arrive in $N + 1$ comply with the implicit forecast of the HP filter. Stating it the other way round: an end-point problem exists only insofar as the stochastic model underlying the filter is a weak representation of the data generating process.

As a standard remedy to the end-point problem, time series are sometimes extended by forecasts,[19] and the filter is applied to the extended series. If the forecast turns out correct *ex post*, there would not be an end-point bias. However, this approach comes with other problems. It is unclear how the filter processes forecast errors, which translate into errors in the trend estimation. Even if the forecast itself is unbiased and the forecast error is a random white noise process, it is unlikely that the implied errors in the computation of the trend share this feature because the filter model differs from that underlying the forecast.

As we have seen, the HP filter is consistent with forecasts derived from its own time series model. Extending the time series on the basis of a different model means that one does not trust the filter model. However, if there are good reasons to assume that there exists a model with a better forecasting performance than the filter model, the former rather than the latter should be applied for the trend-cycle decomposition.

Thus, rendering the filter model more consistent with the data generating process is a more preferable solution to the end-point problem than data extensions on the basis of models inconsistent with the filter. It follows that the end-point problem should be alleviated by improving the forecast performance of the stochastic filter model, *i.e.* its fit to the actual data.

The forecast performance of the filter and the possibilities to adjust it to the data depend mainly on the complexity of the underlying model. The complexity of the stochastic model of the HP filter, for instance, is low: the second order random walk property of the trend is the only prior piece of information that can be exploited

[19] The forecasts are often derived from ARIMA models as for instance in Kaiser and Maravall (1999) and in Denis *et al.* (2002).

for forecasting. Furthermore, the HP filter provides practically no means to adjust it to the data. Hence, its forecast performance cannot be improved.

The TC filter on the other hand provides a somewhat richer stochastic model as it explicitly accounts for the cycle; but does it give better forecasts and what are the empirical implications for the end-of-sample trend-cycle decomposition?

4.2 The forecasting performance of the HP and the TC filter

We investigate now the iterative one-step-ahead forecasts of the TC and HP filters and the EES. Starting with the sample 1970-78, we increase the "last year"s of the sample step by step until 2001, apply the filter on each vintage and compute for each of the filters a series from 1979-2002 of one-step-ahead forecasts $\hat{x}_{s+1|s}$ on the basis of the respective stochastic filter model:

$$\hat{x}_{s+1|s} = \begin{cases} x_s^T + b & \text{for the EES} \\ 2x_s^T - x_{s-1}^T & \text{for the HP filter} \end{cases}$$

where s = 1978...2002. The forecasts generated by the TC filter contains two components: the trend forecast $\hat{x}_{s+1}^T$ generated by the stochastic trend model and cycle forecast $\hat{x}_{s+1}^C$ derived from the stochastic cycle model. Note that only the AR and not the MA part of the stochastic cycle is used to generate the forecast since expected forecast errors are assumed to be equal to zero.

$$\hat{x}_{s+1|s} = \begin{cases} \underbrace{x_s^T + b} + \underbrace{\sum_{i=1}^{2c} a_i x_{s-i+1}^C} & \text{for the TC}(1,c)\text{ filter} \\ \hat{x}_{s+1|s}^T + \hat{x}_{s+1|s}^C & \\ \overbrace{2x_s^T - x_{s-1}^T} + \overbrace{\sum_{i=1}^{2c} a_i x_{s-i+1}^C} & \text{for the TC}(2,c)\text{ filter} \end{cases}$$

where c =1,2 and s = 1978...2001. The quality of the forecasts can be assessed by testing for $b = 1$ and const = 0 in the regression:

$$\Delta x_t = \text{const} + b\Delta\hat{x}_{t|t-1} + u_t \tag{19}$$

In the case of the TC filter, the additional variance explained by stochastic cycle forecast can be assessed by comparing the explained variance in equation (19) to that in the reduced regression:

$$\Delta x^t = \text{const} + b\Delta\hat{x}_{t|t-1}^T + u^t \tag{20}$$

which contains only the trend forecast of the TC filter model.

Table 1 – we present only the euro area results of this test because they are similar for the other countries – shows the result of the forecast regressions, together with some indicators of forecast quality, the root mean square error (RMSE), the mean absolute percentage error (MAPE), Theil's inequality coefficient and the coefficient of correlation between the one-step ahead predictions and actual values.[20] The bias and the variance proportion measure the part of the MSE due to differences in the mean and the variation between the predicted and the actual series. The covariance proportion captures remaining unsystematic forecasting errors. The bias, variance and covariance proportion add up to one. Ideally, the bias and variance proportions should be small so that most of the bias concentrates on the covariance proportion. All filter models predict real GDP growth in the euro area well and are unbiased. The correlation between predicted and actual GDP growth rates increases considerably with the complexity of the underlying filter model; the TC(1,2) and the TC(2,2)-forecast of real GDP growth explain about 80 per cent of actual growth, the EES-forecast only 38 per cent. Furthermore, the stochastic cycle model improves the fit to the data substantially as compared with the forecasts exclusively based on trends. Growth forecasts on the basis of the TC filter variants yield lower RMSE's, lower mean absolute percentage errors and lower Theil inequality statistics than forecasts using the stochastic models HP filter and the EES.

The decomposition of the MSE reveals that it is almost fully explained by the non-systematic covariance component in the case of the TC filter, whereas considerable contributions to the mean square error (13.8 per cent in the case of the HP filter and almost 38 per cent with the EES) derive from differences in variation between predicted and actual growth rates when predictions are based on the HP filter and the EES models.

To conclude, the endogenous stochastic cycle seems to improve the fit of the stochastic filter model to the actual data.[21] Therefore, we expect the TC filter to yield more reliable real time trend/cycle estimations than the EES or the HP filter.

4.3 *The real time reliability of the TC filter*

In order to assess the end-point reliability of trend-cycle decompositions, we generate vintages of trend-cycle estimations by cutting the sample artificially in each year s from 1978-2003 and estimating the trend and the cycle for each sample 1970s. In this way we obtain for each years between 1978-2003 one end-point trend/cycle

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[20] Theil's inequality coefficient is defined as:

$$\sqrt{\frac{MSE}{\sum \hat{x}^2 / n + \sum x^2 / n}}$$

It takes values between 0 and 1, with values closer to unity indicating worse predictors. The indicators used here are described – for instance – in Maddala (1977).

[21] It must be kept in mind, though, that an approach with prior parameterisation cannot deliver an optimal fit.

estimation based on the sample 1970s, the so-called "real-time" estimations $\tilde{x}_s^T, \tilde{x}_s^C$ of the trend and the cycle.[22]

The regression of the real-time cyclical components $\tilde{x}_t^C$ on the "final" results x_t^C of the 2002 vintage:

$$\tilde{x}_t^C = \text{const} + bx_t^C + u_t \tag{21}$$

indicates in how far the real time cyclical components are related to the "true" (the final) ones.

In Rünstler (2002), the "reverse" regression of final on real time results is proposed, which is based on the assumption that deviations of real-time from final results are uncorrelated with real-time results. This property of optimal, linear filters is a necessary condition for unbiased, mimimum mean square errors of the filter components,[23] assuming that the underlying stochastic model is correct. Hence, the test in Rünstler (2002) is based on the idea that the filter makes optimal use of real-time information so that subsequent revisions to initial estimates – once additional information comes in – should be orthogonal to the initial estimates. It can therefore be understood as a misspecification test of the stochastic model underlying the filter. However, as argued above, neither the TC filter, nor the HP filter, nor the EES can be regarded as optimal filters for typical economic time series. Here, we are more interested in the question whether errors are systematically pro-or anti-cyclical when compared to "final" trend deviations and not so much in a specification test for the underlying stochastic model. Under the H0 that errors are not systematically related to "final" results, they should be orthogonal to "final" estimates and the test regression should be specified as in equation (21).

Thus, end point reliability implies that $b = 1$ and $\text{const} = 0$ in equation (21) hold so that real-time cyclical components should be broadly in line with "final" cyclical components. Table 2 presents the results of these regressions, together with the *P*-value for the Wald test of the joint H0: $\text{const} = 0 \wedge b = 1$.

For the HP filter, the H0 must be rejected in all cases. While the constant is not significantly different from zero, b is consistently below 1: the HP filter cyclical components in real-time underestimate the "true" cycle considerably. In addition, the correlations of the real-time with "final" cyclical components are are low; the "true" cycle explains at most 38 per cent of the variance[24] of the cyclical component estimated at real time.

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22 More precisely, these are known as quasi-real time vintages, as the *s*-th vintage does not consist of the data available on period *s*, but of data available in *T*. We thus disregard data revisions.

23 See Priestley (1981), p. 775.

24 The highest coefficient of correlation amounts to 0.617 (in the case of for *IT*) so that the explained variance would be $\rho^2 = 0.38$.

Table 1

Regression of Real GDP Growth on One-step-ahead Forecasts for Real GDP Growth for the Euro Area

Filter		Full forecast: Regression[§] const	Full forecast: Regression[§] *b*	Full forecast: Forecast error Indicators		Full forecast: MSE[†] decomposition		Trend forecast: Regression[§§] const	Trend forecast: Regression[§§] *b*	Trend forecast: Forecast error Indicators		Trend forecast: MSE[†] decomposition	
TC(1,1)	Parameter	0.001	0.939	RMSE[†]	0.007	Bias	0.000	0.000	0.990	RMSE[†]	0.010	Bias	0.002
	Stdv.	0.004	0.16	MAPE	0.378	Variance	0.072	0.008	0.382	MAPE	0.687	Variance	0.340
	F–test[‡]	0.928		Theil	0.156	Covar.	0.928	0.982		Theil	0.223	Covar.	0.658
				Corr.	0.781					Corr.	0.484		
TC(1,2)	Parameter	0.002	0.896	RMSE	0.005	Bias	0.000	0.022	–0.083	RMSE	0.013	Bias	0.008
	Stdv.	0.002	0.091	MAPE	0.208	Variance	0.00	0.011	0.497	MAPE	0.900	Variance	0.267
	F–test[‡]	0.530		Theil	0.108	Covar.	0.999	0.107		Theil	0.279	Covar.	0.726
				Corr.	0.902					Corr.	–0.035		
TC(2,1)	Parameter	0.004	0.818	RMSE	0.006	Bias	0.001	0.000	0.980	RMSE	0.006	Bias	0.008
	Stdv.	0.003	0.11	MAPE	0.276	Variance	0.004	0.003	0.137	MAPE	0.418	Variance	0.070
	F–test[‡]	0.273		Theil	0.138	Covar.	0.996	0.908		Theil	0.135	Covar.	0.922
				Corr.	0.846					Corr.	0.837		
TC(2,2)	Parameter	0.003	0.853	RMSE	0.005	Bias	0.001	0.012	0.363	RMSE	0.013	Bias	0.014
	Stdv.	0.002	0.09	MAPE	0.232	Variance	0.011	0.006	0.231	MAPE	0.987	Variance	0.013
	F–test[‡]	0.283		Theil	0.114	Covar.	0.987	0.033[#]		Theil	0.267	Covar.	0.973
				Corr.	0.896					Corr.	0.317		
HP(30)	Parameter							–0.000	0.987	RMSE	0.008	Bias	0.009
	Stdv.							0.004	0.192	MAPE	0.531	Variance	0.138
	F–test[‡]							0.901		Theil	0.168	Covar.	0.853
										Corr.	0.738		
EES(7)	Parameter							–0.005	1.204	RMSE	0.009	Bias	0.003
	Stdv.							0.007	0.331	MAPE	0.649	Variance	0.379
	F–test[‡]							0.800		Theil	0.203	Covar.	0.618
										Corr.	0.613		

§ $\Delta x_t = \text{const} + b\Delta\hat{x}_{t|t-1} + u_t$ §§ $\Delta x^t = \text{const} + b\Delta\hat{x}^T_{t|t-1} + u^t$

† *MSE*: mean square error; *RMSE*: root mean squared error; *MAPE*: mean absolute percentage error.

† *Theil*: Theil inequality measure; *Corr*: Correlation coefficient.

‡ *P*–value of Wald-test of H0: const $= 0 \wedge b = 1$ # H0 rejected at 5 per cent significance level.

The results are slightly better for the EES. Here, the H0 $\text{const} = 0 \wedge b = 1$ cannot be rejected except in the cases of Italy and the US.[25] The slope parameter is closer to 1 than in the case of the HP filter. In two cases (Italy and the US), the real-time EES estimates are strongly biased, as the constant is significantly different from zero. The coefficient of correlation between the real time and final cyclical components varies between 0.60 and 0.84, which is higher than for the HP filter.

The TC(2, 2) filter turns out best in this exercise. The H0 is never rejected at the 5 per cent level.[26] The slope parameter b is close to one, the constant is not significantly different from zero, and the coefficient of correlation varies between 0.57 and 0.91. Decreasing the order of the cycle while maintaining the order of the trend comes at the cost of a considerable decrease in correlation between real-time and final cyclical components. Decreasing the order of the trend gives rise to rejections of the combined H0 in Spain, France, the euro area and the US. Depending on the time series being filtered, the parameters of the TC filter can to some extent be chosen to adapt the filter to the data generating process.

The underestimation of b gives rise to a pro-cyclical error in the estimation of the trend. This can easily be seen if we approximate the cyclical component by $x_t - x_t^T$. The regression equation $x_t - \tilde{x}_t^T = \text{const} + b(x_t - x_t^T) + u_t$ can be transformed into $x_t^T - \tilde{x}_t^T = \text{const} - (1 - b)(x_t - x_t^T) + u_t$. Values of b between –1 and 1 and different from zero imply that the trend is underestimated in a recession and overestimated in a boom. If $b = 1$ there is no relationship between the cycle and the error in the trend.

Figures 5 and 6 in Appendix 2 compares the errors in the real-time trend with the final cyclical components for the TC(2,2) and the HP(30) filter and the EES(7). As expected, the errors in the real time trend of the TC filter are largely unrelated to the cyclical component. For the HP filter, however, this relationship is strong. The HP filter real-time trend errors approximate very well the final cyclical component. Likewise, the EES induces a pro-cyclical bias in the real-time trend estimations, although the bias is less pronounced than in the case of the HP filter.

An important feature of real-time assessments of the cycle is the behavior around business cycle turning points. Errors in the real-time detection of the ”true” turning points might lead to a misdiagnosis of the current situation. The extent the different approaches to trend-cycle decomposition are prone to errors in the detection of turning points can be assessed by the following indices, which rest on the classification shown in Table 3.

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[25] For the euro area and Spain, it would be rejected at the 10 per cent level.

[26] It would be rejected at the 10 per cent level in Italy and in the US.

Table 2

Regression of the Real-time Cyclical Component on the final Cyclical Component and Correlation Between the Real-time and theFinal Cyclical Component of Real GDP

Filter	Parameter	DE	ES	FR	IT	EURO	US
TC(1,1)	c	1.379	1.896	0.209	–1.919	4.913	46.935
	std. err.†	5.746	2.281	2.982	1.482	12.826	22.257
	beta	0.898	0.999	0.91	1.015	0.914	0.93
	std. err.†	0.295	0.253	0.168	0.145	0.17	0.272
	Ftest‡	0.937	0.703	0.861	0.359	0.861	0.114
	Correlation	0.683	0.572	0.76	0.849	0.74	0.683
TC(1,2)	c	8.813	5.007	1.582	–1.29	21.831	118.695
	std. err.†	9.776	2.728	2.891	1.975	15.047	25.542
	beta	1.424	1.297	1.209	1.317	1.29	1.596
	std. err.†	0.406	0.171	0.093	0.132	0.146	0.209
	Ftest‡	0.436	0.024#	0.025#	0.074	0.047#	0.000#
	Correlation	0.68	0.807	0.913	0.863	0.849	0.86
TC(2,1)	c	0.01	0.125	0.344	–0.703	0.095	3.923
	std. err.†	4.277	1.368	2.567	1.77	10.978	16.373
	beta	0.587	0.637	0.525	0.608	0.538	0.626
	std. err.†	0.24	0.24	0.169	0.186	0.192	0.131
	Ftest‡	0.005#	0.327	0.008#	0.072	0.008#	0.029#
	Correlation	0.561	0.524	0.531	0.527	0.52	0.615
TC(2,2)	c	1.038	1.463	1.591	0.47	7.37	44.273
	std. err.†	9.299	2.845	4.275	2.443	20.188	21.93
	beta	1.354	1.374	1.077	1.355	1.233	1.372
	std. err.†	0.346	0.331	0.189	0.187	0.229	0.196
	Ftest‡	0.235	0.302	0.717	0.055	0.181	0.061
	Correlation	0.67	0.662	0.75	0.804	0.727	0.874
HP(30)	c	–1.086	1.042	0.715	–0.795	1.738	18.927
	std. err.†	5.981	2.001	3.216	1.89	13.613	21.055
	beta	0.422	0.332	0.43	0.503	0.431	0.485
	std. err.†	0.177	0.174	0.135	0.11	0.13	0.095
	Ftest‡	0.005#	0.003#	0.001#	0.000#	0.001#	0.000#
	Correlation	0.471	0.33	0.511	0.617	0.51	0.589
EES(7)	c	5.208	4.002	0.984	–3.451	13.719	95.813
	std. err.†	6.268	2.501	2.808	1.405	12.906	24.857
	beta	0.695	0.741	0.75	0.766	0.717	0.701
	std. err.†	0.225	0.189	0.127	0.088	0.145	0.162
	Ftest‡	0.303	0.067	0.146	0.002#	0.078	0.001#
	Correlation	0.674	0.598	0.765	0.835	0.722	0.709

Equation: $\tilde{x}_t^C = const + bx_t^C + u_t$

† Newey-West corrected standard errors.

‡ P-value of F-test of H0: const $= 0 \wedge b = 1$.

H0 rejected at 5 per cent significance level.

Table 3

Reliability of Signs of Real-time Cyclical Components

		final output gap +	final output gap –	sum
real time output gap	+	N_{++}	N_{+-}	$N_{+.}$
	–	N_{-+}	N_{--}	$N_{-.}$
	sum	$N_{.+}$	$N_{.-}$	$N_{..}$

- The relative share of wrong signs $(N_{[+-]} + N_{[-+]})/N_{[..]}$.
- The information content defined as $I \equiv N_{[++]}/N_{[.+]} + N_{[--]}/N_{[.-]}$. This measure takes values between -1 and 1. Values in the range $0 < I \leq 1$ indicate a positive information content, and $I = 1$ means that the signs of cyclical components in real time and final estimates coincide perfectly. If $-1 \leq I < 0$, there is a systematic bias in the signs of cyclical components in real time.
- The cell counts can be compared with the expected ones under the H0 that cell counts are random: $E(N_{[ij]}) = N_{[i.]}N_{[.j]}/N_{[..]}$, $i, j \in \{+, -\}$. The H0 can be tested, using the test statistic $\Sigma_{i,j\in\{+,-\}}[N_{[ij]} - E(N_{[ij]})]^2 / E(N_{[ij]} \sim x^2(1)$.

Results for these indices for cyclical components of the TC filters with a second order cycle, the HP filter and the EES are shown in Table 4. There is no instance with a negative value for I so that the signs of the real-time cyclical components cannot be regarded biased. The relative share of sign misdiagnoses amounts to roughly 10-25 per cent with the TC filter variants. Signs of cyclical components are likewise often wrongly estimated with the EES except in the case of the US, where the EES gives the highest share (38 per cent) of instances with wrong signs. For the other countries and regions, the HP filter yields the highest shares of wrong signs between 35 and 46 per cent. Correspondingly, the HP filter gives the lowest value for the information content measure I, again with the exception of the US, where the EES performs worse. For all regions except Germany and France, I is generally closer to unity for the TC filter variants. In Germany the EES outperforms both trend variants of the TC filters. In France the EES gives a higher value for I than the TC(1,2) filter. The H0 that the cell counts are random can never be rejected at the 5 per cent level with the HP filter. Only HP filtered real GDP in Germany leads to a rejection of the H0 at the 10 per cent level. According to the χ^2 test, the hypothesis of a random distribution of signs can be rejected at least at the 5 per cent significance level for cyclical components computed with the TC Filter and the EES. All in all, the TC filter generally allows for a more consistent determination of signs of cyclical components in real time than the one-component

Table 4

Sign Tests of Real-time Cyclical Components of Real GDP

Country	Filter	Wrong sign	*I*	Test statistic	*P*-value	Significance[†]
DE	TC(1,2)	0.23	0.55	7.80	0.005	***
	TC(2,2)	0.19	0.62	10.40	0.001	***
	EES(7)	0.15	0.69	12.76	0.000	***
	HP(30)	0.35	0.36	3.31	0.069	*
ES	TC(1,2)	0.19	0.62	10.40	0.001	***
	TC(2,2)	0.27	0.45	5.42	0.020	**
	EES(7)	0.31	0.39	3.94	0.047	**
	HP(30)	0.46	0.08	0.18	0.671	
FR	TC(1,2)	0.08	0.87	19.07	0.000	***
	TC(2,2)	0.23	0.55	7.72	0.005	***
	EES(7)	0.15	0.69	12.76	0.000	***
	HP(30)	0.35	0.31	2.48	0.116	
IT	TC(1,2)	0.08	0.85	18.62	0.000	***
	TC(2,2)	0.15	0.69	13.77	0.000	***
	EES(7)	0.23	0.50	7.10	0.008	***
	HP(30)	0.46	0.10	0.25	0.619	
EURO	TC(1,2)	0.15	0.70	12.83	0.000	***
	TC(2,2)	0.19	0.62	10.40	0.001	***
	EES(7)	0.27	0.46	5.57	0.018	**
	HP(30)	0.38	0.24	1.47	0.225	
US	TC(1,2)	0.19	0.55	10.64	0.001	***
	TC(2,2)	0.12	0.77	16.25	0.000	***
	EES(7)	0.38	0.33	4.54	0.033	**
	HP(30)	0.27	0.45	5.42	0.020	**

[†] *,**,***: significant at 10 per cent, 5 per cent and 1 per cent.

filters. The EES performs remarkable well in this test, while results for the HP filter are less satisfying.

The comparatively weak real time properties of the one-component filters – the HP filter and the EES – derive from the "missing cycle" in these filters. Enhancing these filters with stochastic models for the cycle improves the real-time reliability significantly and removes the pro-cyclical bias in end-of-sample estimates. Obviously, it is not possible to identify the trend at real time in a proper way if a model for the cycle is missing.

5. Conclusion

Univariate trend-cycle decompositions suffer from all too simple implicit models of the data generating process, while more elaborated approaches – as for instance unobserved components models – are not always easily applicable. This paper develops an intermediate approach by generalising the HP filter and incorporating a cyclical component in the model representation of the filter in the time domain. The resulting trend-cycle filter has better end-of-sample properties than the HP filter or the related Extended Exponential Smoothing (EES) procedure. In particular, the pro-cyclicality in end-of-sample trend/cycle estimations, characterising the one-component filters which are based on an implicit model for the trend only with cycle left as a residual from trend-extraction. The incorporation of a cycle model turned out crucial for the favourable properties of the TC filter.[27]

While the TC filter is based on a more complex stochastic model than the EES and the HP filter, its application is almost as simple that of the one-component filters. Once the TC filter has been programmed,[28] it is straightforward to choose the appropriate stochastic trend and cycle models and to obtain the trend-cycle decomposition. It is not necessary to experiment with prior variance restrictions and start values for unobserved variables as it is sometimes required in unobserved components model estimations.

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[27] The trend-cycle filter form cannot be applied to seasonal time series. However, an expansion towards a trend-cycle-season filter or incorporating additional components such as structural breaks is straightforward, see Mohr (2005).

[28] Implementations in EXCEL, EVIEWS 4.x and MatLaB can be obtained from the ECB Working Papers site (http://www.ecb.int/pub/pdf/scpwps/ecbwp499annexes.zip) or from the IDEAS Economics bibliographic database (http://econwpa.wustl.edu:80/eps/em/papers/0508/0508005.zip).

APPENDIX 1
LAG AND DIFFERENCE OPERATORS IN MATRIX FORM

Define the $N \times N$ lag matrix L as:

$$L = \begin{bmatrix} 0 & 0 & \dots & & & 0 \\ 1 & 0 & 0 & \dots & & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ : & & & & & : \\ 0 & \dots & & 1 & & 0 \\ 0 & \dots & & 0 & 1 & 0 \end{bmatrix}$$

The first row of L is zero as in finite samples the d-th lag is not defined for the first d data points. This makes some adaptations to the usual lag- and difference operators necessary. Most of their properties, however, carry over to their matrix representations. Lag and difference matrices have the following properties:

Property 1: The d-th lag in matrix form is defined as $L^d = LL^{d-1}$. It holds that $L^d = L^q L^{d-q}$, for any q, $0 \le q \le d$. For completeness, define $L^0 \equi I$.

Property 2: The lead operator in matrix form is equal to the transpose of L, L'.

Property 3: Denote an $N \times N$ identity-matrix in which the first d rows are filled with zeroes as I_d. Then, $L L' = I_1$ holds. In general, $L^d L^{d'} = I_d$. Furthermore, it holds that $I'_d = I_d$. For any pair (n, m), with $n \ge m$, $I_n I_m = I_n$ holds.

Property 4: The matrix of first differences ∇ can be defined as $\nabla \equiv I_1 (I - L)$. The I_1-matrix renders the first row of ∇ zero, accounting for the fact that the lag of the first data point is not defined. In general we define the d-th difference matrix as $\nabla^d \equiv I_d \nabla \nabla^{d-1}$. Again, this is the same as $\nabla \nabla^{d-1}$ with the first d rows set equal to zero as the d-th lag is not defined for the first d data points. It holds that $\nabla^d = I_d \nabla^q \nabla^{d-q}$, for any q, $0 \le q \le d$. For completeness we define $\nabla^0 \equiv I$.

Property 5:

$$\nabla^{d'} = \begin{cases} L^{d'} \nabla^d & \text{if } d \text{ is even} \\ -L^{d'} \nabla^d & \text{if } d \text{ is odd} \end{cases}$$

Proof:

$$\begin{aligned} \nabla^{d'} &= I^d \underbrace{[I_1(I-L)]' \dots [I_1(I-L)]'}_{d\times} = I^d \underbrace{(I_1 - L)' \dots (I_1 - L)'}_{d\times} \\ &= I^d \underbrace{(I_1' - L') \dots (I_1' - L')}_{d\times} = I^d \underbrace{[L'(L-I)] \dots [L'(L-I)]}_{d\times} \\ &= I^d L^{d'} \underbrace{(L-I) \dots (L-I)}_{d\times} \end{aligned}$$

$$= \begin{cases} L^{d'} I^{d} \underbrace{(I - L)\ldots(I - L)}_{d\times} = L^{d'} \nabla^{d} & \text{if } d \text{ is odd} \\ -L^{d'} I^{d} \underbrace{(I - L)\ldots(I - L)}_{d\times} = -L^{d'} \nabla^{d} & \text{if } d \text{ is even} \end{cases}$$

APPENDIX 2
ADDITIONAL FIGURES

Figure 5

Cyclical Components and Real-time Cyclical Components of Real GDP
(percent of real GDP)

(a) TC(2,2,8) filter

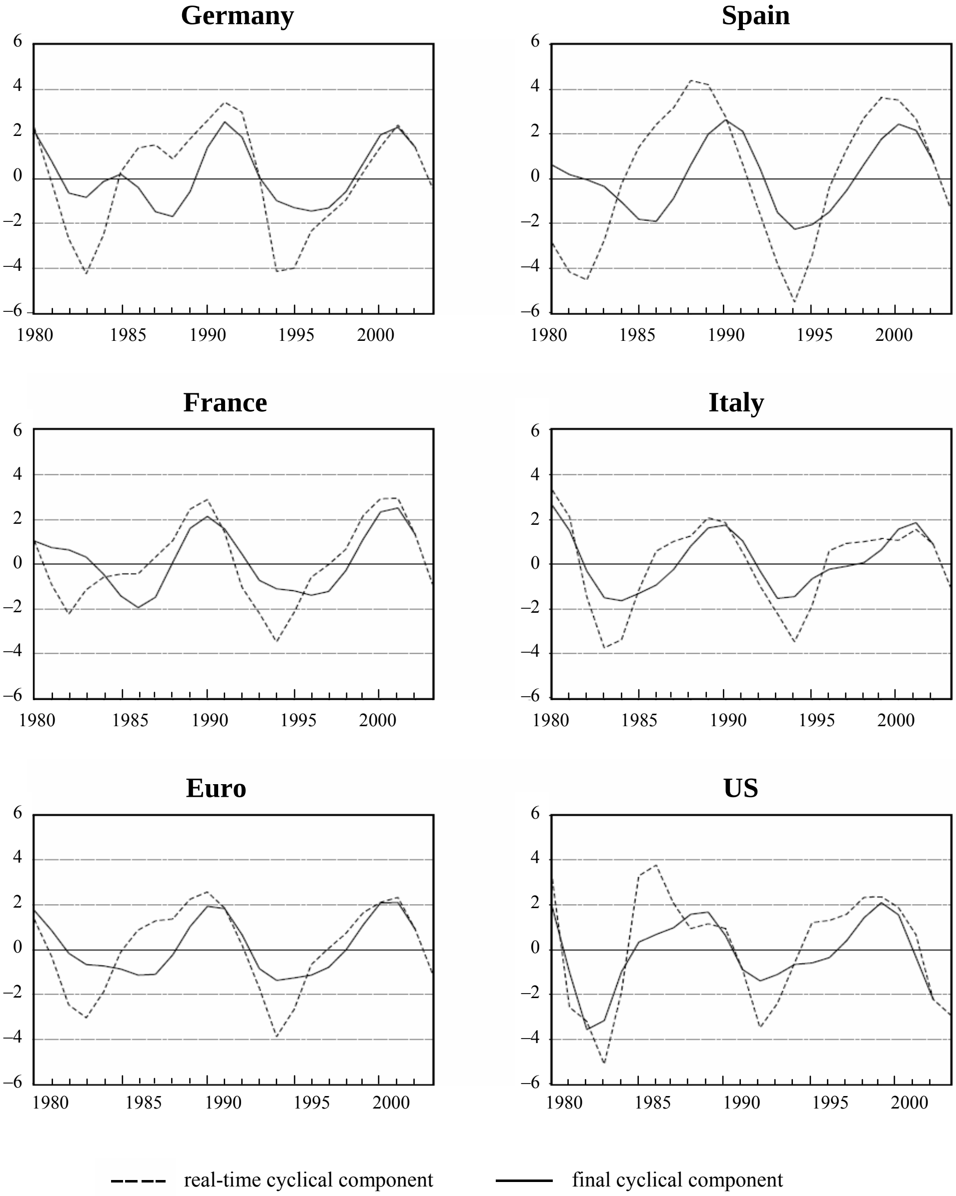


Figure 5 (continued)

Cyclical Components and Real-time Cyclical Components of Real GDP
(percent of real GDP)

(b) HP(30)filter

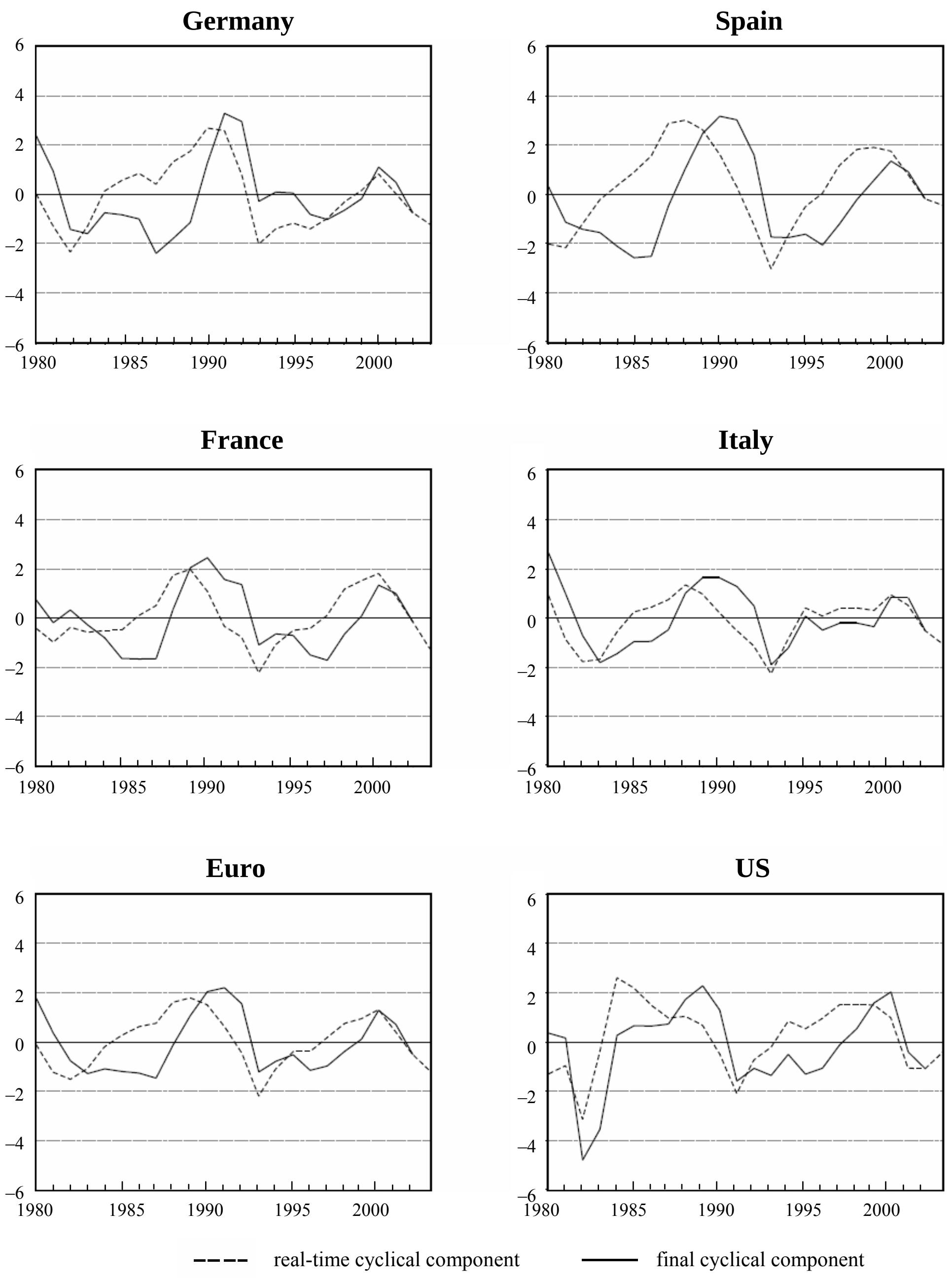


Figure 5 (continued)

Cyclical Components and Real-time Cyclical Components of Real GDP
(percent of real GDP)

(c) EES(7)

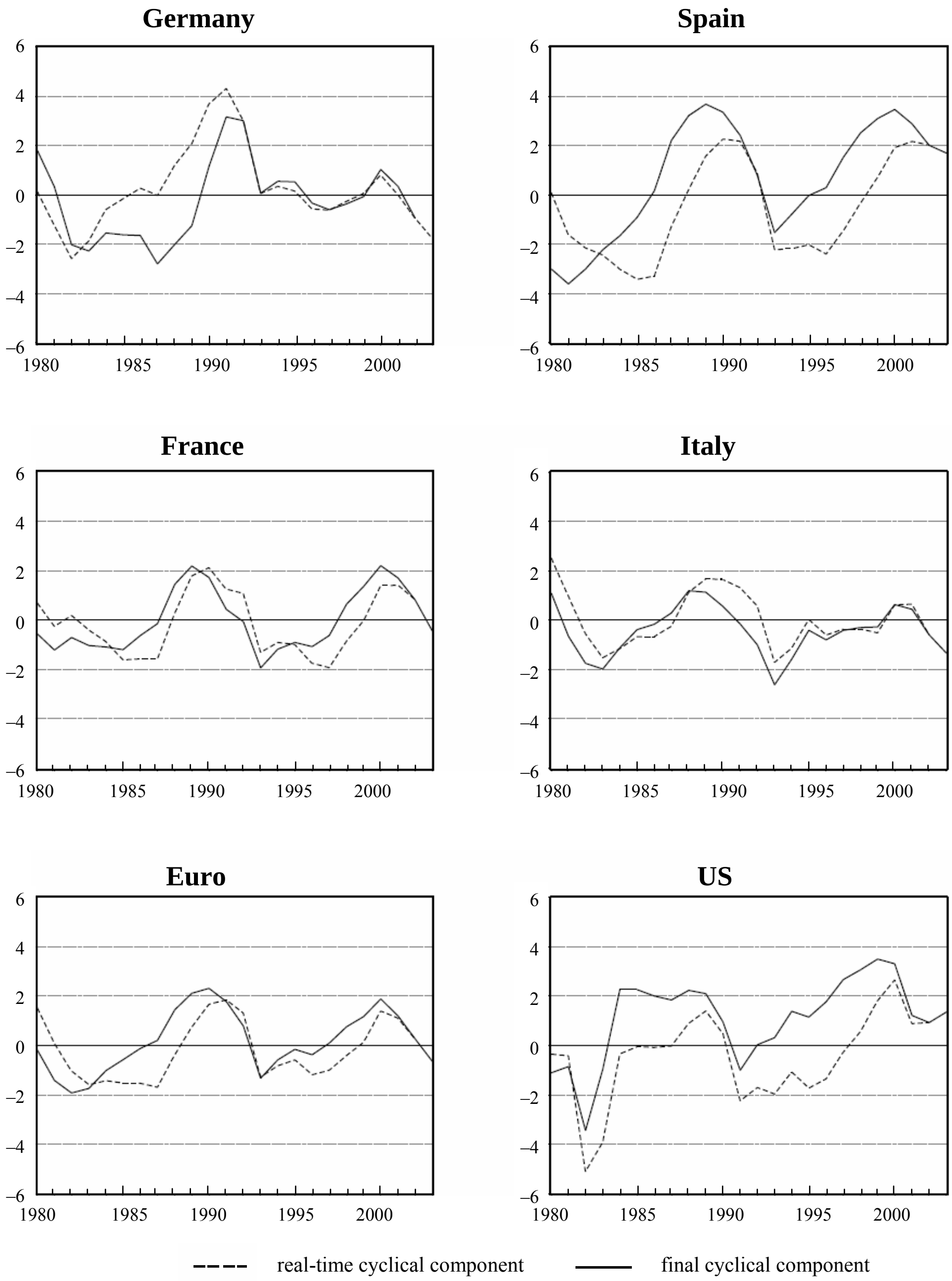


Figure 6

Real-time Minus Final Trend and Final Cyclical Component

(percent of real GDP)

(a) TC(2,2,8) filter

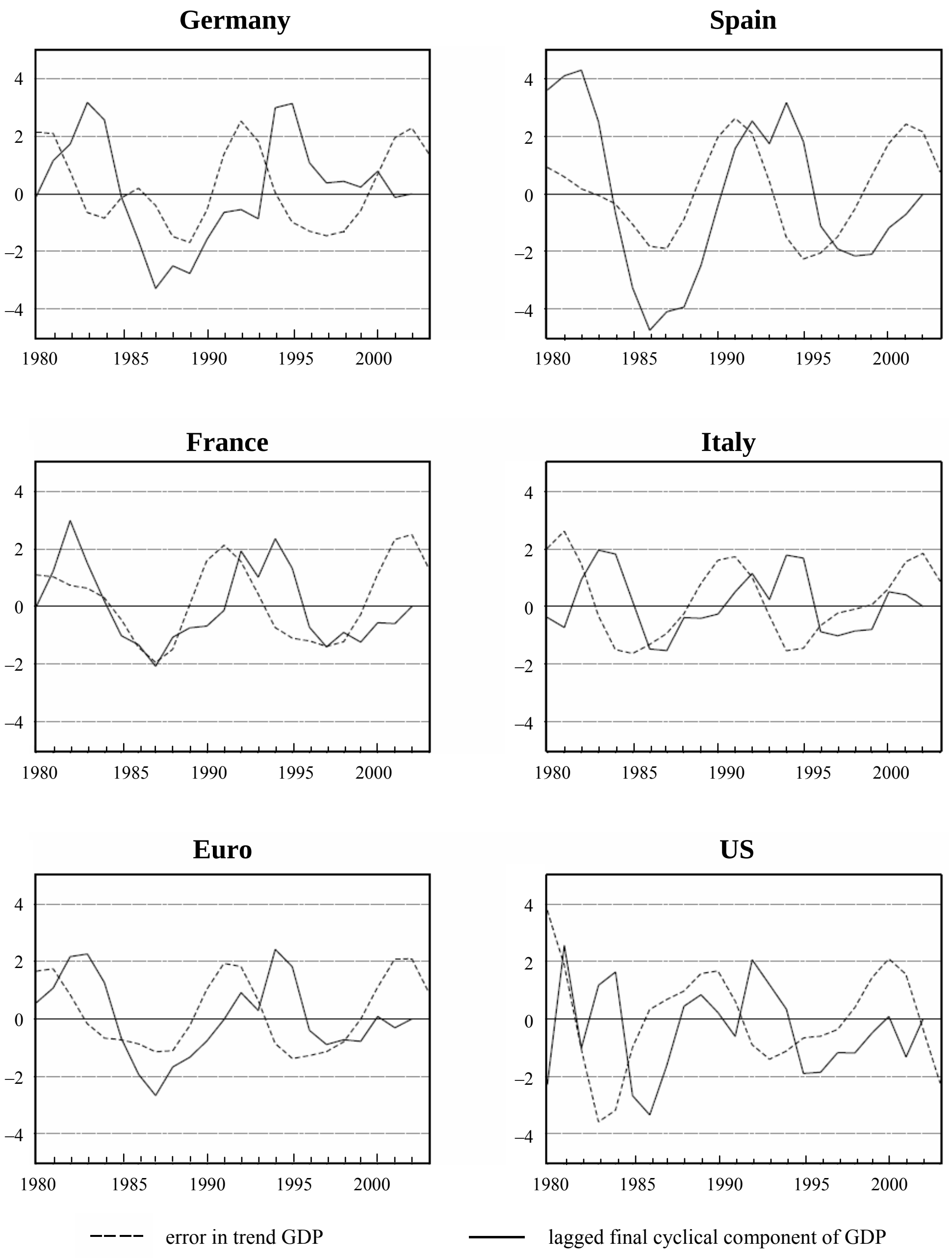


Figure 6 (continued)

Real-time Minus Final Trend and Final Cyclical Component
(percent of real GDP)
(b) HP(30) filter

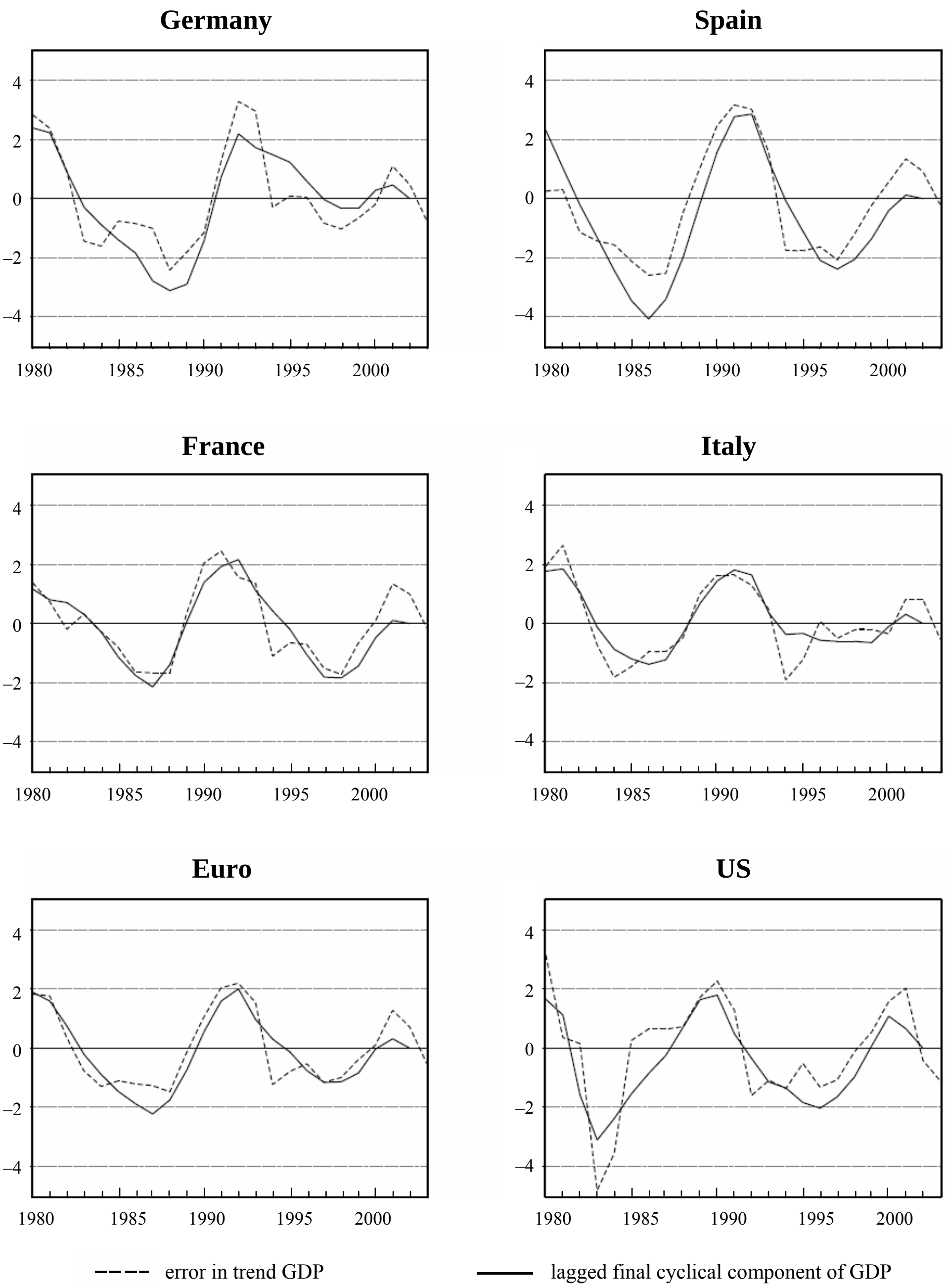
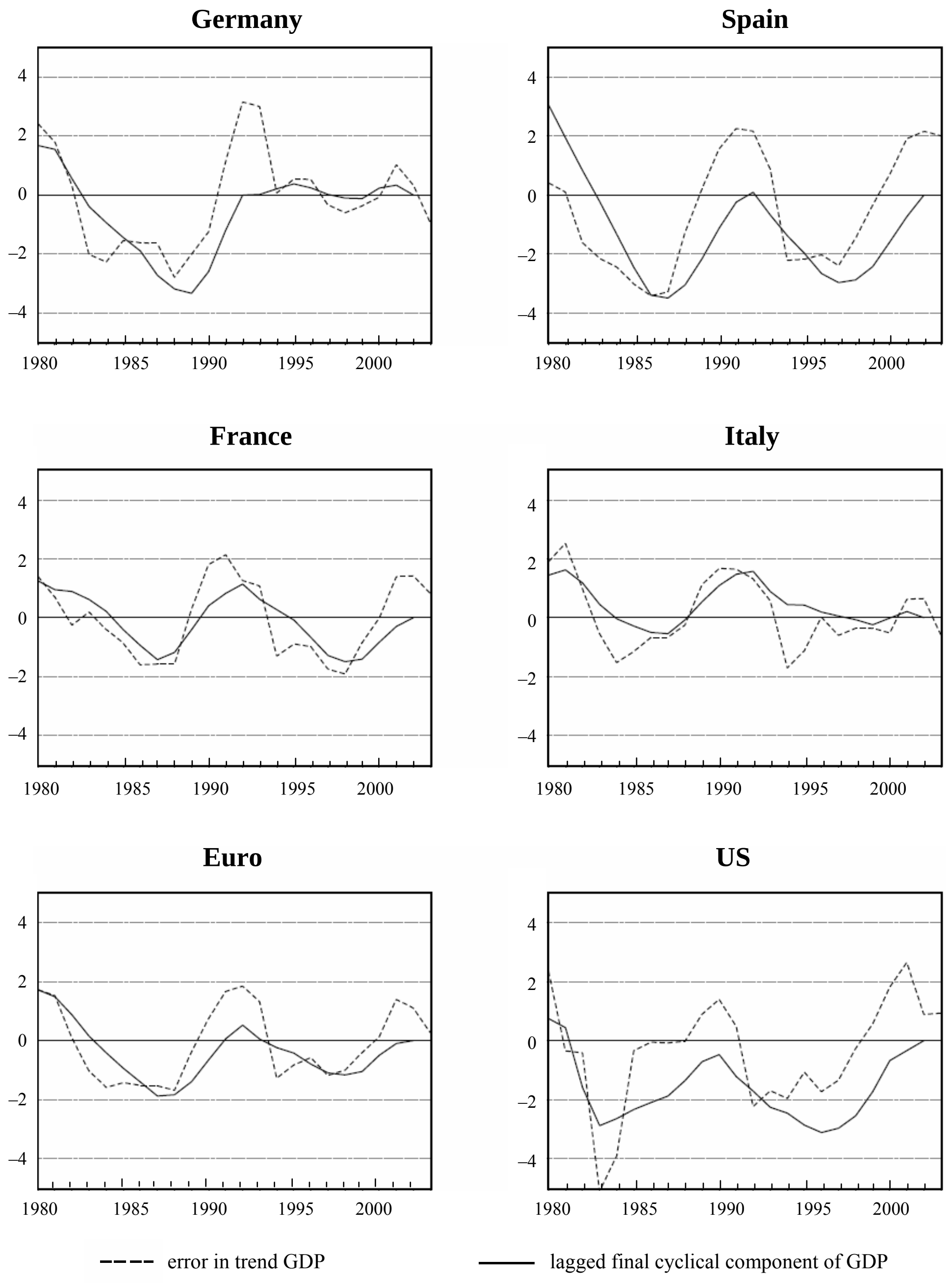


Figure 6 (continued)

Real-time Minus Final Trend and Final Cyclical Component

(percent of real GDP)

(c) EES(7) filter



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