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ON THE DERIVATION OF REDUCED FORMS OF RATIONAL EXPECTATIONS MODELS

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This paper introduces a more intuitive and straightforward method to obtain the reduced forms of linear models containing expectations of the current endogenous variables formed rationally in various previous periods, besides the two proposed by Lucas and Aoki and Canzoneri. This method is then used, with the aid of some examples, to derive the conditions for complete (i.e., in mean and variance) policy ineffectiveness in this kind of models.

1. Introduction

A recent paper by Aoki and Canzoneri (1979), henceforth AC, considers the problem of deriving reduced forms of 'rational expectations' models which include, as explanatory variables, predictions (i.e., 'expectations') of the endogenous variables of the model in a way that these predictions be consistent [i.e., 'rational' in the sense of Muth (1961)] with the predictions of the model itself. Models which contain rational expectations of this kind have been recently proposed in the literature in connection with the hypotheses of efficient markets and natural rates (of various variables), which imply specific constraints on various structural parameters,[1] but are however models of general interest independently of these hypotheses.[2] In their mathematical form the general cases consist of systems of *linear stochastic difference equations*, which, even if linear, 'are difficult to analyze because rational expectations are hard to handle'.[3] The way to analyze their properties has therefore been that of considering their reduced form, that is a form with expectation variables eliminated.

The method generally used to get these reduced forms has been the method of 'undetermined coefficients',[4] which consists in 'guessing' the *general structure of a reduced form* (that differs from the structural form for

[1]See, for one well-known example, Lucas' (1970, 1973) supply equation.
[2]See, for a general discussion of these models, Shiller (1978).
[3]Aoki and Canzoneri (1979, p. 59).
[4]This method was firstly proposed by Lucas (1970).

the elimination of the expectation variables), taking its conditional expectation which is then substituted back in the structural form to get a second reduced form, and then identifying the parameters of this reduced form by means of a comparison with the reduced form initially guessed.

Since this method can involve a good deal of trial and error, AC propose a second method generalizing Muth's (1961) original contribution. Their method involves writing the endogenous variables in *general form* as linear functions of initial conditions and random innovations and taking their conditional expectation which is then substituted in the structural form, to obtain a second system of solution equations that depends only on the initial conditions and the induced autoregressive structure of the disturbances; the parameters of this solution are then determined recursively via a comparison with the original general form and then, through a number of substitutions, the proper reduced form is obtained.

The purpose of this note is to show that there is a third way to get this reduced form; the general idea is to take *directly* the conditional expectation of the *structural form*, substitute it back in the latter and solve for the endogenous variables in terms of their lagged values, the exogenous variables (policy instruments included) and the disturbances only. While this is not novel and is straightforward in the case of AC's general model when there are expectations of the current endogenous variables lagged only one period (i.e., formed on the basis of the last available information set), it can be easily generalized, by proper recursive substitution, to the general case in which expectations formed in various previous periods are present. This will be shown in the next section. The third section will contain a few general comments on the results and on the conditions for policy effectiveness in natural rate *cum* rational expectations models; examples will be presented in the appendix.

2. Derivation of the reduced form of a general class of rational expectations models

The general class of rational expectations models considered by AC[5] is

$$\begin{aligned} z_t &= Az_{t-1} + \sum_{i=1}^{p} B_i z_{t|t-i} + Cx_t + u_t \\ &= K_t z_{t-1} + \sum_{i=1}^{p} B_i z_{t|t-i} + e_t, \end{aligned} \qquad (1)$$

[5] This class of models is general in the sense that all systems of linear equations with a finite sum of lags can be reduced to the structure given by (1); in fact, if y_t are the current endogenous variables and the system contains lagged values of y_t, say $y_{t-1}, \ldots, y_{t-k-1}$, then in (1) $z_t = (y_t', y_{t-1}', \ldots, y_{t-k}')'$, and similarly for x_t.

where

$$x_t = G_t z_{t-1} + v_t, \tag{2}$$

and

$$K_t = A + CG_t, \quad e_t = u_t + Cv_t, \quad z_{t|t-i} = E[z_t | I_{t-i}].$$

z is a vector of endogenous variables, x is a vector of policy instruments, u and v are serially uncorrelated disturbances with zero mean, $z_{t|t-i}$ is the 'rational' expectation of z_t formed at the end of the $(t-i)$th period on the basis of the information set I_{t-i}. The information set consists of all the predetermined variables and the non-stochastic sequence of policy matrices $\{G_t\}$ which describe the systematic part of policy as shown by the (sufficiently general) policy rule (2).[6]

Let us also define for future reference (I being the identity matrix)

$$S_k = \sum_{i=1}^{k} B_i, \qquad D_k = (I - S_k)^{-1},$$

so that[7]

$$I + S_k D_k = I + D_k S_k = D_k. \tag{3}$$

Taking the conditional expectation of (1) given I_{t-p}, we have

$$z_{t|t-p} = K_t z_{t-1|t-p} + S_p z_{t|t-p} = D_p K_t z_{t-1|t-p}, \tag{4a}$$

and, for $j < p$,

$$z_{t|t-j} = K_t z_{t-1|t-j} + S_j z_{t|t-j} + \sum_{i=j+1}^{p} B_i z_{t|t-i}. \tag{4b}$$

Define $\chi(t,j) = z_{t|t-j} - z_{t|t-j-1}$; we then have

$$\chi(t,0) = z_{t|t} - z_{t|t-1} = z_t - z_{t|t-1} = e_t, \tag{5a}$$

and from (4b), for $j > 0$,

$$\chi(t,j) = K_t z_{t-1|t-j} + S_j z_{t|t-j} + \sum_{i=j+1}^{p} B_i z_{t|t-i}$$

[6]Given the fact that I_{t-i} includes z_{t-i}, it follows that $z_{t-i|t-i} = E[z_{t-i} | I_{t-i}] = z_{t-i}$.

[7]By the fact that for any matrix M for which exist $(I-M)^{-1}$, we have $I + M(I-M)^{-1} = I + (I-M)^{-1}M = (I-M)^{-1}$.

$$-\left(K_t z_{t-1|t-j-1} + S_{j+1} z_{t|t-j-1} + \sum_{i=j+2}^{p} B_i z_{t|t-i}\right) \tag{5b}$$

$$= K_t \chi(t-1, j-1) + S_j \chi(t,j) = D_j K_t \chi(t-1, j-1).$$

Also, by sequential substitution,

$$\chi(t,j) = D_j K_t D_{j-1} K_{t-1} \cdot \ldots \cdot D_2 K_{t-j+2} D_1 K_{t-j+1} e_{t-j}. \tag{6}$$

Now, z_t can be expressed as

$$z_t = \sum_{i=0}^{p-1} \chi(t,i) + z_{t|t-p}, \tag{7}$$

and similarly,

$$z_{t-1} = \sum_{i=1}^{p-1} \chi(t-1, i-1) + z_{t-1|t-p}, \tag{8}$$

so that, substituting (4a) and (8) in (7) and using (5b) and (6),

$$z_t = \chi(t,0) + D_p K_t \left(z_{t-1} - \sum_{i=1}^{p-1} \chi(t-1, i-1) \right) + \sum_{i=1}^{p-1} \chi(t,i)$$

$$= D_p K_t z_{t-1} + e_t + \sum_{i=1}^{p-1} (D_i - D_p) K_t \chi(t-1, i-1) \tag{9}$$

$$= D_p K_t z_{t-1} + e_t - \sum_{i=1}^{p-1} D_i \sum_{j=i+1}^{p} B_j D_p P_t^i e_{t-i},$$

where use has been made of the fact, by (3), that

$$D_i - D_p = S_i D_i - S_p D_p = S_i (D_i - D_p) - \sum_{j=i+1}^{p} B_j D_p$$
$$= -D_i \sum_{j=i+1}^{p} B_j D_p, \tag{10}$$

and we have defined

$$P_t^i = K_t D_{i-1} K_{t-1} D_{i-2} \cdot \ldots \cdot D_1 K_{t-i+1}. \tag{11}$$

Substituting again (2) in (9) we then obtain the reduced form we were looking for,

$$z_t = D_p A z_{t-1} + D_p C x_t + \phi_t^p, \tag{12a}$$

$$\phi_t^p = u_t + R^0 v_t + \sum_{i=1}^{p-1} R_t^i (u_{t-i} + C v_{t-i}), \tag{12b}$$

where

$$R^0 = (I - D_p)C = -D_p S_p C, \tag{13a}$$

making use of (3) and

$$R_t^i = -D_i \sum_{j=i+1}^{p} B_j D_p P_t^i. \tag{13b}$$

For $p=1$, the sum on the right side of (12b) is obviously zero and (12a) and (12b) reduce to

$$z_t = D_1 A z_{t-1} + D_1 C x_t + \phi_t^1, \tag{14a}$$

$$\phi_t^1 = u_t + R^0 v_t = u_t - D_1 B_1 C v_t. \tag{14b}$$

The reduced form equations (12a) and (12b) are identical to AC's equations (8a) and (8b); they show that the reduced form of the general model (1) is given by the sum of the reduced form obtained by the perfect foresight version of the model and a serially correlated error structure; while AC's result was obtained for model (1) in the case $v_t = 0$, here the general result is given, and also the R matrices are completely identified in terms of the structural coefficients matrices $A, B_1, \ldots, B_p$, C and G_t; this identification was not provided by AC for the general case of $p > 1$.

3. General remarks

(a) The reduced form equations (12) show then the important result of AC's paper that their non-stochastic components are simply the reduced form equations of the perfect foresight version of the model; for these reduced form equations to be meaningful, the matrix D_p must exist, i.e., $I - \sum_{i=1}^{p} B_i$ must be non-singular, which is the condition for the existence of a unique solution of the perfect foresight version of the model in terms of the predetermined variables.

(b) The method of recursive substitution used in section 2 to derive a reduced form for the general class of rational expectations models (1) can in principle be applied also to models containing expectations of future variables ($z_{t+j|t-i}$), like the one considered in section II of AC's paper; in this case, however, as is well known there is the fundamental problem, observed also by AC, that the reduced form may be not unique ('multiplicity' of equilibria) so that, to get uniqueness, additional conditions have to be introduced, beside those given by the structure of the model and the usual assumptions on its stochastic components.[8]

(c) As AC observe, and as is clear from eqs. (12), (13) and (14), only for the case $p=1$ the reduced form disturbances ϕ_t^p are independent of the policy matrix G_t (so that R^0 does not contain G_t, while R_t^i does), with the consequence that these reduced forms cannot be used as constraints in the usual (dynamic programming) formulation of the optimal control problem. It is interesting to observe that the result of independence of the reduced form disturbances from the policy matrix can be obtained also in the case $p>1$, *provided that the structural form* (1) *contains the lagged vector of policy instruments* x_{t-p+1} *instead of the contemporaneous vector* x_t. This result, which is of interest since it emphasizes the necessity of a proper specification and identification of the actual lag structures existing in real life, is due to the fact that the expectation of z formed at the end of period $t-p$ for period t is based on an information set that contains z_{t-p}, on which depend the policy instruments for period $t-p+1$, as shown by the policy rule (2). The case for $p=1$ is therefore a special case of a more general model containing expectations starting (forward) with $z_{t|t-p}$ and policy variables x_{t-k}, with the constraint that $(t-p)-(t-k)=1$ (so that $k=p-1$); the reduced form of this general model, being independent of the policy matrix, can then be used as a constraint in the usual optimal control problems to obtain a proper policy rule.

(d) This last result also shows that for models incorporating the 'natural rate' hypothesis,[9] anticipated policy may be ineffective only when the policy variables begin to appear in the structural form with a lag, with respect to the endogenous variables they are related to, one period shorter than the maximum lag of the (rational) expectations of the same endogenous variables contained in the model. A special case is obviously that of a model which

[8]See, on this subject, Blanchard and Khan (1980) and Gourieroux, Laffont and Monfort (1979).

[9]This includes an hypothesis of money neutrality for which the (j,k)th element of the matrix D_pC, which measures the effect of the policy instrument 'money supply', x_k, on the level of 'real output', z_j, is zero [see, for example, Lucas (1970)]; in such a case monetary policy would not have any effect, in a perfect foresight model, on the level of real output and, in a rational expectations model, on its expected value. It would also have no effect on the variance of real output if the reduced form residuals were independent of the policy matrices G_t.

contains only x_t and $z_{t|t-1}$. An example for the case of $p=2$ is contained in the appendix.

Appendix

Consider the natural rate *cum* rational expectations model *à la* Lucas (1970, 1973),

$$y_t = \gamma y_{t-1} + \alpha(p_t - p_{t|t-1}) + \beta(p_t - p_{t|t-2}) + u_{1t}, \tag{A.1}$$

$$p_t = x_t - y_t + u_{2t}. \tag{A.2}$$

All variables are in logarithms; y_t is the current level of real output (or the deviation from its 'natural' level), p_t is the current aggregate price level, x_t is a nominal magnitude (for example, the current supply of money) and u_{1t} and u_{2t} are random disturbances with zero means and variances $\sigma^2(u_1)$ and $\sigma^2(u_2)$. Eq. (A.1) is an aggregate supply function identical, for $\beta=0$, to that proposed by Lucas (1973); eq. (A.2) might be interpreted as a special form of an aggregate demand function [see Lucas (1970, 1973)], or as an equation determining the aggregate price level, following, for example, the quantity theory of money.[10]

Obviously, it is not the economic content of this model that is of interest in this paper,[11] but is its general form that can be used to get a deeper understanding of the working of models with rational expectations. This model can in fact be rewritten in the form of the general model (1) of the text as

$$z_t = Az_{t-1} + B_1 z_{t|t-1} + B_2 z_{t|t-2} + Cx_t + u_t, \tag{A.3}$$

where

$$z_t = \begin{bmatrix} y_t \\ p_t \end{bmatrix}, \qquad A = \begin{bmatrix} \gamma/(1+\alpha+\beta) & 0 \\ -\gamma/(1+\alpha+\beta) & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 & -\alpha/(1+\alpha+\beta) \\ 0 & \alpha/(1+\alpha+\beta) \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & -\beta/(1+\alpha+\beta) \\ 0 & \beta/(1+\alpha+\beta) \end{bmatrix},$$

$$C = \begin{bmatrix} (\alpha+\beta)/(1+\alpha+\beta) \\ 1/(1+\alpha+\beta) \end{bmatrix}, \qquad u_t = \begin{bmatrix} (u_{1t} + (\alpha+\beta)u_{2t})/(1+\alpha+\beta) \\ (u_{2t} - u_{1t})/(1+\alpha+\beta) \end{bmatrix}.$$

[10] Constant velocity of circulation of money being normalized, so that its logarithm is zero.

[11] Much criticism can indeed be raised not only with respect to eq. (A.2), but also with respect to equations of the form (A.1) which characterize much of what is now called the 'new neo-classical macroeconomics'.

Assuming that x_t is determined via the policy rule

$$x_t = g_t y_{t-1} + v_t = G_t z_{t-1} + v_t, \tag{A.4}$$

where $G_t = (g_t \quad 0)$ and v_t is a random disturbance with zero mean and variance $\sigma^2(v)$, it is immediately evident that this kind of models contain the *neutrality* hypothesis that x_t cannot have any effect on the expected value of y_t.

The reduced form of (A.3) and (A.4) is in fact, using the results of section 2 [eqs. (12a) and (12b)], equal to

$$z_t = D_2 A z_{t-1} + D_2 C x_t + \phi_t^2, \tag{A.5a}$$

$$\phi_t^2 = u_t - D_2 S_2 C v_t - D_1 B_2 D_2 (A + C G_t)(u_{t-1} + C v_{t-1}), \tag{A.5b}$$

where it is recalled that $D_1 = (I - B_1)^{-1}$, $D_2 = (I - B_1 - B_2)^{-1}$, $S_2 = B_1 + B_2$, and we have that $D_2 C$ is a 2-elements column vector with first element equal to 0 and second element equal to 1.

The variance of y_t can instead be influenced even in this kind of model by changes in the policy parameters, depending on the values of some coefficients. In particular, for $\beta \neq 0$, we have in terms of y_t,

$$y_t = \gamma y_{t-1} + \varepsilon_t - [\beta(\gamma - g_t)/(1+\beta)]\varepsilon_{t-1},$$

where

$$\varepsilon_t = [u_{1t} + (\alpha + \beta)(u_{2t} + v_t)]/(1 + \alpha + \beta),$$

so that the expected value of y_t, being equal to

$$E(y_t) = \gamma y_{t-1},$$

is independent of g_t, while its variance, being equal to[12]

$$\sigma^2(y_t) = \sigma^2(\varepsilon)[1 + \beta^2(\gamma - g_t)^2/(1+\beta)^2],$$

with

$$\sigma^2(\varepsilon) = \{\sigma^2(u_1) + (\alpha+\beta)^2[\sigma^2(u_2) + \sigma^2(v)]\}/(1 + \alpha + \beta)^2,$$

is a function of g_t. Indeed $\sigma^2(y_t)$ is minimum for $g_t = \gamma$.[13]

[12]The hypotheses have been made of independence among u_{1t}, u_{2t} and v_t and of absence of serial correlation.

[13]The model with $p=2$ is still a special case: only g_t is in fact contained in the reduced form residuals so that its 'optimum' value is constant over time (equal in this case to γ).

Were $\beta=0$, so that in (A.1) y_t depended only on the discrepancy $(p_t-p_{t|t-1})$ and not on $(p_t-p_{t|t-2})$, the reduced form for y_t would instead be equal to

$$y_t=y_{t-1}+\varepsilon_t^+,$$

where

$$\varepsilon_t^+=[u_{1t}+\alpha(u_{2t}+v_t)]/(1+\alpha).$$

In this case not only the expected value but also the variance of y_t would be independent of g_t so that x_t would have an effect on y_t only via the unanticipated component v_t.

It is now shown in what follows, as stated in the text, that if the model contained, jointly with $z_{t|t-p}$ (and $p>1$), not x_t but x_{t-k} with $k=p-1$, the previous result for y_t, i.e., its independence of g_t both in mean and variance, would still hold even with $\beta\neq 0$. Since in (A.1) $p=2$, let us consider (A.2) with x_{t-1} replacing x_t. We would have

$$z_t=Az_{t-1}+B_1 z_{t|t-1}+B_2 z_{t|t-2}+Cx_{t-1}+u_t. \tag{A.3$'$}$$

To reconduce (A.3′) and (A.4) to the general form (1) considered in the text, we now write

$$z_t^+=A^+z_{t-1}^+ + B_1^+ z_{t|t-1}^+ + B_2^+ z_{t|t-2}^+ + C^+x_t^+ + u_t^+, \tag{A.6}$$

$$x_t^+=G_t^+ z_{t-1}^+ + v_t^+, \tag{A.7}$$

where

$$z_t^+=\begin{bmatrix} z_t \\ z_{t-1}\end{bmatrix},\qquad A^+=\begin{bmatrix} A & 0 \\ I & 0\end{bmatrix},\qquad B_1^+=\begin{bmatrix} B_1 & 0 \\ 0 & 0\end{bmatrix},$$

$$B_2^+=\begin{bmatrix} B_2 & 0 \\ 0 & 0\end{bmatrix},\qquad C^+=\begin{bmatrix} 0 & C \\ 0 & 0\end{bmatrix},$$

$$x_t^+=\begin{bmatrix} x_t \\ x_{t-1}\end{bmatrix},\qquad u_t^+=\begin{bmatrix} u_t \\ 0\end{bmatrix},\qquad G_t^+=\begin{bmatrix} G_t & 0 \\ 0 & G_{t-1}\end{bmatrix},$$

$$v_t^+=\begin{bmatrix} v_t \\ v_{t-1}\end{bmatrix}.$$

The reduced form of (A.6) and (A.7) is then equal, by (12a) and (12b), to

$$z_t^+ = D_2^+ A^+ z_{t-1}^+ + D_2^+ C^+ x_t^+ + \phi_t^{+2}, \tag{A.8a}$$

$$\phi_t^{+2} = u_t^+ - D_2^+ S_2^+ C^+ v_t^+ - D_1^+ B_2^+ D_2^+ (A^+ + C^+ G_t^+)(u_{t-1}^+ + C^+ v_{t-1}^+), \tag{A.8b}$$

where, obviously,

$$D_1^+ = (I - B_1^+)^{-1}, \quad D_2^+ = (I - B_1^+ - B_2^+)^{-1}, \quad S_2^+ = B_1^+ + B_2^+.$$

It is easy to see that

$$D_1^+ B_2^+ D_2^+ (A^+ + C^+ G_t^+)(u_{t-1}^+ + C^+ v_{t-1}^+)$$

$$= \begin{bmatrix} D_1 B_1 D_2 A & D_1 B_1 D_2 C G_{t-1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_{t-1} + C v_{t-2} \\ 0 \end{bmatrix}$$

$$= D_1 B_1 D_2 A (u_{t-1} + C v_{t-2}),$$

so that in this case the reduced form residuals ϕ are independent of the policy matrices G. Indeed it is easy to show that from (A.8a) and (A.8b) we can write the reduced form for z_t as

$$z_t = D_2 A z_{t-1} + D_2 C x_{t-1} + \phi_t^2, \tag{A.9a}$$

$$\phi_t^2 = u_t - D_2 S_2 C v_{t-1} - D_1 B_1 D_2 A (u_{t-1} + C v_{t-2}). \tag{A.9b}$$

But $D_2 C$ has still the first element equal to 0 and we can therefore rewrite this reduced form in terms of y_t as

$$y_t = \gamma y_{t-1} + \varepsilon_t^{++} - [\beta\gamma/(1+\beta)]\varepsilon_{t-1}^{++},$$

where

$$\varepsilon_t^{++} = [u_{1t} + (\alpha+\beta)(u_{2t} + v_{t-1})]/(1+\alpha+\beta),$$

so that

$$E(y_t) = \gamma y_{t-1},$$

and

$$\sigma^2(y_t) = \sigma^2(\varepsilon)[1 + \beta^2\gamma^2/(1+\beta)^2],$$

are both independent of the process $\{g_t\}$.

It could be easily shown that this independence result would still hold if, instead of replacing x_t with x_{t-1} in (A.2), we had replaced y_{t-1} with y_{t-2} in (A.4). The reduced form residuals will then be independent of the policy coefficients if either the lag of response of y_t to the policy instruments is one period shorter than the maximum lag of the (rational) expectations of y_t contained in the model, or, if the response of y_t to the policy instruments is immediate, the feedback response of x_t to the endogenous variables has a lag equal to the maximum lag of the same expectations.

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