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by Sara Cecchetti and Giovanna Nappo

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A DYNAMIC DEFAULT DEPENDENCE MODEL

by Sara Cecchetti* and Giovanna Nappo†

Abstract

We develop a dynamic multivariate default model for a portfolio of credit-risky assets in which default times are modelled as random variables with possibly different marginal distributions, and Lévy subordinators are used to model the dependence among default times. In particular, we define a cumulative dynamic hazard process as a Lévy subordinator, which allows for jumps and induces positive probabilities of joint defaults. We allow the main asset classes in the portfolio to have different cumulative default probabilities and corresponding different cumulative hazard processes. Under this heterogeneous assumption we compute the portfolio loss distribution in closed form. Using an approximation of the loss distribution, we calibrate the model to the tranches of the iTraxx Europe. Once the multivariate default distribution has been estimated, we analyse the distress dependence in the portfolio by computing indicators of systemic risk, such as the Stability Index, the Distress Dependence Matrix and the Probability of Cascade Effects.

JEL Classification: B26, C02, C53.

Keywords: Lévy subordinators, joint default probability, copula.

Contents

1. Introduction.....	5
2. Credit derivatives.....	7
3. Mathematical preliminaries	8
4. The multivariate default model.....	9
5. Applications.....	20
6. Distress dependence and systemic risk.....	31
7. Conclusions	43
References	44
A Appendix - Mathematical definitions	46
B Appendix - Mathematical proofs	47
C Appendix - Numerical computations.....	61
D Appendix - Other Lévy subordinators	67

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1 Introduction¹

This paper aims to contribute to the literature on the pricing of portfolios of credit derivatives (such as CDOs or basket CDSs) where the goal is to compute the joint probability of default of a portfolio of risky assets. The risk of default of each asset in the portfolio depends mainly on two sources of randomness: an individual risk factor and a common market factor. The latter represents the uncertainty affecting all assets simultaneously. Our objective is to model the aggregate portfolio risk and compute related systemic risk measures. Such main themes are fundamental in finance, both for the valuation of many credit derivatives and for extracting information from market prices that can be relevant from a macro-prudential point of view (such as estimating joint probabilities of default or probabilities of default conditional on other assets being in default). In finance, the market for *credit default swap* (CDS) indexes quickly developed from 2004 on and market operators began to buy and sell so-called CDO tranches for hedging or speculative strategies on the credit derivatives market. The well-known *Gaussian one-factor copula model* of Li (2000) and Hull and White (2004) often cannot explain the spread of the tranches observed on the market (over the counter), partly because it does not attribute proper weights to extreme events. In the last decade, many distributional hypothesis for the dependence structure of the default times have been proposed in the financial literature to try to replicate the tranche spread observed on the market. For example, Andersen et al. (2003) looked at the *Student-t* distribution. Later, Laurent and Gregory (2005) and Andersen and Sidenius (2005) proposed models based on the factor copulas, while Kalemánova et al. (2007) and Eberlein et al. (2008) assumed that synthetic CDOs could be calibrated with the *one-factor Lévy* model using the *normal inverse Gaussian* (NIG) distribution and the more flexible *generalized hyperbolic* (GH) distribution, respectively. In particular, *one-factor Lévy* (or *one-factor infinitely divisible*) models have been introduced by Albrecher et al. (2007).²

The recent international financial crisis has highlighted the lack of correct models for valuing credit derivatives as CDOs. From the theoretical point of view we want to develop a dynamic multivariate default model. Our model is inspired by a recent paper of Mai and Scherer (2009a) that uses a stochastic time change to introduce dependence in a portfolio of credit-risky assets. In that paper the default times are modelled as random variables with possibly different marginal distributions. By restricting the time change to suitable Lévy subordinators the authors can separate the dependence structure and the marginal default probabilities. Using a so-called time normalization they compute the survival copula of all default times. In order to compute the portfolio loss distribution and apply their model to the pricing of CDO tranches, a homogeneous portfolio is assumed, in which all the default times share the same marginal distribution: basically, the time change considered by Mai and Scherer implies that all the default times have the same cumulative hazard rate. Our model develops the ideas of Mai and Scherer (2009a) by assuming possibly different cumulative hazard rates for the default times; we aim to introduce heterogeneity in the model by allowing for a heterogeneous portfolio, as in the implied copula model of Hull and White (2010). In particular, we define and model a cumulative dynamic hazard process as

¹We thank Antonio Di Cesare for his comments and suggestions. The views expressed in the article are those of the authors and do not involve the responsibility of the Bank of Italy.

²Among other relevant papers on this subject are Luciano (2007), Garcia et al. (2009), Donnelly and Embrechts (2010), Masol and Schoutens (2011), and Choros-Tomczyk et al. (2012).

a Lévy subordinator, which allows for jumps and induces positive probabilities of joint defaults, and we model the dependence structure by the implied survival copula, which is related to the choice of the subordinator. In our model we allow the asset classes in the portfolio to have different cumulative default probabilities and corresponding different cumulative hazard processes. We find an analytical closed formula for the distribution of the portfolio loss process under this heterogeneous assumption and we prove an approximation formula for the loss distribution that is useful for empirical applications. Moreover, this model is dynamic in the sense that it allows us to update the portfolio loss distribution at a posterior time, given the portfolio loss distribution at a prior time, by simply computing conditioned default probabilities.

Once we have specified a suitable Lévy subordinator, our model can be calibrated to portfolio CDS spreads and CDO tranche spreads, appropriately choosing the model parameters that determine the dependence structure.

From an empirical point of view we calibrate the parameters of our model to the tranches of the iTraxx Europe, which is a basket of 125 CDSs on European firms. We consider the index as a portfolio and we divide it into two classes: financial firms and non-financial firms. Thanks to our heterogeneous model, once we have estimated the multivariate default distribution of the companies included in the iTraxx, we can follow Segoviano and Goodhart (2009) and Segoviano and Goodhart (2010) and analyse the distress dependence in the portfolio by computing indicators of systemic risk that incorporate changes in distress dependence consistent with the economic cycle. Examples of these stability measures are: 1) the Stability Index, which reflects the expected number of firms becoming distressed given that at least one firm has become distressed; 2) the Distress Dependence Matrix, in which we estimate the set of pairwise conditional probabilities of distress; 3) the Probability of Cascade Effects, that characterizes the likelihood of one or more institutions becoming distressed given that a specific firm becomes distressed. These stability measures can be used to verify which firms are more systemically relevant for the index as a whole.

The paper is organized as follows. In Section 2 we briefly describe the financial products used to calibrate the parameters of our model. Section 3 provides mathematical notions related to the Lévy subordinators.³ In Section 4 we explain the construction of the heterogeneous multivariate default model, we explore the implied dependence structure, and we derive the closed formula for the loss distribution as well as the related approximation that will be used in the applications. Section 5 describes the pricing applications, explaining the pricing formulas, presenting the calibration of the models to iTraxx index market data, and showing the related results. In Section 6 we describe the Distress Dependence measures to which we apply our model, and we show the estimation results at seven significant dates from January 2007 to November 2010. Finally, Section 7 concludes. Mathematical proofs of the theorems⁴ and propositions are shown in Appendix B.

³Other mathematical preliminaries are available in Appendix A.

⁴With the exception of the main theorem, i.e. Theorem 4.1, relating to the portfolio-loss distribution approximation.

2 Credit derivatives

A credit derivative is a derivative security whose payout is conditional on the occurrence of a credit event. It is used primarily to transfer, hedge or manage credit risk. The company for which credit protection is bought is called the reference entity. The credit event is defined with respect to the credit asset(s) issued by a reference entity. If the credit event occurs, a default payment has to be made by one of the counterparties. A credit event is a precisely defined default event, such as bankruptcy, failure to pay, obligation default, repudiation/moratorium, rating downgrade below a given threshold, or changes in the credit spread. A default payment is the payment which has to be made if a credit event happens.

A particular case of credit derivatives is represented by Credit Default Swaps (CDSs). In a single-name CDS the protection seller B agrees to pay the default payment to the protection buyer A if a default occurs. The default payment is structured to replace the loss that a typical lender would incur in the event of a credit event affecting the reference entity. If there is no default of the reference security until the maturity of the default swap, counterpart B pays nothing. On the other hand, the protection buyer A pays a fee for the default protection. In the most common version, the fee is paid at regular intervals until default or maturity. If a default occurs between two fee payment dates, the buyer A has to pay the fraction of the fee payment that has accrued up to the time of default.

2.1 Portfolio credit derivatives and CDOs

When dealing with a portfolio, we need to consider the risk of a clustering of defaults and of joint defaults. Portfolio credit derivatives are instruments used to manage risks of this type.

Collateralized Debt Obligations (CDOs) are financial products designed to securitize portfolios of defaultable assets: loans, bonds or credit default swaps. The assets are sold to a special purpose vehicle (SPV) and investors are offered the opportunity to invest in notes issued by this company. These obligations are collateralized by the underlying debt portfolio. The different notes are structured so as to offer risk/return profiles that are specifically targeted to the risk appetite and investment restrictions of different investor groups. A simple CDO has the following components:

- The underlying portfolio is composed of defaultable assets issued by issuers C_i with notional amounts K_i , $i = 1, \dots, I$. The total notional is $K = \sum_{i=1}^I K_i$.
- The portfolio is transferred to a special created company, the special purpose vehicle (SPV).
- The SPV issues notes:
 - an equity (or first-loss) tranche with notional K_E ;
 - several mezzanine tranches with notional K_{M_1} , K_{M_2} , K_{M_3} , etc.;
 - a senior tranche with notional K_S .
- If during the existence of the CDO one of the bonds in the portfolio defaults, the recovery payments are reinvested in default-free securities or reimbursed.

- At maturity of the CDO, the portfolio is liquidated and the proceeds are distributed to the tranches, according to their seniority ranking.

The key point of the CDO is the final redistribution of the portfolio value according to the seniority of the notes. The senior tranche is served first. If the senior tranche can be fully repaid, the most senior mezzanine tranche is then repaid. If this tranche can also be fully repaid, then the next tranches are paid off in the order of their seniority, until finally the equity tranche is paid whatever is left of the portfolio's value. The payouts are a function of the losses.

1. The first losses hit the equity tranche alone. Until the cumulative loss amount has reached the equity's notional K_E , the other tranches are protected by the equity tranche.
2. Cumulative losses exceeding K_E affect the first mezzanine tranche until its notional is used up.
3. Then the subsequent mezzanine tranches are hit in the order of their seniority.
4. Only when all the other tranches have absorbed their share of the losses will the senior tranche suffer any losses.

In the standard CDOs the underlying portfolio can consist of bonds (collateralized bond obligation, or CBO) or loans (collateralized loan obligation, or CLO). We have a synthetic CDO when credit default swaps are used instead of bonds or loans in the underlying portfolio.

Basically, once a CDO is constructed by partitioning the credit portfolio into tranches with different seniority, a tranche represents a certain loss piece of the overall portfolio which is defined by its lower and upper attachment points. The protection seller receives periodic premium payments depending on the remaining nominal value and the spread of this tranche, while the protection buyer is compensated for the losses affecting this tranche. The pricing of a tranche corresponds to an assessment of the spread such that the expected discounted payment streams of this tranche for the protection buyer and the protection seller agree.

3 Mathematical preliminaries

3.1 Lévy subordinator

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A one-dimensional Lévy process on this probability space is a càdlàg stochastic process $\Lambda = \{\Lambda_t\}_{t \geq 0}$ starting at $\Lambda_0 = 0$, which has independent and stationary increments. A *Lévy subordinator* is a particular Lévy process in which almost all paths are non-decreasing.⁵ It can be shown that a Lévy subordinator has just two characteristics, a drift $\mu \geq 0$ and a positive measure ν (called the *Lévy measure*) on $(0, \infty)$, via the Lévy-Khintchine formula

$$\Psi(\lambda) = \mu\lambda + \int_0^\infty (e^{-\lambda t} - 1) \nu(dt), \quad \lambda \geq 0, \quad t \geq 0.$$

⁵See Appendix A for a more detailed definition, and Applebaum (2004) as a reference for the study of Lévy processes.

Basically, a Lévy subordinator is a process that grows linearly with a constant drift and is affected by random upward jumps. The process drift is $\mu \geq 0$, while the expected number of jumps greater than or equal to x within a unit of time is given by the Lévy measure ν of the interval $[x, \infty)$.

The function Ψ is the *Laplace exponent* of Λ that completely determines the process via its Laplace transform

$$\mathbb{E}(e^{\lambda \Lambda_t}) = e^{t\Psi(\lambda)}.$$

Note that the function Ψ has negative values, $\Psi(0) = 0$ and, unless $\Lambda_t \equiv 0$, is strictly increasing.

In this paper we take into consideration the *Inverse Gaussian subordinator*. Basically, we choose this Lévy subordinator because, as we will see in the following section, it depends on a small number of parameters (that need to be estimated for the practical implementation) and allows us to develop explicit computations. Moreover, this is the subordinator with the best performance in Mai and Scherer (2009a).⁶

3.1.1 The inverse Gaussian subordinator

The *inverse Gaussian (IG) subordinator* $\Lambda^{IG} = \{\Lambda_t^{IG}\}_{t \geq 0}$ belongs to the class of infinite activity subordinators, meaning that processes of this class jump infinitely often within a unit interval of time. The IG Lévy measure as well as the density of the underlying infinitely divisible distribution are well known. In particular, given an IG subordinator with parameters $\eta, \beta > 0$, we have that Λ_t^{IG} follows an Inverse Gaussian $IG(\beta t, \eta)$ -distribution with density

$$f_{IG}(x) = \frac{\beta t}{\sqrt{2\pi}} x^{-\frac{3}{2}} e^{\eta \beta t} e^{-\frac{1}{2} \left(\frac{\beta^2 t^2}{x} + \eta^2 x \right)} \mathbf{1}_{\{x > 0\}}.$$

The corresponding Lévy measure is given by

$$\nu_{IG}(dx) = \frac{1}{\sqrt{2\pi}} \beta x^{-\frac{3}{2}} e^{-\frac{1}{2} \eta^2 x} \mathbf{1}_{\{x > 0\}} dx.$$

4 The multivariate default model

In this section we develop a model that is an extension of the Mai and Scherer model (2009a).

Consider n defaultable firms with random default times $\tau_1, \tau_2, \dots, \tau_n$. In Mai and Scherer (2009a) these default times are supposed to be characterized by individual factors, given by their marginal distribution functions $G_i(t)$ (with $G_i(0) = 0$, $G_i(t) < 1$ for each $t \geq 0$, $i = 1, \dots, n$), and affected by a common factor, a Lévy subordinator Λ_t with Laplace exponent $\Psi(\lambda)$ that infers the dependence structure. In particular, for each firm we consider the related cumulative hazard function $h_i(t) = -\log(1 - G_i(t))$; this function $h_i(t) : [0, \infty) \rightarrow \mathbb{R}$ is non negative, strictly increasing and continuous, with $h_i(0) = 0$ and such that $\lim_{t \rightarrow \infty} h_i(t) = \infty$. The firms' survival functions are defined as $\bar{G}_i(t) := e^{-h_i(t)}$, $t \geq 0$. In

⁶A brief description of the other two subordinators used by Mai and Scherer (2009a) is found in Appendix D, while the results obtained with the calibration of our model using these other two subordinators will be object of further investigation.

the model of Mai and Scherer (2009a) the following *time normalization condition* is needed to separate the marginal distributions and the dependence structure, which is given by a copula that, under such condition, does not depend on the marginal distributions.

Definition 4.1 (Time normalization (TN)). *Let F be a cumulative distribution function with $F(0) = 0$. Let $\Lambda = \{\Lambda_t\}_{t \geq 0}$ be a stochastic process which is almost surely non-decreasing and such that $\Lambda_0 = 0$. We say that Λ satisfies (TN) for the distribution F if $\mathbb{E}[F(\Lambda_t)] = F(t)$, for each $t \geq 0$.*

If, for example, we consider the cumulative distribution function of an exponential random variable with parameter $(-\lambda) > 0$, $F(t) = (1 - e^{\lambda t})\mathbf{1}_{\{t > 0\}}$ and a Lévy subordinator Λ with characteristics (μ, ν) , we have that

$$\Lambda \text{ satisfies (TN) for } F \Leftrightarrow \Psi(\lambda) = \lambda.$$

For each firm the default occurs when the related stochastic process goes beyond a certain threshold. In particular, in order to construct default times that have the pre-specified marginal distributions and the dependence structure given by the subordinator, as threshold factors are considered n exponential times E_i , i.i.d. exponential random variables with parameter 1, also independent on the Lévy subordinator $(\Lambda_t)_{t \geq 0}$ satisfying (TN) for the unit exponential law, and so with Laplace exponent Ψ satisfying $\Psi(-1) = -1$. In Mai and Scherer (2009a) the i -th default time is defined by

$$\tau_i = \inf\{t > 0 : \Lambda_{h_i(t)} > E_i\},$$

and so can be considered the first jump time of a Poisson process with the stochastic clock $\{\Lambda_{h_i(t)}\}_{t \geq 0}$.⁷

4.1 The completely heterogeneous case

In this paper we consider the existence of other individual factors that, together with the individual hazard function and the common subordinator, define the default times as

$$\tau_i = \inf\{t > 0 : a_i \Lambda_{h_i(t)} + b_i h_i(t) > E_i\}.$$

In other words, for each $i = 1, \dots, n$ there exists a Lévy subordinator

$$\Lambda_i(t) = a_i \Lambda_t + b_i t$$

and an increasing function h_i such that

$$\tau_i = \inf\{t > 0 : \Lambda_i(h_i(t)) > E_i\}.$$

Working with the survival distributions, we can compute the marginal distributions and the joint distribution.

⁷See Appendix A for the definition of stochastic clock.

4.1.1 The marginal distributions

Setting

$$G_i(t) := 1 - \bar{G}_i(t) \quad \text{and} \quad \bar{G}_i(t) := e^{-h_i(t)}, \quad i = 1, \dots, n,$$

for each default time the marginal survival distribution is given by

$$\begin{aligned} \bar{F}_i(t) &:= \mathbb{P}(\tau_i > t) = \mathbb{E}(\mathbb{P}(\tau_i > t | \mathcal{F}_\infty^\Lambda)) = \mathbb{E}(e^{-\Lambda_i(h_i(t))}) = \mathbb{E}(e^{-a_i \Lambda_{h_i(t)} - b_i h_i(t)}) \\ &= e^{\Psi(-a_i) h_i(t) - b_i h_i(t)} = e^{-(b_i - \Psi(-a_i)) h_i(t)} = (\bar{G}_i(t))^{b_i - \Psi(-a_i)}. \end{aligned}$$

Remark 4.1. *Let us observe that assuming the parameters constraint $b_i - \Psi(-a_i) = 1$, we would obtain that the default times marginal distributions are in fact $G_i(t)$.*

We can also obtain the inverse function

$$\bar{F}_i^{-1}(u) = h_i^{-1}\left(-\frac{\log u}{b_i - \Psi(-a_i)}\right),$$

as can easily be seen by observing that

$$\begin{aligned} \bar{F}_i^{-1}(u) = t(u) \quad \Leftrightarrow \quad e^{-(b_i - \Psi(-a_i)) h_i(t(u))} = u \quad \Leftrightarrow \quad -(b_i - \Psi(-a_i)) h_i(t(u)) = \log u \\ \Downarrow \\ h_i(t(u)) = -\frac{\log u}{b_i - \Psi(-a_i)} \end{aligned} \tag{1}$$

which means, as h_i is strictly increasing and continuous and so invertible,

$$\bar{F}_i^{-1}(u) = h_i^{-1}\left(-\frac{\log u}{b_i - \Psi(-a_i)}\right).$$

Note that equation (1) is fundamental in computing the survival joint copula, as we will see later.

4.1.2 The joint distribution

Let us define the permutation $\sigma_i(\mathbf{t}) = \sigma_i(t_1, \dots, t_n)$ such that

$$h_{(i)}(\mathbf{t}) := h_{\sigma_i(\mathbf{t})}(t_{\sigma_i(\mathbf{t})})$$

is a reordering of $h_i(t_i)$, which means that

$$h_{(i-1)}(\mathbf{t}) \leq h_{(i)}(\mathbf{t}), \quad i = 1, \dots, n,$$

where we assume by convention $h_{(0)}(\mathbf{t}) = 0$.

Let us also introduce the following notation:

$$\theta_j(\mathbf{t}) = \sum_{i=j}^n a_{\sigma_i(\mathbf{t})}, \quad j = 1, \dots, n$$

Proposition 4.1. *The joint survival distribution is given by*

$$\bar{F}_{\tau_1, \dots, \tau_n}(t_1, \dots, t_n) = e^{-\sum_{j=1}^n (\Psi(-\theta_j(\mathbf{t}) + a_{\sigma_j(\mathbf{t})}) - \Psi(-\theta_j(\mathbf{t}) + b_{\sigma_j(\mathbf{t})}))} h_{\sigma_j(\mathbf{t})}(t_{\sigma_j(\mathbf{t})}). \quad (2)$$

Alternately, introducing the permutation $\sigma_j^{-1}(\mathbf{t})$, i.e. the inverse permutation of $\sigma_j(\mathbf{t})$, we have the following equivalent proposition:

Proposition 4.2. *The joint survival distribution of the default times can be computed as*

$$\begin{aligned} \bar{F}_{\tau_1, \dots, \tau_n}(t_1, \dots, t_n) &= \prod_{j=1}^n (\bar{F}_{\tau_j}(t_j)) \frac{\Psi(-\sum_{i=\sigma_j^{-1}(\mathbf{t})}^n a_{\sigma_i(\mathbf{t}) + a_j}) - [\Psi(-\sum_{i=\sigma_j^{-1}(\mathbf{t})}^n a_{\sigma_i(\mathbf{t})}) - \Psi(-a_j)]}{b_j - \Psi(-a_j)} + 1 \\ &= \prod_{j=1}^n (\bar{F}_{\tau_j}(t_j)) \frac{\Psi(-\theta_{\sigma_j^{-1}(\mathbf{t})}(\mathbf{t}) + a_j) - [\Psi(-\theta_{\sigma_j^{-1}(\mathbf{t})}(\mathbf{t})) - \Psi(-a_j)]}{b_j - \Psi(-a_j)} + 1. \end{aligned} \quad (3)$$

Remark 4.2. *Note that equation (3) can be rewritten in terms of the functions $\bar{G}_i(t)$ as*

$$\bar{F}_{\tau_1, \dots, \tau_n}(t_1, \dots, t_n) = \prod_{j=1}^n (\bar{G}_j(t_j)) \Psi(-\theta_{\sigma_j^{-1}(\mathbf{t})}(\mathbf{t}) + a_j) - \Psi(-\theta_{\sigma_j^{-1}(\mathbf{t})}(\mathbf{t})) + b_j,$$

and in particular, when $b_j - \Psi(-a_j) = 1$, for all $j = 1, \dots, n$

$$\bar{F}_{\tau_1, \dots, \tau_n}(t_1, \dots, t_n) = \prod_{j=1}^n (\bar{G}_j(t_j))^{1 + \Psi(-\theta_{\sigma_j^{-1}(\mathbf{t})}(\mathbf{t}) + a_j) + \Psi(-a_j) - \Psi(-\theta_{\sigma_j^{-1}(\mathbf{t})}(\mathbf{t}))}.$$

If the permutation is uniquely determined, i.e. when $h_i(t_i)$, $i = 1, \dots, n$, are n distinct numbers, then

$$\theta_{\sigma_j^{-1}(\mathbf{t})}(\mathbf{t}) - a_j = \sum_{\substack{i \neq j: \\ h_i(t_i) \geq h_j(t_j)}}^{1, n} a_i \quad \text{and} \quad \theta_{\sigma_j^{-1}(\mathbf{t})}(\mathbf{t}) = \sum_{\substack{i: \\ h_i(t_i) \geq h_j(t_j)}}^{1, n} a_i.$$

If the permutation is not uniquely determined, i.e. when $h_i(t_i)$, $i = 1, \dots, n$, are not n distinct numbers, without loss of generality we can take the permutation with the least number of inversions: more precisely, if we consider a partition of $\{1, 2, \dots, n\}$ in the subsets I , $J = \{j_1, \dots, j_s\}$ and K ⁸, with

$$h_i(t_i) < h_{j_1}(t_{j_1}) = h_{j_2}(t_{j_2}) = \dots = h_{j_s}(t_{j_s}) < h_k(t_k) \quad \forall i \in I, \forall k \in K$$

⁸Note that the sets I , J and K of the partition depend on the vector $\mathbf{t} = (t_1, t_2, \dots, t_n)$.

then we take $\sigma_{|I|+1} < \sigma_{|I|+2} < \dots < \sigma_{|I|+s}$, or equivalently, assuming that $j_1 < j_2 < \dots < j_s$, we take $\sigma_{|I|+1} = j_1, \sigma_{|I|+2} = j_2, \dots, \sigma_{|I|+s} = j_s$. Then we can say that, for all $j = j_l \in J = \{j_1, \dots, j_s\}$

$$\begin{aligned}\theta_{\sigma_j^{-1}(\mathbf{t})}(\mathbf{t}) - a_j &= \sum_{k \in K} a_k + \sum_{v: j_v > j_l}^{1,s} a_{j_v} = \sum_{k \in K} a_k + \sum_{v=l+1}^s a_{j_v}, \\ \theta_{\sigma_j^{-1}(\mathbf{t})}(\mathbf{t}) &= \sum_{k \in K} a_k + \sum_{v: j_v \geq j_l}^{1,s} a_{j_v} = \sum_{k \in K} a_k + \sum_{v=l}^s a_{j_v}.\end{aligned}$$

Furthermore, observe that

$$\bar{G}_j(t_j) = \bar{G}_{j_1}(t_{j_1}) \quad \forall j \in J,$$

so that in the case $b_i - \Psi(-a_i) = 1$, for all $i = 1, \dots, n$,

$$\begin{aligned}\bar{F}_{\tau_1, \dots, \tau_n}(t_1, \dots, t_n) &= \prod_{i \in I \cup K} (\bar{G}_i(t_i))^{1 + \Psi(-\theta_{\sigma_i^{-1}(\mathbf{t})}(\mathbf{t}) + a_i) + \Psi(-a_i) - \Psi(-\theta_{\sigma_i^{-1}(\mathbf{t})}(\mathbf{t}))} \\ &\quad \cdot \prod_{j \in J} (\bar{G}_{j_1}(t_{j_1}))^{1 + \Psi(-\sum_{k \in K} a_k - \sum_{v: j_v > j}^{1,s} a_{j_v}) + \Psi(-a_j) - \Psi(-\sum_{k \in K} a_k - \sum_{v: j_v \geq j}^{1,s} a_{j_v})} \\ &= \prod_{i \in I} (\bar{G}_i(t_i))^{1 + \Psi(-\theta_{\sigma_i^{-1}(\mathbf{t})}(\mathbf{t}) + a_i) + \Psi(-a_i) - \Psi(-\theta_{\sigma_i^{-1}(\mathbf{t})}(\mathbf{t}))} \\ &\quad \cdot (\bar{G}_{j_1}(t_{j_1}))^{|\mathcal{J}| + \sum_{j \in J} \Psi(-a_j) + \Psi(-\sum_{k \in K} a_k) - \Psi(-\sum_{k \in K} a_k - \sum_{j \in J} a_j)} \\ &\quad \cdot \prod_{k \in K} (\bar{G}_k(t_k))^{1 + \Psi(-\theta_{\sigma_k^{-1}(\mathbf{t})}(\mathbf{t}) + a_k) + \Psi(-a_k) - \Psi(-\theta_{\sigma_k^{-1}(\mathbf{t})}(\mathbf{t}))}.\end{aligned}$$

4.1.3 The survival copula

The survival copula⁹ is defined as

$$\begin{aligned}\hat{C}_{\tau_1, \dots, \tau_n}(u_1, \dots, u_n) &= \bar{F}_{\tau_1, \dots, \tau_n}(\bar{F}_{\tau_1}^{-1}(u_1), \dots, \bar{F}_{\tau_n}^{-1}(u_n)) \\ &= \bar{F}_{\tau_1, \dots, \tau_n}(t_1(u_1), \dots, t_n(u_n)) = \bar{F}_{\tau_1, \dots, \tau_n}(\mathbf{t}(\mathbf{u}))\end{aligned}$$

where $\mathbf{t}(\mathbf{u})$ is the vector with components $t_i(u_i)$, by which we mean $\bar{F}_{\tau_i}^{-1}(u_i)$. So we have

$$\hat{C}_{\tau_1, \dots, \tau_n}(u_1, \dots, u_n) = e^{-\sum_{j=1}^n (\Psi(-\theta_j(\mathbf{t}(\mathbf{u})) + a_{\sigma_j(\mathbf{t}(\mathbf{u}))}) - \Psi(-\theta_j(\mathbf{t}(\mathbf{u}))) + b_{\sigma_j(\mathbf{t}(\mathbf{u}))})} h_{\sigma_j(\mathbf{t}(\mathbf{u}))}(t_{\sigma_j(\mathbf{t}(\mathbf{u}))}(u_{\sigma_j(\mathbf{t}(\mathbf{u}))}))$$

⁹See Appendix A for the definition of copula functions and their main properties, and Nelsen (1999) and Cherubini and Vecchiato (2004) for details.

Let us introduce the following notation:

$$\widehat{\sigma}_j(\mathbf{u}) := \sigma_j(\mathbf{t}(\mathbf{u}))$$

and the following characterization, directly in terms of the vector \mathbf{u} :

$$s_i(\mathbf{u}) := -\frac{1}{b_i - \Psi(-a_i)} \log u_i = \log(u_i)^{-\frac{1}{b_i - \Psi(-a_i)}}.$$

Thanks to the relationship (1), to arrange $h_i(t_i(u_i))$ in order of increasing magnitude we can put in an increasing order $-\frac{\log u_i}{b_i - \Psi(-a_i)}$; we consider the permutation $\widehat{\sigma}_j(\mathbf{u})$ as the permutation (not necessarily unique) such that

$$-\frac{\log u_{\widehat{\sigma}_{j-1}(\mathbf{u})}}{b_{\widehat{\sigma}_{j-1}(\mathbf{u})} - \Psi(-a_{\widehat{\sigma}_{j-1}(\mathbf{u})})} \leq -\frac{\log u_{\widehat{\sigma}_j(\mathbf{u})}}{b_{\widehat{\sigma}_j(\mathbf{u})} - \Psi(-a_{\widehat{\sigma}_j(\mathbf{u})})} \quad j = 2, \dots, n.$$

It follows that, considering the following reordering for $s_i(\mathbf{u})$

$$s_{(1)}(\mathbf{u}) \leq s_{(2)}(\mathbf{u}) \leq \dots \leq s_{(n)}(\mathbf{u}),$$

the permutation $\widehat{\sigma}_j(\mathbf{u})$ is defined by

$$-\frac{\log u_{\widehat{\sigma}_j(\mathbf{u})}}{b_{\widehat{\sigma}_j(\mathbf{u})} - \Psi(-a_{\widehat{\sigma}_j(\mathbf{u})})} = s_{(j)}(\mathbf{u}).$$

Let us also introduce the following notation:

$$\widehat{\theta}_j(\mathbf{u}) := \theta_j(\mathbf{t}(\mathbf{u})) = \sum_{i=j}^n a_{\sigma_i(\mathbf{t}(\mathbf{u}))} = \sum_{i=j}^n a_{\widehat{\sigma}_i(\mathbf{u})},$$

For the copula computation we have the following lemma:

Lemma 4.1. *The survival copula of the vector τ_1, \dots, τ_n is given by*

$$\begin{aligned} \widehat{C}_{\tau_1, \dots, \tau_n}(u_1, \dots, u_n) &= e^{\sum_{j=1}^n \left(\Psi(-\widehat{\theta}_j(\mathbf{u}) + a_{\widehat{\sigma}_j(\mathbf{u})}) - \Psi(-\widehat{\theta}_j(\mathbf{u}) + b_{\widehat{\sigma}_j(\mathbf{u})}) \right) \frac{\log u_{\widehat{\sigma}_j(\mathbf{u})}}{b_{\widehat{\sigma}_j(\mathbf{u})} - \Psi(-a_{\widehat{\sigma}_j(\mathbf{u})})}} \\ &= \prod_{j=1}^n u_{\widehat{\sigma}_j(\mathbf{u})}^{\frac{\Psi(-\widehat{\theta}_j(\mathbf{u}) + a_{\widehat{\sigma}_j(\mathbf{u})}) - \Psi(-\widehat{\theta}_j(\mathbf{u}) + b_{\widehat{\sigma}_j(\mathbf{u})})}{b_{\widehat{\sigma}_j(\mathbf{u})} - \Psi(-a_{\widehat{\sigma}_j(\mathbf{u})})}} \\ &= \prod_{j=1}^n \left(\left(u_{\widehat{\sigma}_j(\mathbf{u})} \right)^{\frac{1}{b_{\widehat{\sigma}_j(\mathbf{u})} - \Psi(-a_{\widehat{\sigma}_j(\mathbf{u})})}} \right)^{\Psi(-\widehat{\theta}_j(\mathbf{u}) + a_{\widehat{\sigma}_j(\mathbf{u})}) - \Psi(-\widehat{\theta}_j(\mathbf{u}) + b_{\widehat{\sigma}_j(\mathbf{u})})}. \end{aligned}$$

In our model we will assume the parameter constraint $b_i - \Psi(-a_i) = 1$, by which the marginal survival distributions are exactly \overline{G}_i . We will calibrate the model as in Mai and Scherer (2009a) in two steps: we first calibrate the marginal distributions and then the copula.

4.2 Heterogeneous case with r different homogeneous classes

Let us in general suppose that, according to a certain criterion, our n firms can be divided into r different classes with the related r parameters a_i , b_i and hazard functions h_i . In this case we can consider the first m_1 firms of type 1, other m_2 firms of type 2, and so on up to the remaining m_r firms of class r , with $n = m_1 + m_2 + \dots + m_r$. In other words, denoting $M_1 = m_1$, $M_2 = m_1 + m_2$, $M_\ell = \sum_{i=1}^{\ell} m_i = M_{\ell-1} + m_\ell$, $\ell \leq r$, we assume

$$\left\{ \begin{array}{l} a_i = a^{(1)}, b_i = b^{(1)}, \text{ and } h_i(t) = h^{(1)}(t), \text{ for } i = 1, \dots, m_1 = M_1 \\ a_i = a^{(2)}, b_i = b^{(2)}, \text{ and } h_i(t) = h^{(2)}(t), \text{ for } i = M_1 + 1, \dots, m_1 + m_2 = M_2 \\ \dots \\ a_i = a^{(r)}, b_i = b^{(r)}, \text{ and } h_i(t) = h^{(r)}(t), \text{ for } i = M_{r-1} + 1, \dots, M_r (= n). \end{array} \right.$$

Let us denote the classes $\mathcal{M}_\ell := \{M_{\ell-1} + 1, \dots, M_\ell\}$, for $\ell = 1, \dots, r$, so that the previous assumption may be rewritten shortly as

$$a_i = a^{(\ell)}, \quad b_i = b^{(\ell)}, \quad \text{and} \quad h_i(t) = h^{(\ell)}(t), \quad \text{for } i \in \mathcal{M}_\ell, \quad \ell = 1, \dots, r. \quad (*m)$$

For each firm in \mathcal{M}_ℓ , we denote the common subordinator as

$$\Lambda^{(\ell)}(t) := a^{(\ell)}\Lambda_{h^{(\ell)}(t)} + b^{(\ell)}h^{(\ell)}(t).$$

Furthermore, we assume that

$$b^{(\ell)} - \Psi(-a^{(\ell)}) = 1 \quad \Leftrightarrow \quad b^{(\ell)} = 1 + \Psi(-a^{(\ell)}), \quad \ell = 1, \dots, r. \quad (*p)$$

Remark 4.3. *The previous assumption (*p) is equivalent to the assumption that*

$$\overline{G}_i(t) = e^{-h_i(t)} = \overline{G}^{(\ell)}(t) = e^{-h^{(\ell)}(t)}, \quad \text{for } i \in \mathcal{M}_\ell, \quad \ell = 1, \dots, r.$$

Indeed, as explained in Remark 4.1, if we assume $b_i - \Psi(-a_i) = 1$ for each i , then we get that the survival marginals are $\overline{G}_i(t) = e^{-h_i(t)}$.

*This assumption allows us to calibrate first the marginals, or equivalently the hazard functions $h^{(\ell)}$, and then the coefficients $a^{(\ell)}$ and the Laplace exponent $\Psi(x)$. Indeed, if assumption (*p) does not hold, then for each $i \in \mathcal{M}_\ell$, $\ell = 1, \dots, r$, the survival marginals are given by $(\overline{G}^{(\ell)}(t))^{b^{(\ell)} - \Psi(-a^{(\ell)})} = e^{-[b^{(\ell)} - \Psi(-a^{(\ell)})]h^{(\ell)}(t)}$ and one has to calibrate simultaneously the hazard functions $h^{(\ell)}$, the coefficients $\Psi(-a^{(\ell)})$ and $b^{(\ell)}$.*

Another reasonable assumption about the r classes is that the related hazard functions $h^{(\ell)}(t)$ are such that, for each t ,

$$h^{(1)}(t) \leq h^{(2)}(t) \leq \dots \leq h^{(r)}(t) \quad (*h).$$

Our assumptions imply that, if $i \in \mathcal{M}_\ell$ and $j \in \mathcal{M}_{\ell'}$, with $\ell \leq \ell'$, we have the survival distributions stochastic ordering

$$\mathbb{P}(\tau_i > t) = \overline{G}_i(t) = e^{-h^{(\ell)}(t)} \geq \mathbb{P}(\tau_j > t) = \overline{G}_j(t) = e^{-h^{(\ell')}(t)}.$$

In other words, the default risk classes \mathcal{M}_ℓ are ordered such that the first class is the least risky, while the last class is the most risky.

Remark 4.4. Without the assumption $(*p)$, in order to have the classes ordered in terms of default risk, one could assume, besides $(*h)$,

$$a^{(1)} \leq a^{(2)} \leq \dots \leq a^{(r)} \quad (*a)$$

and

$$b^{(1)} \leq b^{(2)} \leq \dots \leq b^{(r)} \quad (*b)$$

so that, for any $1 \leq \ell \leq \ell' \leq r$,

$$\Lambda^{(\ell)}(t) = a^{(\ell)} \Lambda_{h^{(\ell)}}(t) + b^{(\ell)} h^{(\ell)}(t) \leq a^{(\ell')} \Lambda_{h^{(\ell')}}(t) + b^{(\ell')} h^{(\ell')}(t) = \Lambda^{(\ell')}(t)$$

Then, clearly, under $(*a)$, $(*b)$, and $(*h)$ (but without assuming $(*p)$) we automatically get that, for $0 \leq \ell \leq \ell' \leq r$,

$$\left(\overline{G}^{(\ell)}(t)\right)^{b^{(\ell)} - \Psi(-a^{(\ell)})} = e^{-[b^{(\ell)} - \Psi(-a^{(\ell)})]h^{(\ell)}(t)} \geq \left(\overline{G}^{(\ell')}(t)\right)^{b^{(\ell')} - \Psi(-a^{(\ell')})} = e^{-[b^{(\ell')} - \Psi(-a^{(\ell')})]h^{(\ell')}(t)}.$$

As we will see, our assumptions $(*h)$ and $(*p)$ simplify the computation of the portfolio loss distribution and in particular allow us to compute easily the survival probability involving all the variables τ_i

$$\mathbb{P}(\tau_1 > t, \tau_2 > t, \dots, \tau_n > t).$$

In fact we have the following result:

Proposition 4.3. Under assumptions $(*h)$ and $(*p)$ on the parameter constraints, we get

$$\mathbb{P}(\tau_1 > t, \tau_2 > t, \dots, \tau_n > t) = e^{-\sum_{j=1}^r \left(\Psi(-\sum_{\ell=j+1}^r m_\ell a^{(\ell)}) - \Psi(-\sum_{\ell=j}^r m_\ell a^{(\ell)}) + m_j b^{(j)} \right) h^{(j)}(t)} \quad (4)$$

and more in general, for each $I_j \subseteq \mathcal{M}_j$, $j = 1, \dots, r$,

$$\begin{aligned} & \mathbb{P}(\tau_i > t, \forall i \in I_j, \forall j = 1, \dots, r) \\ &= e^{-\sum_{j=1}^r \left(\Psi(-\sum_{\ell=j+1}^r k_\ell a^{(\ell)}) - \Psi(-\sum_{\ell=j}^r k_\ell a^{(\ell)}) + k_j (\Psi(-a^{(j)}) + 1) \right) h^{(j)}(t)} \end{aligned} \quad (5)$$

where $k_j = |I_j|$.

Let us remark that, due to condition $(*p)$, this formula depends on the parameters $a^{(j)}$ but not on $b^{(j)}$ (so that the number of parameters to estimate is reduced).

4.2.1 The correlation coefficient

An interesting computation involves the default correlation of firms i and j up to time t .

Let us define the stochastic processes $A^i = \{A_t^i\}_{t \geq 0}$ for $i = 1, \dots, n$ by

$$A_t^i := \mathbf{1}_{\{E_i < \Lambda_i(h_i(t))\}}$$

so that the i -th default time can be defined by

$$\tau_i = \inf\{t > 0 : E_i < \Lambda_i(h_i(t))\} = \inf\{t > 0 : A_t^i = 1\}.$$

Proposition 4.4. Consider two firms i and j in the rating classes \mathcal{M}_m and \mathcal{M}_n respectively, with $m < n$; the covariance $Cov[A_t^i, A_t^j]$ is given by

$$Cov[A_t^i, A_t^j] = \bar{G}^{(m)}(t)\bar{G}^{(n)}(t) \left((\bar{G}^{(m)}(t))^{\left(\Psi(-a^{(m)})+\Psi(-a^{(n)})-\Psi(-(a^{(m)}+a^{(n)}))\right)} - 1 \right).$$

while the correlation coefficient $Corr[A_t^i, A_t^j]$ is given by

$$Corr[A_t^i, A_t^j] = \frac{\sqrt{\bar{G}^{(m)}(t)}\sqrt{\bar{G}^{(n)}(t)} \left((\bar{G}^{(m)}(t))^{\left(\Psi(-a^{(m)})+\Psi(-a^{(n)})-\Psi(-(a^{(m)}+a^{(n)}))\right)} - 1 \right)}{\sqrt{1 - \bar{G}^{(m)}(t)}\sqrt{1 - \bar{G}^{(n)}(t)}}.$$

If, instead, i and j ($i \neq j$) are in the same rating class \mathcal{M}_m we have

$$Cov[A_t^i, A_t^j] = \bar{G}^{(m)2}(t) \left((\bar{G}^{(m)}(t))^{(2\Psi(-a^{(m)})-\Psi(-2a^{(m)}))} - 1 \right).$$

and

$$Corr[A_t^i, A_t^j] = \frac{\bar{G}^{(m)}(t) \left((\bar{G}^{(m)}(t))^{(2\Psi(-a^{(m)})-\Psi(-2a^{(m)}))} - 1 \right)}{1 - \bar{G}^{(m)}(t)}.$$

4.2.2 The loss distribution

Let us assume a homogeneous portfolio in which each firm has the same weight. The zero-recovery loss process $L^n = \{L^n(t)\}_{t \geq 0}$ is defined as

$$L^n(t) := \frac{1}{n} \sum_{i=1}^n A_t^i. \quad (6)$$

Thus $L^n(t)$ gives the fraction of defaulted names in the portfolio up to time t .

To compute the portfolio loss distribution we want to compute, for $k \in \{0, \dots, n\}$,

$$\mathbb{P}(nL^n(t) = k)$$

which represents the probability that k of the n institutions in the portfolio default.

We have the following result:

Proposition 4.5. The distribution of $L^n(t)$ is given by

$$\begin{aligned} & \mathbb{P}(nL^n(t) = k) \\ &= \sum_{\substack{0 \leq k_j \leq m_j \\ k_1 + \dots + k_r = n - k}} \dots \sum_{v=0}^k (-1)^v \sum_{\substack{v_1 + \dots + v_r = v \\ 0 \leq v_j \leq m_j - k_j}} \dots \sum_{j=1}^r \prod \frac{m_j!}{k_j! v_j! (m_j - k_j - v_j)!} (\bar{G}^{(j)}(t))^{\Phi^{(j)}}, \end{aligned}$$

where

$$\Phi^{(j)} = \Psi\left(-\sum_{\ell=j+1}^r (k_\ell + v_\ell) a^{(\ell)}\right) - \Psi\left(-\sum_{\ell=j}^r k_\ell a^{(\ell)}\right) + k_j (\Psi(-a^{(j)} + 1)).$$

Once we have computed the closed formula for the portfolio-loss distribution, we also prove an approximation of that formula, valid for large portfolios, that will be fundamental from a practical point of view, as we will see in Section 5.

Theorem 4.1 (Portfolio-loss distribution approximation). *Let us denote $L_{m_i}^{(i)}(t)$ the fraction of defaulted names in the rating class \mathcal{M}_i (with respectively m_i firms) of the portfolio up to time t*

$$L_{m_i}^{(i)}(t) := \frac{1}{m_i} \sum_{j=1}^{m_i} A_t^{M_{i-1}+j}$$

and let us denote

$$L_\infty^{(i)}(t) := 1 - e^{-(a^{(i)}\Lambda_{h^{(i)}}(t) + b^{(i)}h^{(i)}(t))}$$

Under $\mathbb{P}(\cdot | \mathcal{F}_\infty^\Lambda)$,

$$m_i L_{m_i}^{(i)}(t) \sim \text{Bin}(m_i, 1 - e^{-(a^{(i)}\Lambda_{h^{(i)}}(t) + b^{(i)}h^{(i)}(t))}) = \text{Bin}(m_i, 1 - e^{-\Lambda^{(i)}(h^{(i)}(t))}).$$

Moreover, for fixed $t \geq 0$, $L_{m_i}^{(i)}(t)$ tends to the variable $L_\infty^{(i)}(t)$ in L^2 as m_i tends to infinity.

Let us now consider the overall portfolio (6), which can be written as

$$L^n(t) = \frac{1}{n} \sum_{i=1}^r m_i L_{m_i}^{(i)}(t).$$

Let us denote the portfolio loss conditioned average by $\hat{L}^n(t) := \mathbb{E}[L^n(t) | \mathcal{F}_\infty^\Lambda]$.

Then we have

$$\hat{L}^n(t) = \frac{1}{n} \sum_{i=1}^r m_i (1 - e^{-\Lambda^{(i)}(h^{(i)}(t))})$$

and

$$L^n(t) - \hat{L}^n(t) \xrightarrow{n \rightarrow \infty} 0,$$

in L^2 for each t . So we can use $\hat{L}^n(t)$ as an approximation of $L^n(t)$ and in particular, for the approximation error, we have the upper bound of $\sum_{i=1}^r \frac{\sqrt{m_i}}{n}$.

Proof of theorem 4.1. In each rating class \mathcal{M}_i the firms are homogeneous and conditionally independent. So we have that under $\mathbb{P}(\cdot | \mathcal{F}_\infty^\Lambda)$, $\{A_t^{M_{i-1}+j}\}_{j=1, \dots, m_i}$ are independent and follow a Bernoulli distribution with success probability given by

$$\mathbb{P}(A_t^{M_{i-1}+j} = 1 | \mathcal{F}_\infty^\Lambda) = \mathbb{P}(E_j < \Lambda^{(i)}(h^{(i)}(t)) | \mathcal{F}_\infty^\Lambda) = \mathbb{E}[\mathbb{P}(E_j < \Lambda^{(i)}(h^{(i)}(t)) | \Lambda^{(i)}(h^{(i)}(t))] = 1 - e^{-\Lambda^{(i)}(h^{(i)}(t))}$$

for $j = M_{i-1} + 1, \dots, M_i$ and for $i = 1, \dots, r$.

To show the L^2 -convergence of $L_{m_i}^{(i)}(t)$ we compute

$$\mathbb{E}[L_{m_i}^{(i)}(t)] = G^{(i)}(t),$$

$$\begin{aligned}\mathbb{E}[L_{m_i}^{(i)}(t)(1 - e^{-\Lambda^{(i)}(h^{(i)}(t))})] &= \mathbb{E}[(1 - e^{-\Lambda^{(i)}(h^{(i)}(t))})^2], \\ \mathbb{E}[(L_{m_i}^{(i)}(t))^2] &= \frac{G^{(i)}(t)}{m_i} + \frac{m_i - 1}{m_i} \mathbb{E}[(1 - e^{-\Lambda^{(i)}(h^{(i)}(t))})^2].\end{aligned}$$

It thus follows that

$$\begin{aligned}\mathbb{E}[(L_{m_i}^{(i)}(t) - (1 - e^{-\Lambda^{(i)}(h^{(i)}(t))})^2)] &= \mathbb{E}[(L_{m_i}^{(i)}(t))^2] - 2\mathbb{E}[L_{m_i}^{(i)}(t)(1 - e^{-\Lambda^{(i)}(h^{(i)}(t))})] + \mathbb{E}[(1 - e^{-\Lambda^{(i)}(h^{(i)}(t))})^2] \\ &= \frac{G^{(i)}(t)}{m_i} + \frac{m_i - 1}{m_i} \mathbb{E}[(1 - e^{-\Lambda^{(i)}(h^{(i)}(t))})^2] - \mathbb{E}[(1 - e^{-\Lambda^{(i)}(h^{(i)}(t))})^2] \\ &= \frac{1}{m_i} G^{(i)}(t) - \frac{1}{m_i} \mathbb{E}[(1 - e^{-\Lambda^{(i)}(h^{(i)}(t))})^2] \rightarrow_{m_i \rightarrow \infty} 0.\end{aligned}$$

Similarly, for the overall portfolio loss approximation, we want to prove the L^2 convergence. We have

$$L^n(t) = \frac{1}{n} \sum_{l=1}^n A_t^l = \frac{1}{n} \sum_{i=1}^r \sum_{j=1}^{m_i} A_t^{M_{i-1}+j}$$

and so

$$\hat{L}^n(t) = \mathbb{E}[L^n(t) | \mathcal{F}_\infty^\Lambda] = \frac{1}{n} \sum_{i=1}^r \sum_{j=1}^{m_i} \mathbb{E}[A_t^{M_{i-1}+j} | \mathcal{F}_\infty^\Lambda] = \frac{1}{n} \sum_{i=1}^r \sum_{j=1}^{m_i} (1 - e^{-\Lambda^{(i)}(h^{(i)}(t))}) = \frac{1}{n} \sum_{i=1}^r m_i (1 - e^{-\Lambda^{(i)}(h^{(i)}(t))}).$$

$$\begin{aligned}\mathbb{E}[(L^n(t) - \hat{L}^n(t))^2] &= \mathbb{E} \left[\left(\frac{1}{n} \sum_{i=1}^r \frac{m_i}{m_i} \sum_{j=1}^{m_i} A_t^{M_{i-1}+j} - \frac{1}{n} \sum_{i=1}^r m_i (1 - e^{-\Lambda^{(i)}(h^{(i)}(t))}) \right)^2 \right] \\ &= \mathbb{E} \left[\left(\sum_{i=1}^r \frac{m_i}{n} \left(\frac{1}{m_i} \sum_{j=1}^{m_i} A_t^{M_{i-1}+j} - (1 - e^{-\Lambda^{(i)}(h^{(i)}(t))}) \right) \right)^2 \right].\end{aligned}$$

According to our notation we can write the previous formula as

$$\mathbb{E} \left[\left(\sum_{i=1}^r \frac{m_i}{n} (L_{m_i}^{(i)}(t) - L_\infty^{(i)}(t)) \right)^2 \right]$$

Thus, the thesis follows by using the Minkowsky inequality, as we have

$$\left\| \sum_{i=1}^r \frac{m_i}{n} (L_{m_i}^{(i)} - L_\infty^{(i)}) \right\|_{L^2} \leq \sum_{i=1}^r \left\| \frac{m_i}{n} (L_{m_i}^{(i)} - L_\infty^{(i)}) \right\|_{L^2} \rightarrow_{m_i, n \rightarrow \infty} 0.$$

About the upper bound for the approximation error we have

$$\|L_{m_i}^{(i)} - L_\infty^{(i)}\|_{L^2}^2 = \frac{1}{m_i} G^{(i)}(t) - \frac{1}{m_i} \mathbb{E}[(1 - e^{-\Lambda^{(i)}(h^{(i)}(t))})^2] \leq \frac{1}{m_i}$$

and so

$$\left\| \sum_{i=1}^r \frac{m_i}{n} (L_{m_i}^{(i)} - L_{\infty}^{(i)}) \right\|_{L^2} \leq \sum_{i=1}^r \frac{m_i}{n} \sqrt{\frac{1}{m_i}} = \sum_{i=1}^r \frac{\sqrt{m_i}}{n}.$$

□

5 Applications

We apply our dependence model between default times to the iTraxx Europe, which can be considered a synthetic CDO on an equally weighted portfolio of CDS contracts on 125 European firms.

5.1 Pricing CDO tranches

As mentioned before, CDOs are constructed by partitioning the credit portfolio in tranches with different seniority: each tranche represents a certain loss piece of the overall portfolio and is defined via its lower and upper attachment points. In particular, in the iTraxx Europe there are $J = 6$ tranches defined by the following lower and upper attachment points l^j and u^j , $j = 1, \dots, 6$:

$$[0\%, 3\%], [3\%, 6\%], [6\%, 9\%], [9\%, 12\%], [12\%, 22\%], [22\%, 100\%].$$

The protection seller receives periodic premium payments depending on the remaining nominal and the spread of the tranche, while the protection buyer is compensated for losses affecting his tranche. Let us fix a quarterly payment schedule for five years

$$T = \{t_0 = 0 < t_1 < \dots < t_M = 5\} \quad (M = 20)$$

and assume a constant recovery rate $R = 40\%$ for all companies. The loss $L_t^{(j)}$ affecting tranche j up to time t is linked to the overall portfolio loss $(1 - R)L_t^n$ via

$$L_t^{(j)} = \min(\max(0, (1 - R)L_t^n - l^j), u^j - l^j),$$

where $u^j - l^j$ is a cap to the potential loss equal to the whole tranche. The residual nominal value of the portfolio at time t is given by $Nom_t = 1 - L_t^n$, while the residual nominal value of tranche j is $Nom_t^{(j)} = u^j - l^j - L_t^{(j)}$.

Pricing a tranche corresponds to assessing the fair spread such that the expected discounted payment streams of the tranche agree. Defining the expected discounted default leg for tranche j ($EDDL^{(j)}$) as the compensations for defaults that affect tranche j

$$EDDL^{(j)} = \sum_{t_k \in T} e^{-rt_k} (\mathbb{E}[L_{t_k}^{(j)}] - \mathbb{E}[L_{t_{k-1}}^{(j)}]) \quad (7)$$

and the expected discounted premium leg ($EDPL^{(j)}$) as the periodic payments depending on the remaining nominal of tranche j

$$EDPL^{(j)} = \sum_{t_k \in T} \Delta t_k e^{-rt_k} s_T^{(j)} (u^j - l^j - \mathbb{E}[L_{t_k}^{(j)}]/2 - \mathbb{E}[L_{t_{k-1}}^{(j)}]/2) \quad (8)$$

the fair spread is thus given by the following ratio:

$$s_T^{(j)} = \frac{\sum_{t_k \in T} e^{-rt_k} (\mathbb{E}[L_{t_k}^{(j)}] - \mathbb{E}[L_{t_{k-1}}^{(j)}])}{\sum_{t_k \in T} \Delta t_k e^{-rt_k} (u^j - l^j - \mathbb{E}[L_{t_k}^{(j)}]/2 - \mathbb{E}[L_{t_{k-1}}^{(j)}]/2)}, \quad (9)$$

where e^{-rt_k} are the discount factors and $\Delta t_k = t_k - t_{k-1}$.

Up to January 2009 the CDS spread for each tranche was quoted in basis points and the previous equation was true for each tranche except the first, the so-called equity tranche, for which market convention was to use a running spread of 500 basis points plus an upfront payment quoted as a percentage of the nominal; for this first tranche we had

$$UpF^{(1)} = (EDDL^{(1)} - EDPL^{(1)}(500))/u^1, \quad (10)$$

where we denote by $EDPL^{(1)}(500)$, equation (8) with $s_T^{(1)} = 500bp$.

However, in the last four years pricing conventions have changed several times and upfronts of different sizes have been considered even for tranches other than the equity; moreover, sometimes the levels of these tranches have been quoted in percentage of notional, while at other times in basis points. Without entering into details,¹⁰ we can write the upfront for these other tranche respectively as

$$UpF_{perc}^{(j)} = (EDDL^{(j)} - EDPL^{(j)}(fixed))/u^j, \quad (11)$$

$$UpF_{bps}^{(j)} = (EDDL^{(j)} - EDPL^{(j)}(fixed)), \quad (12)$$

where we denote by $EDPL^{(j)}(fixed)$ equation (8) with $s_T^{(j)} = fixed$, and $fixed$ may be equal to 100bp, 300bp, or 500bp, according to the case.

For the pricing of each tranche we need to compute

$$\begin{aligned} \mathbb{E}[L^{(j)}(t)] &= \mathbb{E}[\min(\max(0, (1-R)L^n(t) - l^j), u^j - l^j)] \\ &= \sum_{k=0}^n \mathbb{P}\left(L^n(t) = \frac{k}{n}\right) * \left(\min\left(\max\left(0, (1-R)\frac{k}{n} - l^j\right), u^j - l^j\right)\right). \end{aligned} \quad (13)$$

We can use either the exact formula above or the formula obtained with the approximation $\hat{L}^n(t)$ for the portfolio loss distribution:

$$\mathbb{E}[L^{(j)}(t)] \approx \mathbb{E}\left[\min\left(\max\left(0, (1-R)\left(\frac{1}{n} \sum_{i=1}^r m_i (1 - e^{-\Lambda^{(i)}(h^{(i)}(t))})\right) - l^j\right), u^j - l^j\right)\right]. \quad (14)$$

In both cases we need to specify the Lévy subordinator that we use to model the dependence structure.

¹⁰See the Markit website for details: <http://www.creditfixings.com/CreditEventAuctions/itraxx.jsp>. (See also Section 5.3.)

5.2 The Lévy subordinator

For our applications we use the *inverse Gaussian subordinator* described in Section 3. Let us recall that this Lévy subordinator only depends on two parameters (η, β) . In case of only two classes ($r = 2$), we can rewrite equation (14) as

$$\begin{aligned} \mathbb{E}[L^{(j)}(t)] &\approx \mathbb{E} \left[\min \left(\max \left(0, (1-R) \left(\frac{m_1}{n} (1 - e^{-a^{(1)}\Lambda_{h^{(1)}(t)} - b^{(1)}h^{(1)}(t)}) + \frac{m_2}{n} (1 - e^{-a^{(2)}\Lambda_{h^{(2)}(t)} - b^{(2)}h^{(2)}(t)}) \right) - l^j \right), u^j - l^j \right) \right] \\ &\approx \mathbb{E} \left[\min \left(\max \left(0, (1-R) \left(1 - \frac{m_1}{n} e^{-a^{(1)}\Lambda_{h^{(1)}(t)} - b^{(1)}h^{(1)}(t)} - \frac{m_2}{n} e^{-a^{(2)}\Lambda_{h^{(2)}(t)} - b^{(2)}h^{(2)}(t)} \right) - l^j \right), u^j - l^j \right) \right]. \end{aligned} \quad (15)$$

The previous expected value corresponds to a double integral in the case of the inverse Gaussian subordinator.

We can write

$$\Lambda_{h^{(2)}(t)} = \Lambda_{h^{(1)}(t)} + (\Lambda_{h^{(2)}(t)} - \Lambda_{h^{(1)}(t)})$$

and considering that $h^{(1)}(t) < h^{(2)}(t)$, observe that, since the subordinator is a process with stationary increments, we have

$$\Lambda_{h^{(2)}(t)} - \Lambda_{h^{(1)}(t)} \sim \Lambda_{(h^{(2)}(t) - h^{(1)}(t))}.$$

Finally, also taking into account the independent increments property, we have

$$\Lambda_{h^{(1)}(t)} \perp \Lambda_{h^{(2)}(t)} - \Lambda_{h^{(1)}(t)}.$$

Let us denote

$$\tilde{\Lambda}(t) := \Lambda_{h^{(1)}(t)}, \quad (16)$$

$$\Delta\tilde{\Lambda}(t) := \Lambda_{h^{(2)}(t)} - \Lambda_{h^{(1)}(t)}, \quad (17)$$

and

$$F(t, x, y) := \min \left(\max \left(0, (1-R) \left(1 - \frac{m_1}{n} e^{-a^{(1)}x - b^{(1)}h^{(1)}(t)} - \frac{m_2}{n} e^{-a^{(2)}x - a^{(2)}y - b^{(2)}h^{(2)}(t)} \right) - l^j \right), u^j - l^j \right). \quad (18)$$

With this notation we can rewrite equation (15) as

$$\mathbb{E}[F(t, \tilde{\Lambda}(t), \Delta\tilde{\Lambda}(t))] = \int_0^\infty \int_0^\infty F(t, x, y) \cdot f_{\tilde{\Lambda}}(x) \cdot f_{\Delta\tilde{\Lambda}}(y) dx dy, \quad (19)$$

where $f_{\tilde{\Lambda}}$ and $f_{\Delta\tilde{\Lambda}}$ denote the density functions of $\tilde{\Lambda}$ and $\Delta\tilde{\Lambda}$ respectively. In the case of the Inverse Gaussian subordinator, we have

$$\tilde{\Lambda}(t) \sim IG(\beta h^{(1)}(t), \eta) \quad (20)$$

and

$$\Delta\tilde{\Lambda}(t) \sim IG(\beta(h^{(2)}(t) - h^{(1)}(t)), \eta) \quad (21)$$

so that

$$f_{\tilde{\Lambda}}(x) = \frac{\beta(h^{(1)}(t))}{\sqrt{2\pi}} x^{-\frac{3}{2}} e^{\eta\beta(h^{(1)}(t))} e^{-\frac{1}{2} \left(\frac{\beta^2(h^{(1)}(t))^2}{x} + \eta^2 x \right)} \mathbf{1}_{\{x>0\}}$$

and

$$f_{\Delta\tilde{\Lambda}}(y) = \frac{\beta(h^{(2)}(t) - h^{(1)}(t))}{\sqrt{2\pi}} y^{-\frac{3}{2}} e^{\eta\beta(h^{(2)}(t) - h^{(1)}(t))} e^{-\frac{1}{2} \left(\frac{\beta^2(h^{(2)}(t) - h^{(1)}(t))^2}{y} + \eta^2 y \right)} \mathbf{1}_{\{y>0\}}.$$

As the numerical integration of the double integral in equation (19) may be difficult (mainly because the upper integration extremes are infinite), we use a proper truncation. For the case of the Inverse Gaussian subordinator, we consider the following truncation:¹¹

$$\mathbb{E}[F(t, \tilde{\Lambda}(t), \Delta\tilde{\Lambda}(t))] = \int_0^{M_x(h^{(1)}(t))} \int_0^{M_y(h^{(2)}(t) - h^{(1)}(t))} F(t, x, y) \cdot f_{\tilde{\Lambda}}(x) \cdot f_{\Delta\tilde{\Lambda}}(y) dx dy, \quad (22)$$

where

$$M_x(h^{(1)}(t)) := \inf_m \left\{ \frac{e^{-\frac{1}{2}\eta^2 m}}{m^{\frac{1}{2}}} \leq 10^{-4} \frac{\sqrt{2\pi}}{2\beta h^{(1)}(t) e^{\beta\eta h^{(1)}(t)}} \right\}$$

and

$$M_y(h^{(2)}(t) - h^{(1)}(t)) := \inf_m \left\{ \frac{e^{-\frac{1}{2}\eta^2 m}}{m^{\frac{1}{2}}} \leq 10^{-4} \frac{\sqrt{2\pi}}{2\beta(h^{(2)}(t) - h^{(1)}(t)) e^{\beta\eta(h^{(2)}(t) - h^{(1)}(t))}} \right\}.$$

In this way we know a priori that the error deriving from the truncation is always inferior¹² to $2 * 10^{-4}$.

5.2.1 Parameter constraints

Whatever the choice of the subordinator, in general the dependence structure is determined by the pair of parameters (η, β) , while μ is indirectly specified by the Lévy measure of the subordinator via the (TN) condition for the unit exponential distribution.

In particular, in this paper, having defined the subordinator for each class as $\Lambda^{(i)}(h^{(i)}(t)) = a^{(i)}\Lambda_{h^{(i)}(t)} + b^{(i)}h^{(i)}(t)$, without loss of generality we can assume $\mu = 0$. Indeed, denoting

$$\Psi_0(x) := \Psi(x) - \mu x$$

and

$$b_0^{(i)} := b^{(i)} + \mu a^{(i)} (\geq 0),$$

then, taking into account that

$$\Psi_0(-a^{(i)}) := \Psi(-a^{(i)}) + \mu a^{(i)},$$

the parameter constraint $b^{(i)} - \Psi(-a^{(i)}) = 1$ is equivalent to $b_0^{(i)} - \Psi_0(-a^{(i)}) = 1$. At this point we can consider $\mu = 0$ and therefore $\Psi_0(x) = \Psi(x)$ and $b_0^{(i)} = b^{(i)}$. Finally, the condition $b_0^{(i)} \geq 0$ holds if and only if

$$\Psi_0(-a^{(i)}) \geq -1,$$

¹¹An alternative method could be to perform a change of variable so that the domain of integration is finite: this method is developed in Appendix C.

¹²See Appendix C for details.

which gives the constraint on the parameters $a^{(i)}$.

Once we have (η, β) , we can compute the Laplace exponent $\Psi_0(x)$ for the subordinator. Then the constraint on the value of $\Psi_0(-a^{(i)})$ is translated into a constraint for the parameters (η, β) .

For the inverse Gaussian subordinator we have:

$$\begin{aligned}\Psi_{0,IG}(-a^{(i)}) &= \int_0^\infty (e^{-a^{(i)}s} - 1)\nu_{IG}(ds) \\ &= \frac{1}{\sqrt{2\pi}}\beta \int_0^\infty (e^{-a^{(i)}s} - 1)s^{-\frac{3}{2}}e^{-\frac{1}{2}\eta^2s}ds \\ &= \beta \left(\eta - \sqrt{2a^{(i)} + \eta^2} \right).\end{aligned}$$

The constraint $\Psi_0(-a^{(i)}) \geq -1$ is translated into the following constraint for (η, β) :

$$\eta > 0, \quad 0 < \beta \leq \frac{\sqrt{2\pi}}{\int_0^\infty (1 - e^{-a^{(i)}s})s^{-\frac{3}{2}}e^{-\frac{1}{2}\eta^2s}ds} = \frac{1}{-\eta + \sqrt{2a^{(i)} + \eta^2}}$$

for each $i = 1, \dots, r$, and so, denoting $a_{\max} := \max\{a^{(i)}\}_{i=1, \dots, r}$, we need to impose the constraint

$$\eta > 0, \quad 0 < \beta \leq \frac{1}{-\eta + \sqrt{2a_{\max} + \eta^2}}.$$

5.3 The calibration to iTraxx quotes

According to the iTraxx conventions the payment streams for the calibration are quarterly premium payments with the previously specified attachment points for the tranches. The discount factors required as input are obtained from risk-free par yields.¹³ The market quotes to which the model is calibrated comprise the portfolio CDS spreads with maturities 3 and 5 years and the spreads for the tranches. In particular, we use the CDS spreads to calibrate the marginal distributions and the tranche spreads to calibrate the subordinator parameters. Seven data,¹⁴ from June 2007 to December 2010 (this time interval comprises Lehman Brothers' bankruptcy and the start of the recent financial crisis) are used from the 9th series of iTraxx Europe with maturity 5 years, and a calibration is run for each of the selected days. Different pricing formulae for the different tranches have been considered during this time

¹³Source: Datastream. This choice is common in the literature and is the same that Mai and Scherer (2009a) use; an alternative choice could be the LIBOR curve. However, different curves should not change the substance of the results.

¹⁴Free data are available on the website <http://www.creditfixings.com/CreditEventAuctions/itraxx.jsp> with only bimonthly frequency. We report only the results related to some significant dates, but we have calibrated our model in all the available bimonthly dates in the time series considered, and the results are coherent with those reported.

interval, as in the last four years the iTraxx price maker Markit has changed the rules of the tranche pricing convention several times, according to market conditions.

We divide our basket of 125 firms into two classes considering the class of the 25 financial firms and the class of the remaining 100 firms.¹⁵

The calibration procedure can be split into two different steps: the first step involves the calibration of the marginal distributions, while the second step involves the calibration of the parameters of the subordinator, related to the dependence structure.

For the marginal distributions G_i , $\{i = 1, \dots, 125\}$, we assume a piecewise linear intensity. In particular, we consider a CDS with a maturity of 3 years and a CDS with a maturity of 5 years. For each firm we have

$$1 - G_i(t) = e^{-h_i(t)} = e^{-\int_0^t \lambda_i(s) ds}$$

where we assume the default intensity $h_i(t)$ given by

$$h_i(t) = \int_0^t \lambda_i(s) ds = \int_0^t (\lambda_i^3 \min\{s, 3\} + \lambda_i^5 (s - 3) \mathbf{1}_{\{s > 3\}}) ds,$$

with λ_i^3 and λ_i^5 being positive intensity parameters that are calibrated to the portfolio-CDS spreads for the 3-year and 5-year contracts, respectively. To be consistent with the assumption *(*m)* (i.e. the assumption that the firms in the same class have the same hazard rate), we need to assume $\lambda_i^3 = \lambda_{(\ell)}^3$ and $\lambda_i^5 = \lambda_{(\ell)}^5$, $i \in \mathcal{M}_\ell$, $\ell = 1, 2$, and we estimate these parameters by using least square method. In particular we compute the 3-year CDS spread and the 5-year CDS spread using the discrete formula

$$modelCDSspread_{(\ell)}^{(3)} = \frac{\frac{1-R}{(1+r_1)} p_1^{(\ell),3} + \frac{1-R}{(1+r_2)^2} p_2^{(\ell),3} + \frac{1-R}{(1+r_3)^3} p_3^{(\ell),3}}{1 + \frac{1-p_1^{(\ell),3}}{1+r_1} + \frac{1-p_1^{(\ell),3}-p_2^{(\ell),3}}{(1+r_2)^2}}$$

and

$$modelCDSspread_{(\ell)}^{(5)} = \frac{\frac{1-R}{(1+r_1)} p_1^{(\ell),3} + \frac{1-R}{(1+r_2)^2} p_2^{(\ell),3} + \frac{1-R}{(1+r_3)^3} p_3^{(\ell),3} + \frac{1-R}{(1+r_4)^4} p_4^{(\ell),5} + \frac{1-R}{(1+r_5)^5} p_5^{(\ell),5}}{1 + \frac{1-p_1^{(\ell),3}}{1+r_1} + \frac{1-p_1^{(\ell),3}-p_2^{(\ell),3}}{(1+r_2)^2} + \frac{1-p_1^{(\ell),3}-p_2^{(\ell),3}-p_3^{(\ell),3}}{(1+r_3)^3} + \frac{1-p_1^{(\ell),3}-p_2^{(\ell),3}-p_3^{(\ell),3}-p_4^{(\ell),5}}{(1+r_4)^4}}.$$

where

- $p_t^{(\ell),\cdot}$, $t \geq 1$, are the discrete default probabilities for the event “For firm i in the class \mathcal{M}_ℓ there will be default in year t ”, that we compute from our $G_{(\ell)}(t)$ as

$$p_t^{(\ell),\cdot} = G_{(\ell)}(t) - G_{(\ell)}(t-1)$$

with $G_{(\ell)}(0) = 0$;

¹⁵Two other criteria to divide the portfolio into two classes could be to consider firms with higher average spread and firms with lower average spread, or to consider firms with lower rating and firms with higher rating. The choice of two classes is mainly for computational issues; however, a calibration with a higher number of classes would be possible and could be the object of further applications.

- $r_t, t \geq 1$, are the risk free interest rates with maturity t used for the discount factors: we used the discount factors related to the Germany zero curves downloaded from Datastream.

Note that, due to the assumption on the form of $h(t)$, we can see that $p_t^{(\ell),3}$ is a function of $\lambda_{(\ell)}^3$ for $t \leq 3$, and $p_t^{(\ell),5}$ is a function of $(\lambda_{(\ell)}^3, \lambda_{(\ell)}^5)$ for $t > 3$; it follows that we can consider $modelCDSspread_{(\ell)}^{(3)}$ and $modelCDSspread_{(\ell)}^{(5)}$ as functions of $\lambda_{(\ell)}^3$, and $(\lambda_{(\ell)}^3, \lambda_{(\ell)}^5)$, respectively:

$$modelCDSspread_{(\ell)}^{(3)} = modelCDSspread^{(3)}(\lambda_{(\ell)}^3),$$

and

$$modelCDSspread_{(\ell)}^{(5)} = modelCDSspread^{(5)}(\lambda_{(\ell)}^3, \lambda_{(\ell)}^5).$$

We first estimate the intensity parameters $\lambda_{(\ell)}^3$ for the two classes as

$$\lambda_{(\ell)}^3 = argmin_{\lambda^3} \frac{1}{m_\ell} \sum_{i=1}^{m_\ell} (marketCDSspread_i^{(3)} - modelCDSspread^{(3)}(\lambda^3))^2,$$

then, using the obtained value $\lambda_{(\ell)}^3$, we estimate $\lambda_{(\ell)}^5$ as

$$\lambda_{(\ell)}^5 = argmin_{\lambda^5} \frac{1}{m_\ell} \sum_{i=1}^{m_\ell} (marketCDSspread_i^{(5)} - modelCDSspread^{(5)}(\lambda_{(\ell)}^3, \lambda^5))^2.$$

The second step involves the calibration of the parameters of the subordinator for the two classes in which the iTraxx underlying firms are divided. The intensity parameters for each class, $\lambda_{(\ell)}^3$ and $\lambda_{(\ell)}^5$, are thus fixed and from them we compute the hazard rates $h^{(\ell)}(t)$ for each class.¹⁶ Mai and Scherer (2009a) consider an alternative procedure that consists in directly calibrating the average hazard rate to the iTraxx index market price. However, we have chosen this procedure for two main reasons:

1. The first is that we divide our portfolio into the financial class and the non-financial class, and while the iTraxx Financial Index exists, there is no a index for non-financial institutions: therefore we do not have the related market price against which to calibrate the average hazard rate for the non-financial class.
2. The second reason is that in this way we take into account the financial feature of the iTraxx that the iTraxx price is slightly lower than the average of the CDS prices of the underlying institutions. To understand this inequality let us consider the following example in which we assume that the iTraxx has only two underlying institutions with related CDS prices equal to 100 and 20: two protection sellers sell respectively 1 iTraxx and 2 underlying CDSs; let us assume that the institution with

¹⁶Another procedure could be to calibrate the parameters λ_i^3 and λ_i^5 for each firm by imposing that $modelCDSspread^{(3)}(\lambda_i^3) = marketCDSspread_i^{(3)}$ and $modelCDSspread^{(5)}(\lambda_i^3, \lambda_i^5) = marketCDSspread_i^{(5)}$ to compute the related $h_i(t)$; then we could compute $h^{(\ell)}(t)$ for the two considered classes by taking the average value in each class, as $h^{(\ell)}(t) = \frac{1}{m_\ell} \sum_{i=1}^{m_\ell} h_i(t)$.

a CDS price (=100), that is higher than the average of the CDS prices (=60) defaults: the seller of the iTraxx pays $\frac{1}{2} \cdot 60$ to the buyer, while the seller of the single contracts pays $\frac{1}{2} \cdot 100$; let us now assume that the institution with a lower CDS price (=20) than the average of the CDS prices (=60) defaults: the seller of the iTraxx pays $\frac{1}{2} \cdot 60$ to the buyer, while the seller of the single contracts pays $\frac{1}{2} \cdot 20$; as the CDS with price 100 is much more likely to default, on average the default seller of the iTraxx will have to pay less than the protection seller of the single contracts: as a compensation for this the iTraxx price is lower than the average of the CDS prices.

The parameters of the subordinator, (η, β) , specifying the dependence, as well as the parameter values $a^{(i)}$, are calibrated to observed market spreads of the tranches of the CDO. We follow the procedure suggested by Mai and Scherer (2009a). Without loss of generality we fix $a^{(1)} = 1$ and we consider different values for $a^{(2)}$ (from 0.6 to 1.4). For each couple $(a^{(1)}, a^{(2)})$ we define a grid for η . Given η , our subordinator parameter constraint (deriving from the (TN) condition) defines an interval for β . On this interval, β is chosen so that the upfront payment of the tranche equity is perfectly matched. Finally, among the possible parameters obtained, $(a^{(2)}, \eta, \beta)$ are chosen to minimize the sum of square deviations of market to model spreads over all tranches¹⁷ $j = 2, \dots, 5$, i.e. the root mean squared error (*RMSE*). For this we solve, using Matlab, the minimization problem $\min_{(\eta, \beta, a^{(2)})} RMSE$, with the root mean squared error defined as

$$RMSE := \sqrt{\frac{1}{4} \sum_{j=2}^5 |marketspread^{(j)} - s_T^{(j)}|^2} \quad (23)$$

where $marketspread^{(j)}$ is observed on Markit and $s_T^{(j)}$ is computed using (9), (11), or (12), according to the pricing convention for the related tranche in the different times. In particular:

- until January 2009: the levels for 0-3 tranches are upfront with a fixed 500bps spread and are quoted in terms of percentage of notional; the levels for all the other tranches are in bps with no fixed running spread.
- from March 2009 to January 2010: the levels for 0-3, 3-6 and 6-9 tranches are upfront with a fixed 500bps spread and are quoted in terms of percentage of notional; the levels for all the other tranches are in bps with no fixed running spread.
- in March 2010: tranches 0-3 are upfront with a fixed 500bps spread and are quoted in terms of percentage of notional; tranches 3-6 and 6-9 are upfront with a fixed 300bps spread and are quoted in bps; tranches 9-12 and 12-22 are upfronts with a fixed 100bps spread and are quoted in bps.
- from May 2010: tranches 0-3, 3-6 and 6-9 are upfront with a fixed 500bps spread and are quoted in percentage of notional; tranches 9-12 and 12-22 are upfronts with a fixed 100bps spread and are quoted in bps.

¹⁷We do not consider the super senior tranche [22%, 100%] as this tranche is traded very rarely and we have no market quotes for it.

5.3.1 Results

Figure 1 shows the hazard rates $h^{(F)}(t)$ and $h^{(NF)}(t)$ obtained for the two classes (the financial and the non-financial firms, respectively) at five different dates, once the intensity parameters $\lambda_{(\ell)}^3$ and $\lambda_{(\ell)}^5$ have been calibrated and consequently the default intensities for each class $h^{(\ell)}(t)$ computed. We can see the inversion of average riskiness of the two classes: in 2007 the financial class was less risky than the non-financial and our results show that this was the case up to November 2009; starting in January 2010, the non-financial class became the least risky. The hazard rates of both classes reached their peaks in the period after Lehman-Brothers' bankruptcy, decreasing back down in 2009 and increasing again after the beginning of the recent sovereign debt crisis in the euro area.

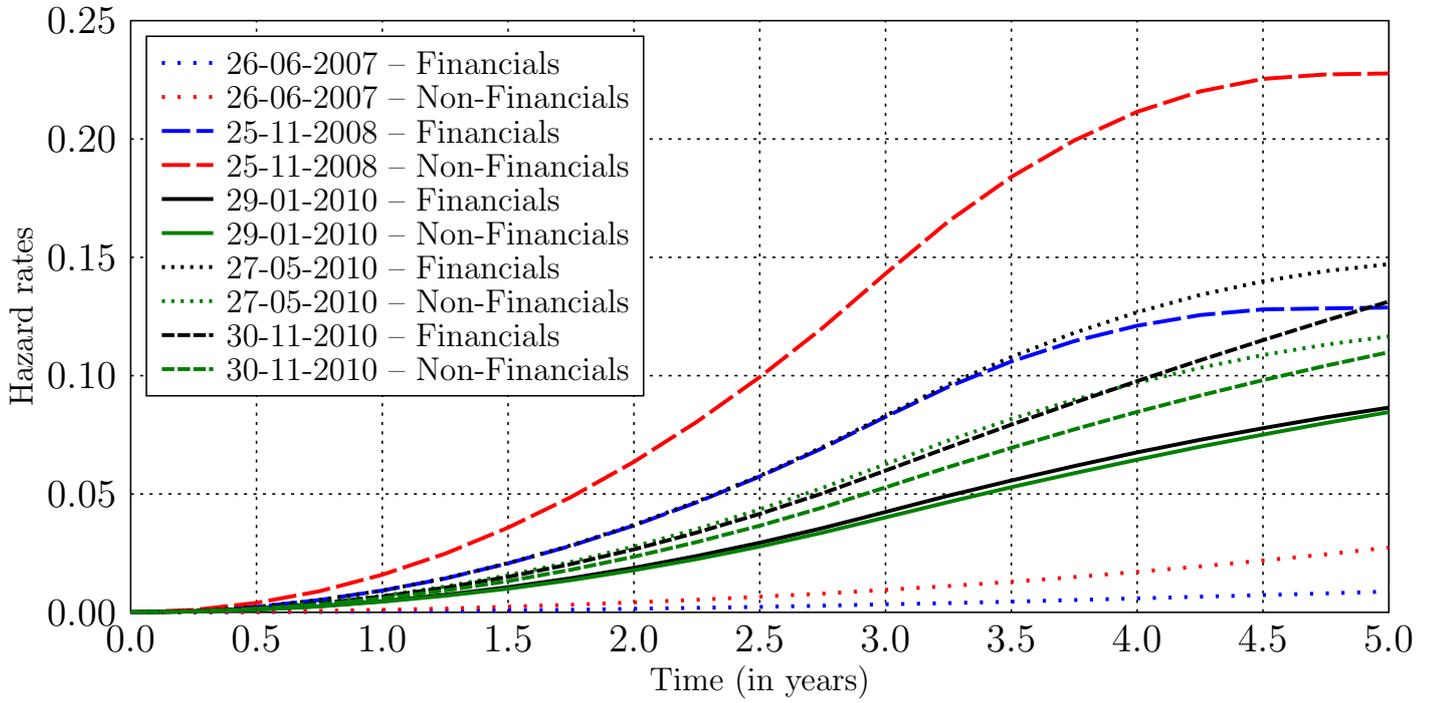
The results of the calibration of the parameters of the subordinator, at seven significant dates, are shown in Table 1, which contains the market quotes for the upfront payments (in per cent) and the tranche spreads (in basis points) (denoted by $s^i, i = 1, \dots, 5$), and the related estimated quotes given by the inverse Gaussian model.¹⁸ We also report the deviations of model to market spreads, i.e. the root mean squared errors ($RMSE$, in basis points) as defined in equation (23).

Let us note that the equity tranche is always perfectly matched, due to the estimation procedure for the parameters η and β . Unfortunately, the price errors relating to the other tranches are larger in times of financial distress, but this may be explained by considering that in these periods the tranches are less liquid and sometimes their reported prices are not actual prices but interpolated values of more liquid tranches.¹⁹ Moreover, these errors could be further minimized by considering a larger range for the parameters of the subordinator.

¹⁸The fitted dependence parameters $(a^{(1)}, a^{(2)}, \eta, \beta)$, calibrated for the implemented model, are available by contacting the authors.

¹⁹It would be interesting to compare the results obtained with our model to those that could be obtained using other models in the literature, such as Albrecher et al. (2007), Hofert and Scherer (2011), or Kalemánova et al. (2007). However, this comparison was made by Mai and Sherer (2009a) using the results for June 2007. Our results for periods in which the tranches were liquid are very satisfying and comparable to those obtained by Mai and Scherer (2009a).

Figure 1: Hazard rates for the two classes at different dates



Date	s^1	s^2	s^3	s^4	s^5	RMSE
2007-06-26						
Market price	11.87%	63.70bp	37.74bp	7.35bp	3.18bp	
Model price	11.87%	64.01bp	36.21bp	24.00bp	12.37bp	27.7bp
2008-05-30						
Market price	34.03%	300.92bp	188.92bp	126.96bp	61.63bp	
Model price	34.03%	276.00bp	191.00bp	154.00bp	114.00bp	32.0bp
2008-11-25						
Market price	64.03%	1175.83bp	600.56bp	325.00bp	127.33bp	
Model price	64.03%	1077.00bp	731.00bp	578.00bp	419.00bp	210.0bp
2009-11-30						
Market price	36.27%	-0.73%	-7.80%	134.81bp	54.31bp	
Model price	36.27%	-0.73%	-7.73%	253.00bp	173.00bp	84.0bp
2010-01-29						
Market price	28.81%	-4.00%	-10.48%	104.83bp	41.33bp	
Model price	28.81%	-4.24%	-9.69%	225.00bp	160.00bp	94.0bp
2010-05-27						
Market price	41.39%	7.43%	3.45%	189.75bp	77.30bp	
Model price	41.39%	11.38%	1.21%	39.00bp	71.00bp	239.0bp
2010-11-30						
Market price	29.80%	0.08%	-2.29%	106.50bp	49.67bp	
Model price	29.80%	5.17%	-2.25%	34.00bp	62.00bp	257.0bp

Table 1: Calibration results for the parameter values

6 Distress dependence and systemic risk

From the multivariate default distribution it is possible to calculate the joint probability of distress, which is the probability of all the institutions in the system (portfolio) becoming distressed (i.e. the tail risk of the system). It is an empirical fact that the banks in the system experience large losses simultaneously in times of financial distress. In particular, in such periods the financial system's joint probability of distress may experience larger non linear increases than those experienced by the average probabilities of default of the individual institutions. After having estimated the multivariate default distribution of the companies included in the iTraxx, we follow Segoviano and Goodhart (2010) to analyse the distress dependence in the portfolio by computing a set of indicators of systemic risk. In particular, we estimate three stability measures that incorporate changes in distress dependence that we show to be consistent with the economic cycle. The stability measures that we use are:

1. *The Stability Index*;
2. *The Distress Dependence Matrix*;
3. *The Probability of Cascade Effects*.

Once these stability measures have been computed we could employ them to verify which firms are more systemically relevant for the index as a whole.

6.1 The Stability Index

The Stability Index (SI) is a measure of the tail risk of the system, i.e. the common distress of the financial institutions in the system. The SI is based on the conditional expectation²⁰ and measures the expected number of institutions that fall into distress given that at least one specific institution has become distressed (i.e. defaults). The SI represents a probability measure that conditions any institution becoming distressed without indicating the specific bank. In the simplest case of two financial institutions with default times τ_i and τ_j , let κ_t stand for the number of the institutions in default at time t , i.e. $\kappa_t = \mathbf{1}_{\{\tau_i \leq t\}} + \mathbf{1}_{\{\tau_j \leq t\}}$. Our extreme linkage indicator is the conditional expectation $\mathbb{E}[\kappa_t | \kappa_t \geq 1]$. From elementary probability theory we have

$$\begin{aligned} SI = \mathbb{E}[\kappa_t | \kappa_t \geq 1] &= \frac{\mathbb{P}\{\tau_i \leq t, \tau_j > t\} + \mathbb{P}\{\tau_i > t, \tau_j \leq t\} + 2\mathbb{P}\{\tau_i \leq t, \tau_j \leq t\}}{\mathbb{P}\{\tau_i \leq t \text{ or } \tau_j \leq t\}} \\ &= \frac{\mathbb{P}\{\tau_i \leq t\} + \mathbb{P}\{\tau_j \leq t\}}{1 - \mathbb{P}\{\tau_i > t, \tau_j > t\}}. \end{aligned}$$

This measure can be interpreted as a relative measure of the *system linkage*: the system linkage is weak when SI is close to 1, while the system linkage increases as the value of the SI increases.

In our portfolio of 125 firms corresponding to the institutions in the iTraxx, we consider the classes of the 25 financial institutions and 100 non-financial institutions and denote by \mathcal{M}_1 and \mathcal{M}_2 the class with

²⁰See the measure developed by Huang (1992).

lower and higher average hazard rates respectively and with m_1 and m_2 the number of elements in each class. In practice, the least risky class \mathcal{M}_1 corresponds to the financial class up to the end of 2009, while since January 2010 the financial class has become riskier than the class of non-financial institutions. Let us denote by $X_t^{(\ell)} := \sum_{i=1}^{m_\ell} \mathbf{1}_{\{\tau_i^{(\ell)} \leq t\}}$, $\ell = 1, 2$ (where τ_i^ℓ represents the default time of institution i in the class \mathcal{M}_ℓ). We can compute the stability index in the following cases:

$$\mathbb{E}[X_{t'}^{(i)} | X_t^{(j)} \geq 1] \quad i, j = 1, 2 \quad t \leq t'. \quad (24)$$

In this sense the model is dynamic, allowing us to estimate the number of defaults at a posterior date t' given the number of defaults at a prior date t , and in general to update the portfolio loss distribution at t' given the portfolio loss distribution at t .

Proposition 6.1. *In the different cases we have respectively*

$$\begin{aligned} 1) - 2) \quad & \mathbb{E}[X_{t'}^{(\ell)} | X_t^{(\ell)} \geq 1] \quad \ell = 1, 2 \\ & = m_\ell \cdot \frac{1 - e^{-h^{(\ell)}(t')}(b^{(\ell)} - \Psi(-a^{(\ell)}))}{1 - e^{-h^{(\ell)}(t)}(m_\ell b^{(\ell)} - \Psi(-m_\ell a^{(\ell)}))} \\ & - m_\ell \cdot \frac{e^{-h^{(\ell)}(t)}(m_\ell b^{(\ell)} - \Psi(-m_\ell a^{(\ell)})) \left(1 - e^{-\left(h^{(\ell)}(t') - h^{(\ell)}(t)\right)}\right)}{1 - e^{-h^{(\ell)}(t)}(m_\ell b^{(\ell)} - \Psi(-m_\ell a^{(\ell)}))} \\ & = m_\ell \cdot \frac{1 - \bar{G}^{(\ell)}(t') - (\bar{G}^{(\ell)}(t))^{m_\ell + m_\ell \Psi(-a^{(\ell)}) - \Psi(-m_\ell a^{(\ell)})} \left(1 - e^{-\left(h^{(\ell)}(t') - h^{(\ell)}(t)\right)}\right)}{1 - (\bar{G}^{(\ell)}(t))^{m_\ell + m_\ell \Psi(-a^{(\ell)}) - \Psi(-m_\ell a^{(\ell)})}}. \end{aligned} \quad (25)$$

$$\begin{aligned}
3) \quad & \mathbb{E}[X_{t'}^{(1)} | X_t^{(2)} \geq 1] \\
&= m_1 \cdot \frac{1 - e^{-h^{(1)}(t') \left(b^{(1)} - \Psi(-a^{(1)}) \right)} - e^{-h^{(2)}(t) \left(m_2 b^{(2)} - \Psi(-m_2 a^{(2)}) \right)}}{1 - e^{-h^{(2)}(t) \left(m_2 b^{(2)} - \Psi(-m_2 a^{(2)}) \right)}} \\
&+ m_1 \cdot \frac{e^{-h^{(1)}(t') \left(b^{(1)} - \Psi(-a^{(1)}) \right)} e^{-h^{(2)}(t) \left(\Psi(-a^{(1)}) + m_2 b^{(2)} - \Psi(-a^{(1)} - m_2 a^{(2)}) \right)}}{1 - e^{-h^{(2)}(t) \left(m_2 b^{(2)} - \Psi(-m_2 a^{(2)}) \right)}} \cdot \mathbf{1}_{\{h^{(1)}(t') \geq h^{(2)}(t)\}} \\
&+ m_1 \cdot \frac{e^{-h^{(1)}(t') \left(b^{(1)} + \Psi(-m_2 a^{(2)}) - \Psi(-a^{(1)} - m_2 a^{(2)}) \right)} e^{-h^{(2)}(t) \left(m_2 b^{(2)} - \Psi(-m_2 a^{(2)}) \right)}}{1 - e^{-h^{(2)}(t) \left(m_2 b^{(2)} - \Psi(-m_2 a^{(2)}) \right)}} \cdot \mathbf{1}_{\{h^{(1)}(t') < h^{(2)}(t)\}} \\
&= m_1 \cdot \frac{1 - \bar{G}^{(1)}(t') - \left(\bar{G}^{(2)}(t) \right)^{m_2 + m_2 \Psi(-a^{(2)}) - \Psi(-m_2 a^{(2)})}}{1 - \left(\bar{G}^{(2)}(t) \right)^{m_2 + m_2 \Psi(-a^{(2)}) - \Psi(-m_2 a^{(2)})}} \\
&+ m_1 \cdot \frac{\bar{G}^{(1)}(t') \left(\bar{G}^{(2)}(t) \right)^{\Psi(-a^{(1)}) + m_2 + m_2 \Psi(-a^{(2)}) - \Psi(-a^{(1)} - m_2 a^{(2)})}}{1 - \left(\bar{G}^{(2)}(t) \right)^{m_2 + m_2 \Psi(-a^{(2)}) - \Psi(-m_2 a^{(2)})}} \cdot \mathbf{1}_{\{\bar{G}^{(1)}(t') \leq \bar{G}^{(2)}(t)\}} \\
&+ m_1 \cdot \frac{\left(\bar{G}^{(1)}(t') \right)^{1 + \Psi(-a^{(1)}) + \Psi(-m_2 a^{(2)}) - \Psi(-a^{(1)} - m_2 a^{(2)})} \left(\bar{G}^{(2)}(t) \right)^{m_2 + m_2 \Psi(-a^{(2)}) - \Psi(-m_2 a^{(2)})}}{1 - \left(\bar{G}^{(2)}(t) \right)^{m_2 + m_2 \Psi(-a^{(2)}) - \Psi(-m_2 a^{(2)})}} \cdot \mathbf{1}_{\{\bar{G}^{(1)}(t') > \bar{G}^{(2)}(t)\}}. \tag{26}
\end{aligned}$$

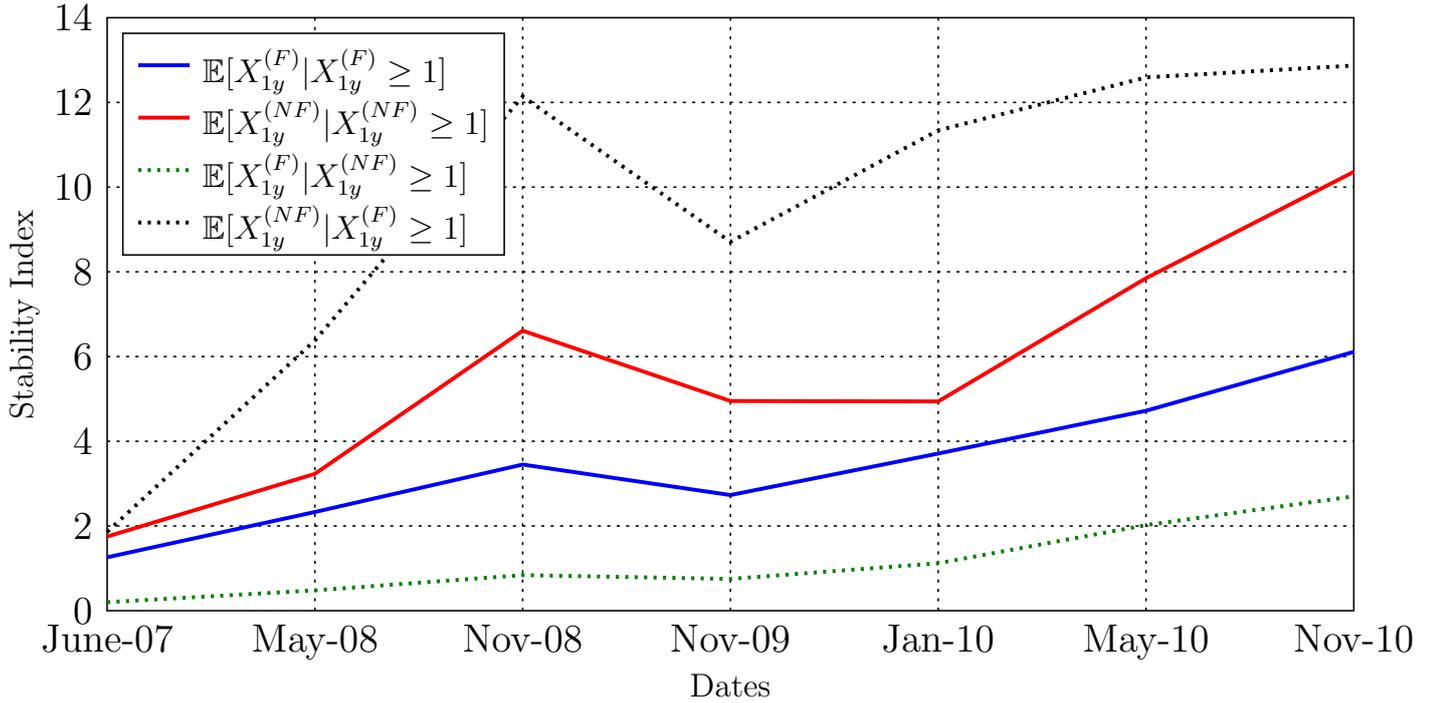
$$\begin{aligned}
4) \quad & \mathbb{E}[X_{t'}^{(2)} | X_t^{(1)} \geq 1] \\
&= m_2 \cdot \frac{1 - e^{-h^{(2)}(t') \left(b^{(2)} - \Psi(-a^{(2)}) \right)} - e^{-h^{(1)}(t) \left(m_1 b^{(1)} - \Psi(-m_1 a^{(1)}) \right)}}{1 - e^{-h^{(1)}(t) \left(m_1 b^{(1)} - \Psi(-m_1 a^{(1)}) \right)}} \\
&+ m_2 \cdot \frac{e^{-h^{(2)}(t') \left(b^{(2)} - \Psi(-a^{(2)}) \right)} e^{-h^{(1)}(t) \left(\Psi(-a^{(2)}) + m_1 b^{(1)} - \Psi(-a^{(2)} - m_1 a^{(1)}) \right)}}{1 - e^{-h^{(1)}(t) \left(m_1 b^{(1)} - \Psi(-m_1 a^{(1)}) \right)}}. \tag{27} \\
&= m_2 \cdot \frac{1 - \bar{G}^{(2)}(t') - \left(\bar{G}^{(1)}(t) \right)^{m_1 + m_1 \Psi(-a^{(1)}) - \Psi(-m_1 a^{(1)})}}{1 - \left(\bar{G}^{(1)}(t) \right)^{m_1 + m_1 \Psi(-a^{(1)}) - \Psi(-m_1 a^{(1)})}} \\
&+ m_2 \cdot \frac{\bar{G}^{(2)}(t') \left(\bar{G}^{(1)}(t) \right)^{\Psi(-a^{(2)}) + m_1 + m_1 \Psi(-a^{(1)}) - \Psi(-a^{(2)} - m_1 a^{(1)})}}{1 - \left(\bar{G}^{(1)}(t) \right)^{m_1 + m_1 \Psi(-a^{(1)}) - \Psi(-m_1 a^{(1)})}}.
\end{aligned}$$

(See Appendix B for the proof.)

If we denote by \mathcal{M}_F the class of financial institutions and by \mathcal{M}_{NF} the class of non-financial institutions and denote by $X_t^{(\ell)} := \sum_{i=1}^{m_\ell} \mathbf{1}_{\{\tau_i^{(\ell)} \leq t\}}$, $\ell = F, NF$ (where τ_i^ℓ represents the default time of institution i in the class \mathcal{M}_ℓ), we can also compute the SI as $\mathbb{E}[X_{t'}^{(\ell')} | X_t^{(\ell)} \geq 1]$ for $\ell' \neq$ or $= \ell$.

Figure 2 represents the Stability Index evaluated at seven different dates, using the results of Proposition 6.1 and the calibration for the values of the parameters (see Table 1); we consider the case in which time t' is equal to time t and equal to 1year.

Figure 2: Stability Index at different dates



6.2 The Distress Dependence Matrix

The Distress Dependence between two companies is the probability that one institution becomes distressed conditional on the other becoming distressed. This measure is useful to analyse financial stability as it can measure the time-varying inter-linkages between two institutions or two groups of companies, showing how spillover effects may change over time. The distress dependence matrix is a matrix which collects pairwise probabilities of financial institutions experiencing distress conditional on other institutions being in distress. It thus accounts for the relationship between the institutions. For each pair of institutions in the portfolio, we estimate the pairwise conditional probabilities of distress based on market data: the probability of distress of institution i at time t' conditional on institution j becoming

distressed at time t ($t \leq t'$) is computed as

$$\mathbb{P}(\tau_i \leq t' | \tau_j \leq t) = \frac{\mathbb{P}(\tau_i \leq t', \tau_j \leq t)}{\mathbb{P}(\tau_j \leq t)}.$$

Even if conditional probabilities do not imply causation, this set of pairwise conditional probabilities can provide important insights into interlinkages and the likelihood of contagion between the institutions in the system.

Let us denote $\mathbb{P}(\tau_i \leq t' | \tau_j \leq t) := \mathbb{P}(\text{Firm}i(t') | \text{Firm}j(t))$; the pairwise conditional probabilities of distress are represented in the following Distress Dependence Matrix: basically, the elements of the matrix show the conditional probabilities of distress of the institution in the row at time t' , given that the institution in the column falls into distress at time t .

	Firm 1	Firm i ($i = 2, \dots, 124$)	Firm 125
Firm 1	1	$\mathbb{P}(\text{Firm}1(t') \text{Firm}i(t))$	$\mathbb{P}(\text{Firm}1(t') \text{Firm}125(t))$
Firm i ($i = 2, \dots, 124$)	$\mathbb{P}(\text{Firm}i(t') \text{Firm}1(t))$	1	$\mathbb{P}(\text{Firm}i(t') \text{Firm}125(t))$
Firm 125	$\mathbb{P}(\text{Firm}125(t') \text{Firm}1(t))$	$\mathbb{P}(\text{Firm}125(t') \text{Firm}i(t))$	1

Anyway, as we have divided our portfolio into two classes, in practice we can get the following 2×2 matrix:

	Firms in class \mathcal{M}_1	Firms in class \mathcal{M}_2
Firms in class \mathcal{M}_1	$\mathbb{P}(\tau_i^{(1)} \leq t' \tau_j^{(1)} \leq t)$	$\mathbb{P}(\tau_i^{(1)} \leq t' \tau_j^{(2)} \leq t)$
Firms in class \mathcal{M}_2	$\mathbb{P}(\tau_i^{(2)} \leq t' \tau_j^{(1)} \leq t)$	$\mathbb{P}(\tau_i^{(2)} \leq t' \tau_j^{(2)} \leq t)$

where $\mathbb{P}(\tau_i^{(1)} \leq t' | \tau_j^{(1)} \leq t)$ is the probability that institution i in class \mathcal{M}_1 becomes distressed at time t' given that institution j in the same class \mathcal{M}_1 becomes distressed at time t , while $\mathbb{P}(\tau_i^{(1)} \leq t' | \tau_j^{(2)} \leq t)$ is the probability that one specific institution (institution i) in class \mathcal{M}_1 becomes distressed at t' given that one specific institution (institution j) in class \mathcal{M}_2 becomes distressed at t .

Proposition 6.2. *We have*

$$\begin{aligned}
1) - 2) \quad \mathbb{P}(\tau_i^{(\ell)} \leq t' | \tau_j^{(\ell)} \leq t) &= \frac{1 - e^{-h^{(\ell)}(t') (b^{(\ell)} - \Psi(-a^{(\ell)}))} - e^{-h^{(\ell)}(t) (b^{(\ell)} - \Psi(-a^{(\ell)}))}}{1 - e^{-h^{(\ell)}(t) (b^{(\ell)} - \Psi(-a^{(\ell)}))}} & \ell = 1, 2 \\
&+ \frac{e^{-h^{(\ell)}(t) (2b^{(\ell)} - \Psi(-2a^{(\ell)}))} e^{-(h^{(\ell)}(t') - h^{(\ell)}(t)) (b^{(\ell)} - \Psi(-a^{(\ell)}))}}{1 - e^{-h^{(\ell)}(t) (b^{(\ell)} - \Psi(-a^{(\ell)}))}} \\
&= \frac{1 - \overline{G}^{(\ell)}(t') - \overline{G}^{(\ell)}(t) + (\overline{G}^{(\ell)}(t))^{1+2\Psi(-a^{(\ell)}) - \Psi(-2a^{(\ell)})} \overline{G}^{(\ell)}(t')}{1 - \overline{G}^{(\ell)}(t)}. \tag{28}
\end{aligned}$$

$$\begin{aligned}
3) \quad \mathbb{P}(\tau_i^{(1)} \leq t' | \tau_j^{(2)} \leq t) &= \frac{1 - e^{-h^{(1)}(t') (b^{(1)} - \Psi(-a^{(1)}))} - e^{-h^{(2)}(t) (b^{(2)} - \Psi(-a^{(2)}))}}{1 - e^{-h^{(2)}(t) (b^{(2)} - \Psi(-a^{(2)}))}} \\
&+ \frac{e^{-h^{(1)}(t') (b^{(1)} - \Psi(-a^{(1)}))} e^{-h^{(2)}(t) (\Psi(-a^{(1)}) + b^{(2)} - \Psi(-a^{(1)} - a^{(2)}))}}{1 - e^{-h^{(2)}(t) (b^{(2)} - \Psi(-a^{(2)}))}} \cdot \mathbf{1}_{\{h^{(1)}(t') \geq h^{(2)}(t)\}} \\
&+ \frac{e^{-h^{(1)}(t') (b^{(1)} + \Psi(-a^{(2)}) - \Psi(-a^{(1)} - a^{(2)}))} e^{-h^{(2)}(t) (b^{(2)} - \Psi(-a^{(2)}))}}{1 - e^{-h^{(2)}(t) (b^{(2)} - \Psi(-a^{(2)}))}} \cdot \mathbf{1}_{\{h^{(1)}(t') < h^{(2)}(t)\}}. \\
&= \frac{1 - \overline{G}^{(1)}(t') - \overline{G}^{(2)}(t)}{1 - \overline{G}^{(2)}(t)} + \frac{\overline{G}^{(1)}(t') (\overline{G}^{(2)}(t))^{1 + \Psi(-a^{(1)}) + \Psi(-a^{(2)}) - \Psi(-a^{(1)} - a^{(2)})}}{1 - \overline{G}^{(2)}(t)} \cdot \mathbf{1}_{\{\overline{G}^{(1)}(t') \leq \overline{G}^{(2)}(t)\}} \\
&+ \frac{(\overline{G}^{(1)}(t'))^{1 + \Psi(-a^{(1)}) + \Psi(-a^{(2)}) - \Psi(-a^{(1)} - a^{(2)})} \overline{G}^{(2)}(t)}{1 - \overline{G}^{(2)}(t)} \cdot \mathbf{1}_{\{\overline{G}^{(1)}(t') > \overline{G}^{(2)}(t)\}}. \tag{29}
\end{aligned}$$

$$\begin{aligned}
4) \quad \mathbb{P}(\tau_i^{(2)} \leq t' | \tau_j^{(1)} \leq t) &= \frac{1 - e^{-h^{(2)}(t') (b^{(2)} - \Psi(-a^{(2)}))} - e^{-h^{(1)}(t) (b^{(1)} - \Psi(-a^{(1)}))}}{1 - e^{-h^{(1)}(t) (b^{(1)} - \Psi(-a^{(1)}))}} \\
&+ \frac{e^{-h^{(2)}(t') (b^{(2)} - \Psi(-a^{(2)}))} e^{-h^{(1)}(t) (\Psi(-a^{(2)}) + b^{(1)} - \Psi(-a^{(2)} - a^{(1)}))}}{1 - e^{-h^{(1)}(t) (b^{(1)} - \Psi(-a^{(1)}))}} \\
&= \frac{1 - \overline{G}^{(2)}(t') - \overline{G}^{(1)}(t)}{1 - \overline{G}^{(1)}(t)} + \frac{\overline{G}^{(2)}(t') (\overline{G}^{(1)}(t))^{\Psi(-a^{(2)}) + 1 + \Psi(-a^{(1)}) - \Psi(-a^{(2)} - a^{(1)})}}{1 - \overline{G}^{(1)}(t)}. \tag{30}
\end{aligned}$$

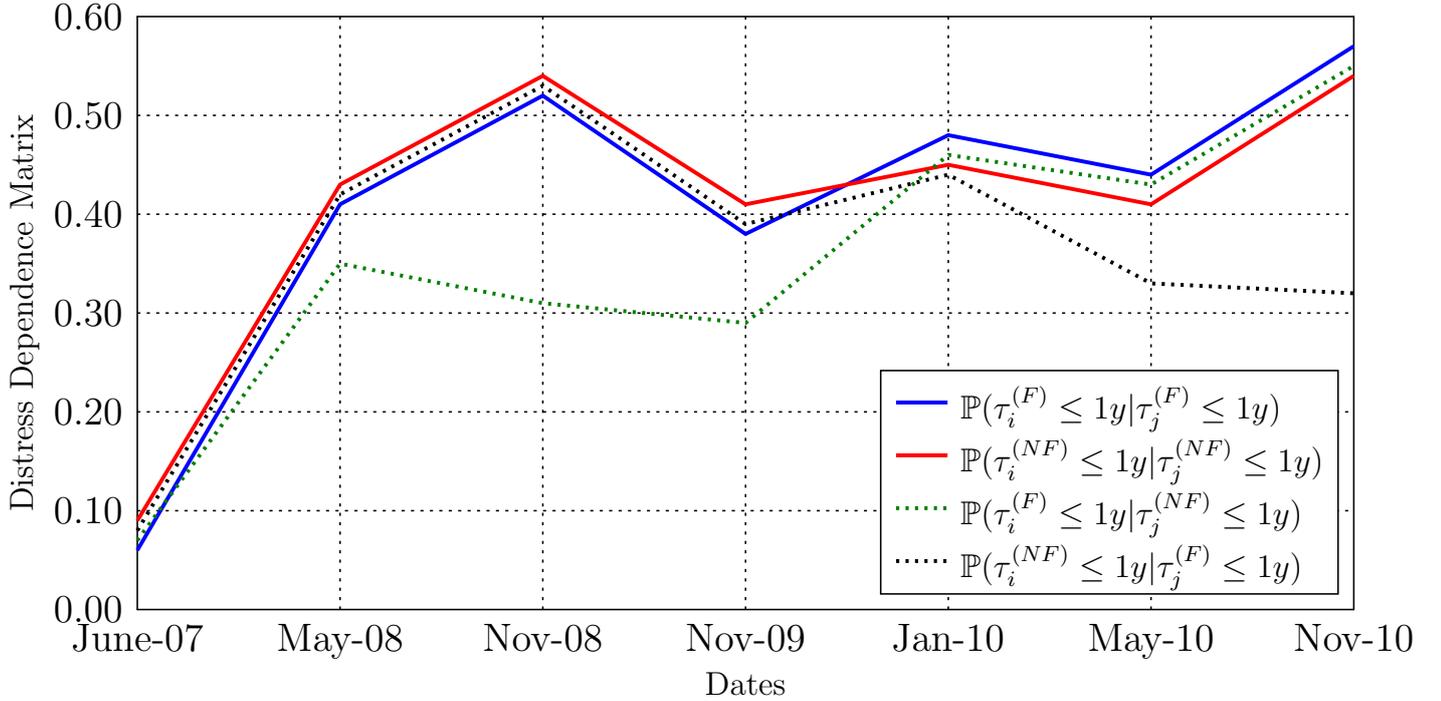
(See Appendix B for the proof.)

With the same notation used for the Stability Index, we can compute

	Firms in class \mathcal{M}_F	Firms in class \mathcal{M}_{NF}
Firms in class \mathcal{M}_F	$\mathbb{P}(\tau_i^{(F)} \leq t' \tau_j^{(F)} \leq t)$	$\mathbb{P}(\tau_i^{(F)} \leq t' \tau_j^{(NF)} \leq t)$
Firms in class \mathcal{M}_{NF}	$\mathbb{P}(\tau_i^{(NF)} \leq t' \tau_j^{(F)} \leq t)$	$\mathbb{P}(\tau_i^{(NF)} \leq t' \tau_j^{(NF)} \leq t)$

Figure 3 represents the entries of this Distress Dependence Matrix at seven different significant dates, computed by using Proposition 6.2 and the calibration results for the values of the parameters (see Table 1). We consider the case $t' = t = 1year$.

Figure 3: Distress Dependence Matrix at different dates



6.3 The Probability of Cascade Effects

The Probability of Cascade Effects is an indicator that measures the likelihood that at least one institution becomes distressed given that a specific institution becomes distressed. In this way it quantifies the potential “cascade” effects in the system given the distress of a specific institution and so this measure

can be considered an indicator that allows us to quantify the systemic importance of a specific institution if it becomes distressed. For this systemic risk indicator we could divide our portfolio into four groups (having performed the calibration for only two groups, we describe this case only from a theoretical point of view): the group of financial institutions with lower spread (denoted by (F,L)), the group of financial institutions with higher spread (denoted by (F,H)), the group of non-financial institutions with lower spread (denoted by (N,L)) and the group of non-financial institutions with higher spread (denoted by (N,H)).

In general we could denote by

- $\tau_i^{(F,L)}$ the default time of institution i in the class of financial firms with lower spread,
- $\tau_i^{(F,H)}$ the default time of institution i in the class of financial firms with higher spread,
- $\tau_i^{(N,L)}$ the default time of institution i in the class of non-financial firms with lower spread,
- $\tau_i^{(N,H)}$ the default time of institution i in the class of non-financial firms with higher spread,

and consequently define

$$\begin{aligned} X_t^{(F,L)} &= \sum_i \mathbf{1}_{(\tau_i^{(F,L)} \leq t)} \\ X_t^{(F,H)} &= \sum_i \mathbf{1}_{(\tau_i^{(F,H)} \leq t)} \\ X_t^{(N,L)} &= \sum_i \mathbf{1}_{(\tau_i^{(N,L)} \leq t)} \\ X_t^{(N,H)} &= \sum_i \mathbf{1}_{(\tau_i^{(N,H)} \leq t)}. \end{aligned}$$

We could compute the probability of cascade effects by computing

$$\mathbb{P}[X_{t'}^{(k,s)} \geq 1 | \tau_i^{(k',s')} \leq t],$$

where $k, k' \in \{F, N\}$ and $s, s' \in \{L, H\}$. For example

$$\mathbb{P}[X_{t'}^{(N,L)} \geq 1 | \tau_i^{(F,H)} \leq t] = \mathbb{P}\left[\min_j \tau_j^{(N,L)} \leq t' | \tau_i^{(F,H)} \leq t\right]. \quad (31)$$

In the simpler case of the two classes (class with lower hazard rate (1) and class with higher hazard rate (2)), we denote $X_t^{(1)} := \sum_i \mathbf{1}_{(\tau_i^{(1)} \leq t)}$ and $X_t^{(2)} := \sum_i \mathbf{1}_{(\tau_i^{(2)} \leq t)}$ and we can compute the probability of cascade effects by computing

$$\mathbb{P}[X_{t'}^{(j)} \geq 1 | \tau_i^{(k)} \leq t] \quad j, k = 1, 2, \quad i \in \mathcal{M}_k, \quad t \leq t'.$$

Proposition 6.3. *In the different cases we have*

$$1) - 2) \quad \mathbb{P}[X_{t'}^{(\ell)} \geq 1 | \tau_i^{(\ell)} \leq t] = \mathbb{P}\left[\min_j \tau_j^{(\ell)} \leq t' | \tau_i^{(\ell)} \leq t\right] = 1 \quad \ell = 1, 2. \quad (32)$$

$$\begin{aligned} 3) \quad \mathbb{P}[X_{t'}^{(1)} \geq 1 | \tau_i^{(2)} \leq t] &= \mathbb{P}\left[\min_j \tau_j^{(1)} \leq t' | \tau_i^{(2)} \leq t\right] \\ &= \frac{1 - e^{-h^{(2)}(t)(b^{(2)} - \Psi(-a^{(2)}))} - e^{-h^{(1)}(t')(m_1 b_1 - \Psi(-m_1 a^1))}}{1 - e^{-h^{(2)}(t)(b^{(2)} - \Psi(-a^{(2)}))}} \\ &+ \frac{e^{-h^{(1)}(t')(m_1 b^{(1)} - \Psi(-m_1 a^{(1)}))} e^{-h^{(2)}(t)(b^{(2)} + \Psi(-m_1 a^{(1)}) - \Psi(-a^{(2)} - m_1 a_1))}}{1 - e^{-h^{(2)}(t)(b^{(2)} - \Psi(-a^{(2)}))}} \mathbf{1}_{\{h^{(1)}(t') \geq h^{(2)}(t)\}} \\ &+ \frac{e^{-h^{(1)}(t')(\Psi(-a^{(2)}) + m_1 b^{(1)} - \Psi(-m_1 a^{(1)} - a^{(2)}))} e^{-h^{(2)}(t)(b^{(2)} - \Psi(-a^{(2)}))}}{1 - e^{-h^{(2)}(t)(b^{(2)} - \Psi(-a^{(2)}))}} \mathbf{1}_{\{h^{(1)}(t') < h^{(2)}(t)\}} \\ &= \frac{1 - \bar{G}^{(2)}(t) - (\bar{G}^{(1)}(t'))^{m_1 + m_1 \Psi(-a^{(1)}) - \Psi(-m_1 a^1)}}{1 - \bar{G}^{(2)}(t)} \\ &+ \frac{(\bar{G}^{(1)}(t'))^{m_1 + m_1 \Psi(-a^{(1)}) - \Psi(-m_1 a^{(1)})} (\bar{G}^{(2)}(t))^{1 + \Psi(-a^{(2)}) + \Psi(-m_1 a^{(1)}) - \Psi(-a^{(2)} - m_1 a^{(1)})}}{1 - \bar{G}^{(2)}(t)} \mathbf{1}_{\{\bar{G}^{(1)}(t') \leq \bar{G}^{(2)}(t)\}} \\ &+ \frac{(\bar{G}^{(1)}(t'))^{\Psi(-a^{(2)}) + m_1 + m_1 \Psi(-a^{(1)}) - \Psi(-m_1 a^{(1)} - a^{(2)})} \bar{G}^{(2)}(t)}{1 - \bar{G}^{(2)}(t)} \mathbf{1}_{\{\bar{G}^{(1)}(t') > \bar{G}^{(2)}(t)\}}. \end{aligned} \quad (33)$$

$$\begin{aligned} 4) \quad \mathbb{P}[X_{t'}^{(2)} \geq 1 | \tau_i^{(1)} \leq t] &= \mathbb{P}\left[\min_j \tau_j^{(2)} \leq t' | \tau_i^{(1)} \leq t\right] \\ &= \frac{1 - e^{-h^{(1)}(t)(b^{(1)} - \Psi(-a^{(1)}))} - e^{-h^{(2)}(t')(m_2 b_2 - \Psi(-m_2 a^{(2)}))}}{1 - e^{-h^{(1)}(t)(b^{(1)} - \Psi(-a^{(1)}))}} \\ &+ \frac{e^{-h^{(2)}(t')(m_2 b^{(2)} - \Psi(-m_2 a^{(2)}))} e^{-h^{(1)}(t)(b^{(1)} + \Psi(-m_2 a^{(2)}) - \Psi(-a^{(1)} - m_2 a^{(2)}))}}{1 - e^{-h^{(1)}(t)(b^{(1)} - \Psi(-a^{(1)}))}} \\ &= \frac{1 - \bar{G}^{(1)}(t) - (\bar{G}^{(2)}(t'))^{m_2 + m_2 \Psi(-a^{(2)}) - \Psi(-m_2 a^2)}}{1 - \bar{G}^{(1)}(t)} \\ &+ \frac{(\bar{G}^{(2)}(t'))^{m_2 + m_2 \Psi(-a^{(2)}) - \Psi(-m_2 a^{(2)})} (\bar{G}^{(1)}(t))^{1 + \Psi(-a^{(1)}) + \Psi(-m_2 a^{(2)}) - \Psi(-a^{(1)} - m_2 a^{(2)})}}{1 - \bar{G}^{(1)}(t)}. \end{aligned} \quad (34)$$

The results in equation (32) are obvious, while to prove the conditioned probabilities in equations (33) and (34) we can follow the same kind of computation to get equation (26) (see (B.5)).

As for the previous measures, if we denote by

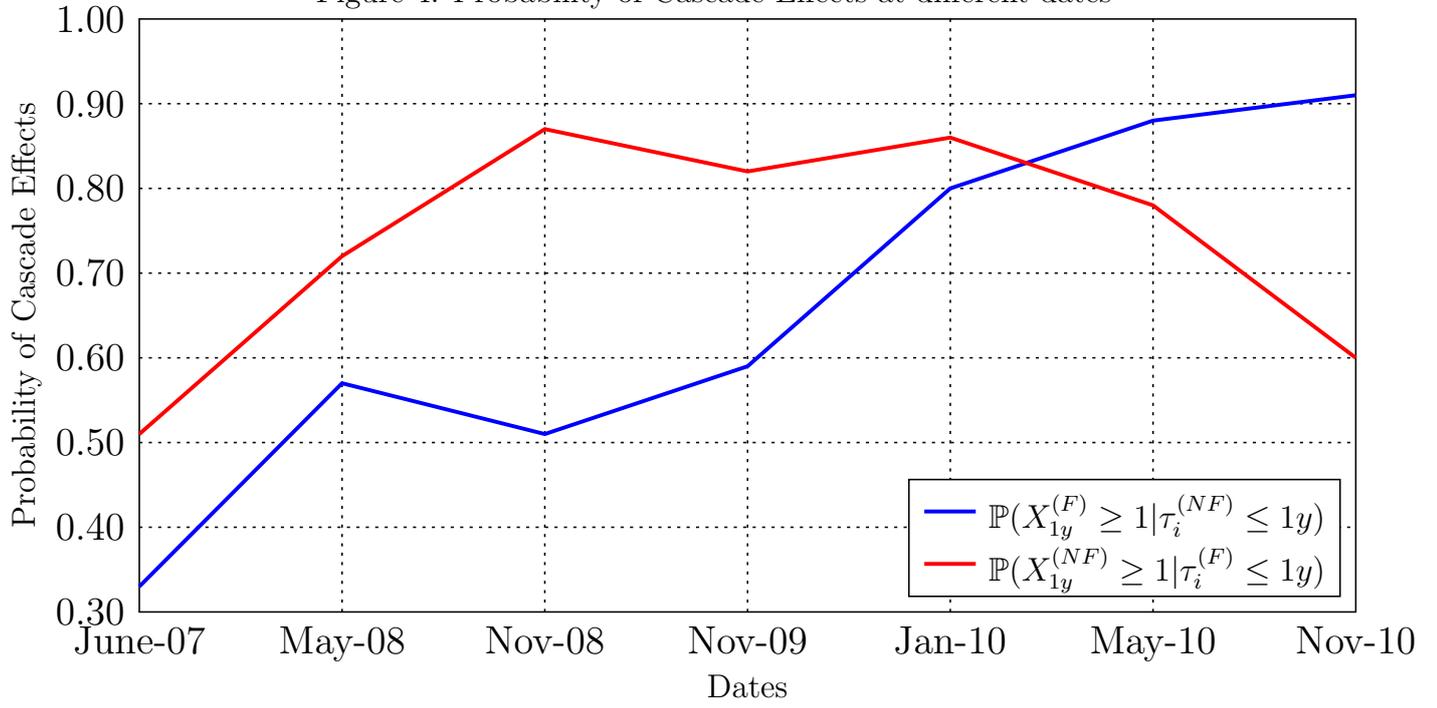
$$X_t^{(\ell)} = \sum_i \mathbf{1}_{(\tau_i^{(\ell)} \leq t)}, \quad \ell = F, NF.$$

we can compute the PCE as

$$\mathbb{P}[X_{t'}^{(\ell)} \geq 1 | \tau_i^{(\ell')} \leq t], \quad \ell, \ell' = F, NF, \quad \ell \neq \ell', \quad t \leq t'.$$

Figure 4 represents the Probability of Cascade Effects in these two cases at seven different significant dates, computed by using Proposition 6.3 and the calibration results for the values of the parameters (see Table 1). We consider the case $t' = t = 1year$.

Figure 4: Probability of Cascade Effects at different dates



6.4 The evidence from these measures

Results for the different systemic risk measures show that the probability of contagion was higher at the end of 2010 compared with the post-Lehman period, even if the individual default probabilities were lower (as previously shown in Figure 1).

In particular, all the four stability indices considered (Figure 2) show that the expected number of institutions of a certain class (financial or non-financial) that would fall into distress within one year, given the default of at least one specific institution in the same class or in the other class, reached high values in November 2008, that is after the Lehman bankruptcy. These indices decreased afterwards but started an upward trend after the beginning of the sovereign debt crisis in the euro area, reaching maximum values in November 2010 (that is, the last date considered). The stability indices also show that the contagion from a financial institution has always had more influence, inducing a higher expected number of defaults than contagion from a non-financial institution. Let us note that even if the expected number of non-financial institutions that would go into distress is higher than the number of financial institutions, in percentage terms the results are comparable (the two classes having 100 and 25 components, respectively).

Moreover, the conditioned default probabilities expressed by the distress dependence matrix (Figure 3) reached their peaks in November 2010, after the high values registered in the post-Lehman bankruptcy period; an exception is the the default probability of a non-financial institution given the default of a financial institution, which reached its maximum value in November 2008: this evidence may be justified by the high default probability of financial institutions in the last period (November 2010), which appears in the denominator of the conditioned default probability computation.

Analogous considerations are valid for the probability of cascade effects (Figure 4): in fact, while the probability of having at least one default in the class of financial institutions, given the default of a non-financial institution, increased considerably at the end of the time period considered and reached its peak in November 2010, vice versa the probability of having at least one default of a non-financial institution given the default of a financial institution decreased considerably after the maximum values reached in November 2008 and January 2010.

The following table shows some results relating to the risk measures considered, estimated at two different significant dates: in November 2008, that is after the Lehman bankruptcy, and in November 2010, after the start of the sovereign debt crisis in the euro area.

	November 2008	November 2010
$\mathbb{E}[X_{1y}^{(F)} X_{1y}^{(F)} \geq 1]$	3	6
$\mathbb{E}[X_{1y}^{(NF)} X_{1y}^{(NF)} \geq 1]$	7	10
$\mathbb{P}(\tau_i^{(F)} \leq 1y \tau_j^{(F)} \leq 1y)$	52%	57%
$\mathbb{P}(X_{1y}^{(F)} \geq 1 \tau_i^{(NF)} \leq 1y)$	51%	91%

7 Conclusions

We constructed a multivariate default times model for a portfolio of assets exposed to credit risk using a conditional independence approach with a stochastic time-change as common factor. We kept the dependence structure separate from the parameters of the marginal default probabilities by choosing a suitable Lévy subordinator as stochastic clock. Thanks to this separation between the univariate marginals and the dependence structure, we could explicitly compute the implied copula of the default times. Under the assumption of a heterogeneous portfolio, we obtained a closed formula for the portfolio loss distribution and we presented an approximation for large portfolios. We demonstrated the model efficiency by calibrating it to the market quotes of the CDSs and the tranches of the iTraxx index, using the Inverse Gaussian subordinator. To this end we considered the iTraxx index a portfolio and we divided it into two classes: financial institutions and non-financial institutions. The fit to market data is satisfactory, but it could be improved by considering a larger range in the calibration of the parameters. Although the calibration results are comparable with the results of the homogeneous model,²¹ the main contribution of our heterogeneous model is to allow us to compute three measures of portfolio systemic risk, monitoring the Portfolio Distress Dependence through time. The results in the time interval considered seem to be coherent with the evolution of tensions on the financial markets in recent years and show that the proposed model provides accurate estimates of the Distress Dependence of a portfolio. In particular, this analysis shows that during the recent financial crisis the likelihood of contagion between the two classes increased considerably. The model could thus be used to monitor the evolution of risk measures based on the distribution of the joint default probability.

²¹Actually we cannot compare our results perfectly with those of Mai and Scherer as their market data for the third tranche are different from the Bloomberg market data that we use and because the procedure that we follow for the calibration of the hazard rates is not the same one that they use.

References

- [1] Albrecher, H., Ladoucette, S. and Schoutens, W. (2007), “A Generic One-Factor Lévy Model for Pricing Synthetic CDOs.”, *Advances in Mathematical Finance*, Springer, 259-277.
- [2] Andersen, L., Sidenius, J. and Basu, S. (2003), “All Your Hedges in One Basket”, *Risk*, November, 67-72.
- [3] Andersen, L. and Sidenius, J. (2005), “Extensions to the Gaussian Copula: Random Recovery and Random Factor Loadings.”, *Journal of Credit Risk*, 1 (1), 29-70.
- [4] Applebaum, D. (2004), *Lévy Processes and Stochastic Calculus*. Cambridge Studies in Advanced Mathematics.
- [5] Cherubini, U., Luciano and E., Vecchiato, W. (2004), *Copula Methods in Finance*. Wiley Finance Series.
- [6] Choros-Tomczyk, B., Haerdle, W. and Overbeck, L. (2012), “Copula Dynamics in CDOs.”, *Working Paper*, Humboldt-Universität Berlin.
- [7] Donnelly, C. and Embrechts, P. (2010), “The Devil is in the Tails: Actuarial Mathematics and the Subprime Mortgage Crisis.”, *ASTIN Bulletin*, 40 (1), 1-33.
- [8] Eberlein, E., Frey, R. and Hammerstein, E. (2008), “Advanced Credit Portfolio Modeling and CDO Pricing.”, *Mathematics - Key technology for the future*, Springer, 253-279.
- [9] Garcia, J., Goossens, S., Masol, V. and Shoutens, W. (2009), “Lévy Base Correlation.”, *Wilmott Journal*, 1 (2), 95-100.
- [10] Hofert, M. and Scherer, M. (2011), “CDO Pricing with Nested Archimedean Copulas.”, *Quantitative Finance*, 11 (5), 775-787.
- [11] Huang, X. (1992), “Statistics of Extreme Values”, *Tinbergen Institute Research Series, Ph.D. Thesis No. 22*. Erasmus University.
- [12] Hull, J. and White, A. (2004), “Valuation of a CDO and an n-th to Default CDS Without Monte Carlo Simulation”, *The Journal of Derivatives*, 2, 8-23.
- [13] Hull, J. and White, A. (2010), “An Improved Implied Copula Model and its Application to the Valuation of Bespoke CDO Tranches”, *Journal of Investment Management*, Third Quarter, 2010.
- [14] Kalemánova, A., Schmid, B. and Werner, JR. (2007), “The Normal Inverse Gaussian Distribution for Synthetic CDO Pricing”, *Journal of Derivatives*, 14(3), 80-93.
- [15] Laurent, J.P. and Gregory, J. (2005), “Basket Default Swaps, CDOs and Factor Copulas”, *Journal of Risk*, 7 (4), 103-122.

- [16] Li, D. X. (2000), “On Default Correlation: a Copula Approach”, *Journal of Fixed Income*, Vol.9, Issue 4, 43-54.
- [17] Luciano, E. (2007), “Copula-based Default Dependence Modelling: Where do We Stand?”, *Credit Risk: Models, derivatives and management*. Chapman & Hall. 327-347.
- [18] Mai, J.F. and Scherer, M. (2009a), “A Tractable Multivariate Default Model Based on a Stochastic Time-Change”, *International Journal of Theoretical and Applied Finance*, Vol. 12, No. 2, 227-249.
- [19] Mai, J.F. and Scherer, M. (2009b), “Lévy-Frailty Copulas”, *Journal of Multivariate Analysis*, 100, 1567-1585.
- [20] Masol, V. and Schoutens, W. (2011), “Comparing Alternative Lévy Base Correlation Models for Pricing and Hedging CDO Tranches”, *Quantitative Finance*, 11 (5), 763-773.
- [21] Nelsen, R. B. (1999), *An Introduction to Copulas*. Springer, New York.
- [22] Segoviano, M. A. and Goodhart, C. (2009), “Banking Stability Measures”, *IMF Working Papers*.
- [23] Segoviano, M. A. and Goodhart, C. (2010), “Distress Dependence and Financial Stability”, *Central Bank of Chile Working Papers*.

A Appendix: Mathematical definitions

Copula functions

Definition A.1 (Copula function). *A copula is an n -dimensional distribution function $C : [0, 1]^n \rightarrow [0, 1]$ of a random vector (U_1, \dots, U_n) , where the marginal law of U_i is the uniform distribution on $[0, 1]$ for all $i \in \{1, \dots, n\}$.*

Copula functions are very popular in the study of multivariate distribution functions thanks to their role in imposing a dependence structure on predetermined marginal distributions. Their importance derives from *Sklar's theorem*, which proves that any multivariate distribution function can be characterized by a copula and that copula functions, together with univariate marginal distribution functions, can be used to construct multivariate distribution functions.

Theorem A.1 (Sklar's theorem). *Let H be an n -dimensional distribution function with marginals F_1, \dots, F_n .*

Then an n -copula C exists such that, for each $\mathbf{x} \in \mathbf{R}^n$,

$$H(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)).$$

If the marginals F_1, \dots, F_n are all continuous, then C is unique; otherwise C is univocally determined on $(\text{Ran}F_1 \times \text{Ran}F_2 \times \dots \times \text{Ran}F_n)$ (where $\text{Ran}F_i$ denotes the range of F_i). Conversely, if C is an n -copula and F_1, \dots, F_n are distribution functions, then the function H defined above is an n -dimensional distribution function with marginals F_1, \dots, F_n .

The proof of this theorem can be found in [21].

The main feature of Sklar's theorem is that for continuous multivariate distribution functions, the univariate marginals and the multivariate dependence structure can be separated and the dependence structure can be represented by a copula.

Let F be an univariate distribution function. Let us recall that the generalized inverse of F is defined as $F^{-1}(t) = \inf\{x \in \mathbf{R} \mid F(x) \geq t\}$ for each t in $[0, 1]$, with the usual convention that $\inf(\emptyset) = -\infty$.

An important corollary of Sklar's theorem, which is fundamental in the study of copulas and their applications, is the following:

Corollary A.1. *Let H be an n -dimensional distribution function with continuous marginals $1_1, \dots, 1_n$ and copula C . Then for each $\mathbf{u} \in [0, 1]^n$,*

$$C(u_1, \dots, u_n) = H(F_1^{-1}(u_1), \dots, F_n^{-1}(u_n)).$$

Lévy processes

Definition A.2 (Lévy process). *A Lévy process is any continuous-time stochastic process $X = \{X_t : t \geq 0\}$ such that*

1. $X_0 = 0$ almost surely;
2. It has independent increments: for any $n \geq 2$ and $0 \leq t_1 < t_2 < \dots < t_n < \infty$, the increments $X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent;
3. It has stationary increments: for any $s < t$, $X_t - X_s$ is equal in distribution to X_{t-s} ;
4. $t \rightarrow X_t$ is almost surely right continuous with left limits.

The best-known examples of Lévy processes are the Wiener process and the Poisson process.

Stochastic clock

Definition A.3 (Stochastic clock). Consider a stochastic process $T(t)$ with increasing paths and starting from 0. Consider another random process $X(s)$. When we consider the process $Y(t) = X(T(t))$, i.e. we compute the process X in the random time $T(t)$, then $T(t)$ is called “random clock”.

B Appendix: Mathematical proofs

Proof of Proposition 4.1

By conditioning with respect to $\mathcal{F}_\infty^\Lambda$ we have

$$\begin{aligned}
\bar{F}_{\tau_1, \dots, \tau_n}(t_1, \dots, t_n) &:= \mathbb{P}(\tau_i > t_i, i = 1, \dots, n) = \mathbb{E}(\mathbb{P}(\tau_i > t_i, i = 1, \dots, n | \mathcal{F}_\infty^\Lambda)) \\
&= \mathbb{E}\left(\prod_{i=1}^n e^{-\Lambda_i(h_i(t_i))}\right) = \mathbb{E}\left(\prod_{i=1}^n e^{-a_i \Lambda_{h_i(t_i)} - b_i h_i(t_i)}\right) \\
&= \mathbb{E}\left(\prod_{i=1}^n e^{-a_i \Lambda_{h_i(t_i)}}\right) e^{-\sum_{i=1}^n b_i h_i(t_i)}.
\end{aligned}$$

To compute the expected value

$$\mathbb{E}\left(\prod_{i=1}^n e^{-a_i \Lambda_{h_i(t_i)}}\right) = \mathbb{E}\left(e^{-\sum_{i=1}^n a_i \Lambda_{h_i(t_i)}}\right)$$

we consider the permutation $\sigma_i(\mathbf{t}) = \sigma_i(t_1, \dots, t_n)$ such that

$$h_{(i)}(\mathbf{t}) := h_{\sigma_i(\mathbf{t})}(t_{\sigma_i(\mathbf{t})})$$

is a reordering of $h_i(t_i)$, which means that

$$h_{(i-1)}(\mathbf{t}) \leq h_{(i)}(\mathbf{t}), \quad i = 1, \dots, n,$$

where we assume by convention $h_{(0)}(\mathbf{t}) = 0$.

By using the notation

$$\theta_j(\mathbf{t}) = \sum_{i=j}^n a_{\sigma_i(\mathbf{t})}, \quad j = 1, \dots, n$$

we have that

$$\theta_{j+1}(\mathbf{t}) = \theta_j(\mathbf{t}) - a_{\sigma_j(\mathbf{t})}, \quad j = 1, \dots, n$$

where we assume by convention that $\theta_{n+1}(\mathbf{t}) = 0$.

So we have

$$\begin{aligned} \sum_{i=1}^n a_i \Lambda_{h_i(t_i)} &= \sum_{i=1}^n a_{\sigma_i(\mathbf{t})} \Lambda_{h_{\sigma_i(\mathbf{t})}(t_{\sigma_i(\mathbf{t})})} = \sum_{i=1}^n a_{\sigma_i(\mathbf{t})} \Lambda_{h_{(i)}(\mathbf{t})} \\ &= \sum_{i=1}^n a_{\sigma_i(\mathbf{t})} \sum_{j=1}^i (\Lambda_{h_{(j)}(\mathbf{t})} - \Lambda_{h_{(j-1)}(\mathbf{t})}) \\ &= \sum_{j=1}^n (\Lambda_{h_{(j)}(\mathbf{t})} - \Lambda_{h_{(j-1)}(\mathbf{t})}) \sum_{i=j}^n a_{\sigma_i(\mathbf{t})} \\ &= \sum_{j=1}^n (\Lambda_{h_{(j)}(\mathbf{t})} - \Lambda_{h_{(j-1)}(\mathbf{t})}) \theta_j(\mathbf{t}). \end{aligned}$$

Now, as a subordinator is a process with independent increments, we can compute

$$\begin{aligned} \mathbb{E}\left(\prod_{i=1}^n e^{-a_i \Lambda_{h_i(t_i)}}\right) &= \mathbb{E}\left(e^{-\sum_{i=1}^n a_i \Lambda_{h_i(t_i)}}\right) = \mathbb{E}\left(e^{-\sum_{j=1}^n (\Lambda_{h_{(j)}(\mathbf{t})} - \Lambda_{h_{(j-1)}(\mathbf{t})}) \theta_j(\mathbf{t})}\right) \\ &= \prod_{j=1}^n \mathbb{E}\left(e^{-\theta_j(\mathbf{t}) (\Lambda_{h_{(j)}(\mathbf{t})} - \Lambda_{h_{(j-1)}(\mathbf{t})})}\right) = \prod_{j=1}^n e^{\Psi(-\theta_j(\mathbf{t})) (h_{(j)}(\mathbf{t}) - h_{(j-1)}(\mathbf{t}))} \\ &= e^{\sum_{j=1}^n h_{(j)}(\mathbf{t}) (\Psi(-\theta_j(\mathbf{t})) - \Psi(-\theta_{j+1}(\mathbf{t})))} \end{aligned}$$

where in the last equation we consider that $h_{(0)}(\mathbf{t}) = 0$, $\theta_{n+1}(\mathbf{t}) = 0$ and so $\Psi(-\theta_{n+1}(\mathbf{t})) = 0$. Finally, considering also the relationship between $\theta_{j+1}(\mathbf{t})$ and $\theta_j(\mathbf{t})$, we can compute

$$\begin{aligned}
\overline{F}_{\tau_1, \dots, \tau_n}(t_1, \dots, t_n) &:= \mathbb{P}(\tau_i > t_i, i = 1, \dots, n) = \mathbb{E}\left(\prod_{i=1}^n e^{-a_i \Lambda_{h_i(t_i)}}\right) e^{-\sum_{i=1}^n b_i h_i(t_i)} \\
&= e^{-\sum_{j=1}^n h_{(j)}(\mathbf{t}) \left(\Psi(-\theta_{j+1}(\mathbf{t})) - \Psi(-\theta_j(\mathbf{t}))\right)} e^{-\sum_{i=1}^n b_i h_i(t_i)} \\
&= e^{-\sum_{j=1}^n h_{(j)}(\mathbf{t}) \left(\Psi(-\theta_j(\mathbf{t}) + a_{\sigma_j(\mathbf{t})}) - \Psi(-\theta_j(\mathbf{t}))\right)} e^{-\sum_{i=1}^n b_{\sigma_i(\mathbf{t})} h_{\sigma_i(\mathbf{t})}(t_{\sigma_i(\mathbf{t})})} \\
&= e^{-\sum_{j=1}^n \left(\Psi(-\theta_j(\mathbf{t}) + a_{\sigma_j(\mathbf{t})}) - \Psi(-\theta_j(\mathbf{t})) + b_{\sigma_j(\mathbf{t})}\right) h_{\sigma_j(\mathbf{t})}(t_{\sigma_j(\mathbf{t})})}.
\end{aligned}$$

Note that, for the monotonicity property of the Laplace exponent, we have

$$\Psi(-\theta_j(\mathbf{t}) + a_{\sigma_j(\mathbf{t})}) - \Psi(-\theta_j(\mathbf{t})) + b_{\sigma_j(\mathbf{t})} > 0.$$

□

Proof of Proposition 4.2

$$\begin{aligned}
\overline{F}_{\tau_1, \dots, \tau_n}(t_1, \dots, t_n) &= e^{-\sum_{j=1}^n \left(\Psi(-\theta_{\sigma_j^{-1}(\mathbf{t})}(\mathbf{t}) + a_j) - \Psi(-\theta_{\sigma_j^{-1}(\mathbf{t})}(\mathbf{t})) + b_j\right) h_j(t_j)} \\
&= \prod_{j=1}^n \overline{G}_j(t_j)^{\Psi(-\theta_{\sigma_j^{-1}(\mathbf{t})}(\mathbf{t}) + a_j) - \Psi(-\theta_{\sigma_j^{-1}(\mathbf{t})}(\mathbf{t})) + b_j} \\
&= \prod_{j=1}^n \overline{G}_j(t_j)^{\Psi(-\theta_{\sigma_j^{-1}(\mathbf{t})}(\mathbf{t}) + a_j) - \Psi(-\theta_{\sigma_j^{-1}(\mathbf{t})}(\mathbf{t})) + \Psi(-a_j) + b_j - \Psi(-a_j)} \\
&= \prod_{j=1}^n \left(\overline{G}_j(t_j)^{b_j - \Psi(-a_j)}\right)^{\frac{\Psi(-\theta_{\sigma_j^{-1}(\mathbf{t})}(\mathbf{t}) + a_j) - [\Psi(-\theta_{\sigma_j^{-1}(\mathbf{t})}(\mathbf{t})) - \Psi(-a_j)]}{b_j - \Psi(-a_j)}} + 1} \\
&= \prod_{j=1}^n \left(\overline{G}_{\tau_j}(t_j)\right)^{\frac{\Psi(-\theta_{\sigma_j^{-1}(\mathbf{t})}(\mathbf{t}) + a_j) - [\Psi(-\theta_{\sigma_j^{-1}(\mathbf{t})}(\mathbf{t})) - \Psi(-a_j)]}{b_j - \Psi(-a_j)}} + 1}.
\end{aligned}$$

□

Proof of Proposition 4.3

We have

$$\begin{aligned}
\sum_{i=1}^n a_i \Lambda_{h_i(t)} &= \sum_{\ell=1}^r m_\ell a^{(\ell)} \Lambda_{h^{(\ell)}(t)} \\
&= \sum_{\ell=1}^r m_\ell a^{(\ell)} \sum_{j=1}^i (\Lambda_{h^{(j)}(t)} - \Lambda_{h^{(j-1)}(t)}) \\
&= \sum_{j=1}^r (\Lambda_{h^{(j)}(t)} - \Lambda_{h^{(j-1)}(t)}) \sum_{i=j}^r m_\ell a^{(\ell)} \\
&= \sum_{j=1}^n (\Lambda_{h^{(j)}(t)} - \Lambda_{h^{(j-1)}(t)}) \theta_j^r,
\end{aligned}$$

where

$$\theta_j^r := \sum_{\ell=j}^r m_\ell a^{(\ell)}.$$

We can thus compute

$$\begin{aligned}
\mathbb{P}(\tau_1 > t, \tau_2 > t, \dots, \tau_n > t) &= \mathbb{E}\left(\prod_{i=1}^n e^{-a_i \Lambda_{h_i(t)}}\right) e^{-\sum_{i=1}^n b_i h_i(t)} \\
&= e^{-\sum_{j=1}^r h^{(j)}(t)} \left(\Psi(-\theta_j^r + m_j a^{(j)}) - \Psi(-\theta_j^r)\right) e^{-\sum_{i=1}^r b^{(i)} h^{(i)}(t)} \\
&= e^{-\sum_{j=1}^r \left(\Psi(-\theta_j^r + m_j a^{(j)}) - \Psi(-\theta_j^r) + m_j b^{(j)}\right) h^{(j)}(t)} \\
&= e^{-\sum_{j=1}^r \left(\Psi(-\sum_{\ell=j+1}^r m_\ell a^{(\ell)}) - \Psi(-\sum_{\ell=j}^r m_\ell a^{(\ell)}) + m_j b^{(j)}\right) h^{(j)}(t)}, \quad (\text{B.1})
\end{aligned}$$

i.e. formula (4) and similarly

$$\begin{aligned}
&\mathbb{P}(\tau_i > t, \forall i \in I_j, \forall j = 1, \dots, r) \\
&= e^{-\sum_{j=1}^r \left(\Psi(-\sum_{\ell=j+1}^r k_\ell a^{(\ell)}) - \Psi(-\sum_{\ell=j}^r k_\ell a^{(\ell)}) + k_j b^{(j)}\right) h^{(j)}(t)} \\
&= e^{-\sum_{j=1}^r \left(\Psi(-\sum_{\ell=j+1}^r k_\ell a^{(\ell)}) - \Psi(-\sum_{\ell=j}^r k_\ell a^{(\ell)}) + k_j (\Psi(-a^{(j)}) + 1)\right) h^{(j)}(t)} \quad (\text{B.2})
\end{aligned}$$

where $k_j = |I_j|$, i.e. formula (5). □

Proof of Proposition 4.4

Before starting the proof of the proposition, let us note that the parameter constraint $(*p)$ simplifies the formula for the survival copula (see Lemma 4.1): first of all, with this constraint, the numbers

$u_{\hat{\sigma}_j(\mathbf{u})}$ are in decreasing order, i.e., $u_{\hat{\sigma}_1(\mathbf{u})} \geq u_{\hat{\sigma}_2(\mathbf{u})} \geq \dots \geq u_{\hat{\sigma}_n(\mathbf{u})}$, and furthermore $\widehat{C}_{\tau_1, \dots, \tau_n}(u_1, \dots, u_n) = \prod_{j=1}^n u_{\hat{\sigma}_j(\mathbf{u})}^{1-\alpha_j(\mathbf{u})}$, where

$$\alpha_j(\mathbf{u}) := \Psi(-\widehat{\theta}_j(\mathbf{u})) - \Psi(-\widehat{\theta}_j(\mathbf{u}) + a_{\hat{\sigma}_j(\mathbf{u})}) - \Psi(-a_{\hat{\sigma}_j(\mathbf{u})}) = \int_0^\infty (1 - e^{-\widehat{\theta}_j(\mathbf{u}) + a_{\hat{\sigma}_j(\mathbf{u})}s})(1 - e^{-a_{\hat{\sigma}_j(\mathbf{u})}s})\nu(ds)$$

is positive and does not depend on the value of μ . Moreover, the parameter constraint implies that $\alpha_j(\mathbf{u}) \leq 1$. We have in fact

$$\alpha_j(\mathbf{u}) \leq \int_0^\infty (1 - e^{-a_{\hat{\sigma}_j(\mathbf{u})}s})\nu(ds) = -\Psi(-a_{\hat{\sigma}_j(\mathbf{u})}) - \mu a_{\hat{\sigma}_j(\mathbf{u})} = 1 - b_{\hat{\sigma}_j(\mathbf{u})} - \mu a_{\hat{\sigma}_j(\mathbf{u})} \leq 1.$$

Observing that $\widehat{\theta}_n(\mathbf{u}) = a_{\hat{\sigma}_n(\mathbf{u})}$, $\Psi(0) = 0$ and $\widehat{\theta}_n(\mathbf{u}) = a_{\hat{\sigma}_n(\mathbf{u})} + a_{\hat{\sigma}_{n-1}(\mathbf{u})}$, we have respectively

$$\begin{aligned} \alpha_n(\mathbf{u}) &= \Psi(-\widehat{\theta}_n(\mathbf{u})) - \Psi(-\widehat{\theta}_n(\mathbf{u}) + a_{\hat{\sigma}_n(\mathbf{u})}) - \Psi(-a_{\hat{\sigma}_n(\mathbf{u})}) = 0, \\ \alpha_{n-1}(\mathbf{u}) &= \Psi(-\widehat{\theta}_{n-1}(\mathbf{u})) - \Psi(-\widehat{\theta}_{n-1}(\mathbf{u}) + a_{\hat{\sigma}_{n-1}(\mathbf{u})}) - \Psi(-a_{\hat{\sigma}_{n-1}(\mathbf{u})}) \\ &= \Psi(-a_{\hat{\sigma}_n(\mathbf{u})} - a_{\hat{\sigma}_{n-1}(\mathbf{u})}) - \Psi(-a_{\hat{\sigma}_n(\mathbf{u})}) - \Psi(-a_{\hat{\sigma}_{n-1}(\mathbf{u})}). \end{aligned}$$

As a consequence, we can immediately deduce that the survival copula of τ_i and τ_j is

$$\widehat{C}_{\tau_i, \tau_j}(u, v) = (u \vee v)^{1-\alpha_{i,j}} (u \wedge v),$$

where

$$\alpha_{i,j} := \Psi(-a_i - a_j) - \Psi(-a_i) - \Psi(-a_j) \in [0, 1],$$

that is a Cuadras-Augé copula.²² Indeed, by taking $u_i = u$, $u_j = v$ and $u_k = 1$ for $k \neq i, j$, we get $u_{\hat{\sigma}_j(\mathbf{u})} = 1$ for $j = 1, \dots, n-2$, $u_{\hat{\sigma}_{n-1}(\mathbf{u})} = u \vee v$, $u_{\hat{\sigma}_n(\mathbf{u})} = u \wedge v$.

At this point the proof of the proposition is straightforward. First of all observe that for two different indexes i and j and two times t_i and t_j , we have

$$\begin{aligned} \text{Cov}[A_{t_i}^i, A_{t_j}^j] &= \text{Cov}[1 - A_{t_i}^i, 1 - A_{t_j}^j] \\ &= \mathbb{P}(\tau_i > t_i, \tau_j > t_j) - \mathbb{P}(\tau_i > t_i)\mathbb{P}(\tau_j > t_j) \\ &= \overline{F}_{\tau_i, \tau_j}(t_i, t_j) - \overline{G}_i(t_i)\overline{G}_j(t_j) \\ &= \widehat{C}_{\tau_i, \tau_j}(\overline{G}_i(t_i), \overline{G}_j(t_j)) - \overline{G}_i(t_i)\overline{G}_j(t_j) \end{aligned}$$

Then as the survival copula $\widehat{C}_{\tau_i, \tau_j}$ is a Cuadras-Augé copula

$$\begin{aligned} &= (\overline{G}_i(t_i) \vee \overline{G}_j(t_j))^{1-\alpha_{i,j}} (\overline{G}_i(t_i) \wedge \overline{G}_j(t_j)) - \overline{G}_i(t_i)\overline{G}_j(t_j) \\ &= \overline{G}_i(t_i)\overline{G}_j(t_j) \left((\overline{G}_i(t_i) \vee \overline{G}_j(t_j))^{-\alpha_{i,j}} - 1 \right) \end{aligned}$$

²²See Mai and Scherer (2009b) for details.

where $\alpha_{i,j} := \Psi(-a_i - a_j) - \Psi(-a_i) - \Psi(-a_j) \in [0, 1]$. Furthermore, obviously

$$\begin{aligned} \text{Var}(A_t^i) &= \text{Var}(1 - A_t^i) \\ &= \mathbb{P}(\tau_i > t)(1 - \mathbb{P}(\tau_i > t)) = \overline{G}_i(t)(1 - \overline{G}_i(t)). \end{aligned}$$

From the above formulas the results immediately follows. Indeed in the case $i \in \mathcal{M}_m, j \in \mathcal{M}_n$, with $m < n$, we have

$$\begin{aligned} a_i &= a^{(m)} & a_j &= a^{(n)}, \\ b_i &= b^{(m)} & b_j &= b^{(n)}, \\ h_i &= h^{(m)} \leq h_j = h^{(n)}, \\ \overline{G}_i(t) &= \overline{G}^{(m)}(t) = e^{-h^{(m)}(t)}, & \overline{G}_j(t) &= \overline{G}^{(n)}(t) = e^{-h^{(n)}(t)}, \\ \overline{G}^{(m)}(t) &= \max(\overline{G}^{(m)}(t), \overline{G}^{(n)}(t)), \end{aligned}$$

and

$$\alpha_{i,j} = \Psi(-a^{(m)} - a^{(n)}) - \Psi(-a^{(m)}) - \Psi(-a^{(n)})$$

while in the case $i, j \in \mathcal{M}_m$, in the previous formula we have obviously $a^{(m)} = a^{(n)}, \overline{G}^{(m)}(t) = \overline{G}^{(n)}(t)$, and $\alpha_{i,j} = \Psi(-2a^{(m)}) - 2\Psi(-a^{(m)})$.

□

Proof of Proposition 4.5

To compute the portfolio loss distribution we use formulas (4) and (5). We have

$$\mathbb{P}(nL^n(t) = n - k) = \sum_{\substack{0 \leq k_j \leq m_j \\ k_1 + \dots + k_r = k}} \dots \sum \mathbb{P}(\forall j = 1, \dots, r \exists I_j : |I_j| = k_j, \tau_i > t, \forall i \in I_j, ; \tau_{i'} \leq t, \forall i' \in \mathcal{M}_j \setminus I_j)$$

where $\mathcal{M}_j = \{M_{j-1} + 1, \dots, M_j\}$

$$= \sum_{\substack{0 \leq k_j \leq m_j \\ k_1 + \dots + k_r = k}} \dots \sum \prod_{j=1}^r \binom{m_j}{k_j} \mathbb{P}(\tau_i > t, \tau_{i'} \leq t, M_{j-1} < i \leq M_{j-1} + k_j, M_{j-1} + k_j < i' \leq M_j, j = 1, \dots, r)$$

Considering that

$$\begin{aligned} &\mathbb{P}(\tau_i > t, \tau_{i'} \leq t, M_{j-1} < i \leq M_{j-1} + k_j, M_{j-1} + k_j < i' \leq M_j, j = 1, \dots, r) \\ &= \mathbb{P}(A \cap B^c) = \mathbb{P}(A) - \mathbb{P}(A \cap B) \end{aligned}$$

where

$$A = \{\tau_i > t, M_{j-1} < i \leq M_{j-1} + k_j, j = 1, \dots, r\}$$

and

$$B^c = \{\tau_{i'} \leq t, M_{j-1} + k_j < i' \leq M_j, j = 1, \dots, r\}$$

so that

$$B = \bigcup_{j=1, \dots, r} \bigcup_{M_{j-1} + k_j < i' \leq M_j} \{\tau_{i'} > t\}$$

we have

$$\mathbb{P}(A) - \mathbb{P}(A \cap B) = \mathbb{P}(A) - \mathbb{P}\left(\bigcup_{j=1, \dots, r} \bigcup_{M_{j-1} + k_j < i' \leq M_j} A \cap \{\tau_{i'} > t\}\right)$$

and, using the inclusion/exclusion formula, we get

$$= \mathbb{P}(A) - \sum_{v=1}^{n-k} \sum_{\substack{v_1 + \dots + v_r = v \\ 0 \leq v_j \leq m_j - k_j}} \dots \sum_{j=1}^r (-1)^{v+1} \binom{m_j - k_j}{v_j} \mathbb{P}(A, \tau_{i'} > t, M_{j-1} + k_j < i' \leq M_{j-1} + k_j + v_j, j = 1, \dots, r)$$

(where by $\{A, \tau_{i'} > t, \text{etc.}\}$ we mean $A \cap \{\tau_{i'} > t, \text{etc.}\}$)

$$\sum_{v=0}^{n-k} \sum_{\substack{v_1 + \dots + v_r = v \\ 0 \leq v_j \leq m_j - k_j}} \dots \sum_{j=1}^r (-1)^v \binom{m_j - k_j}{v_j} \mathbb{P}(A, \tau_{i'} > t, M_{j-1} + k_j < i' \leq M_{j-1} + k_j + v_j, j = 1, \dots, r)$$

having used

$$\mathbb{P}(A) = \sum_{\substack{v_1 + \dots + v_r = 0 \\ 0 \leq v_j \leq m_j - k_j}} \dots \sum_{j=1}^r (-1)^0 \binom{m_j - k_j}{v_j} \mathbb{P}(A, \tau_{i'} > t, M_{j-1} + k_j < i' \leq M_{j-1} + k_j + v_j, j = 1, \dots, r)$$

$$\mathbb{P}(A) = \sum_{v_j=0, j=1, \dots, r} \dots \sum_{j=1}^r (-1)^0 \binom{m_j - k_j}{0} \mathbb{P}(A, \tau_{i'} > t, M_{j-1} + k_j < i' \leq M_{j-1} + k_j + 0, j = 1, \dots, r)$$

where the condition $\tau_{i'} > t, M_{j-1} + k_j < i' \leq M_{j-1} + k_j + 0$ is pointless as it does not involve any index i' .

With our assumption $(*h)$ on the functions $h^{(i)}(t)$ we obtain

$$\begin{aligned}
& \mathbb{P}(nL^n(t) = n - k) \\
&= \sum_{\substack{0 \leq k_j \leq m_j \\ k_1 + \dots + k_r = k}} \dots \sum_{j=1}^r \prod_{j=1}^r \binom{m_j}{k_j} \mathbb{P}(\tau_i > t, \tau_{i'} \leq t, M_{j-1} < i \leq M_{j-1} + k_j, M_{j-1} + k_j < i' \leq M_j, j = 1, \dots, r) \\
&= \sum_{\substack{0 \leq k_j \leq m_j \\ k_1 + \dots + k_r = k}} \dots \sum_{v=0}^{n-k} \sum_{\substack{v_1 + \dots + v_r = v \\ 0 \leq v_j \leq m_j - k_j}} \dots \sum_{j=1}^r \prod_{j=1}^r \binom{m_j}{k_j} \prod_{j=1}^r \binom{m_j - k_j}{v_j} \mathbb{P}(\tau_i > t, M_{j-1} < i \leq M_{j-1} + k_j + v_j, j = 1, \dots, r) \\
&= \sum_{\substack{0 \leq k_j \leq m_j \\ k_1 + \dots + k_r = k}} \dots \sum_{v=0}^{n-k} \sum_{\substack{v_1 + \dots + v_r = v \\ 0 \leq v_j \leq m_j - k_j}} \dots \sum_{j=1}^r \prod_{j=1}^r \frac{m_j!}{k_j! v_j! (m_j - k_j - v_j)!} \mathbb{P}(\tau_i > t, M_{j-1} < i \leq M_{j-1} + k_j + v_j, j = 1, \dots, r) \\
&= \sum_{\substack{0 \leq k_j \leq m_j \\ k_1 + \dots + k_r = k}} \dots \sum_{v=0}^{n-k} \sum_{\substack{v_1 + \dots + v_r = v \\ 0 \leq v_j \leq m_j - k_j}} \dots \prod_{j=1}^r \frac{m_j!}{k_j! v_j! (m_j - k_j - v_j)!} \\
&\quad e^{-\sum_{j'=1}^r \left(\Psi(-\sum_{\ell=j'+1}^r (k_\ell + v_\ell) a^{(\ell)}) - \Psi(-\sum_{\ell=j'}^r k_\ell a^{(\ell)}) + k_{j'} b^{(j')} \right) v^{(j')}(t)}
\end{aligned}$$

and in the same way we get

$$\begin{aligned}
& \mathbb{P}(nL^n(t) = k) \\
&= \sum_{\substack{0 \leq k_j \leq m_j \\ k_1 + \dots + k_r = n - k}} \dots \sum_{j=1}^r \prod_{j=1}^r \binom{m_j}{k_j} \mathbb{P}(\tau_i > t, \tau_{i'} \leq t, M_{j-1} < i \leq M_{j-1} + k_j, M_{j-1} + k_j < i' \leq M_j, j = 1, \dots, r) \\
&= \sum_{\substack{0 \leq k_j \leq m_j \\ k_1 + \dots + k_r = n - k}} \dots \sum_{v=0}^k \sum_{\substack{v_1 + \dots + v_r = v \\ 0 \leq v_j \leq m_j - k_j}} \dots \sum_{j=1}^r \prod_{j=1}^r \binom{m_j}{k_j} \prod_{j=1}^r \binom{m_j - k_j}{v_j} \mathbb{P}(\tau_i > t, M_{j-1} < i \leq M_{j-1} + k_j + v_j, j = 1, \dots, r) \\
&= \sum_{\substack{0 \leq k_j \leq m_j \\ k_1 + \dots + k_r = n - k}} \dots \sum_{v=0}^k \sum_{\substack{v_1 + \dots + v_r = v \\ 0 \leq v_j \leq m_j - k_j}} \dots \prod_{j=1}^r \frac{m_j!}{k_j! v_j! (m_j - k_j - v_j)!} \mathbb{P}(\tau_i > t, M_{j-1} < i \leq M_{j-1} + k_j + v_j, j = 1, \dots, r) \\
&= \sum_{\substack{0 \leq k_j \leq m_j \\ k_1 + \dots + k_r = n - k}} \dots \sum_{v=0}^k \sum_{\substack{v_1 + \dots + v_r = v \\ 0 \leq v_j \leq m_j - k_j}} \dots \prod_{j=1}^r \frac{m_j!}{k_j! v_j! (m_j - k_j - v_j)!} \\
&\quad e^{-\sum_{j'=1}^r \left(\Psi(-\sum_{\ell=j'+1}^r (k_\ell + v_\ell) a^{(\ell)}) - \Psi(-\sum_{\ell=j'}^r k_\ell a^{(\ell)}) + k_{j'} b^{(j')} \right) v^{(j')}(t)}.
\end{aligned}$$

Let us note that the final exponential factor can be written as

$$\begin{aligned}
& e^{-\sum_{j'=1}^r \left(\Psi(-\sum_{\ell=j'+1}^r (k_\ell + v_\ell) a^{(\ell)}) - \Psi(-\sum_{\ell=j'}^r k_\ell a^{(\ell)}) + k_{j'} b^{(j')} \right) v^{(j')}(t)} \\
&= \prod_{j'=1}^r \left(e^{-v^{(j')}(t)} \right)^{\Psi(-\sum_{\ell=j'+1}^r (k_\ell + v_\ell) a^{(\ell)}) - \Psi(-\sum_{\ell=j'}^r k_\ell a^{(\ell)}) + k_{j'} b^{(j')}} \\
&= \prod_{j'=1}^r \left(\overline{G}^{(j')}(t) \right)^{\Psi(-\sum_{\ell=j'+1}^r (k_\ell + v_\ell) a^{(\ell)}) - \Psi(-\sum_{\ell=j'}^r k_\ell a^{(\ell)}) + k_{j'} b^{(j')}}
\end{aligned}$$

so the final result is

$$\begin{aligned}
& \mathbb{P}(nL^n(t) = k) \\
&= \sum_{\substack{0 \leq k_j \leq m_j \\ k_1 + \dots + k_r = n-k}} \dots \sum_{v=0}^k (-1)^v \sum_{\substack{v_1 + \dots + v_r = v \\ 0 \leq v_j \leq m_j - k_j}} \dots \sum_{j=1}^r \prod_{j=1}^r \frac{m_j!}{k_j! v_j! (m_j - k_j - v_j)!} \left(\overline{G}^{(j)}(t) \right)^{\Psi(-\sum_{\ell=j+1}^r (k_\ell + v_\ell) a^{(\ell)}) - \Psi(-\sum_{\ell=j}^r k_\ell a^{(\ell)}) + k_j b^{(j)}}.
\end{aligned}$$

□

Proof of Proposition 6.1

First of all we observe that, for $\ell = 1, 2$,

$$X^{(\ell)}(t) = \sum_{i=1}^{m_\ell} \mathbf{1}_{\{\tau_i^{(\ell)} \leq t\}}$$

and

$$\{X^{(\ell)}(t) \geq 1\} = \left\{ \min_{1 \leq j \leq m_\ell} \tau_j^{(\ell)} \leq t \right\}.$$

In the first case we have (we prove only the first case, $\ell = 1$, since the proof for $\ell = 2$ is identical)

$$\begin{aligned}
\mathbb{E}[X_{t'}^{(1)} | X_t^{(1)} \geq 1] &= \sum_{i=1}^{m_1} \mathbb{P}(\tau_i^{(1)} \leq t' | \min_{1 \leq j \leq m_1} \tau_j^{(1)} \leq t) = \sum_{i=1}^{m_1} \frac{\mathbb{P}(\tau_i^{(1)} \leq t', \min_{1 \leq j \leq m_1} \tau_j^{(1)} \leq t)}{\mathbb{P}(\min_{1 \leq j \leq m_1} \tau_j^{(1)} \leq t)} \\
&= \sum_{i=1}^{m_1} \frac{\mathbb{P}(\{\tau_i^{(1)} \leq t'\} \cap (\cup_{j=1}^{m_1} \{\tau_j^1 \leq t\}))}{\mathbb{P}(\min_{1 \leq j \leq m_1} \tau_j^{(1)} \leq t)} = \sum_{i=1}^{m_1} \frac{\mathbb{P}(\{\tau_i^{(1)} \leq t'\} \cap (\cap_{j=1}^{m_1} \{\tau_j^1 > t\})^c)}{\mathbb{P}(\min_{1 \leq j \leq m_1} \tau_j^{(1)} \leq t)} \\
&= \sum_{i=1}^{m_1} \frac{\mathbb{P}(\{\tau_i^{(1)} \leq t'\}) - \mathbb{P}(\{\tau_i^{(1)} \leq t'\} \cap (\cap_{j=1}^{m_1} \{\tau_j^1 > t\}))}{\mathbb{P}(\min_{1 \leq j \leq m_1} \tau_j^{(1)} \leq t)} \\
&= \sum_{i=1}^{m_1} \frac{\mathbb{P}(\{\tau_i^{(1)} \leq t'\}) - \mathbb{P}(\{t < \tau_i^{(1)} \leq t'\} \cap (\cap_{j \neq i}^{1, m_1} \{\tau_j^1 > t\}))}{\mathbb{P}(\min_{1 \leq j \leq m_1} \tau_j^{(1)} \leq t)}
\end{aligned}$$

$$\begin{aligned}
&= m_1 \cdot \frac{\mathbb{E}\left[1 - e^{-a^{(1)}\Lambda_{h^{(1)}}(t')} - b^{(1)}h^{(1)}(t')\right]}{\mathbb{E}\left[1 - e^{-m_1 a^{(1)}\Lambda_{h^{(1)}}(t)} - m_1 b^{(1)}h^{(1)}(t)\right]} \\
&- m_1 \cdot \frac{\mathbb{E}\left[\left(e^{-a^{(1)}\Lambda_{h^{(1)}}(t)} - b^{(1)}h^{(1)}(t) - e^{-a^{(1)}\Lambda_{h^{(1)}}(t')} - b^{(1)}h^{(1)}(t')\right) e^{-(m_1-1)a^{(1)}\Lambda_{h^{(1)}}(t)} - (m_1-1)b^{(1)}h^{(1)}(t)\right]}{\mathbb{E}\left[1 - e^{-m_1 a^{(1)}\Lambda_{h^{(1)}}(t)} - m_1 b^{(1)}h^{(1)}(t)\right]} \\
&= m_1 \cdot \frac{1 - e^{-b^{(1)}h^{(1)}(t')} e^{h^{(1)}(t')\Psi(-a^{(1)})}}{1 - e^{-h^{(1)}(t)(m_1 b^{(1)} - \Psi(-m_1 a^{(1)}))}} \\
&- m_1 \cdot \frac{e^{-h^{(1)}(t)(m_1 + m_1 \Psi(-a^{(1)}) - \Psi(-m_1 a^{(1)}))} \left(1 - e^{-(h^{(1)}(t') - h^{(1)}(t))}\right)}{1 - e^{-h^{(1)}(t)(m_1 b^{(1)} - \Psi(-m_1 a^{(1)}))}}
\end{aligned}$$

where we have used the fact that

$$\begin{aligned}
&\mathbb{E}\left[\left(e^{-a^{(1)}\Lambda_{h^{(1)}}(t)} - b^{(1)}h^{(1)}(t) - e^{-a^{(1)}\Lambda_{h^{(1)}}(t')} - b^{(1)}h^{(1)}(t')\right) e^{-(m_1-1)a^{(1)}\Lambda_{h^{(1)}}(t)} - (m_1-1)b^{(1)}h^{(1)}(t)\right] \\
&= \mathbb{E}\left[e^{-a^{(1)}\Lambda_{h^{(1)}}(t)} - b^{(1)}h^{(1)}(t) - e^{-(m_1-1)a^{(1)}\Lambda_{h^{(1)}}(t)} e^{-(m_1-1)b^{(1)}h^{(1)}(t)}\right] \\
&- \mathbb{E}\left[e^{-a^{(1)}(\Lambda_{h^{(1)}}(t') - \Lambda_{h^{(1)}}(t) + \Lambda_{h^{(1)}}(t))} - b^{(1)}(h^{(1)}(t') - h^{(1)}(t) + h^{(1)}(t)) e^{-(m_1-1)a^{(1)}\Lambda_{h^{(1)}}(t)} - (m_1-1)b^{(1)}h^{(1)}(t)\right] \\
&= \mathbb{E}\left[e^{-m_1(b^{(1)}h^{(1)}(t) + a^{(1)}\Lambda_{h^{(1)}}(t))}\right] \\
&- \mathbb{E}\left[e^{-a^{(1)}(\Lambda_{h^{(1)}}(t') - \Lambda_{h^{(1)}}(t))} - b^{(1)}(h^{(1)}(t') - h^{(1)}(t)) e^{-m_1 a^{(1)}\Lambda_{h^{(1)}}(t)} - m_1 b^{(1)}h^{(1)}(t)\right] \\
&= e^{-m_1 b^{(1)}h^{(1)}(t)} e^{h^{(1)}(t)\Psi(-m_1 a^{(1)})} \\
&- e^{-b^{(1)}(h^{(1)}(t') - h^{(1)}(t)) - m_1 b^{(1)}h^{(1)}(t)} e^{(h^{(1)}(t') - h^{(1)}(t))\Psi(-a^{(1)}) + h^{(1)}(t)\Psi(-m_1 a^{(1)})} \\
&= e^{-m_1 b^{(1)}h^{(1)}(t)} e^{h^{(1)}(t)\Psi(-m_1 a^{(1)})} \left(1 - e^{-(h^{(1)}(t') - h^{(1)}(t))(b^{(1)} - \Psi(-a^{(1)}))}\right) \\
&= e^{-h^{(1)}(t)(m_1 b^{(1)} - \Psi(-m_1 a^{(1)}))} \left(1 - e^{-(h^{(1)}(t') - h^{(1)}(t))(b^{(1)} - \Psi(-a^{(1)}))}\right) \\
&= e^{-h^{(1)}(t)(m_1 + m_1 \Psi(-a^{(1)}) - \Psi(-m_1 a^{(1)}))} \left(1 - e^{-(h^{(1)}(t') - h^{(1)}(t))}\right).
\end{aligned}$$

In the third case we have

$$\begin{aligned}
\mathbb{E}[X_{t'}^{(1)} | X_t^{(2)} \geq 1] &= \sum_{i=1}^{m_1} \mathbb{P}\left(\tau_i^{(1)} \leq t' \mid \min_{1 \leq j \leq m_2} \tau_j^{(2)} \leq t\right) \\
&= \sum_{i=1}^{m_1} \frac{\mathbb{P}\left(\{\tau_i^{(1)} \leq t'\} \cap \{\min_{1 \leq j \leq m_2} \tau_j^{(2)} \leq t\}\right)}{\mathbb{P}\left(\min_{1 \leq j \leq m_2} \tau_j^{(2)} \leq t\right)}. \tag{B.3}
\end{aligned}$$

The denominator of equation (B.3) can be written as

$$\begin{aligned}
\mathbb{P}\left(\min_{1 \leq j \leq m_2} \tau_j^{(2)} \leq t\right) &= 1 - \mathbb{P}\left(\min_{1 \leq j \leq m_2} \tau_j^{(2)} > t\right) = 1 - \mathbb{P}\left(\bigcap_{j=1}^{m_2} \{\tau_j^{(2)} > t\}\right) \\
&= 1 - \mathbb{E}\left[\mathbb{P}\left(\bigcap_{j=1}^{m_2} \{\tau_j^{(2)} > t\} \mid \infty_\infty^\Lambda\right)\right] = 1 - \mathbb{E}\left[\mathbb{P}\left(\tau_j^{(2)} > t \mid \infty_\infty^\Lambda\right)^{m_2}\right] \\
&= 1 - \mathbb{E}\left[\left(e^{-a^{(2)}\Lambda_{h^{(2)}(t)} - b^{(2)}h^{(2)}(t)}\right)^{m_2}\right] \\
&= 1 - \mathbb{E}\left[e^{-m_2 a^{(2)}\Lambda_{h^{(2)}(t)} - m_2 b^{(2)}h^{(2)}(t)}\right].
\end{aligned} \tag{B.4}$$

In the numerator we have instead

$$\begin{aligned}
\mathbb{P}\left(\{\tau_i^{(1)} \leq t'\} \cap \{\min_{1 \leq j \leq m_2} \tau_j^{(2)} \leq t\}\right) &= \mathbb{P}(\tau_i^{(1)} \leq t') - \mathbb{P}\left(\{\tau_i^{(1)} \leq t'\} \cap \bigcap_{j=1}^{m_2} \{\tau_j^{(2)} > t\}\right) \\
&= 1 - \mathbb{P}(\tau_i^{(1)} > t') - \mathbb{P}\left(\{\tau_i^{(1)} \leq t'\} \cap \bigcap_{j=1}^{m_2} \{\tau_j^{(2)} > t\}\right) \\
&= 1 - \mathbb{P}(\tau_i^{(1)} > t') - \mathbb{E}\left[\mathbb{P}\left(\{\tau_i^{(1)} \leq t'\} \cap \bigcap_{j=1}^{m_2} \{\tau_j^{(2)} > t\} \mid \infty_\infty^\Lambda\right)\right] \\
&= 1 - \mathbb{E}\left[e^{-a^{(1)}\Lambda_{h^{(1)}(t')} - b^{(1)}h^{(1)}(t')}\right] - \mathbb{E}\left[e^{-m_2 a^{(2)}\Lambda_{h^{(2)}(t)} - m_2 b^{(2)}h^{(2)}(t)}\right] \\
&\quad + \mathbb{E}\left[e^{-a^{(1)}\Lambda_{h^{(1)}(t')} - b^{(1)}h^{(1)}(t')} e^{-m_2 a^{(2)}\Lambda_{h^{(2)}(t)} - m_2 b^{(2)}h^{(2)}(t)}\right].
\end{aligned} \tag{B.5}$$

We can thus rewrite equation (B.3) as

$$\begin{aligned}
&\mathbb{E}[X_{t'}^{(1)} \mid X_t^{(2)} \geq 1] \\
&= m_1 \cdot \frac{1 - \mathbb{E}\left[e^{-a^{(1)}\Lambda_{h^{(1)}(t')} - b^{(1)}h^{(1)}(t')}\right] - \mathbb{E}\left[e^{-m_2 a^{(2)}\Lambda_{h^{(2)}(t)} - m_2 b^{(2)}h^{(2)}(t)}\right]}{\mathbb{E}\left[1 - e^{-m_2 a^{(2)}\Lambda_{h^{(2)}(t)} - m_2 b^{(2)}h^{(2)}(t)}\right]} \\
&+ m_1 \cdot \frac{\mathbb{E}\left[e^{-a^{(1)}\Lambda_{h^{(1)}(t')} - b^{(1)}h^{(1)}(t')} e^{-m_2 a^{(2)}\Lambda_{h^{(2)}(t)} - m_2 b^{(2)}h^{(2)}(t)}\right]}{\mathbb{E}\left[1 - e^{-m_2 a^{(2)}\Lambda_{h^{(2)}(t)} - m_2 b^{(2)}h^{(2)}(t)}\right]} \cdot \mathbf{1}_{\{h^{(1)}(t') \geq h^{(2)}(t)\}} \\
&+ m_1 \cdot \frac{\mathbb{E}\left[e^{-a^{(1)}\Lambda_{h^{(1)}(t')} - b^{(1)}h^{(1)}(t')} e^{-m_2 a^{(2)}\Lambda_{h^{(2)}(t)} - m_2 b^{(2)}h^{(2)}(t)}\right]}{\mathbb{E}\left[1 - e^{-m_2 a^{(2)}\Lambda_{h^{(2)}(t)} - m_2 b^{(2)}h^{(2)}(t)}\right]} \cdot \mathbf{1}_{\{h^{(1)}(t') < h^{(2)}(t)\}}
\end{aligned}$$

$$\begin{aligned}
&= m_1 \cdot \frac{1 - e^{-h^{(1)}(t')}(b^{(1)} - \Psi(-a^{(1)})) - e^{-h^{(2)}(t)}(m_2 b^{(2)} - \Psi(-m_2 a^{(2)}))}{1 - e^{-h^{(2)}(t)}(m_2 b^{(2)} - \Psi(-m_2 a^{(2)}))} \\
&+ m_1 \cdot \frac{e^{(h^{(1)}(t') - h^{(2)}(t))\Psi(-a^{(1)})} e^{h^{(2)}(t)\Psi(-a^{(1)} - m_2 a^{(2)})} e^{-h^{(1)}(t')b^{(1)} - m_2 b^{(2)}h^{(2)}(t)}}{1 - e^{-h^{(2)}(t)}(m_2 b^{(2)} - \Psi(-m_2 a^{(2)}))} \cdot \mathbf{1}_{\{h^{(1)}(t') \geq h^{(2)}(t)\}} \\
&+ m_1 \cdot \frac{e^{h^{(1)}(t')\Psi(-a^{(1)} - m_2 a^{(2)})} e^{(h^{(2)}(t) - h^{(1)}(t'))\Psi(-m_2 a^{(2)})} e^{-h^{(1)}(t')b^{(1)} - m_2 b^{(2)}h^{(2)}(t)}}{1 - e^{-h^{(2)}(t)}(m_2 b^{(2)} - \Psi(-m_2 a^{(2)}))} \cdot \mathbf{1}_{\{h^{(1)}(t') < h^{(2)}(t)\}} \\
&= m_1 \cdot \frac{1 - e^{-h^{(1)}(t')}(b^{(1)} - \Psi(-a^{(1)})) - e^{-h^{(2)}(t)}(m_2 b^{(2)} - \Psi(-m_2 a^{(2)}))}{1 - e^{-h^{(2)}(t)}(m_2 b^{(2)} - \Psi(-m_2 a^{(2)}))} \\
&+ m_1 \cdot \frac{e^{-h^{(1)}(t')}(b^{(1)} - \Psi(-a^{(1)})) e^{-h^{(2)}(t)}(\Psi(-a^{(1)}) + m_2 b^{(2)} - \Psi(-a^{(1)} - m_2 a^{(2)}))}{1 - e^{-h^{(2)}(t)}(m_2 b^{(2)} - \Psi(-m_2 a^{(2)}))} \cdot \mathbf{1}_{\{h^{(1)}(t') \geq h^{(2)}(t)\}} \\
&+ m_1 \cdot \frac{e^{-h^{(1)}(t')}(b^{(1)} + \Psi(-m_2 a^{(2)}) - \Psi(-a^{(1)} - m_2 a^{(2)})) e^{-h^{(2)}(t)}(m_2 b^{(2)} - \Psi(-m_2 a^{(2)}))}{1 - e^{-h^{(2)}(t)}(m_2 b^{(2)} - \Psi(-m_2 a^{(2)}))} \cdot \mathbf{1}_{\{h^{(1)}(t') < h^{(2)}(t)\}}.
\end{aligned}$$

Finally in the fourth case, by interchanging the role of the firm in class \mathcal{M}_1 and \mathcal{M}_2 , and taking into account that $h^{(1)}(t) \leq h^{(2)}(t) \leq h^{(2)}(t')$ for $t \leq t'$, we have

$$\begin{aligned}
&\mathbb{E}[X_{t'}^{(2)} | X_t^{(1)} \geq 1] \\
&= m_2 \cdot \frac{1 - \mathbb{E}[e^{-a^{(2)}\Lambda_{h^{(2)}(t')} - b^{(2)}h^{(2)}(t')}] - \mathbb{E}[(1 - e^{-a^{(2)}\Lambda_{h^{(2)}(t')} - b^{(2)}h^{(2)}(t')})e^{-m_1 a^{(1)}\Lambda_{h^{(1)}(t)} - m_1 b^{(1)}h^{(1)}(t)}]}{\mathbb{E}\left[1 - e^{-m_1 a^{(1)}\Lambda_{h^{(1)}(t)} - m_1 b^{(1)}h^{(1)}(t)}\right]} \\
&= m_2 \cdot \frac{1 - e^{-h^{(2)}(t')}(b^{(2)} - \Psi(-a^{(2)})) - e^{-h^{(1)}(t)}(m_1 b^{(1)} - \Psi(-m_1 a^{(1)}))}{1 - e^{-h^{(1)}(t)}(m_1 b^{(1)} - \Psi(-m_1 a^{(1)}))} \\
&+ m_2 \cdot \frac{e^{-h^{(2)}(t')}(b^{(2)} - \Psi(-a^{(2)})) e^{-h^{(1)}(t)}(\Psi(-a^{(2)}) + m_1 b^{(1)} - \Psi(-a^{(2)} - m_1 a^{(1)}))}{1 - e^{-h^{(1)}(t)}(m_1 b^{(1)} - \Psi(-m_1 a^{(1)}))}.
\end{aligned} \tag{B.6}$$

□

Proof of Proposition 6.2

In the first case we have (we prove only the first case, $\ell = 1$, since the proof for $\ell = 2$ is identical)

$$\begin{aligned}
\mathbb{P}(\tau_i^{(1)} \leq t' | \tau_j^{(1)} \leq t) &= \frac{\mathbb{P}(\{\tau_i^{(1)} \leq t'\} \cap \{\tau_j^{(1)} \leq t\})}{\mathbb{P}(\tau_j^{(1)} \leq t)} \\
&= \frac{\mathbb{E}\left[\left(1 - e^{-a^{(1)}\Lambda_{h^{(1)}(t')} - b^{(1)}h^{(1)}(t')}\right)\left(1 - e^{-a^{(1)}\Lambda_{h^{(1)}(t)} - b^{(1)}h^{(1)}(t)}\right)\right]}{\mathbb{E}\left[1 - e^{-a^{(1)}\Lambda_{h^{(1)}(t)} - b^{(1)}h^{(1)}(t)}\right]} \\
&= \frac{\mathbb{E}\left[1 - e^{-a^{(1)}\Lambda_{h^{(1)}(t')} - b^{(1)}h^{(1)}(t')} - e^{-a^{(1)}\Lambda_{h^{(1)}(t)} - b^{(1)}h^{(1)}(t)}\right]}{\mathbb{E}\left[1 - e^{-a^{(1)}\Lambda_{h^{(1)}(t)} - b^{(1)}h^{(1)}(t)}\right]} \\
&+ \frac{\mathbb{E}\left[e^{-a^{(1)}\Lambda_{h^{(1)}(t')} - b^{(1)}h^{(1)}(t')} e^{-a^{(1)}\Lambda_{h^{(1)}(t)} - b^{(1)}h^{(1)}(t)}\right]}{\mathbb{E}\left[1 - e^{-a^{(1)}\Lambda_{h^{(1)}(t)} - b^{(1)}h^{(1)}(t)}\right]} \\
&= \frac{1 - e^{-h^{(1)}(t')}\left(b^{(1)} - \Psi(-a^{(1)})\right) - e^{-h^{(1)}(t)}\left(b^{(1)} - \Psi(-a^{(1)})\right)}{1 - e^{-h^{(1)}(t)}\left(b^{(1)} - \Psi(-a^{(1)})\right)} \\
&+ \frac{e^{-h^{(1)}(t)}\left(2b^{(1)} - \Psi(-2a^{(1)})\right) e^{-(h^{(1)}(t') - h^{(1)}(t))\left(b^{(1)} - \Psi(-a^{(1)})\right)}}{1 - e^{-h^{(1)}(t)}\left(b^{(1)} - \Psi(-a^{(1)})\right)}.
\end{aligned} \tag{B.7}$$

where we have used the fact that

$$\begin{aligned}
&\mathbb{E}\left[e^{-a^{(1)}\Lambda_{h^{(1)}(t')} - b^{(1)}h^{(1)}(t')} e^{-a^{(1)}\Lambda_{h^{(1)}(t)} - b^{(1)}h^{(1)}(t)}\right] \\
&= \mathbb{E}\left[e^{-a^{(1)}(\Lambda_{h^{(1)}(t')} - \Lambda_{h^{(1)}(t)})} e^{-2a^{(1)}\Lambda_{h^{(1)}(t)} - b^{(1)}(h^{(1)}(t') - h^{(1)}(t))} e^{-2b^{(1)}h^{(1)}(t)}\right].
\end{aligned}$$

In the third case we have

$$\mathbb{P}(\tau_i^{(1)} \leq t' | \tau_j^{(2)} \leq t) = \frac{\mathbb{P}(\{\tau_i^{(1)} \leq t'\} \cap \{\tau_j^{(2)} \leq t\})}{\mathbb{P}(\tau_j^{(2)} \leq t)}. \tag{B.8}$$

The denominator of equation (B.8) can be written as

$$\begin{aligned}
\mathbb{P}(\tau_j^{(2)} \leq t) &= 1 - \mathbb{P}(\tau_j^{(2)} > t) = 1 - \mathbb{E}\left[\mathbb{P}(\tau_j^{(2)} > t | \mathcal{F}_\infty^\Lambda)\right] \\
&= 1 - \mathbb{E}\left[e^{-a^{(2)}\Lambda_{h^{(2)}(t)} - b^{(2)}h^{(2)}(t)}\right] \\
&= 1 - e^{-h^{(2)}(t)(b^{(2)} - \Psi(-a^{(2)}))}.
\end{aligned} \tag{B.9}$$

In the numerator we have instead

$$\begin{aligned}
\mathbb{P}\left(\{\tau_i^{(1)} \leq t'\} \cap \{\tau_j^{(2)} \leq t\}\right) &= \mathbb{E}\left[\mathbb{P}\left(\{\tau_i^{(1)} \leq t'\} \cap \{\tau_j^{(2)} \leq t\} \mid \mathcal{F}_\infty^\Lambda\right)\right] \\
&= \mathbb{E}\left[\left(1 - e^{-a^{(1)}\Lambda_{h^{(1)}(t')} - b^{(1)}h^{(1)}(t')}\right)\left(1 - e^{-a^{(2)}\Lambda_{h^{(2)}(t)} - b^{(2)}h^{(2)}(t)}\right)\right] \\
&= \mathbb{E}\left[1 - e^{-a^{(1)}\Lambda_{h^{(1)}(t')} - b^{(1)}h^{(1)}(t')} - e^{-a^{(2)}\Lambda_{h^{(2)}(t)} - b^{(2)}h^{(2)}(t)}\right] \\
&+ \mathbb{E}\left[\left(e^{-a^{(1)}\Lambda_{h^{(1)}(t')} - b^{(1)}h^{(1)}(t')}\right)\left(e^{-a^{(2)}\Lambda_{h^{(2)}(t)} - b^{(2)}h^{(2)}(t)}\right)\right] \\
&= 1 - \mathbb{E}\left[e^{-a^{(1)}\Lambda_{h^{(1)}(t')} - b^{(1)}h^{(1)}(t')}\right] - \mathbb{E}\left[e^{-a^{(2)}\Lambda_{h^{(2)}(t)} - b^{(2)}h^{(2)}(t)}\right] \\
&+ \mathbb{E}\left[e^{-a^{(1)}\Lambda_{h^{(1)}(t')} - b^{(1)}h^{(1)}(t')} e^{-a^{(2)}\Lambda_{h^{(2)}(t)} - b^{(2)}h^{(2)}(t)}\right] \cdot \mathbf{1}_{\{h^{(1)}(t') \geq h^{(2)}(t)\}} \\
&+ \mathbb{E}\left[e^{-a^{(1)}\Lambda_{h^{(1)}(t')} - b^{(1)}h^{(1)}(t')} e^{-a^{(2)}\Lambda_{h^{(2)}(t)} - b^{(2)}h^{(2)}(t)}\right] \cdot \mathbf{1}_{\{h^{(1)}(t') < h^{(2)}(t)\}} \\
&= 1 - e^{-h^{(1)}(t')\left(b^{(1)} - \Psi(-a^{(1)})\right)} - e^{-h^{(2)}(t)\left(b^{(2)} - \Psi(-a^{(2)})\right)} \\
&+ e^{h^{(1)}(t') - h^{(2)}(t)} e^{\Psi(-a^{(1)})} e^{h^{(2)}(t) \Psi(-a^{(1)} - a^{(2)})} e^{-b^{(1)}h^{(1)}(t') - b^{(2)}h^{(2)}(t)} \cdot \mathbf{1}_{\{h^{(1)}(t') \geq h^{(2)}(t)\}} \\
&+ e^{h^{(1)}(t') \Psi(-a^{(1)} - a^{(2)})} e^{h^{(2)}(t) - h^{(1)}(t')} e^{\Psi(-a^{(2)})} e^{-b^{(1)}h^{(1)}(t') - b^{(2)}h^{(2)}(t)} \cdot \mathbf{1}_{\{h^{(1)}(t') < h^{(2)}(t)\}} \\
&= 1 - e^{-h^{(1)}(t')\left(b^{(1)} - \Psi(-a^{(1)})\right)} - e^{-h^{(2)}(t)\left(b^{(2)} - \Psi(-a^{(2)})\right)} \\
&+ e^{-h^{(1)}(t')\left(b^{(1)} - \Psi(-a^{(1)})\right)} e^{-h^{(2)}(t)\left(\Psi(-a^{(1)}) + b^{(2)} - \Psi(-a^{(1)} - a^{(2)})\right)} \cdot \mathbf{1}_{\{h^{(1)}(t') \geq h^{(2)}(t)\}} \\
&+ e^{-h^{(1)}(t')\left(b^{(1)} + \Psi(-a^{(2)}) - \Psi(-a^{(1)} - a^{(2)})\right)} e^{-h^{(2)}(t)\left(b^{(2)} - \Psi(-a^{(2)})\right)} \cdot \mathbf{1}_{\{h^{(1)}(t') < h^{(2)}(t)\}}.
\end{aligned} \tag{B.10}$$

We can thus rewrite equation (B.8) as

$$\begin{aligned}
\mathbb{P}(\tau_i^{(1)} \leq t' \mid \tau_j^{(2)} \leq t) &= \frac{1 - e^{-h^{(1)}(t')\left(b^{(1)} - \Psi(-a^{(1)})\right)} - e^{-h^{(2)}(t)\left(b^{(2)} - \Psi(-a^{(2)})\right)}}{1 - e^{-h^{(2)}(t)\left(b^{(2)} - \Psi(-a^{(2)})\right)}} \\
&+ \frac{e^{-h^{(1)}(t')\left(b^{(1)} - \Psi(-a^{(1)})\right)} e^{-h^{(2)}(t)\left(\Psi(-a^{(1)}) + b^{(2)} - \Psi(-a^{(1)} - a^{(2)})\right)}}{1 - e^{-h^{(2)}(t)\left(b^{(2)} - \Psi(-a^{(2)})\right)}} \cdot \mathbf{1}_{\{h^{(1)}(t') \geq h^{(2)}(t)\}} \\
&+ \frac{e^{-h^{(1)}(t')\left(b^{(1)} + \Psi(-a^{(2)}) - \Psi(-a^{(1)} - a^{(2)})\right)} e^{-h^{(2)}(t)\left(b^{(2)} - \Psi(-a^{(2)})\right)}}{1 - e^{-h^{(2)}(t)\left(b^{(2)} - \Psi(-a^{(2)})\right)}} \cdot \mathbf{1}_{\{h^{(1)}(t') < h^{(2)}(t)\}}.
\end{aligned} \tag{B.11}$$

Finally in the fourth case, by interchanging the role of the firm in class \mathcal{M}_1 and \mathcal{M}_2 and taking into

account that $h^{(1)}(t) \leq h^{(2)}(t) \leq h^{(2)}(t')$ for $t \leq t'$, we have

$$\begin{aligned}
& \mathbb{P}(\tau_i^{(2)} \leq t' | \tau_i^{(1)} \leq t) \\
&= \frac{1 - \mathbb{E}[e^{-a^{(2)}\Lambda_{h^{(2)}}(t')} - b^{(2)}h^{(2)}(t')] - \mathbb{E}[(1 - e^{-a^{(2)}\Lambda_{h^{(2)}}(t')} - b^{(2)}h^{(2)}(t'))e^{-a^{(1)}\Lambda_{h^{(1)}}(t)} - b^{(1)}h^{(1)}(t)]]}{\mathbb{E}\left[1 - e^{-a^{(1)}\Lambda_{h^{(1)}}(t)} - b^{(1)}h^{(1)}(t)\right]} \\
&= \frac{1 - e^{-h^{(2)}(t')(b^{(2)} - \Psi(-a^{(2)}))} - e^{-h^{(1)}(t)(b^{(1)} - \Psi(-a^{(1)}))}}{1 - e^{-h^{(1)}(t)(b^{(1)} - \Psi(-a^{(1)}))}} + \frac{e^{-h^{(2)}(t')(b^{(2)} - \Psi(-a^{(2)}))} e^{-h^{(1)}(t)(\Psi(-a^{(2)}) + b^{(1)} - \Psi(-a^{(2)} - a^{(1)}))}}{1 - e^{-h^{(1)}(t)(b^{(1)} - \Psi(-a^{(1)}))}}.
\end{aligned} \tag{B.12}$$

□

C Appendix: Numerical computations

Truncation error

To compute the expectation in (19) we compute

$$\mathbb{E}[F(t, \tilde{\Lambda}(t), \Delta\tilde{\Lambda}(t))] = \int_0^{M_x(h^{(1)}(t))} \int_0^{M_y(h^{(2)}(t) - h^{(1)}(t))} F(t, x, y) \cdot f_{\tilde{\Lambda}}(x) \cdot f_{\Delta\tilde{\Lambda}}(y) dx dy, \tag{C.1}$$

where $F(t, x, y)$ is the function defined in (18), by properly choosing the upper integration extremes so that the truncation error is less than 2×10^{-4} . To do so we consider that in general, for a non-negative function $F(x, y)$, we can write

$$\begin{aligned}
\mathbb{E}[F(X, Y)] &= \int_0^\infty dx \int_0^\infty dy F(x, y) f_X(x) f_Y(y) \\
&= \int_0^{M_X} dx \int_0^{M_Y} dy F(x, y) f_X(x) f_Y(y) \\
&+ \int_{M_X}^\infty dx \int_0^\infty dy F(x, y) f_X(x) f_Y(y) \\
&+ \int_0^\infty dx \int_{M_Y}^\infty dy F(x, y) f_X(x) f_Y(y) \\
&- \int_{M_X}^\infty dx \int_{M_Y}^\infty dy F(x, y) f_X(x) f_Y(y).
\end{aligned} \tag{C.2}$$

So we have

$$\begin{aligned}
0 \leq \mathbb{E}[F(X, Y)] - \int_0^{M_X} dx \int_0^{M_Y} dy F(x, y) f_X(x) f_Y(y) &\leq \int_{M_X}^\infty dx \int_0^\infty dy F(x, y) f_X(x) f_Y(y) \\
&+ \int_0^\infty dx \int_{M_Y}^\infty dy F(x, y) f_X(x) f_Y(y).
\end{aligned} \tag{C.3}$$

In our case, being $0 \leq F(t, x, y) \leq 1$, we get the following inequality:

$$0 \leq \mathbb{E}[F(t, \tilde{\Lambda}(t), \Delta\tilde{\Lambda}(t))] - \int_0^{M_X} dx \int_0^{M_Y} dy F(t, x, y) \cdot f_{\tilde{\Lambda}}(x) \cdot f_{\Delta\tilde{\Lambda}}(y) \leq \int_{M_X}^\infty f_{\tilde{\Lambda}}(x) dx + \int_{M_Y}^\infty f_{\Delta\tilde{\Lambda}}(y) dy. \tag{C.4}$$

In the case of the inverse Gaussian subordinator, being

$$X \sim IG(\beta h^{(1)}(t), \eta)$$

and

$$Y \sim IG(\beta(h^{(2)}(t) - h^{(1)}(t)), \eta),$$

we have

$$\begin{aligned} \int_{M_X}^{\infty} f_{\tilde{\Lambda}}(x) dx &= \int_{M_X}^{\infty} \frac{\beta(h^{(1)}(t))}{\sqrt{2\pi}} x^{-\frac{3}{2}} e^{\eta\beta(h^{(1)}(t))} e^{-\frac{1}{2}\left(\frac{\beta^2(h^{(1)}(t))^2}{x} + \eta^2 x\right)} \mathbf{1}_{\{x>0\}} dx \\ &\leq \frac{\beta(h^{(1)}(t))e^{\eta\beta(h^{(1)}(t))}}{\sqrt{2\pi}} \int_{M_X}^{\infty} \frac{1}{x^{3/2}} e^{-\frac{1}{2}\eta^2 M_X} \\ &= \frac{\beta(h^{(1)}(t))e^{\eta\beta(h^{(1)}(t))}}{\sqrt{2\pi}} \cdot \frac{2}{M_X^{1/2}} e^{-\frac{1}{2}\eta^2 M_X} \end{aligned} \quad (\text{C.5})$$

and

$$\begin{aligned} \int_{M_Y}^{\infty} f_{\Delta\tilde{\Lambda}}(y) dy &= \int_{M_Y}^{\infty} \frac{\beta(h^{(2)}(t) - h^{(1)}(t))}{\sqrt{2\pi}} y^{-\frac{3}{2}} e^{\eta\beta(h^{(2)}(t) - h^{(1)}(t))} e^{-\frac{1}{2}\left(\frac{\beta^2(h^{(2)}(t) - h^{(1)}(t))^2}{y} + \eta^2 y\right)} \mathbf{1}_{\{y>0\}} dy \\ &\leq \frac{\beta(h^{(2)}(t) - h^{(1)}(t))e^{\eta\beta(h^{(2)}(t) - h^{(1)}(t))}}{\sqrt{2\pi}} \int_{M_Y}^{\infty} \frac{1}{y^{3/2}} e^{-\frac{1}{2}\eta^2 M_Y} \\ &= \frac{\beta(h^{(2)}(t) - h^{(1)}(t))e^{\eta\beta(h^{(2)}(t) - h^{(1)}(t))}}{\sqrt{2\pi}} \cdot \frac{2}{M_Y^{1/2}} e^{-\frac{1}{2}\eta^2 M_Y} \\ &= \frac{\beta(h^{(2)}(t) - h^{(1)}(t))e^{\eta\beta(h^{(2)}(t) - h^{(1)}(t))}}{\sqrt{2\pi}} \cdot \frac{2}{M_Y^{1/2}} e^{-\frac{1}{2}\eta^2 M_Y} \end{aligned} \quad (\text{C.6})$$

We can easily verify that, by choosing

$$M_X = \inf_m \left\{ \frac{e^{-\frac{1}{2}\eta^2 m}}{m^{\frac{1}{2}}} \leq 10^{-4} \frac{\sqrt{2\pi}}{2\beta h^{(1)}(t) e^{\beta\eta h^{(1)}(t)}} \right\}$$

and

$$M_Y = \inf_m \left\{ \frac{e^{-\frac{1}{2}\eta^2 m}}{m^{\frac{1}{2}}} \leq 10^{-4} \frac{\sqrt{2\pi}}{2\beta(h^{(2)}(t) - h^{(1)}(t)) e^{\beta\eta(h^{(2)}(t) - h^{(1)}(t))}} \right\},$$

we have

$$\int_{M_X}^{\infty} f_X(x) \leq 10^{-4}$$

and

$$\int_{M_Y}^{\infty} f_Y(y) dy \leq 10^{-4}$$

so that

$$\mathbb{E}[F(t, \tilde{\Lambda}(t), \Delta\tilde{\Lambda}(t))] - \int_0^{M_X} dx \int_0^{M_Y} dy F(t, x, y) \cdot f_{\tilde{\Lambda}}(x) \cdot f_{\Delta\tilde{\Lambda}}(y) < 2 \times 10^{-4}.$$

Integration on a bounded domain

The aim of this paragraph is to rewrite (13) as an integral on a bounded domain. First of all let us consider that the function denoted in (13) can also be written as

$$\begin{aligned}\mathbb{E}[L^{(j)}(t)] &= \mathbb{E}[\min(\max(0, (1-R)L^n(t) - l^j), u^j - l^j)] \\ &= \mathbb{E}\left[\left((1-R)L^n(t) - l^j\right)\mathbf{1}_{\left\{\frac{l^j}{1-R} < L^n(t) \leq \frac{u^j}{1-R}\right\}} + (u^j - l^j)\mathbf{1}_{\left\{\frac{u^j}{1-R} < L^n(t)\right\}}\right],\end{aligned}\tag{C.7}$$

as we have

- $L^n(t) > \frac{u^j}{1-R} \iff (1-R)L^n(t) > u^j > l^j$

so that, in this case,

$$\max(0, (1-R)L^n(t) - l^j) = (1-R)L^n(t) - l^j$$

and

$$\min((1-R)L^n(t) - l^j, (u^j - l^j)) = u^j - l^j;$$

- $\frac{l^j}{1-R} < L^n(t) \leq \frac{u^j}{1-R} \iff l^j < (1-R)L^n(t) \leq u^j$

so that, in this case,

$$\max(0, (1-R)L^n(t) - l^j) = (1-R)L^n(t) - l^j$$

and

$$\min((1-R)L^n(t) - l^j, (u^j - l^j)) = (1-R)L^n(t) - l^j;$$

- $L^n(t) \leq \frac{l^j}{1-R} \iff (1-R)L^n(t) \leq l^j$

so that, in this case,

$$\max(0, (1-R)L^n(t) - l^j) = 0$$

and

$$\min(0, u^j - l^j) = 0.$$

To perform numerical computation we use a change of variables so that the double integral in equation (19) is on a compact support. Let us denote

$$X = X(x) := e^{-a^{(1)}x - b^{(1)}h^{(1)}(t)}$$

and

$$Y = Y(x, y) := e^{-a^{(2)}y - a^{(2)}x - b^{(2)}h^{(2)}(t)}.$$

We can easily verify that

$$\lim_{x \rightarrow \infty} X(x) = 0, \quad \lim_{x \rightarrow 0} X(x) = e^{-b^{(1)}h^{(1)}(t)},$$

and, for all x ,

$$\lim_{y \rightarrow \infty} Y(x, y) = 0, \quad \lim_{y \rightarrow 0} Y(x, y) = e^{-b^{(2)}h^{(2)}(t) - a^{(2)}x},$$

so that for all x and y

$$0 < X(x) < e^{-b^{(1)}h^{(1)}(t)} < 1 \quad (\text{C.8})$$

and

$$0 < Y(x, y) < e^{-b^{(2)}h^{(2)}(t) - a^{(2)}x} < 1. \quad (\text{C.9})$$

This change of variable is basically given by the bivariate function Φ that relates the variables (x, y) and (X, Y) by

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \Phi \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e^{-a^{(1)}x - b^{(1)}h^{(1)}(t)} \\ e^{-a^{(2)}y - a^{(2)}x - b^{(2)}h^{(2)}(t)} \end{bmatrix},$$

and consequently

$$\begin{bmatrix} x \\ y \end{bmatrix} = \Phi^{-1} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} -\frac{1}{a^{(1)}}(\log X + b^{(1)}h^{(1)}(t)) \\ -\frac{1}{a^{(2)}}(\log Y + b^{(2)}h^{(2)}(t)) + \frac{1}{a^{(1)}}(\log X + b^{(1)}h^{(1)}(t)) \end{bmatrix}.$$

Taking into account the above formulas, (C.8) and (C.9), the integration domain can be expressed as

$$0 < X < e^{-b^{(1)}h^{(1)}(t)} < 1 \quad (\text{C.10})$$

and

$$0 < Y < e^{-b^{(2)}h^{(2)}(t) + \frac{a^{(2)}}{a^{(1)}}(\log X + b^{(1)}h^{(1)}(t))} < 1. \quad (\text{C.11})$$

In order to compute equation (19) in terms of the new variables we need to compute

$$|J_{\Phi^{-1}}(X, Y)| = \left| \begin{bmatrix} -\frac{1}{a^{(1)}X} & 0 \\ \frac{1}{a^{(1)}X} & -\frac{1}{a^{(2)}Y} \end{bmatrix} \right| = \frac{1}{a^{(1)}a^{(2)}XY}.$$

By denoting $f(x, y) := f_{\tilde{\lambda}}(x) \cdot f_{\Delta\tilde{\lambda}}(y)$ in equation (19), we can thus finally rewrite the expected value in (19) in terms of (X, Y) as

$$\int_0^1 \int_0^1 \tilde{F}(t, X, Y) \cdot f(\Phi^{-1}(X, Y)) \cdot |J_{\Phi^{-1}}(X, Y)| dX dY, \quad (\text{C.12})$$

where

$$\tilde{F}(t, X, Y) = \min \left(\max \left(0, (1 - R) \left(1 - \frac{m_1}{n} X - \frac{m_2}{n} Y \right) - l^j \right), u^j - l^j \right)$$

and

$$f(\Phi^{-1}(X, Y)) = f_{\tilde{\lambda}} \left(-\frac{1}{a^{(1)}}(\log X + b^{(1)}h^{(1)}(t)) \right) \cdot f_{\Delta\tilde{\lambda}} \left(-\frac{1}{a^{(2)}}(\log Y + b^{(2)}h^{(2)}(t)) + \frac{1}{a^{(1)}}(\log X + b^{(1)}h^{(1)}(t)) \right).$$

In particular, in case of the inverse Gaussian subordinator, considering (20) and (21), we have

$$\begin{aligned}
f_{\tilde{\Lambda}}\left(-\frac{1}{a^{(1)}}(\log X + b^{(1)}h^{(1)}(t))\right) &= \frac{\beta(h^{(1)}(t))}{\sqrt{2\pi}}\left(-\frac{1}{a^{(1)}}(\log X + b^{(1)}h^{(1)}(t))\right)^{-\frac{3}{2}} \\
&\times e^{\eta\beta(h^{(1)}(t))}e^{\frac{a^{(1)}}{2}\left(\frac{\beta^2(h^{(1)}(t))^2}{\log X + b^{(1)}h^{(1)}(t)}\right)} \\
&\times e^{\frac{\eta^2}{2a^{(1)}}(\log X + b^{(1)}h^{(1)}(t))} \\
&\times \mathbf{1}_{\left\{-\frac{1}{a^{(1)}}(\log X + b^{(1)}h^{(1)}(t)) > 0\right\}}
\end{aligned} \tag{C.13}$$

and

$$\begin{aligned}
f_{\Delta\tilde{\Lambda}}\left(-\frac{1}{a^{(2)}}(\log Y + b^{(2)}h^{(2)}(t)) + \frac{1}{a^{(1)}}(\log X + b^{(1)}h^{(1)}(t))\right) &= \frac{\beta(h^{(2)}(t) - h^{(1)}(t))}{\sqrt{2\pi}} \\
&\times \left(-\frac{1}{a^{(2)}}(\log Y + b^{(2)}h^{(2)}(t)) + \frac{1}{a^{(1)}}(\log X + b^{(1)}h^{(1)}(t))\right)^{-\frac{3}{2}} \\
&\times e^{\eta\beta(h^{(2)}(t) - h^{(1)}(t))} \\
&\times e^{-\frac{\beta^2(h^{(2)}(t) - h^{(1)}(t))^2}{2\left(-\frac{1}{a^{(2)}}(\log Y + b^{(2)}h^{(2)}(t)) + \frac{1}{a^{(1)}}(\log X + b^{(1)}h^{(1)}(t))\right)}} \\
&\times e^{-\frac{\eta^2}{2}\left(-\frac{1}{a^{(2)}}(\log Y + b^{(2)}h^{(2)}(t)) + \frac{1}{a^{(1)}}(\log X + b^{(1)}h^{(1)}(t))\right)} \\
&\times \mathbf{1}_{\left\{-\frac{1}{a^{(2)}}(\log Y + b^{(2)}h^{(2)}(t)) + \frac{1}{a^{(1)}}(\log X + b^{(1)}h^{(1)}(t)) > 0\right\}}.
\end{aligned} \tag{C.14}$$

In terms of the new random variables

$$\begin{aligned}
X(\omega) &:= X(\tilde{\Lambda}(\omega)), \\
Y(\omega) &:= Y(\tilde{\Lambda}(\omega), \Delta\tilde{\Lambda}(\omega)),
\end{aligned}$$

and with some manipulation equation (C.7) can be rewritten as

$$\begin{aligned}
\mathbb{E}[L^{(j)}(t)] &\approx \mathbb{E}\left[\left((1-R)\left(1 - \frac{m_1}{n}X(\omega) - \frac{m_2}{n}Y(\omega)\right) - l_j\right)\mathbf{1}_{\left\{\frac{l_j}{1-R} < 1 - \frac{m_1}{n}X(\omega) - \frac{m_2}{n}Y(\omega) \leq \frac{u^j}{1-R}\right\}}\right. \\
&\quad \left. + (u^j - l^j)\mathbf{1}_{\left\{\frac{u^j}{1-R} < 1 - \frac{m_1}{n}X(\omega) - \frac{m_2}{n}Y(\omega)\right\}}\right] \\
&= (1-R-l^j)\mathbb{E}\left[\mathbf{1}_{\left\{1 - \frac{u^j}{1-R} \leq \frac{m_1}{n}X(\omega) + \frac{m_2}{n}Y(\omega) \leq 1 - \frac{l^j}{1-R}\right\}}\right] \\
&\quad - (1-R)\mathbb{E}\left[\left(\frac{m_1}{n}X(\omega) + \frac{m_2}{n}Y(\omega)\right)\mathbf{1}_{\left\{1 - \frac{u^j}{1-R} \leq \frac{m_1}{n}X(\omega) + \frac{m_2}{n}Y(\omega) \leq 1 - \frac{l^j}{1-R}\right\}}\right] \\
&\quad + (u^j - l^j)\mathbb{E}\left[\mathbf{1}_{\left\{\frac{m_1}{n}X(\omega) + \frac{m_2}{n}Y(\omega) \leq 1 - \frac{u^j}{1-R}\right\}}\right].
\end{aligned} \tag{C.15}$$

Using equation (C.12), (C.10) and (C.11), the latter expectation becomes

$$\begin{aligned}
\mathbb{E}[L^{(j)}(t)] &\approx (1 - R - l^j) \int_0^{e^{-b^{(1)}h^{(1)}(t)}} dX \int_0^{e^{-b^{(2)}h^{(2)}(t) + \frac{a^{(2)}}{a^{(1)}}(\log X + b^{(1)}h^{(1)}(t))}} dY \mathbf{1}_{\left\{1 - \frac{u_j}{1-R} \leq \frac{m_1}{n}X + \frac{m_2}{n}Y \leq 1 - \frac{l^j}{1-R}\right\}} \\
&\quad \times f_{\bar{\Lambda}}\left(-\frac{1}{a^{(1)}}(\log X + b^{(1)}h^{(1)}(t))\right) \\
&\quad \times f_{\Delta\bar{\Lambda}}\left(-\frac{1}{a^{(2)}}(\log Y + b^{(2)}h^{(2)}(t)) + \frac{1}{a^{(1)}}(\log X + b^{(1)}h^{(1)}(t))\right) \\
&\quad \times \frac{1}{a^{(1)}a^{(2)}XY} \\
&- (1 - R) \int_0^{e^{-b^{(1)}h^{(1)}(t)}} dX \int_0^{e^{-b^{(2)}h^{(2)}(t) + \frac{a^{(2)}}{a^{(1)}}(\log X + b^{(1)}h^{(1)}(t))}} dY \left(\frac{m_1}{n}X + \frac{m_2}{n}Y\right) \\
&\quad \times \mathbf{1}_{\left\{1 - \frac{u_j}{1-R} \leq \frac{m_1}{n}X + \frac{m_2}{n}Y \leq 1 - \frac{l^j}{1-R}\right\}} \\
&\quad \times f_{\bar{\Lambda}}\left(-\frac{1}{a^{(1)}}(\log X + b^{(1)}h^{(1)}(t))\right) \\
&\quad \times f_{\Delta\bar{\Lambda}}\left(-\frac{1}{a^{(2)}}(\log Y + b^{(2)}h^{(2)}(t)) + \frac{1}{a^{(1)}}(\log X + b^{(1)}h^{(1)}(t))\right) \\
&\quad \times \frac{1}{a^{(1)}a^{(2)}XY} \\
&+ (u^j - l^j) \int_0^{e^{-b^{(1)}h^{(1)}(t)}} dX \int_0^{e^{-b^{(2)}h^{(2)}(t) + \frac{a^{(2)}}{a^{(1)}}(\log X + b^{(1)}h^{(1)}(t))}} dY \mathbf{1}_{\left\{\left(\frac{m_1}{n}X + \frac{m_2}{n}Y\right) \leq 1 - \frac{u_j}{1-R}\right\}} \\
&\quad \times f_{\bar{\Lambda}}\left(-\frac{1}{a^{(1)}}(\log X + b^{(1)}h^{(1)}(t))\right) \\
&\quad \times f_{\Delta\bar{\Lambda}}\left(-\frac{1}{a^{(2)}}(\log Y + b^{(2)}h^{(2)}(t)) + \frac{1}{a^{(1)}}(\log X + b^{(1)}h^{(1)}(t))\right) \\
&\quad \times \frac{1}{a^{(1)}a^{(2)}XY}.
\end{aligned} \tag{C.16}$$

The previous equation can also be written as

$$\begin{aligned}
\mathbb{E}[L^{(j)}(t)] &\approx (1 - R - l^j) \int_0^1 dX \int_0^1 dY \mathbf{1}_{\left\{1 - \frac{u_j}{1-R} \leq \frac{m_1}{n} X + \frac{m_2}{n} Y \leq 1 - \frac{l^j}{1-R}\right\}} \\
&\quad \times f_{\tilde{\Lambda}} \left(-\frac{1}{a^{(1)}} (\log X + b^{(1)} h^{(1)}(t)) \right) \\
&\quad \times f_{\Delta \tilde{\Lambda}} \left(-\frac{1}{a^{(2)}} (\log Y + b^{(2)} h^{(2)}(t)) + \frac{1}{a^{(1)}} (\log X + b^{(1)} h^{(1)}(t)) \right) \\
&\quad \times \frac{1}{a^{(1)} a^{(2)} XY} \\
&\quad \times \mathbf{1}_{\left\{0 < X < e^{-b^{(1)} h^{(1)}(t)}, 0 < Y < e^{-b^{(2)} h^{(2)}(t) + \frac{a^{(2)}}{a^{(1)}} (\log X + b^{(1)} h^{(1)}(t))}\right\}} \\
&- (1 - R) \int_0^1 dX \int_0^1 dY \left(\frac{m_1}{n} X + \frac{m_2}{n} Y \right) \\
&\quad \times \mathbf{1}_{\left\{1 - \frac{u_j}{1-R} \leq \frac{m_1}{n} X + \frac{m_2}{n} Y \leq 1 - \frac{l^j}{1-R}\right\}} \\
&\quad \times f_{\tilde{\Lambda}} \left(-\frac{1}{a^{(1)}} (\log X + b^{(1)} h^{(1)}(t)) \right) \\
&\quad \times f_{\Delta \tilde{\Lambda}} \left(-\frac{1}{a^{(2)}} (\log Y + b^{(2)} h^{(2)}(t)) + \frac{1}{a^{(1)}} (\log X + b^{(1)} h^{(1)}(t)) \right) \\
&\quad \times \frac{1}{a^{(1)} a^{(2)} XY} \\
&\quad \times \mathbf{1}_{\left\{0 < X < e^{-b^{(1)} h^{(1)}(t)}, 0 < Y < e^{-b^{(2)} h^{(2)}(t) + \frac{a^{(2)}}{a^{(1)}} (\log X + b^{(1)} h^{(1)}(t))}\right\}} \\
&+ (u^j - l^j) \int_0^1 dX \int_0^1 dY \mathbf{1}_{\left\{\left(\frac{m_1}{n} X + \frac{m_2}{n} Y\right) \leq 1 - \frac{u^j}{1-R}\right\}} \\
&\quad \times f_{\tilde{\Lambda}} \left(-\frac{1}{a^{(1)}} (\log X + b^{(1)} h^{(1)}(t)) \right) \\
&\quad \times f_{\Delta \tilde{\Lambda}} \left(-\frac{1}{a^{(2)}} (\log Y + b^{(2)} h^{(2)}(t)) + \frac{1}{a^{(1)}} (\log X + b^{(1)} h^{(1)}(t)) \right) \\
&\quad \times \frac{1}{a^{(1)} a^{(2)} XY} \\
&\quad \times \mathbf{1}_{\left\{0 < X < e^{-b^{(1)} h^{(1)}(t)}, 0 < Y < e^{-b^{(2)} h^{(2)}(t) + \frac{a^{(2)}}{a^{(1)}} (\log X + b^{(1)} h^{(1)}(t))}\right\}}.
\end{aligned} \tag{C.17}$$

D Appendix: Other Lévy subordinators

Two other subordinators considered by Mai and Sherer (2009a)²³ are

- the *gamma subordinator*, which, like the inverse Gaussian subordinator, is in the class of the infinite activity subordinators;
- the *compound Poisson subordinator*.

²³Other subordinators will be object of further investigation.

The gamma subordinator

The *gamma* (Γ) *subordinator* $\Lambda^\Gamma = \{\Lambda_t^\Gamma\}_{t \geq 0}$ is another Lévy process that belongs to the class of infinite activity subordinators. In a gamma subordinator with parameters $\eta, \beta > 0$, Λ_y^Γ follows a gamma $\Gamma(\beta t, \eta)$ -distribution with density

$$f_\Gamma(x) = \frac{\eta^{\beta t}}{\Gamma(\beta t)} x^{\beta t - 1} e^{-\eta x} \mathbf{1}_{\{x > 0\}},$$

where

$$\Gamma(y) = \int_0^\infty t^{y-1} e^{-t} dt.$$

The corresponding Lévy measure is given by

$$\nu_\Gamma(dx) = \beta e^{-\eta x} \frac{1}{x} \mathbf{1}_{\{x > 0\}} dx.$$

The compound Poisson subordinator

The *compound Poisson subordinator* is of the form

$$\Lambda_t = \mu t + \sum_{k=1}^{N_t} J_k,$$

where $\{J_k\}_{k \in \mathbb{N}}$ are i.i.d. random variables with a cumulative distribution function D with support on the positive axis, and $N = \{N_t\}_{t \geq 0}$ is a Poisson process with intensity β which is independent of $\{J_k\}_{k \in \mathbb{N}}$. The Lévy measure corresponding to this Lévy subordinator has the form $\nu_P(dy) = \beta dD(y)$. This subordinator has upward jumps of random magnitude J_k and the expected number of jumps within a unit time interval is β . For instance, one can assume that D is the exponential distribution with parameter $\eta > 0$, so that the subordinator depends only on parameters η and β .

The Laplace exponent and the parameter constraint for these two subordinators are the following.

For the gamma subordinator we have:

$$\begin{aligned} \Psi_{0,\Gamma}(-a^{(i)}) &= \int_0^\infty (e^{-a^{(i)}s} - 1) \nu_\Gamma(ds) \\ &= \beta \int_0^\infty (e^{-a^{(i)}s} - 1) \frac{1}{s} e^{-\eta s} ds \\ &= \beta \ln \left(\frac{\eta}{a^{(i)} + \eta} \right) \end{aligned}$$

and the constraint $\Psi_0(-a^{(i)}) \geq -1$ is translated into the following constraint for (η, β) :

$$\eta > 0, \quad 0 < \beta \leq \frac{1}{\int_0^\infty (1 - e^{-a^{(i)}s}) \frac{1}{s} e^{-\eta s} ds} = \frac{1}{\ln \left[\frac{a^{(i)} + \eta}{\eta} \right]}$$

for each $i = 1, \dots, r$, and so, as $a_{\max} = \max \{a^{(i)}\}_{i=1, \dots, r}$, we want

$$\eta > 0, \quad 0 < \beta \leq \frac{1}{\ln \left[\frac{a_{\max} + \eta}{\eta} \right]}.$$

For the compound Poisson subordinator, we have

$$\begin{aligned}
\Psi_{0,P}(-a^{(i)}) &= \int_0^\infty (e^{-a^{(i)}s} - 1) \nu_P(ds) \\
&= \beta \int_0^\infty (e^{-a^{(i)}y} - 1) dD(y) \\
&= \beta \mathbb{E}[e^{-a^{(i)}J_1} - 1] \\
&= -\frac{a^{(i)}\beta}{a^{(i)} + \eta}.
\end{aligned}$$

and the constraint for (η, β) becomes

$$\eta > 0, \quad 0 < \beta \leq \frac{1}{1 - \mathbb{E}[e^{-a_{\max}J_1}]} = \frac{a_{\max} + \eta}{a_{\max}}.$$

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2009

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2010

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2011

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2012

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- I. BUONO and G. LALANNE, *The effect of the Uruguay Round on the intensive and extensive margins of trade*, Journal of International Economics, **TD No. 835 (February 2011)**.
- G. BARONE, R. FELICI and M. PAGNINI, *Switching costs in local credit markets*, International Journal of Industrial Organization, **TD No. 760 (June 2010)**.
- E. COCOZZA and P. PISELLI, *Testing for east-west contagion in the European banking sector during the financial crisis*, in R. Matoušek; D. Stavárek (eds.), *Financial Integration in the European Union*, Taylor & Francis, **TD No. 790 (February 2011)**.
- A. DE SOCIO, *Squeezing liquidity in a “lemons market” or asking liquidity “on tap”*, Journal of Banking and Finance, **TD No. 819 (September 2011)**.
- M. AFFINITO, *Do interbank customer relationships exist? And how did they function in the crisis? Learning from Italy*, Journal of Banking and Finance, **TD No. 826 (October 2011)**.
- O. BLANCHARD and M. RIGGI, *Why are the 2000s so different from the 1970s? A structural interpretation of changes in the macroeconomic effects of oil prices*, Journal of the European Economic Association, **TD No. 835 (November 2011)**.
- S. FEDERICO, *Industry dynamics and competition from low-wage countries: evidence on Italy*, Oxford Bulletin of Economics and Statistics, **TD No. 877 (settembre 2012)**.