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Testing for Stochastic Trends in Series with Structural Breaks

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TESTING FOR STOCHASTIC TRENDS IN SERIES WITH STRUCTURAL BREAKS

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Abstract

This paper considers the problem of testing for the presence of stochastic trends in multivariate time series with structural breaks. The breakpoints are assumed to be known. The testing framework is the multivariate Locally Best Invariant test and the common trend test of Nyblom and Harvey (2000). The asymptotic distributions of the test statistics are derived under a general specification of the deterministic component, which allows for structural breaks as a particular case. Asymptotic critical values are provided for the case of a single breakpoint. A modified statistic is then proposed, the asymptotic distribution of which is independent of the breakpoint location and belongs to the Cramér-von Mises family. This modification is particularly advantageous in the case of multiple breakpoints. It is also shown that the asymptotic distributions of the test statistics are unchanged when seasonal dummy variables and/or weakly dependent exogenous regressors are included. Finally, as an example, the tests are applied to UK macroeconomic data and to data on road casualties in Great Britain.

JEL classification: C12, C32.

Keywords: cointegration, common trends, Cramér-von Mises distribution, locally best invariant test, structural breaks.

Contents

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1. Introduction[1]

A common empirical issue is that of trying to establish whether a time series is trend stationary or integrated of order one. This is routinely done in the Dickey-Fuller (or autoregressive) framework, where under the null hypothesis the series is integrated and under the alternative hypothesis it is stationary. Perron (1989) shows that in the presence of structural breaks the Dickey-Fuller test is biased towards not rejecting the null hypothesis and he therefore provides appropriate critical values for this situation; additional results are obtained by Banerjee et al. (1992), Perron and Vogelsang (1992a,b) and Zivot and Andrews (1992).

An alternative way of testing trend stationarity is the KPSS test of Kwiatowski et al. (1992), which reverses the role of the null and alternative hypotheses of the Dickey-Fuller procedure. It extends the locally best invariant test for the presence of a random walk component proposed by Nyblom and Makelainen (1983) and Nyblom (1986). Paralleling Perron's argument, Busetti and Harvey (2000) modified the KPSS test to cover the case of series with structural breaks, since the KPSS test is also consistent against the trend-break hypothesis; see also Lee et al. (1997).

The multivariate generalizations of the Dickey-Fuller and the KPSS tests are, in some sense, the tests proposed by Johansen (1988, 1991) and by Nyblom and Harvey (2000) respectively. While structural breaks in Johansen's procedure have been considered by Inoue (1999), this paper deals with breaks in the Nyblom-Harvey framework. The existence and location of the breakpoints are assumed to be known.

The framework is an unobserved component model. The presence of a random walk component (stochastic trend) in an otherwise covariance stationary time series makes it integrated of order one. In the multivariate case the vector time series will be cointegrated if the covariance matrix of the disturbance term driving the multivariate random walk is not of full rank.

The simplest case is the univariate random walk plus noise model. Under Gaussianity, Nyblom and Makelainen (1983) obtain the locally best invariant (LBI) test for the null

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hypothesis that the series is white noise against the alternative hypothesis that a random walk component is present. The test will be hereafter denoted as NM test. The asymptotic distribution of the NM statistic under the null hypothesis is the well known Cramér-von Mises distribution, as in Anderson and Darling (1952). Nyblom (1986) then augments the model by including a deterministic linear trend; in this case the statistic converges to what is sometimes called second level Cramér-von Mises distribution.

When the random walk is embedded within a general stationary process, the NM statistic can be corrected nonparametrically to produce a test statistic with the same asymptotic distribution. This is done in Kwiatowski et al. (1992) and it is usually denoted as KPSS test. The same correction applies for a model with a deterministic linear trend.

The NM/KPSS test is extended to the multivariate case in Nyblom (1989) and Nyblom and Harvey (2000), where the LBI test for the presence of a multivariate random walk component is derived under the assumptions of Gaussianity and white noise innovations. A test for common trends is also proposed by Nyblom and Harvey (2000). Their null hypothesis is a specified value for the rank of the covariance matrix of the disturbances driving the random walk component. Under the alternative hypothesis the rank is bigger. Since the existence of common trends implies cointegration, this also tests the dimension of the cointegration space.

As already mentioned, the NM/KPSS test contrasts with the tests in the unit root literature (Dickey-Fuller test and modifications) because it reverses the roles of the null and the alternative hypotheses. Similarly, the common trend test of Nyblom and Harvey (2000) contrasts with the tests of Johansen (1988,1991).

The objective of this paper is to extend the multivariate NM/KPSS test and the common trend test of Nyblom and Harvey (2000) to cover, in particular, the case of series with one or more structural breaks (which we will often refer to as the ”breaking trends” case). The issue is important because these tests are also consistent against shifts in the series, as indirectly showed in an early work of Gardner (1969) and later in Nyblom (1989) and Lee et al. (1997). Thus the failure to take into account a structural break is likely to produce evidence of nonstationarity (or unit root) for series that are actually stationary. This problem was highlighted by Perron (1989) in the context of the unit root literature. In our setting, Busetti and Harvey (2000) analyze the case of a scalar time series. Here we also generalize some results of that study. The location of the breakpoints is assumed to be known, e.g. they may correspond to exogenous events that affect the behavior of the series.

The paper is organized as follows. Section 2 reviews the multivariate LBI statistic and derives its asymptotic distribution, under both the null and the alternative hypotheses, for a general specification of the deterministic component, which includes breaking trends as a particular case. The breaking trends case is analyzed in Section 3, where upper tail percentage points from the distribution under the null hypothesis are tabulated across a range of the breakpoint location parameter. A modified test statistic is then introduced, whose asymptotic distribution belongs to the Cramér-von Mises family and is independent of the breakpoint parameter. The advantage of this modification is that it allows the construction of a test in the case of multiple breakpoints. Section 4 considers tests for the presence of a certain number of common trends and provides the critical values for the breaking trends case. This also corresponds to testing the dimension of the cointegration space. Section 5 shows that adding a deterministic seasonal component and/or weakly dependent exogenous regressors to the model does not affect the asymptotic distribution of the tests considered in the previous Sections. Finally, Section 6 presents some applications and Section 7 concludes.

Note that throughout the paper we will often refer to the (multivariate) KPSS test as a test for nonstationarity, although Lee and Schmidt (1996) showed that it can also be used to detect long memory in a stationary time series.

We use standard notation: $\xrightarrow{p}$ and $\xrightarrow{d}$ indicate convergence in probability and convergence in distribution, respectively; $\Rightarrow$ stands for weak convergence to a stochastic process; $\|\cdot\|$ is the euclidean norm and $1(\cdot)$ is the indicator variable

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2. The multivariate LBI test for nonstationarity

Let $\mathbf{y}_t$ be a vector of N time series. Assume $\mathbf{y}_t$ is generated by the model

$$\mathbf{y}_t = \boldsymbol{\beta}'\mathbf{x}_t + \boldsymbol{\mu}_t + \boldsymbol{\varepsilon}_t, \tag{1}$$

$$\boldsymbol{\mu}_t = \boldsymbol{\mu}_{t-1} + \boldsymbol{\eta}_t, \tag{2}$$

$$(3) \qquad \boldsymbol{\eta}_t \sim i.i.d.(\mathbf{0}, \boldsymbol{\Sigma}_\eta),$$

where $\mathbf{x}_t$ is a p-dimensional vector of $non-stochastic$ regressors (including a constant term), $\boldsymbol{\beta}$ is a pxN matrix of parameters, $\boldsymbol{\mu}_t$ is a multivariate random walk (stochastic trend) with $\boldsymbol{\mu}_0 = \mathbf{0}$, and for the time being $\boldsymbol{\varepsilon}_t$ is a white noise disturbance term, independent of $\boldsymbol{\eta}_s$ for all t and s. When $\mathbf{x}_t = 1$ for every t, Harvey (1989) refers to (1)-(3) as the multivariate local level model. The notation $i.i.d.$ stands for independent and identically distributed.

Under the further assumption of Gaussianity, Nyblom and Harvey (2000) show that the Locally Best Invariant (LBI) test for H$_0$: $\boldsymbol{\Sigma}_\eta = 0$ against H$_A$: $\boldsymbol{\Sigma}_\eta = q\boldsymbol{\Sigma}_\varepsilon$, where $\boldsymbol{\Sigma}_\varepsilon := Var(\boldsymbol{\varepsilon}_t)$ and $q > 0,$ has a rejection region of the form

$$(4) \qquad tr\left[\mathbf{S}^{-1}\mathbf{C}\right] > c,$$

where $\mathbf{C} = T^{-2}\sum_{t=1}^{T}\left[\sum_{s=1}^{t}\mathbf{e}_t\right]\left[\sum_{s=1}^{t}\mathbf{e}_t\right]'$, $\mathbf{S} = T^{-1}\sum_{t=1}^{T}\mathbf{e}_t\mathbf{e}_t'$, $\mathbf{e}_t$'s are the OLS residuals from regressing $\mathbf{y}_t$ on $\mathbf{x}_t$ and c is an appropriate critical value. The test is invariant with respect to the affine linear transformation $\mathbf{y}_t \longmapsto \mathbf{P}\mathbf{y}_t + \mathbf{A}\mathbf{x}_t,$ where $\mathbf{P}$ is a nonsingular NxN matrix and $\mathbf{A}$ is an arbitrary Nxp matrix.

Under the null hypothesis the model does not contain a stochastic trend component. Under the alternative hypothesis $\boldsymbol{\Sigma}_\eta$ is proportional to $\boldsymbol{\Sigma}_\varepsilon$, i.e. the model is "homogeneous" in the sense of Harvey (1989, chapter 8). Thus, the test maximizes the local power against homogeneous alternatives. However the test is also consistent against the more general alternative hypothesis H$_A$: $rank(\boldsymbol{\Sigma}_\eta) > 0.$

Nyblom and Harvey (2000) then concentrate on the cases $\mathbf{x}_t = 1$ and $\mathbf{x}_t = (1, t)'$, i.e. on the null hypothesis of stationarity around a constant level and a linear trend respectively. In this paper, we consider a more general form for the regressors $\mathbf{x}_t,$ that covers the case of breaking trends.

The case of serial dependence in the disturbance term $\boldsymbol{\varepsilon}_t$ can be treated by correcting the statistic (4) nonparametrically by replacing $\mathbf{S}$ with a consistent estimator of the long run variance of $\boldsymbol{\varepsilon}_t$. This correction is often termed KPSS correction, after Kwiatowski et al. (1992).

The proposed estimator has the form

$$\widehat{\mathbf{\Omega}}(m) = \sum_{\tau=-m}^{m} w(\tau, m)\widehat{\mathbf{\Gamma}}(\tau),$$

where $w(\tau, m)$ is a weighting function and

$$\widehat{\mathbf{\Gamma}}(\tau) = T^{-1} \sum_{t=\tau+1}^{T} \mathbf{e}_t\mathbf{e}'_{t-\tau}$$

is the sample autocovariance at lag τ. In this paper we use $w(\tau, m) = 1 - |\tau|/(m+1)$, i.e. the simple Bartlett kernel; other possibilities are examined in Andrews (1991).

The corrected statistic is then

$$\xi_N = tr\left(\widehat{\mathbf{\Omega}}(m)^{-1}\mathbf{C}\right). \tag{5}$$

The next proposition gives the asymptotic distribution of ξ_N under the null hypothesis and for an assumption on the regressors $\mathbf{x}_t$ which includes breaking trends as a particular case. In the next Section we will analyze the breaking trends case in detail, providing upper tail percentage points from that distribution.

Assumption 1. The regressors $\mathbf{x}_t$ are non-stochastic and there existd a scaling matrix $\boldsymbol{\delta}_T$ and a bounded piecewise continuous function $\mathbf{x}(r)$ such that (i) $\boldsymbol{\delta}_T\mathbf{x}_{[Tr]} \to \mathbf{x}(r)$ as $T \to \infty$ uniformly in $r \in [0,1]$, and (ii) $\int_0^1 \mathbf{x}(r)\mathbf{x}(r)'dr$ is positive definite.

Assumption 2. The vector process $\{\boldsymbol{\varepsilon}_t\}$ satisfies the following assumptions: (i) $\mathrm{E}(\varepsilon_{jt}) = 0$, $j = 1, ..., N$, $t = 1, ..., T$; (ii) $\sup_t \mathrm{E}|\varepsilon_{jt}|^{2\beta} < \infty$, $j = 1, ..., N$, $\beta > 2$; (iii) $\{\boldsymbol{\varepsilon}_t\}$ is strong mixing with mixing coefficients α_h that satisfy $\sum_{h=1}^{\infty} \alpha_h^{1-2/\beta} < \infty$;(iv) $\mathbf{\Omega} = \lim T^{-1}\mathrm{E}\left(\sum_{t=1}^{T} \boldsymbol{\varepsilon}_t \sum_{t=1}^{T} \boldsymbol{\varepsilon}'_t\right)$exists and is positive definite.

Assumption 3. $m \to \infty$ as $T \to \infty$ such that $m = o\left(T^{1/4}\right)$.

Proposition 1 *Let $\mathbf{y}_t$ be generated by the model (1)-(3) under assumptions 1-3. Then under* $H_0 : \mathbf{\Sigma}_\eta = 0$

$$\xi_N \xrightarrow{d} \int_0^1 \mathbf{B}^X(r)'\mathbf{B}^X(r)dr, \tag{6}$$

where $\mathbf{B}^X(r) = \mathbf{W}(r) - \left(\int_0^1 \mathbf{x}(r)d\mathbf{W}(r)'\right)' \left(\int_0^1 \mathbf{x}(r)\mathbf{x}(r)'dr\right)^{-1} \int_0^r \mathbf{x}(s)ds,\, r \in [0,1]$, *with* $\mathbf{W}(\cdot)$ *being a standard vector Wiener process of dimension* N.

The proof is provided in Appendix A. It is a straightforward extension of the one in Nyblom and Harvey (2000), where only the cases $\mathbf{x}_t = 1$ and $\mathbf{x}_t = (1,t)'$ are considered. In those cases $\mathbf{B}^X(r)$ reduces respectively to a standard Brownian bridge, denoted as $\mathbf{B}^1(r)$, and to a second level Brownian bridge, denoted as $\mathbf{B}^2(r)$, where

$$\mathbf{B}^1(r) = \mathbf{W}(r) - r\mathbf{W}(1), \tag{7}$$

$$\mathbf{B}^2(r) = \mathbf{W}(r) - r\mathbf{W}(1) + 6r(1-r)\left\{\tfrac{1}{2}\mathbf{W}(1) - \int_0^1 \mathbf{W}(s)ds\right\}. \tag{8}$$

More generally, when $\mathbf{x}_t$ contains all the first h powers of t, i.e. from t^0 to t^{h-1}, $\mathbf{B}^X(r)$ is a (multivariate) h^{th}-level Brownian bridge as in McNeill (1978) and the distribution of $\int_0^1 \mathbf{B}^X(r)'\mathbf{B}^X(r)dr$ is called h^{th}-level Cramér-von Mises distribution with N degrees of freedom. For $N = 1$ percentage points are tabulated in Anderson and Darling (1952), MacNeill (1978), Nyblom and Mäkeläinen (1983), Nyblom (1986), Nabeya and Tanaka (1988), Kwiatkowski *et al.* (1992); when $N > 1$ percentage points are tabulated in Nyblom (1989), Canova and Hansen (1995) and Nyblom and Harvey (2000).

In our case the process $\mathbf{B}^X(r)$ is more general as since it includes, for example, the case of breaking trends. We will call this process a generalized Brownian bridge.

Assumption 1 follows Phillips and Xiao (1998). Note that it excludes the dummy variables used to model seasonal effects; however in Section 5 we will show that adding these dummies does not affect the limiting distribution. Assumption 2 permits a fairly general correlation structure for the disturbances ε_t, which can also be heteroschedastic. Assumption 2 is sufficient for applying the invariance principle and, together with assumption 3, for the consistency of $\widehat{\mathbf{\Omega}}(m)$; see Phillips (1987). Note that imposing stronger conditions on ε_t would allow faster rates for m. For example, in the classical spectral theory of stationary process only $m = o(T)$ is required. In practice the rate $m = o(T^{1/2})$ can be satisfactory under both the null and the alternative hypothesis; see Kwiatowski et al. (1992).

The percentage points from the distribution of (6) can be used to construct an asymptotically valid test for the null hypothesis of $\mathrm{H}_0 : \mathbf{\Sigma}_\eta = 0$ against $\mathrm{H}_A : rank(\mathbf{\Sigma}_\eta) > 0$. If $\boldsymbol{\varepsilon}_t$ is a white noise process, then the test is asymptotically equivalent to the LBI test (4).

Under the alternative hypothesis $\mathrm{H}_A : rank(\mathbf{\Sigma}_\eta) = K > 0$ the statistic diverges, so the test is consistent. The asymptotic distribution of $(m/T)\,\xi_N$ under H_A is established by the following proposition.

Proposition 2 *Let $\mathbf{y}_t$ be generated by the model (1)-(3) under assumptions 1-3. Then under $H_A : rank(\mathbf{\Sigma}_\eta) = K > 0$*

$$(m/T)\,\xi_N \xrightarrow{d} tr\left\{\left(\int_0^1 \mathbf{W}^X(s)\mathbf{W}^X(s)'ds\right)^{-1}\int_0^1\left(\int_0^r \mathbf{W}^X(s)ds\right)\left(\int_0^r \mathbf{W}^X(s)ds\right)'dr\right\},$$

where $\mathbf{W}^X(r) = \mathbf{W}(r) - \int_0^1 \mathbf{W}(r)\mathbf{x}(r)'dr\left(\int_0^1 \mathbf{x}(r)\mathbf{x}(r)'dr\right)^{-1}\mathbf{x}(r),\ r \in [0,1]$, and $\mathbf{W}(\cdot)$ is a standard vector Wiener process of dimension K.

The proof is provided in Appendix B. The process $\mathbf{W}^X(\cdot)$ is the projection in $L_2\,[0,1]$ of a Wiener process onto the space orthogonal to the span of $\mathbf{x}(\cdot)$. For $\mathbf{x}_t = 1$ it becomes the demeaned Wiener process $\mathbf{W}(r) - \int_0^1 \mathbf{W}(r)dr$, and for $\mathbf{x}_t = (1,t)'$ it becomes the detrended Wiener process $\mathbf{W}(r) + (6r-4)\int_0^1 \mathbf{W}(r)dr - (12r-6)\int_0^1 r\mathbf{W}(r)dr$.

3. Testing for nonstationarity in the presence of structural breaks

The multivariate NM/KPSS test proposed by Nyblom and Harvey (2000), based on the statistic (4)/(5) with $\mathbf{x}_t = 1$ or $\mathbf{x}_t = (1,t)'$, is also consistent against the alternative hypothesis of a one-time shift in the deterministic trend. This is indirectly shown in Gardner (1969) who derives the NM statistic in a Bayesian framework to detect a break in an otherwise i.i.d. series. Later Nyblom (1989) derives (4) as an LM statistic to test for a general form of parameter constancy in the mean of the series, namely for cases in which, under the alternative hypothesis, the mean is a martingale (which includes both the cases of random walk and single shift at a randomly chosen point). Lee et al. (1997) then show directly that the KPSS statistic diverges when there is a structural break in a (trend stationary) process.

Therefore it seems important to study the case of breaking trends when testing for the presence of a random walk component. For a univariate series such study has been conducted in Busetti and Harvey (2000).

Suppose that there is a shift in the deterministic trend of the series at time $T_1 = \lambda T$, $\lambda \in (0,1)$. We assume that the breakpoint λ is exogenous and known. In other words, the existence and location of a structural break is not of concern. Instead, we focus on series that are subject to a break in the trend due to some exogenous event, such as a change in economic policy.

We consider the model (1)-(3) under four different specifications of the deterministic trend. Let

$$\mathbf{x}_t^i(\lambda) = \begin{cases} (1, w_t(\lambda))' & i = 1, \\ (1, t, w_t(\lambda), tw_t(\lambda))' & i = 2, \\ (1, t, w_t(\lambda))' & i = 2a, \\ (1, t, (t - \lambda T)w_t(\lambda))' & i = 2b, \end{cases} \tag{9}$$

where $w_t(\lambda) = 1\,(t > \lambda T)$. Case 1 corresponds to a level break with no slope and case 2 to a structural break in both the level and the slope; in cases 2a and 2b the break occurs only in the level and only in the slope, respectively.

Since $\mathbf{x}_{[Tr]}^i(\lambda) \to \mathbf{x}^i(r;\lambda)$, defined by

$$\mathbf{x}^i(r;\lambda) = \begin{cases} (1, w(r;\lambda))' & i = 1, \\ (1, r, w(r;\lambda), w(r;\lambda))' & i = 2, \\ (1, r, w(r;\lambda))' & i = 2a, \\ (1, r, (r - \lambda)w(r;\lambda))' & i = 2b, \end{cases} \tag{10}$$

with $w(r;\lambda) = 1(r > \lambda)$, assumption 1 holds. Therefore we can apply proposition 1. Call $\xi_N^i(\lambda)$ the statistic (5) constructed using $\mathbf{x}_t^i(\lambda)$ as regressors, $i = 1, 2, 2a, 2b$. Then under H_0

$$\xi_N^i(\lambda) \xrightarrow{d} \int_0^1 \mathbf{B}^i(r;\lambda)'\mathbf{B}^i(r;\lambda)dr, \tag{11}$$

where the generalized Brownian bridge $\mathbf{B}^i(r;\lambda)$ is defined in Appendix C.

The form of the processes $\mathbf{B}^i(r;\lambda)$, $i = 1, 2, 2a, 2b$, has been derived in Busetti and Harvey (2000) who consider the same problem for a univariate time series. For example $\mathbf{B}^1(r;\lambda)$ can be interpreted as two adjacent independent Brownian bridges over the intervals $[0,\lambda]$ and $[\lambda, 1]$ respectively.

The upper tail percentage points for the distribution of (11) when $\lambda = 0.002$, 0.1, 0.2, 0.3, 0.4, 0.5 are reported in the first column (labelled $K = 0$) of Tables 1, 2, 3 and 4. The values for $\lambda > 0.5$ are not reported since the distribution is symmetric around $\lambda = 0.5$. These percentiles are obtained by simulating the processes $\mathbf{B}^i(r;\lambda)$, $i = 1, 2, 2a, 2b$ using a sample size of 1,000 and 100,000 replications. We use the random number generator of the matrix programming language Ox; see Doornik (1998). Note that the values for $\lambda = 0.002$ closely agree with the values reported in Nyblom and Harvey (2000), which refer to the case $\lambda = 0$ and were obtained using the series expansion of the distribution.

The statistic $\xi^i_N(\lambda)$ can then be used to construct an asymptotically valid test for nonstationarity in a model with breaking trends, using the first column of Tables 1, 2, 3 and 4 to select the appropriate critical values.

The test based on $\xi^i_N(\lambda)$ has desirable properties, including being asymptotically equivalent to the LBI test for a Gaussian model with serially independent disturbances. However the critical values depend on the breakpoint parameter λ.

In the following subSection we propose a modified version of the statistic $\xi^i_N(\lambda)$ for which the asymptotic distribution is independent of λ. This extends to multiple breakpoints and allows us to tabulate critical values across multiple dimensions of the breakpoint location. The rationale behind this modified statistic is to exploit the additivity property of the Cramér-von Mises distributions; see also Busetti and Harvey (2000).

3.1 *A modified statistic*

Here we restrict our attention to the cases $i = 1$ and $i = 2$ of the deterministic component $\mathbf{x}^i_t(\lambda)$ in (9). Denote by $\mathbf{e}^i_t(\lambda)$ the residuals from the OLS regression of $\mathbf{y}_t$ on $\mathbf{x}^i_t(\lambda)$, $i = 1, 2$. Note that the orthogonality conditions for the residuals allow us to write

$$\sum_{t=1}^{T}\left(\sum_{s=1}^{t}\mathbf{e}^i_s(\lambda)\right)\left(\sum_{s=1}^{t}\mathbf{e}^i_s(\lambda)\right)' =$$

$$= \sum_{t=1}^{T_1}\left(\sum_{s=1}^{t}\mathbf{e}^i_s(\lambda)\right)\left(\sum_{s=1}^{t}\mathbf{e}^i_s(\lambda)\right)' + \sum_{t=T_1+1}^{T}\left(\sum_{s=T_1+1}^{t}\mathbf{e}^i_s(\lambda)\right)\left(\sum_{s=T_1+1}^{t}\mathbf{e}^i_s(\lambda)\right)', \quad i = 1, 2,$$

because the sum of residuals in each of the two subsamples {1,...,T_1} and $\{T_1 + 1, ..., T\}$ is zero. Essentially the idea is to take the sum of the two statistics (5) applied to each subsample.

In particular, we consider the statistic $\xi_N^{*i}(\lambda)$ defined as

(12) $$\xi_N^{*i}(\lambda) = tr\left[\widehat{\mathbf{\Omega}}(m)^{-1}\left(\mathbf{C}^1(\lambda)+\mathbf{C}^2(\lambda)\right)\right], \qquad i=1,2,$$

where

$$\mathbf{C}^1(\lambda) = T_1{}^{-2}\sum_{t=1}^{T_1}\left(\sum_{s=1}^{t}\mathbf{e}_s^i(\lambda)\right)\left(\sum_{s=1}^{t}\mathbf{e}_s^i(\lambda)\right)',$$
$$\mathbf{C}^2(\lambda) = (T-T_1)^{-2}\sum_{t=T_1+1}^{T}\left(\sum_{s=T_1+1}^{t}\mathbf{e}_s^i(\lambda)\right)\left(\sum_{s=T_1+1}^{t}\mathbf{e}_s^i(\lambda)\right)',$$

and $\widehat{\mathbf{\Omega}}(m)$ defined as in Section 2.

Proposition 3 *Let $\mathbf{y}_t$ be generated by the model (1)-(3) under assumptions 2-3 and with the regressors defined by (9). Then under $H_0 : \mathbf{\Sigma}_\eta = 0$*

(13) $$\xi_N^{*i}(\lambda) \xrightarrow{d} \int_0^1 \mathbf{B}^i(r)'\mathbf{B}^i(r)dr, \qquad i=1,2,$$

where $\mathbf{B}^1(r)$ and $\mathbf{B}^2(r)$ are respectively a standard vector Brownian bridge and a second level standard vector Brownian bridge of dimension $2N$.

The proof is in Appendix D. The random variable to which $\xi_N^{*i}(\lambda)$ converge has a Cramér-von Mises distribution with $2N$ degrees of freedom. As previously stated, percentage points from this distribution are tabulated in Nyblom (1989), Canova and Hansen (1995) and Nyblom and Harvey (2000). They can be used to construct an alternative test for $\mathrm{H}_0 : \mathbf{\Sigma}_\eta = 0$ against $\mathrm{H}_A : rank(\mathbf{\Sigma}_\eta) > 0$ in a model with breaking trends. This test coincides with the LBI test (11) when $\lambda = 0.5$, since $\xi_N^{*i}(0.5) = 0.25\xi_N^i(0.5)$.[2] For other values of λ the test is of course consistent; furthermore the simulation results of Busetti and Harvey (2000), relating to the univariate version of the test, show that it only suffers a small loss in power compared with the LBI test.

The attraction of the test (12) is that it can be easily generalized to the case of two or more breaks in the deterministic trend. Let there be two structural breaks at time $T_1 = \lambda_1 T$ and $T_2 = \lambda_2 T$. Then we can base the test for nonstationarity on the statistic $\xi_N^{*i}(\lambda_1, \lambda_2) =$

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[2] From the results of appendix D it follows that for $i = 1, 2$ the asymptotic distribution of $\xi_N^i(\lambda)$ may also be also represented as the distribution of a weighted sum of two Cramer-von Mises random variables with N degrees of freedom, where the weights are λ^2 and $(1-\lambda)^2$.

$tr\left[\widehat{\mathbf{\Omega}}(m)^{-1}\left(\mathbf{C}^1(\lambda_1,\lambda_2)+\mathbf{C}^2(\lambda_1,\lambda_2)+\mathbf{C}^3(\lambda_1,\lambda_2)\right)\right]$, $i = 1,2,$ which will be defined by generalizing (12) in an obvious way. Its asymptotic distribution is Cramér-von Mises with $3N$ degrees of freedom.

Another case may be covered by the statistic $\xi_N^{*i}(\lambda_1,\lambda_2)$, namely when some of the N series break at point λ_1 and some at point λ_2. Constructing the statistic by forcing all the N series to have two breakpoints gives rise to a valid test, though not efficient. On the other hand, for this situation the literature does not seem to provide alternative procedures.

4. Testing for the presence of common trends

In this Section we consider the model (1)-(3) under the null hypothesis $\mathrm{H}_0 : rank(\mathbf{\Sigma}_\eta) = K,$ with $0 \leq K < N,$ which corresponds to nonstationarity with K stochastic trends. The alternative hypothesis is $\mathrm{H}_A : rank(\mathbf{\Sigma}_\eta) > K.$

The existence of K common trends implies the existence of $R = N - K$ cointegration relationships, i.e. there is a RxN (full rank) matrix $\mathbf{A}$ such that $\mathbf{Ay}_t$ is stationary. If we knew $\mathbf{A}$, we could test H_0 against H_A by applying statistic (5) to $\mathbf{Ay}_t$. But since $\mathbf{A}$ is unknown, one way of proceeding is to take the minimum of (5) over the set of the RxN matrices $\mathbf{A}$. The minimum is given by the sum of the R smallest eigenvalues of $\widehat{\mathbf{\Omega}}(m)^{-1}\mathbf{C}$; see Nyblom and Harvey (2000).

The statistic we use is then

(14) $$\xi_{K,N} = \sum_{j=K+1}^{N} \ell_j,$$

where $\ell_1 \geq \ell_2 \geq ... \geq \ell_N \geq 0$ are the N ordered eigenvalues of $\widehat{\mathbf{\Omega}}(m)^{-1}\mathbf{C}$. Note that $\xi_{0,N}$ corresponds to statistic (5).

Proposition 4 *Let $\mathbf{y}_t$ be generated by the model (1)-(3) under assumptions 1-3. Then under $H_0 : rank(\mathbf{\Sigma}_\eta) = K$*

(15) $$\xi_{K,N} \xrightarrow{d} tr\left(\mathbf{C}_{22}^{*} - \mathbf{C}_{12}^{*\prime}\mathbf{C}_{11}^{*-1}\mathbf{C}_{12}^{*}\right),$$

where the stochastic matrices $\mathbf{C}_{ij}^{*}$, $i,j=1,2,$ *are defined as*

$$\begin{aligned}
\mathbf{C}_{11}^{*} &:= \int_0^1 \left(\int_0^r \mathbf{W}^X(s)ds\right)\left(\int_0^r \mathbf{W}^X(s)ds\right)' dr, \\
\mathbf{C}_{12}^{*} &:= \int_0^1 \left(\int_0^r \mathbf{W}^X(s)ds\right)\mathbf{B}^X(r)'dr, \\
\mathbf{C}_{22}^{*} &:= \int_0^1 \mathbf{B}^X(r)\mathbf{B}^X(r)'dr,
\end{aligned}$$

where $\mathbf{W}^X(r)$ *is a* K*-dimensional vector process as defined in proposition 2.2 and* $\mathbf{B}^X(r)$ *is an* R*-dimensional vector process as defined in proposition 2.1.*

The proof is in Appendix E. Under $\mathrm{H}_A : rank(\mathbf{\Sigma}_\eta) > K$, the statistic $\xi_{K,N}$ diverges to infinity as it contains at least one eigenvalue that is $O_p(T/m)$; see Appendix B.

When the regressors are defined by (9), the upper tail percentage points of the distribution of $\xi_{K,N}$ under H_0 are provided in Tables 1, 2, 3 and 4 for a range of values of the breakpoint parameter λ. These are obtained simulating the stochastic processes involved using a sample size of 1,000 and 100,000 replications.

These tables can then be used to construct an asymptotically valid test for $\mathrm{H}_0 : rank(\mathbf{\Sigma}_\eta) = K$ against $\mathrm{H}_A : rank(\mathbf{\Sigma}_\eta) > K$ in the breaking trends case. Note that this is also a test on the dimension of the cointegration space, i.e. a test of

H_0 : there are R cointegration relationships

against

H_A : there are fewer than R cointegration relationships.

This is to be contrasted with the Johansen type tests of Inoue (1999), where under the alternative hypothesis there are more than R cointegration relationships (i.e. under the null hypothesis the model is more nonstationary). In addition, Inoue's tests are not directly comparable with ours since they are obtained by taking the supremum of Johansen's statistics with respect to the breakpoint location, i.e. they are unconditional to the existence of a break. Another difference is that, unlike here, they require fitting a set of statistical models to the data, namely fitting a vector autoregression for each possible breakpoint location.

Unfortunately, the test for $K > 0$ cannot be modified along the lines of the previous Section to yield a statistic whose asymptotic distribution is free of λ. Note also that this model of breaking trends implies that in general the long run equilibrium relation has been subject to a shift at time T_1.

5. Seasonal effects and weakly dependent exogenous regressors

In this Section we will show that augmenting the model (1)-(3) by including deterministic seasonality and/or weakly dependent exogenous regressors does not affect the asymptotic distributions of the test statistics of the previous sections. For simplicity we only consider the test for nonstationarity of Section 2.

We replace equation (1) by

$$\mathbf{y}_t = \boldsymbol{\beta}'\mathbf{x}_t + \boldsymbol{\gamma}'\mathbf{z}_t + \boldsymbol{\mu}_t + \boldsymbol{\varepsilon}_t, \tag{16}$$

where $\mathbf{z}_t$ are additional regressors and $\boldsymbol{\gamma}$ the corresponding coefficients.

Assumption 4. Either [A] or [B] below holds.

[A] $\mathbf{z}_t$ is a zero mean second order stationary process such that: (A1) $\sum_{t=1}^{\infty} \|\mathrm{E}\,(\mathbf{z}_t\mathbf{z}_1')\| < \infty$, (A2) $\mathrm{E}(\mathbf{z}_t\boldsymbol{\varepsilon}_t') = \mathbf{0}$, (A3) $\lim T^{-1}\mathrm{E}\left(\sum_{t=1}^{T} \mathbf{z}_t\varepsilon_{it}\right)\left(\sum_{t=1}^{T} \mathbf{z}_t\varepsilon_{it}\right)' < \infty$, $i = 1, ..., N$.

[B] $\mathbf{z}_t = (z_{1t}, ..., z_{s-1,t})'$ is a set of $s-1$ deterministic seasonal dummy variables, defined by

$$z_{ht} = \begin{cases} 1 & t = h + ns, \\ 0 & t \neq h + ns, \\ -1 & t = ns, \end{cases} \tag{17}$$

for $n = 0, 1, 2, ...$ Furthermore the function $\mathbf{x}(r)$ of assumption 1 is of bounded variation, i.e. there exists $M < \infty$ such that for every finite partition of the unit interval $0 = r_0 < r_1 < ... < r_n = 1$, $\sum_{i=1}^{n} \|\mathbf{x}(r_i) - \mathbf{x}(r_{i-1})\| < M$.

In assumption 4[A], $\mathbf{z}_t$ is a weakly dependent process (in second order sense) since 4[A1] implies a finite spectrum at the origin. 4[A2] is an exogeneity condition. The zero mean assumption is innocuous since an intercept is included among the other regressors $\mathbf{x}_t$.

In assumption 4[B] the seasonal dummies sum to zero over s periods, so they represent the relative seasonal deviation from the common mean. A reparametrization of the seasonal effect could be used. The bounded variation condition is of course satisfied in the breaking trends case.

Let the statistic ξ_N of Section 2, equation (5), be constructed using the residuals from regressing $\mathbf{y}_t$ on $(\mathbf{x}_t', \mathbf{z}_t')'$. Then the following proposition holds.

Proposition 5 *Let $\mathbf{y}_t$ be generated by the model (16),(2)-(3) under assumptions 1-4. Then, under $H_0 : \mathbf{\Sigma}_\eta = 0,$ the asymptotic distribution of ξ_N is the one defined in proposition 2.1.*

The proof is in Appendix F. Proposition 5 implies that we can use the tests of Sections 2 and 3 to test for nonstationarity in models containing a deterministic seasonal component and/or weakly dependent regressors too. The same result carries over to the common trend test of Section 4. The proof is not provided but follows similar lines.

When the seasonality is stochastic and possibly there are seasonal unit roots, the strategy advocated by Harvey and Streibel (1997) can be applied. Their idea was to replace the nonparametric treatment of the serial correlation in $\boldsymbol{\varepsilon}_t$ with a fully parametric one and it can be extended to deal with stochastic seasonality. Note that this approach requires the extra effort of fitting a statistical model to the data, but such effort is rewarded with higher power and better size of the test.

The Harvey-Streibel approach may be summarized as follows. First, fully parametrize the model to account for serial correlation and possibly stochastic seasonality. Then estimate it under the alternative hypothesis of nonstationarity, using the Kalman filter to construct the likelihood function. Finally insert these estimates, re-run the Kalman filter under the null hypothesis and construct the LBI statistic (4) using the Kalman filter innovations (whose correlation structure has been consistently estimated). Harvey and Streibel (1997) show that the distribution of this statistic is asymptotically equivalent to the distribution of (4). They also compare their procedure with the KPSS correction and show, through simulation experiments, that their procedure permits large gains in the power of the test and a more reliable size.

The case of I(1) regressors $\mathbf{z}_t$ is examined in Choi and Ahn (1995). They consider testing for stationarity of the errors in multiple equations with integrated variables. In their framework stationarity of the errors corresponds to cointegration between regressands and regressors.

6. Examples

The use of the tests is illustrated with two examples, one using UK macroeconomic data and the other using data on road casualties in Great Britain. The latter dataset was used by Harvey and Durbin (1986) to study the effect of the introduction of the seat belt law and is provided with the program STAMP 5.0 of Koopman et al. (1995).

Figures 1a, 1b and 1c plot the logarithms of UK gross domestic product, consumption and investment for the period 1960-1990, together with a fitted broken trend. The data are quarterly, seasonally adjusted and at constant 1990 prices. The source is the Central Statistical Office.

We have assumed that there was an exogenous structural break around 1979-1980, as the plot of the series, that of investment in particular, seem to reveal. The exogeneity of the break may be due to the following two reasons: first, the oil shock of 1979-1980 may have caused a drop in the series; second, in 1979 a right-wing government with Margaret Thatcher as Prime Minister was elected and this may have determined the higher growth rates of investment thereafter. Thus, we have chosen to fit a broken linear trend to the data, with a break in the second quarter of 1980.[3]

We consider the case of a break in both level and slope and we apply the common trend test of Section 4 to the trivariate series of (log) GDP, consumption and investment. The results are displayed in Table 5 for various values of the lag length parameter m. The breakpoint is at the second quarter of 1980, corresponding to a value of 0.66 for λ. Note that the statistic for $K = 0$ is equivalent to the nonstationarity test statistic of Section 3, case 2. The critical values for the case when there is no break are taken from Nyblom and Harvey (2000), those for the case of structural break are taken from Table 2, N=3 and λ=0.3.

Standard macroeconomic arguments would suggest the existence of one stochastic trend among the variables, probably representing the effect of technological progress. However Table 5 shows that the null hypothesis of one common trend $(K = 1)$ is rejected at 5 per cent significance level when the structural break of 1979-1980 is not considered. Indeed, even the hypothesis of two trends seems to be rejected. On the contrary, fitting a broken deterministic

[3] Of course there might have been other events that, on a priori grounds, could have determined a break in the series. However, since the purpose of this section is only to illustrate the use of the tests, we do not delve deeply into the issue of the exogeneity of the breakpoint.

trend to the data results in non rejection of $H_0 : K = 1$ at 10 per cent level of significance. The result is obtained for $m \geq 4$; note also that the KPSS correction has little adverse effect on the value of the statistic as m grows.

Now consider the series of the logarithm of front and rear passengers killed or seriously injured (KSI) in road accidents, as displayed in Figure 2. The data are monthly, not seasonally-adjusted, and cover the period January 1969 to December 1984. These data were used by Harvey and Durbin (1986) to assess the effect of the seat belt law, which made it compulsory for front-seat passengers to wear seat belts after 31 January 1983. Clearly, there is an exogenous structural break in the series of front-seat passengers but not in the series of rear passengers. As explained in Section 3, it is possible to apply our tests by forcing both series to break in February 1983.

Harvey and Durbin (1986) show that a reasonable univariate time series model for the KSI series would be the simple random walk plus noise and a seasonal component, with seasonality being fixed. Given the nature of the data, it would seem plausible that in a multivariate model the random walk component has dimension 1.

Table 6 shows the results of applying the common trend test of Section 4 to the bivariate series of front and rear-seat passengers. A slope component is not included. The breakpoint parameter λ takes the value 0.88, corresponding to February 1983. The statistic is computed including the set of 11 dummy variables as regressors to account for seasonality. As explained in Section 5 the asymptotic distribution is not affected.

If we do not consider the break (first two rows of the Table), we end up rejecting the null hypothesis of one common trend, $K = 1,$ at 5 per cent level of significance even for very large values of the lag truncation parameter m ($m = 14$ corresponds to the square root of the sample size). On the other hand, forcing both series to break at 1983.2 results in not rejecting the hypothesis of one common trend even for $m = 1.$ Note that since each series can be modelled as a univariate random walk plus noise, considering small values of m seems appropriate. Finally, the modified test of nonstationarity (fifth row of the Table) confirms the finding that $K > 0.$

7. Concluding remarks

This paper has considered tests for the presence of stochastic trends in multivariate time series with structural breaks. The asymptotic distributions of the statistics have been derived and the critical values have been tabulated. Testing for the existence of a certain number of stochastic trends may also be interpreted as testing the dimension of the cointegration space. Note that this implies a shift in the cointegration relation at the time of the break.

The basic nonstationarity test has also been modified to handle the case of multiple breakpoints. Interestingly, the asymptotic distribution of this modified statistic is independent of the breakpoint location parameters.

Throughout the paper, the breakpoint is assumed to be known. If this is not the case, it seems preferable to establish whether or not a break exists for each series in turn. For a univariate model Busetti and Harvey (2000) have proposed an inf-type statistic for nonstationarity that covers the case of an unknown breakpoint. Alternatively, one could adopt a two step strategy, where first the breakpoint is estimated and then this estimate is used to compute our tests. However, since the properties of the tests will be affected in a complicated way, this strategy is left for future research.

Table 1

N	Lambda	K=0			K=1			K=2			K=3			K=4			K=5		
		0,9	0,95	0,99	0,9	0,95	0,99	0,9	0,95	0,99	0,9	0,95	0,99	0,9	0,95	0,99	0,9	0,95	0,99
	λ=0.002	0,349	0,464	0,746															
	λ=0.1	0,284	0,375	0,604															
	λ=0.2	0,231	0,302	0,484															
1	λ=0.3	0,187	0,243	0,380															
	λ=0.4	0,160	0,201	0,303															
	λ=0.5	0,152	0,187	0,271															
	λ=0.002	0,606	0,748	1,078	0,163	0,222	0,396												
	λ=0.1	0,494	0,608	0,876	0,134	0,181	0,323												
	λ=0.2	0,402	0,492	0,704	0,114	0,151	0,266												
2	λ=0.3	0,329	0,398	0,558	0,105	0,136	0,230												
	λ=0.4	0,282	0,334	0,454	0,108	0,137	0,213												
	λ=0.5	0,265	0,309	0,409	0,111	0,140	0,213												
	λ=0.002	0,837	0,999	1,357	0,296	0,381	0,622	0,093	0,120	0,202									
	λ=0.1	0,685	0,815	1,106	0,244	0,313	0,510	0,078	0,100	0,166									
	λ=0.2	0,558	0,660	0,897	0,206	0,262	0,419	0,070	0,088	0,143									
3	λ=0.3	0,459	0,538	0,718	0,190	0,235	0,357	0,072	0,089	0,137									
	λ=0.4	0,395	0,453	0,587	0,194	0,232	0,324	0,075	0,093	0,145									
	λ=0.5	0,371	0,422	0,532	0,198	0,234	0,316	0,075	0,094	0,149									
	λ=0.002	1,057	1,232	1,610	0,422	0,528	0,825	0,169	0,208	0,322	0,062	0,078	0,121						
	λ=0.1	0,868	1,004	1,316	0,348	0,435	0,681	0,141	0,172	0,265	0,053	0,065	0,101						
	λ=0.2	0,708	0,820	1,068	0,295	0,365	0,553	0,127	0,153	0,230	0,051	0,062	0,092						
4	λ=0.3	0,583	0,668	0,859	0,272	0,327	0,473	0,129	0,152	0,219	0,053	0,065	0,096						
	λ=0.4	0,502	0,568	0,712	0,273	0,318	0,426	0,134	0,160	0,228	0,054	0,066	0,100						
	λ=0.5	0,473	0,530	0,651	0,277	0,319	0,411	0,136	0,162	0,231	0,054	0,066	0,100						
	λ=0.002	1,273	1,465	1,871	0,548	0,678	1,012	0,240	0,289	0,431	0,113	0,134	0,193	0,047	0,056	0,084			
	λ=0.1	1,046	1,196	1,527	0,451	0,557	0,830	0,201	0,241	0,362	0,096	0,113	0,161	0,040	0,048	0,070			
	λ=0.2	0,855	0,979	1,239	0,383	0,468	0,681	0,181	0,213	0,314	0,092	0,107	0,148	0,040	0,048	0,068			
5	λ=0.3	0,705	0,797	0,999	0,352	0,418	0,581	0,183	0,212	0,296	0,096	0,112	0,152	0,041	0,049	0,071			
	λ=0.4	0,607	0,679	0,831	0,351	0,402	0,523	0,191	0,222	0,304	0,097	0,114	0,158	0,041	0,050	0,072			
	λ=0.5	0,573	0,635	0,762	0,354	0,400	0,500	0,194	0,227	0,307	0,097	0,114	0,160	0,041	0,049	0,072			
	λ=0.002	1,484	1,686	2,124	0,673	0,823	1,192	0,311	0,369	0,542	0,161	0,188	0,259	0,084	0,097	0,133	0,037	0,044	0,062
	λ=0.1	1,215	1,378	1,727	0,553	0,678	0,981	0,260	0,308	0,451	0,137	0,158	0,218	0,073	0,084	0,113	0,032	0,038	0,053
	λ=0.2	0,996	1,126	1,396	0,470	0,567	0,811	0,235	0,274	0,392	0,132	0,150	0,199	0,073	0,083	0,110	0,033	0,039	0,054
6	λ=0.3	0,824	0,922	1,133	0,433	0,509	0,691	0,236	0,272	0,370	0,137	0,156	0,203	0,075	0,086	0,114	0,033	0,039	0,055
	λ=0.4	0,712	0,787	0,946	0,426	0,484	0,617	0,246	0,283	0,375	0,139	0,159	0,213	0,075	0,086	0,116	0,033	0,039	0,055
	λ=0.5	0,672	0,737	0,871	0,429	0,479	0,589	0,251	0,289	0,378	0,139	0,160	0,217	0,075	0,086	0,116	0,033	0,039	0,055

UPPER TAIL PERCENTAGE POINTS: i=1

Table 2

N	Lambda	K=0			K=1			K=2			K=3			K=4			K=5		
		0,9	0,95	0,99	0,9	0,95	0,99	0,9	0,95	0,99	0,9	0,95	0,99	0,9	0,95	0,99	0,9	0,95	0,99
	λ=0.002	0,119	0,148	0,217															
	λ=0.1	0,097	0,121	0,178															
	λ=0.2	0,079	0,097	0,142															
1	λ=0.3	0,065	0,079	0,113															
	λ=0.4	0,056	0,066	0,090															
	λ=0.5	0,052	0,061	0,082															
	λ=0.002	0,210	0,245	0,330	0,084	0,105	0,160												
	λ=0.1	0,172	0,201	0,267	0,069	0,086	0,130												
	λ=0.2	0,140	0,163	0,215	0,057	0,070	0,104												
2	λ=0.3	0,116	0,133	0,174	0,049	0,059	0,086												
	λ=0.4	0,099	0,112	0,142	0,045	0,053	0,073												
	λ=0.5	0,094	0,105	0,131	0,045	0,052	0,070												
	λ=0.002	0,294	0,335	0,427	0,151	0,178	0,245	0,061	0,075	0,113									
	λ=0.1	0,242	0,275	0,349	0,124	0,146	0,200	0,050	0,061	0,092									
	λ=0.2	0,197	0,224	0,281	0,102	0,120	0,162	0,042	0,051	0,075									
3	λ=0.3	0,163	0,184	0,228	0,088	0,102	0,134	0,038	0,046	0,065									
	λ=0.4	0,141	0,157	0,189	0,082	0,092	0,117	0,038	0,044	0,061									
	λ=0.5	0,134	0,147	0,175	0,080	0,090	0,112	0,038	0,044	0,060									
	λ=0.002	0,376	0,422	0,521	0,214	0,246	0,322	0,110	0,129	0,177	0,046	0,056	0,082						
	λ=0.1	0,309	0,345	0,425	0,176	0,201	0,262	0,090	0,106	0,146	0,038	0,046	0,067						
	λ=0.2	0,252	0,281	0,343	0,145	0,166	0,214	0,076	0,089	0,120	0,033	0,039	0,056						
4	λ=0.3	0,209	0,232	0,281	0,126	0,142	0,180	0,069	0,079	0,103	0,031	0,037	0,051						
	λ=0.4	0,182	0,199	0,235	0,116	0,129	0,158	0,068	0,076	0,096	0,031	0,037	0,050						
	λ=0.5	0,173	0,187	0,218	0,114	0,126	0,151	0,068	0,077	0,095	0,032	0,037	0,050						
	λ=0.002	0,455	0,504	0,613	0,274	0,311	0,397	0,156	0,179	0,236	0,083	0,096	0,129	0,036	0,043	0,062			
	λ=0.1	0,373	0,414	0,500	0,226	0,255	0,323	0,129	0,147	0,195	0,069	0,080	0,106	0,030	0,036	0,051			
	λ=0.2	0,306	0,337	0,405	0,187	0,211	0,265	0,109	0,124	0,161	0,060	0,069	0,090	0,027	0,031	0,043			
5	λ=0.3	0,255	0,278	0,330	0,162	0,180	0,223	0,099	0,111	0,140	0,057	0,064	0,082	0,027	0,031	0,041			
	λ=0.4	0,222	0,240	0,278	0,150	0,164	0,196	0,097	0,107	0,130	0,057	0,064	0,080	0,027	0,031	0,042			
	λ=0.5	0,211	0,226	0,259	0,148	0,160	0,187	0,097	0,107	0,128	0,058	0,065	0,081	0,027	0,031	0,042			
	λ=0.002	0,532	0,585	0,700	0,334	0,375	0,469	0,202	0,228	0,293	0,119	0,135	0,175	0,066	0,075	0,099	0,030	0,035	0,048
	λ=0.1	0,438	0,481	0,571	0,275	0,308	0,384	0,166	0,189	0,241	0,099	0,111	0,144	0,055	0,063	0,081	0,025	0,029	0,040
	λ=0.2	0,358	0,392	0,463	0,228	0,254	0,313	0,141	0,158	0,200	0,086	0,096	0,122	0,049	0,055	0,071	0,023	0,026	0,035
6	λ=0.3	0,299	0,325	0,380	0,198	0,218	0,263	0,129	0,143	0,175	0,082	0,090	0,112	0,048	0,054	0,067	0,023	0,026	0,035
	λ=0.4	0,261	0,280	0,321	0,183	0,199	0,234	0,125	0,137	0,163	0,082	0,090	0,109	0,049	0,055	0,067	0,023	0,027	0,035
	λ=0.5	0,248	0,265	0,299	0,180	0,194	0,223	0,125	0,136	0,160	0,082	0,091	0,109	0,049	0,055	0,069	0,023	0,027	0,035

UPPER TAIL PERCENTAGE POINTS: i=2

Table 3

		K=0			K=1			K=2			K=3			K=4			K=5		
N	**Lambda**	**0,9**	**0,95**	**0,99**	**0,9**	**0,95**	**0,99**	**0,9**	**0,95**	**0,99**	**0,9**	**0,95**	**0,99**	**0,9**	**0,95**	**0,99**	**0,9**	**0,95**	**0,99**
	λ=0.002	0,119	0,148	0,217															
	λ=0.1	0,098	0,122	0,179															
	λ=0.2	0,085	0,103	0,148															
1	λ=0.3	0,086	0,103	0,142															
	λ=0.4	0,097	0,120	0,180															
	λ=0.5	0,105	0,134	0,205															
	λ=0.002	0,210	0,245	0,330	0,084	0,105	0,160												
	λ=0.1	0,174	0,204	0,270	0,070	0,087	0,131												
	λ=0.2	0,152	0,175	0,228	0,065	0,079	0,114												
2	λ=0.3	0,151	0,173	0,222	0,068	0,083	0,117												
	λ=0.4	0,172	0,202	0,270	0,069	0,085	0,129												
	λ=0.5	0,188	0,223	0,306	0,067	0,084	0,133												
	λ=0.002	0,294	0,335	0,427	0,151	0,178	0,245	0,061	0,075	0,113									
	λ=0.1	0,245	0,278	0,352	0,127	0,149	0,202	0,052	0,063	0,093									
	λ=0.2	0,214	0,241	0,299	0,117	0,135	0,179	0,051	0,061	0,087									
3	λ=0.3	0,214	0,239	0,292	0,121	0,140	0,182	0,053	0,064	0,092									
	λ=0.4	0,243	0,276	0,352	0,124	0,146	0,200	0,051	0,062	0,092									
	λ=0.5	0,263	0,303	0,395	0,122	0,146	0,212	0,050	0,060	0,089									
	λ=0.002	0,376	0,422	0,521	0,214	0,246	0,322	0,110	0,129	0,177	0,046	0,056	0,082						
	λ=0.1	0,313	0,350	0,430	0,179	0,205	0,266	0,093	0,109	0,149	0,040	0,048	0,068						
	λ=0.2	0,275	0,304	0,367	0,166	0,187	0,237	0,092	0,105	0,139	0,041	0,049	0,068						
4	λ=0.3	0,274	0,301	0,360	0,171	0,193	0,239	0,095	0,109	0,143	0,041	0,049	0,071						
	λ=0.4	0,310	0,348	0,429	0,176	0,202	0,266	0,092	0,107	0,145	0,040	0,048	0,068						
	λ=0.5	0,336	0,380	0,481	0,175	0,205	0,282	0,090	0,104	0,143	0,040	0,047	0,066						
	λ=0.002	0,455	0,504	0,613	0,274	0,312	0,397	0,156	0,179	0,236	0,083	0,096	0,129	0,036	0,043	0,062			
	λ=0.1	0,379	0,419	0,506	0,231	0,260	0,328	0,133	0,152	0,199	0,072	0,083	0,110	0,032	0,038	0,053			
	λ=0.2	0,333	0,364	0,433	0,214	0,238	0,292	0,130	0,146	0,186	0,074	0,084	0,108	0,033	0,039	0,054			
5	λ=0.3	0,333	0,362	0,425	0,220	0,244	0,295	0,134	0,152	0,192	0,074	0,085	0,112	0,033	0,039	0,054			
	λ=0.4	0,376	0,416	0,503	0,227	0,257	0,328	0,131	0,149	0,194	0,072	0,083	0,109	0,033	0,038	0,053			
	λ=0.5	0,407	0,454	0,562	0,227	0,262	0,350	0,128	0,146	0,193	0,072	0,082	0,107	0,032	0,038	0,052			
	λ=0.002	0,532	0,585	0,700	0,334	0,375	0,469	0,202	0,228	0,293	0,119	0,135	0,175	0,066	0,075	0,099	0,030	0,035	0,048
	λ=0.1	0,445	0,488	0,578	0,281	0,314	0,389	0,172	0,194	0,247	0,103	0,116	0,149	0,058	0,066	0,085	0,027	0,031	0,042
	λ=0.2	0,391	0,425	0,497	0,260	0,287	0,347	0,168	0,187	0,230	0,105	0,118	0,146	0,060	0,068	0,087	0,028	0,032	0,044
6	λ=0.3	0,390	0,422	0,489	0,267	0,293	0,347	0,173	0,193	0,237	0,106	0,119	0,152	0,060	0,068	0,088	0,027	0,032	0,043
	λ=0.4	0,441	0,482	0,575	0,277	0,309	0,386	0,169	0,190	0,243	0,103	0,115	0,147	0,059	0,067	0,085	0,027	0,032	0,043
	λ=0.5	0,476	0,527	0,640	0,278	0,320	0,414	0,166	0,187	0,242	0,102	0,114	0,145	0,059	0,067	0,085	0,027	0,032	0,043

UPPER TAIL PERCENTAGE POINTS: i=2a

Table 4

N	Lambda	K=0			K=1			K=2			K=3			K=4			K=5		
		0,9	0,95	0,99	0,9	0,95	0,99	0,9	0,95	0,99	0,9	0,95	0,99	0,9	0,95	0,99	0,9	0,95	0,99
	λ=0.002	0,119	0,148	0,218															
	λ=0.1	0,103	0,127	0,186															
	λ=0.2	0,089	0,110	0,161															
1	λ=0.3	0,078	0,096	0,138															
	λ=0.4	0,071	0,086	0,123															
	λ=0.5	0,069	0,083	0,118															
	λ=0.002	0,211	0,246	0,329	0,084	0,106	0,160												
	λ=0.1	0,182	0,213	0,284	0,073	0,090	0,137												
	λ=0.2	0,158	0,183	0,242	0,064	0,079	0,117												
2	λ=0.3	0,139	0,161	0,211	0,058	0,070	0,103												
	λ=0.4	0,127	0,146	0,189	0,055	0,066	0,095												
	λ=0.5	0,123	0,141	0,182	0,054	0,065	0,092												
	λ=0.002	0,295	0,336	0,428	0,151	0,178	0,245	0,061	0,075	0,114									
	λ=0.1	0,255	0,290	0,369	0,131	0,154	0,210	0,053	0,065	0,097									
	λ=0.2	0,222	0,252	0,317	0,115	0,135	0,182	0,047	0,057	0,084									
3	λ=0.3	0,196	0,221	0,277	0,104	0,121	0,162	0,044	0,053	0,076									
	λ=0.4	0,179	0,202	0,249	0,099	0,113	0,149	0,043	0,052	0,073									
	λ=0.5	0,174	0,195	0,239	0,097	0,111	0,145	0,043	0,052	0,073									
	λ=0.002	0,376	0,422	0,522	0,214	0,247	0,322	0,110	0,129	0,178	0,046	0,056	0,082						
	λ=0.1	0,326	0,364	0,451	0,186	0,213	0,278	0,096	0,112	0,155	0,040	0,048	0,070						
	λ=0.2	0,283	0,316	0,389	0,163	0,186	0,240	0,085	0,099	0,134	0,036	0,044	0,062						
4	λ=0.3	0,251	0,279	0,338	0,148	0,168	0,214	0,080	0,092	0,122	0,035	0,042	0,059						
	λ=0.4	0,231	0,255	0,306	0,141	0,158	0,200	0,079	0,090	0,116	0,035	0,042	0,058						
	λ=0.5	0,224	0,246	0,296	0,139	0,155	0,193	0,078	0,089	0,115	0,035	0,042	0,058						
	λ=0.002	0,456	0,505	0,614	0,275	0,312	0,398	0,156	0,180	0,237	0,084	0,097	0,130	0,036	0,043	0,062			
	λ=0.1	0,394	0,437	0,530	0,239	0,270	0,341	0,136	0,156	0,206	0,073	0,084	0,112	0,032	0,038	0,053			
	λ=0.2	0,343	0,378	0,455	0,210	0,237	0,297	0,122	0,139	0,180	0,066	0,076	0,100	0,030	0,035	0,048			
5	λ=0.3	0,305	0,334	0,398	0,191	0,214	0,265	0,115	0,129	0,165	0,064	0,073	0,095	0,029	0,034	0,047			
	λ=0.4	0,280	0,306	0,361	0,181	0,201	0,247	0,112	0,125	0,157	0,064	0,073	0,093	0,029	0,034	0,047			
	λ=0.5	0,272	0,296	0,349	0,179	0,197	0,240	0,112	0,125	0,155	0,064	0,073	0,093	0,030	0,035	0,047			
	λ=0.002	0,534	0,587	0,702	0,334	0,376	0,469	0,202	0,229	0,294	0,119	0,135	0,176	0,066	0,076	0,099	0,030	0,035	0,048
	λ=0.1	0,462	0,507	0,605	0,290	0,325	0,405	0,176	0,198	0,254	0,104	0,118	0,152	0,058	0,067	0,086	0,026	0,031	0,042
	λ=0.2	0,403	0,441	0,521	0,256	0,285	0,353	0,158	0,177	0,225	0,095	0,107	0,136	0,054	0,061	0,078	0,025	0,029	0,039
6	λ=0.3	0,358	0,389	0,457	0,233	0,258	0,316	0,148	0,165	0,206	0,092	0,103	0,128	0,054	0,060	0,076	0,025	0,029	0,039
	λ=0.4	0,330	0,357	0,415	0,221	0,243	0,292	0,145	0,160	0,196	0,092	0,102	0,126	0,054	0,060	0,076	0,025	0,029	0,039
	λ=0.5	0,320	0,346	0,401	0,218	0,239	0,284	0,144	0,159	0,193	0,092	0,102	0,125	0,054	0,060	0,076	0,025	0,029	0,039

UPPER TAIL PERCENTAGE POINTS: i=2b

Table 5

COMMON TREND TEST FOR UK QUARTERLY SERIES OF Y,C,I (1960-1990)

		m=1	**m=2**	**m=3**	**m=4**	**m=5**	**m=8**	**m=11**	**10%**	**5%**
	K=0	2.043	1.409	1.089	0.896	0.768	0.558	0.457	0.296	0.332
No break	K=1	0.845	0.584	0.453	0.374	0.322	0.238	0.199	0.151	0.180
	K=2	0.180	0.129	0.102	0.087	0.077	0.062	0.057	0.061	0.075
	K=0	0.694	0.494	0.392	0.331	0.291	0.228	0.207	0.163	0.184
Break 1980.2	K=1	0.159	0.116	0.094	0.082	0.074	0.066	0.074	0.088	0.102
(lambda = 0.66)	K=2	0.038	0.030	0.026	0.024	0.023	0.025	0.032	0.038	0.046

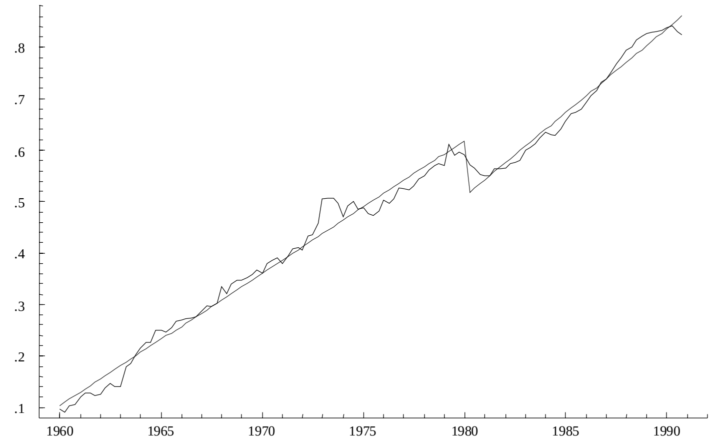
Table 6

COMMON TREND TEST AND MODIFIED TEST FOR KSI FRONT AND REAR PASSENGERS (1969-1984)

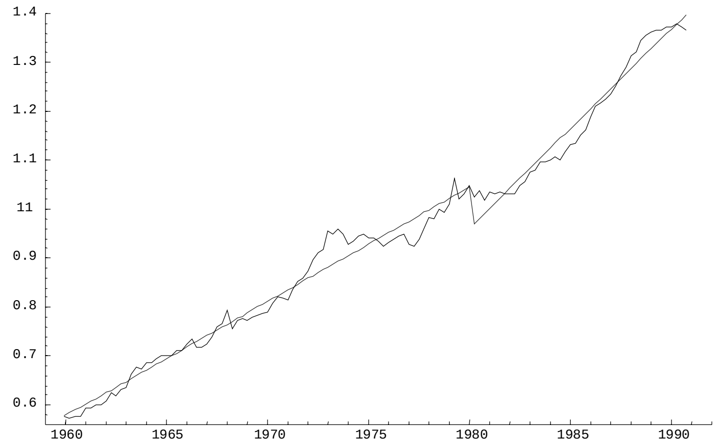
		m=0	**m=1**	**m=2**	**m=3**	**m=4**	**m=5**	**m=14**	**10%**	**5%**
No break	K=0	13.002	7.210	5.081	3.955	3.265	2.785	1.535	0.596	0.746
	K=1	1.121	0.855	0.694	0.585	0.513	0.454	0.274	0.156	0.212
Break 1983.2	K=0	7.992	4.640	3.339	2.640	2.197	1.889	0.881	0.494	0.608
(lambda = 0.88)	K=1	0.184	0.171	0.161	0.151	0.146	0.139	0.107	0.134	0.181
Modified Test		10.667	6.255	4.537	3.608	3.023	2.612	1.257	0.607	0.748

Figure 1

1a) UK GDP WITH FITTED TREND



1b) UK CONSUMPTION WITH FITTED TREND



1c) UK INVESTMENT WITH FITTED TREND

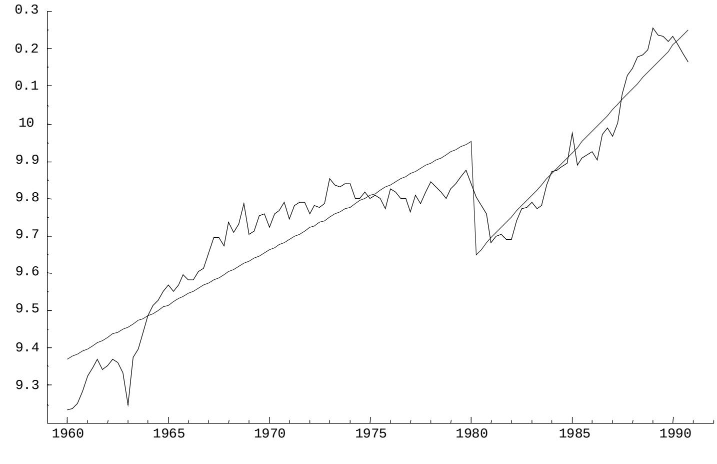
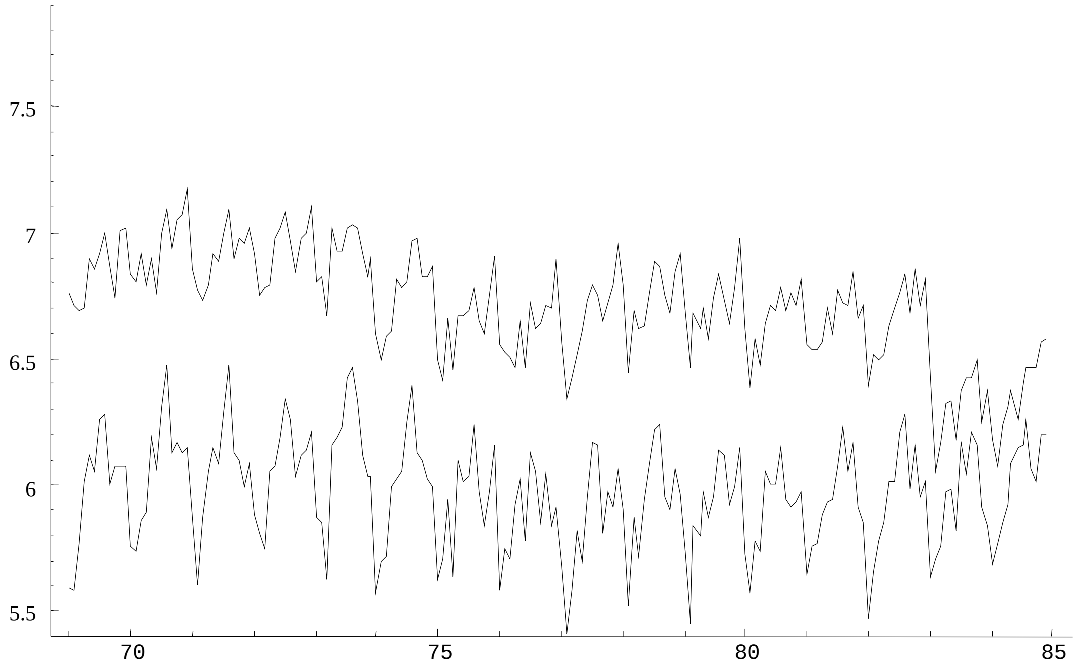


Figure 2

FRONT AND REAR PASSENGERS KILLED OR SERIOUSLY INJURED IN ROAD ACCIDENTS, 1969-1983



APPENDIX

A. Proof of proposition 1

Under H_0 and assumptions 1-3, $\widehat{\mathbf{\Omega}}(m) \xrightarrow{p} \mathbf{\Omega}$. By assumption 2, the long-run variance of the disturbances $\mathbf{\Omega}$ is finite and of full rank. Hence there exists a nonsingular matrix $\mathbf{P}$ such that $\mathbf{P\Omega P}' = \mathbf{I}$ and $\mathbf{P\Sigma}_\eta\mathbf{P}' = diag(q_1, q_2, ..., q_N)$, where the q_j's are the N roots of $|\mathbf{\Sigma}_\eta - q\mathbf{\Omega}| = 0$; see Rao (1973, p.41).

Since the test statistic ξ_N is invariant to premultiplying the observations $\mathbf{y}_t$ by an arbitrary nonsingular NxN matrix **P,** without loss of generality we can restrict ourselves to the case of $\mathbf{\Omega} = \mathbf{I}$ and $\mathbf{\Sigma}_\eta = diag(q_1, q_2, ..., q_N)$.

Then $\xi_N = tr\left((\mathbf{I}+o_p(1))^{-1}\mathbf{C}\right)$.

Since $\mathbf{e}_t = \boldsymbol{\varepsilon}_t - \sum_{t=1}^T \boldsymbol{\varepsilon}_t\mathbf{x}_t'\left(\sum_{t=1}^T \mathbf{x}_t\mathbf{x}_t'\right)^{-1}\mathbf{x}_t$, it follows that under assumptions 1-2

$$T^{-\frac{1}{2}}\sum_{t=1}^{[Tr]}\mathbf{e}_t \Rightarrow \mathbf{W}(r) - \left(\int_0^1 \mathbf{x}(r)d\mathbf{W}(r)'\right)'\left(\int_0^1 \mathbf{x}(r)\mathbf{x}(r)'dr\right)^{-1}\int_0^r \mathbf{x}(s)ds, \quad r\in[0,1], \tag{A.1}$$

where $\mathbf{W}(r)$ is a standard vector Wiener process of dimension N. We denote the process on the right-hand side of (A.1) as $\mathbf{B}^X(r)$, and we call it a generalized Brownian bridge. Thus, by the continuous mapping theorem and using the definition of **C**, $\xi_N \xrightarrow{d} \int_0^1 \mathbf{B}^X(r)'\mathbf{B}^X(r)dr$.

B. Proof of proposition 2

Let assumptions 1-3 hold. Without loss of generality we again restrict ourselves to the case of $\mathbf{\Omega} = \mathbf{I}$ and $\mathbf{\Sigma}_\eta = diag(q_1, ..., q_N)$. The hypothesis $\mathrm{H}_A : rank(\mathbf{\Sigma}_\eta) = K$ can be equivalently formulated as $\mathrm{H}_A : q_j > 0$ for $j = 1, ..., K$ and $q_j = 0$ for $j = K+1, ..., N$.

Consider the OLS residuals for the j-equation under H_A,

$$e_{jt} = (\mu_{jt} + \varepsilon_{jt}) - \mathbf{x}_t'\left(\sum_{t=1}^T \mathbf{x}_t\mathbf{x}_t'\right)^{-1}\sum_{t=1}^T \mathbf{x}_t(\mu_{jt}+\varepsilon_{jt}), \qquad j = 1, ..., K,$$

and

$$e_{jt} = \varepsilon_{jt} - \mathbf{x}_t'\left(\sum_{t=1}^T \mathbf{x}_t\mathbf{x}_t'\right)^{-1}\sum_{t=1}^T \mathbf{x}_t\varepsilon_{jt}, \qquad j = K+1, ..., N,$$

since $\mu_{jt} = 0$ for $j = K+1, ..., N$, $t = 1, ..., T$.

Then, the following weak convergence results hold:

$$T^{-1/2} e_{j,[Tr]} \Rightarrow q_j W_j^X(r), \qquad j = 1, ..., K, \tag{B.1}$$

$$T^{-3/2} \sum_{t=1}^{[Tr]} e_{jt} \Rightarrow q_j \int_0^r W_j^X(s) ds, \qquad j = 1, ..., K, \tag{B.2}$$

$$T^{-1/2} \sum_{t=1}^{[Tr]} e_{jt} \Rightarrow B_j^X(r), \qquad j = K+1, ..., N, \tag{B.3}$$

where the processes $W_j^X(r)$ and $B_j^X(r)$, $r \in [0,1]$, are the one-dimensional version of the processes defined in the statement of propositions 1 and 2, and are uncorrelated across j (and with each other). The results (B.1)-(B.3) are obtained applying the invariance principle to the partial sums of $\boldsymbol{\varepsilon}_t$ and $\boldsymbol{\eta}_t$ and using assumption 1 of non-stochastic regressors.

Denote the (j,h)-elements of the matrices $\mathbf{C}$, $\widehat{\mathbf{\Omega}}(m)$ by c_{jh}, $\widehat{\omega}_{jh}$. Then we have the following further asymptotic results:

$$T^{-2} c_{jh} \xrightarrow{d} (q_j q_h)^{1/2} \int_0^1 \left(\int_0^r W_j^X(s) ds \int_0^r W_h^X(s) ds \right) dr, \qquad j, h \leq K, \tag{B.4}$$

$$T^{-1} c_{jh} \xrightarrow{d} q_j^{1/2} \int_0^1 \left(\int_0^r W_j^X(s) ds \right) B_h^X(r) dr, \qquad j \leq K,\ h > K, \tag{B.5}$$

$$c_{jh} \xrightarrow{d} \int_0^1 B_j^X(r) B_h^X(r) dr, \qquad j, h > K, \tag{B.6}$$

$$(mT)^{-1} \widehat{\omega}_{jh} \xrightarrow{d} (q_j q_h)^{1/2} \int_0^1 W_j^X(r) W_h^X(r) dr, \qquad j, h \leq K, \tag{B.7}$$

$$\widehat{\omega}_{jh} = O_p(m), \qquad j \leq K,\ h > K, \tag{B.8}$$

$$\widehat{\omega}_{jh} \xrightarrow{p} 1(j = h), \qquad j, h > K. \tag{B.9}$$

(B.4)-(B.6) result directly from the application of the continuous mapping theorem; (B.7) corresponds to equation (23) of Kwiatowski et al. (1992); (B.9) holds because of the consistency of the long run variance estimator and (B.8) because $T^{-1}\sum_{t=\tau+1}^{T} e_{tj}e_{t-\tau,h} = O_p(1)$ uniformly in $\tau = -m, -m+1, ..., m$.

Our test statistic is defined as the trace of $\widehat{\mathbf{\Omega}}(m)^{-1}\mathbf{C}$, i.e. the sum of its eigenvalues. Let $\ell_1 \geq \ell_2 \geq ... \geq \ell_N \geq 0$ be the N ordered eigenvalues of $\widehat{\mathbf{\Omega}}(m)^{-1}\mathbf{C}$. Using the results (B.4)-(B.9) we will show that, under $\mathrm{H}_A : rank(\mathbf{\Sigma}_\eta) = K$, K eigenvalues are $O_p(T/m)$ and $N-K$ eigenvalues are $O_p(1)$. Thus asymptotically the distribution of the statistic coincides with the distribution of the sum of those K asymptotically bigger eigenvalues; see also Nyblom and Harvey (2000).

Partition $\widehat{\mathbf{\Omega}}(m)$ and $\mathbf{C}$ as

$$\widehat{\mathbf{\Omega}}(m) = \begin{bmatrix} \widehat{\mathbf{\Omega}}_{11} & \widehat{\mathbf{\Omega}}_{12} \\ \widehat{\mathbf{\Omega}}_{12}{}' & \widehat{\mathbf{\Omega}}_{22} \end{bmatrix} \text{ and } \mathbf{C} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{12}{}' & \mathbf{C}_{22} \end{bmatrix},$$

where $\widehat{\mathbf{\Omega}}_{11}$ and $\mathbf{C}_{11}$ are KxK. The eigenvalues of $\widehat{\mathbf{\Omega}}(m)^{-1}\mathbf{C}$ solve the determinantal equation

$$\begin{aligned} 0 &= \left|\mathbf{C} - \ell_j \widehat{\mathbf{\Omega}}(m)\right| \\ &= \left|\mathbf{C}_{11} - \ell_j \widehat{\mathbf{\Omega}}_{11}\right| \\ &\quad \times \left|\mathbf{C}_{22} - \ell_j \widehat{\mathbf{\Omega}}_{22} - \left(\mathbf{C}_{12} - \ell_j \widehat{\mathbf{\Omega}}_{12}\right)' \left(\mathbf{C}_{11} - \ell_j \widehat{\mathbf{\Omega}}_{11}\right)^{-1} \left(\mathbf{C}_{12} - \ell_j \widehat{\mathbf{\Omega}}_{12}\right)\right|, \end{aligned}$$

see Rao (1973, p.32). Then using (B.4)-(B.9) we see that the roots of the first determinant are $O_p(T/m)$, whereas the roots of the second determinant are $O_p(1)$ and asymptotically equivalent to the eigenvalues of $(\mathbf{C}_{22} - \mathbf{C}_{12}'\mathbf{C}_{11}^{-1}\mathbf{C}_{12})$; see Nyblom and Harvey (2000) for further details.

Therefore $(m/T)\,\xi_N \xrightarrow{d} tr\left(\mathbf{\Omega}_{11}^{*-1}\mathbf{C}_{11}^{*}\right)$, where

$$\begin{aligned} \mathbf{C}_{11}^{*} &: = \int_0^1 \left(\int_0^r \mathbf{W}^X(s)ds\right)\left(\int_0^r \mathbf{W}^X(s)ds\right)' dr, \\ \mathbf{\Omega}_{11}^{*} &: = \int_0^1 \mathbf{W}^X(r)\mathbf{W}^X(r)'dr, \end{aligned}$$

with $\mathbf{W}^X(r)$ being the K-dimensional vector process defined in the statement of the proposition.

C. The form of the generalized Brownian bridge for the statistic (11).

$$\mathbf{B}^1(r,\lambda) = \begin{cases} \mathbf{W}(r) - \frac{r}{\lambda}\mathbf{W}(\lambda) & \text{for } 0 \leq r \leq \lambda \\ (\mathbf{W}(r) - \mathbf{W}(\lambda)) - \frac{r-\lambda}{1-\lambda}(\mathbf{W}(1) - \mathbf{W}(\lambda)) & \text{for } \lambda < r \leq 1 \end{cases}$$

$$\mathbf{B}^2(r,\lambda) = \begin{cases} \mathbf{W}(r) - \frac{r}{\lambda}\mathbf{W}(\lambda) \\ \quad -\frac{6}{\lambda^3}r(r-\lambda)\left[\int_0^\lambda r d\mathbf{W}(r) - \frac{\lambda}{2}\mathbf{W}(\lambda)\right] & \text{for } 0 \leq r \leq \lambda \\ (\mathbf{W}(r) - \mathbf{W}(\lambda)) - \frac{r-\lambda}{1-\lambda}(\mathbf{W}(1) - \mathbf{W}(\lambda)) - \\ \quad -\frac{6}{(1-\lambda)^3}(r-1)(r-\lambda)\cdot \\ \quad \cdot\left[\int_\lambda^1 r d\mathbf{W}(r) - \frac{1+\lambda}{2}(\mathbf{W}(1) - \mathbf{W}(\lambda))\right] & \text{for } \lambda < r \leq 1 \end{cases}$$

$$\mathbf{B}^{2a}(r,\lambda) = \begin{cases} \mathbf{W}(r) - \frac{r}{\lambda}\mathbf{W}(\lambda) \\ \quad -\frac{6}{1-3\lambda+3\lambda^2}r(r-\lambda) \\ \quad \cdot\left[\int_0^1 r d\mathbf{W}(r) - \frac{\lambda}{2}\mathbf{W}(\lambda) - \frac{1+\lambda}{2}(\mathbf{W}(1) - \mathbf{W}(\lambda))\right] & \text{for } 0 \leq r \leq \lambda \\ (\mathbf{W}(r) - \mathbf{W}(\lambda)) - \frac{r-\lambda}{1-\lambda}(\mathbf{W}(1) - \mathbf{W}(\lambda)) \\ \quad -\frac{6}{1-3\lambda+3\lambda^2}r(r-\lambda)\cdot \\ \quad \cdot\left[\int_0^1 r d\mathbf{W}(r) - \frac{\lambda}{2}\mathbf{W}(\lambda) - \frac{1+\lambda}{2}(\mathbf{W}(1) - \mathbf{W}(\lambda))\right] & \text{for } \lambda < r \leq 1 \end{cases}$$

$$\mathbf{B}^{2b}(r,\lambda) = \begin{cases} \mathbf{W}(r) - r\mathbf{W}(1) - \frac{3}{\lambda^3(1-\lambda)^3} \\ \quad \cdot\left\{\left(a\frac{r^2}{2} - a\lambda r + \frac{r}{2}\left(a\lambda^2 - b(1-\lambda)^2\right)\right)\mathbf{J}_1\right. \\ \quad \left. + \left(b\frac{r^2}{2} - b\lambda r + \frac{r}{2}\left(b\lambda^2 - c(1-\lambda)^2\right)\right)\mathbf{J}_2\right\} \\ \qquad \text{for } 0 \leq r \leq \lambda \\ \mathbf{W}(r) - r\mathbf{W}(1) - \frac{3}{\lambda^3(1-\lambda)^3} \\ \quad \cdot\left\{\left(-a\frac{\lambda^2}{2} + b\frac{r^2-\lambda^2}{2} - b\lambda(r-\lambda) + \frac{r}{2}\left(a\lambda^2 - b(1-\lambda)^2\right)\right)\mathbf{J}_1\right. \\ \quad \left. + \left(-b\frac{\lambda^2}{2} + c\frac{r^2-\lambda^2}{2} - c\lambda(r-\lambda) + \frac{r}{2}\left(b\lambda^2 - c(1-\lambda)^2\right)\right)\mathbf{J}_2\right\} \\ \qquad \text{for } \lambda < r \leq 1 \end{cases}$$

where $\mathbf{W}(r)$, $r \in [0,1]$, is an N-dimensional standard Wiener process and

$$
\begin{aligned}
&a = (1-\lambda)^3(1+3\lambda),\\
&b = -3\lambda^2(1-\lambda)^2,\\
&c = \lambda^3(4-3\lambda),\\
&\mathbf{J}_1 = \textstyle\int_0^\lambda r d\mathbf{W}(r) - \lambda \mathbf{W}(\lambda) + \frac{\lambda^2}{2}\mathbf{W}(1),\\
&\mathbf{J}_2 = \textstyle\int_\lambda^1 r d\mathbf{W}(r) - \lambda\left(\mathbf{W}(1) - \mathbf{W}(\lambda)\right) - \frac{(1-\lambda)^2}{2}\mathbf{W}(1).
\end{aligned}
$$

D. Proof of proposition 3

Let $\mathbf{z}_t = 1$ for the case $i = 1$ and let $\mathbf{z}_t = (1,t)'$ for the case $i = 2$; drop the superscript i for simplicity. The model can be parametrized equivalently using $\mathbf{x}_t(\lambda) = (\mathbf{x}_{1t}(\lambda)', \mathbf{x}_{2t}(\lambda)')'$ as regressors, where $\mathbf{x}_{1t}(\lambda) = \mathbf{z}_t \cdot 1\left(t \leq \lambda T\right)$ and $\mathbf{x}_{2t} = \mathbf{z}_t \cdot 1\left(t > \lambda T\right)$. Then the model is orthogonal and in order to obtain the OLS residuals we can consider the two subsamples $\{1,...,T_1\}$,$\{T_1+1,...,T\}$ separately and in each of them regress $\mathbf{y}_t$ on $\mathbf{z}_t$.

Again, without loss of generality and under assumption 2, we can restrict ourselves to the case of $\mathbf{\Omega} = \mathbf{I}$ and $\mathbf{\Sigma}_\eta = diag(q_1, q_2, ..., q_N)$. Then, under assumptions 2-3, we can write $\xi_N^{i*}(\lambda) = tr\left[\left(\mathbf{I} + op(1)\right)^{-1}\left(\mathbf{C}^1(\lambda) + \mathbf{C}^2(\lambda)\right)\right]$. Under H_0, the invariance principle can be applied in each subsample, yielding

$$T_1^{-1/2}\sum_{t=1}^{[T_1 r]} \mathbf{e}_t(\lambda) \Rightarrow \mathbf{B}_1^i(r),$$

and

$$(T-T_1)^{-1/2}\sum_{t=1}^{[(T-T_1)r]} \mathbf{e}_{t+T_1}(\lambda) \Rightarrow \mathbf{B}_2^i(r), \qquad j = 1,2,...,N,$$

where $\mathbf{B}_1^i(r)$ and $\mathbf{B}_2^i(r)$ are N-dimensional independent standard vector Brownian bridges for $i = 1$ and second-level Brownian bridges for $i = 2$. Independence holds because the two Brownian bridges are the limit of partial sum processes containing non-overlapping subsets of disturbances.

By the continuous mapping theorem it then follows that $\xi_N^{i*}(\lambda)$ converges to the sum of two independent random variables, each with a Cramér-von Mises distribution with N degrees of freedom. The limiting distribution is then Cramér-von Mises with $2N$ degrees of freedom; see Nyblom (1989) and Busetti and Harvey (2000) for details on the additivity property of Cramér-von Mises random variables.

E. Proof of proposition 4

From appendix B we know that under assumptions 1-3 and under $rank(\mathbf{\Sigma}_\eta) = K$, the R smallest eigenvalues of $\widehat{\mathbf{\Omega}}^{-1}\mathbf{C}$ are asymptotically equivalent to the eigenvalues of $\mathbf{C}_{22} - \mathbf{C}_{12}{}'\mathbf{C}_{11}{}^{-1}\mathbf{C}_{12}$. Therefore, using the results (B.4)-(B.6),

$$\begin{aligned}
&T^{-2}\mathbf{C}_{11} \xrightarrow{d} \mathbf{Q}^{\frac{1}{2}}\mathbf{C}^*_{11}\mathbf{Q}^{\frac{1}{2}},\\
&T^{-1}\mathbf{C}_{12} \xrightarrow{d} \mathbf{Q}^{\frac{1}{2}}\mathbf{C}^*_{12},\\
&\mathbf{C}_{22} \xrightarrow{d} \mathbf{C}^*_{22},
\end{aligned}$$

where $\mathbf{C}^*_{ij}$, $i = 1, 2$, are defined in the statement of proposition 4 and $\mathbf{Q}^{\frac{1}{2}} = diag\left(q_1^{\frac{1}{2}}, ..., q_k^{\frac{1}{2}}\right)$. Thus, by the continuous mapping theorem,

$$\xi_{K,N} \xrightarrow{d} tr\left(\mathbf{C}^*_{22} - \mathbf{C}^{*\prime}_{12}\mathbf{C}^{*-1}_{11}\mathbf{C}^*_{12}\right).$$

F. Proof of proposition 5

Without loss of generality we assume $\boldsymbol{\delta}_T = \mathbf{I}$ in assumption 1 and $\mathbf{\Omega} = \mathbf{I}$ in assumption 2. The standardized partial sums of OLS residuals can be written as

$$T^{-\frac{1}{2}}\sum_{t=1}^{[Tr]}\mathbf{e}_t = T^{-\frac{1}{2}}\sum_{t=1}^{[Tr]}\boldsymbol{\varepsilon}_t - T^{\frac{1}{2}}(\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta})'T^{-1}\sum_{t=1}^{[Tr]}\mathbf{x}_t - T^{\frac{1}{2}}(\widehat{\boldsymbol{\gamma}}-\boldsymbol{\gamma})'T^{-1}\sum_{t=1}^{[Tr]}\mathbf{z}_t,\ \ r \in [0,1],$$

where

$$\begin{pmatrix} \widehat{\boldsymbol{\beta}}-\boldsymbol{\beta} \\ \widehat{\boldsymbol{\gamma}}-\boldsymbol{\gamma} \end{pmatrix} = \begin{pmatrix} \sum_{t=1}^T \mathbf{x}_t\mathbf{x}_t' & \sum_{t=1}^T \mathbf{x}_t\mathbf{z}_t' \\ \sum_{t=1}^T \mathbf{z}_t\mathbf{x}_t' & \sum_{t=1}^T \mathbf{z}_t\mathbf{z}_t' \end{pmatrix}^{-1} \begin{pmatrix} \sum_{t=1}^T \mathbf{x}_t\boldsymbol{\varepsilon}_t' \\ \sum_{t=1}^T \mathbf{z}_t\boldsymbol{\varepsilon}_t' \end{pmatrix}.$$

If we show that

$$T^{-1}\sum_{t=1}^{T}\mathbf{x}_t\mathbf{z}_t' \xrightarrow{p} \mathbf{0} \tag{F.1}$$

and

$$T^{\frac{1}{2}}(\widehat{\boldsymbol{\gamma}}-\boldsymbol{\gamma})'T^{-1}\sum_{t=1}^{[Tr]}\mathbf{z}_t \xrightarrow{p} \mathbf{0} \text{ uniformly in } r \in [0,1], \tag{F.2}$$

then $T^{-\frac{1}{2}}\sum_{t=1}^{[Tr]}\mathbf{e}_t \Rightarrow \mathbf{B}^X(r)$, the generalized Brownian bridge defined in proposition 2, and thus proposition 5 follows by applying the continuous mapping theorem.

Consider assumption 4[A] first. Condition (F.1) above holds because for each z_{it} element of $\mathbf{z}_t$ and each x_{jt} element of $\mathbf{x}_t$ we have

$$\mathrm{E}\left|T^{-1}\sum_{t=1}^{T} z_{it}x_{jt}\right|^2 \leq \max_{t,s}|x_{jt}x_{js}|\,T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}|\mathrm{E}\left(z_{it}z_{is}\right)| \to 0$$

by assumption 4[A1] and by recalling that without loss of generality x_{it} is assumed bounded throughout this proof. Then from assumption 4[A] it follows that $T^{-1}\sum_{t=1}^{T}\mathbf{z}_t\mathbf{z}_t' = O_p(1)$, $T^{-\frac{1}{2}}\sum_{t=1}^{T}\mathbf{z}_t\boldsymbol{\varepsilon}_t' = O_p(1)$ and $T^{-1}\sum_{t=1}^{[Tr]}\mathbf{z}_t = o_p(1)$ uniformly in $r \in [0,1]$; thus condition (F.2) holds too.

Now consider assumption 4[B]. To check condition (F.1) first note that $\sum_{t=1}^{T} z_{ht} = 1\left(T \neq ns,\ n = 1,2,...\right)$, for each $h = 1,...,s-1$. Then, using the summation by parts argument,

$$\begin{aligned}\left\|T^{-1}\sum_{t=1}^{T}\mathbf{x}_t z_{ht}\right\| &\leq \left\|T^{-1}\sum_{t=1}^{T-1}\left(\mathbf{x}_{t+1}-\mathbf{x}_t\right)\sum_{j=1}^{t}\mathbf{z}_{hj}\right\| + \left\|T^{-1}\mathbf{x}_T\sum_{t=1}^{T} z_{ht}\right\| \\ &\leq T^{-1}\sum_{t=1}^{T-1}\|\mathbf{x}_{t+1}-\mathbf{x}_t\| + T^{-1}\|\mathbf{x}_T\| \\ &\to 0 \quad \text{as } T \to \infty\end{aligned}$$

by the assumption of bounded variation. Condition (F.2) holds since $T^{-1}\sum_{t=1}^{T}\mathbf{z}_t\mathbf{z}_t' \to s^{-1}\boldsymbol{\Sigma}$, where $[\boldsymbol{\Sigma}]_{hl} = 1 + 1(h = l)$, and clearly $T^{-\frac{3}{2}}\sum_{t=1}^{T}\mathbf{z}_t\boldsymbol{\varepsilon}_t' = o_p(1)$.

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