Competitive Equilibrium in Asset Markets with Adverse Selection*

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Abstract

We develop a theory of equilibrium in asset markets with adverse selection. Traders can buy and sell an asset at any price. Sellers recognize that their trades may be rationed if they ask for a high price, while buyers recognize that they can only get a high quality good by paying a high price. These beliefs are consistent with rational behavior by the traders on the other side of the market. In the resulting equilibrium, the existence of low-quality assets reduces the liquidity and price-dividend ratio in the market for high quality assets. A larger player who purchases and destroys all the low quality assets will improve the liquidity and raise the price-dividend ratio for the remaining assets.

1 Introduction

This paper develops a dynamic equilibrium model of asset markets with adverse selection. Sellers can attempt to sell a durable asset at any price. Buyers must form rational expectations about the type of asset that is available at each price. In equilibrium, sellers are rationed by a shortage of buyers at all prices except the lowest one, and it is increasingly difficult to sell an asset at higher prices. This keeps the owners of low quality assets from trying to sell them at high prices. On the other hand, the owners of high quality assets are willing to set a high price despite the low sale probability because the asset is worth more to them if they fail to sell it.

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Our model offers an abstract view of an illiquid asset market, for example the market for asset-backed securities during the 2007–2008 financial crisis. Prior to the crisis, market participants viewed AAA securities as a safe investment, indistinguishable from Treasuries; indeed, they were treated as such by banking regulators. In the early stages of the crisis, investors started to recognize that some of these securities were likely to pay less than face value. Moreover, it was difficult to determine the exact assets that backed each individual security. Anticipating that she might later have to sell it, at this point it started to pay for the owner of an asset to learn its quality. On the other hand, it did not pay potential buyers to investigate the quality of all possible assets because they did not know which assets would later be for sale. This created an adverse selection problem, where sellers have superior information than buyers, as in the classic market for lemons (Akerlof, 1970).[1]

We predict that within an asset class, such as AAA-rated mortgage backed securities, a seller should always be able to sell an asset at a sufficiently low price. However, the owners of good quality assets will choose to hold out for a higher price, recognizing that there will be a shortage of buyers at that price and so it will take time to sell the asset. Moreover, the price that buyers are willing to pay for a high quality asset will be depressed because the market is less liquid. That is, even if a buyer somehow understood that a particular mortgage-backed security would pay the promised dividends with certainty, he would pay less for it than for a Treasury because he would anticipate having trouble reselling the MBS to future buyers who don't have his information. Illiquidity therefore serves to further depress asset prices. In particular, the ability of sellers to learn the quality of their assets will depress the liquidity and may depress the value of all securities even if the average the quality is unchanged.

An obvious solution to this problem is to have a third party evaluate the quality of the assets. Indeed, this is the role that the rating agencies were supposed to play. But the rating agencies lost their credibility during the crisis and there was no one with the reputation and capability to take their place. We find instead that there may have been a role for an investor with deep pockets, such as a government, to purchase low quality assets and alleviate the illiquidity of high quality ones. In particular, suppose the government stood ready to buy all assets at a moderate price. Any asset which the seller believed was worth less than that price, even if fully liquid, would be sold to the government, which in turn would take a loss on its purchases. The elimination of trade in low quality assets moderates the adverse selection problem. This makes all other assets more liquid and more expensive. Thus asset purchases can potentially alleviate both illiquidity and insolvency.

Our model is deliberately stylized. Assets are perfectly durable and pay a constant

[1]For a detailed description of the first phase of the crisis and an analysis of the source of the adverse selection problem, see Gorton (2008).

dividend, a nondurable consumption good. Better quality assets pay a higher dividend. Individuals are risk-neutral and have a discount factor that shifts randomly over time, creating a reason for trade. The only permissible trades are between the consumption good and the asset. Still, we believe this framework is useful for capturing our main idea that illiquidity may serve to separate high and low quality assets. In particular, it is a dynamic general equilibrium model in which the distribution of asset holdings evolves endogenously over time as individuals trade and experience preference shocks. We define a competitive equilibrium in this environment and prove that it is unique. In equilibrium, higher quality assets trade at a higher price but with a lower probability. Indeed, the expected revenue from selling a an asset, the product of its price and trading probability, is decreasing in the quality of the asset.

We also show that the trading frictions in this environment do not depend on any assumptions about the frequency of trading opportunities. Even with continuous trading opportunities, there are not enough buyers in the market for high quality assets and so it takes a real amount of calendar time to sell at a high price. This is in contrast to models that emphasize illiquidity in asset markets due to search frictions, such as Duffie, Gârleanu and Pedersen (2005), Weill (2008), and Lagos and Rocheteau (2009), where the economy converges to the frictionless outcome when the time between trading opportunities goes to zero. In our adverse selection economy, real trading delays are essential for separating the good assets from the bad ones. Of course, in reality adverse selection and search frictions may coexist in a market, and it is indeed straightforward to introduce search into our framework (Guerrieri, Shimer and Wright, 2010; Chang, 2010).

There is a large literature on dynamic adverse selection models. In many cases, the authors implicitly assume that all trades must take place at one price, so there is necessarily a pooling equilibrium (e.g. Eisfeldt, 2004; Kurlat, 2009; Daley and Green, 2010; Chari, Shourideh and Zetlin-Jones, 2010). This implies that sellers choose not to sell some assets because the price is too low, and so in a sense these models also deliver illiquidity. In our model, in contrast, sellers try to sell all their assets, but most only sell with some probability between 0 and 1. Moreover, our model allows for the possibility that a seller can demand a high price for her asset, something that models which impose a uniform price cannot address. At the end of our paper, we compare our notion of equilibrium to an environment in which we impose that all trades must take place at a common price. A number of substantive results differ. For example, our equilibrium is unique while equilibrium is generally not unique in the pooling environment. In addition, public asset purchases, a policy intervention that is important in practice, are more effective in our environment than in the economy with pooling.

This paper builds on our previous work with Randall Wright, Guerrieri, Shimer and Wright (2010). It also complements a contemporaneous paper by Chang (2010). There are a number of small differences between that paper and this one. For example, we look at an environment in which individuals may later want to resell assets that they purchase today. This means that buyers care about the liquidity of the asset and so liquidity affects the equilibrium price-dividend ratio. It follows that interventions in the market which boost liquidity may also raise asset prices. We allow individuals to hold multiple assets, although that turns out to be inessential for our analysis. We also focus more explicitly on general equilibrium, allowing for the possibility that buyers may be driven to a corner in which they do not consume anything. Still, both papers leverage our earlier research to study separating equilibria in a dynamic adverse selection environment.

This paper proceeds as follows. Section 2 describes our basic model. Section 3 describes the individual's problem and shows how to express it recursively. Section 4 defines equilibrium and establishes existence and uniqueness. Section 5 provides closed-form solutions for a version of the model with a continuum of assets. Section 6 extends the model to have persistent preference shocks and then shows that the frictions survive in the continuous time limit. Section 7 discusses how our model can generate fire sales following the revelation of some information and how illiquidity and insolvency can be alleviated through an asset purchase program, although the program necessarily loses money. Section 8 compares the implications of our equilibrium concept to an environment in which a pooling equilibrium is imposed.

2 Model

There is a unit measure of risk-neutral individuals. In each period t, they can be in one of two states, $s_t \in \{l, h\}$, which determines their discount factor β_{s_t} between periods t and $t+1$. We assume $0 < \beta_l < \beta_h < 1$. The preference shock is independent across individuals and for now we assume that it is also independent over time. Thus π_s denotes the probability that an individual is in state $s \in \{l, h\}$ in any period, and it is also the fraction of individuals who are in state s in any period. Let $\bar{\beta} = \pi_l \beta_l + \pi_h \beta_h$ denote the expected discount factor. For any particular individual, let $s^t \equiv \{s_0, \dots, s_t\}$ denote the history of states through period t.

There is a finite number of different types of assets, distinguished by their type $j \in \{1, \dots, J\}$. Assets are perfectly durable and so their supply is fixed; let K_j denote the measure of type j assets in the economy. Each type j asset produces δ_j units of a homogeneous, nondurable consumption good each period, and so aggregate consumption $\sum_{j=1}^{J} \delta_j K_j$ is fixed. Without loss of generality, assume that higher type assets produce more of the consumption good, $0 \leq \delta_1 < \cdots < \delta_J$. The assumption that there is a finite number of asset types simplifies our notation, but in Section 5, we discuss the limiting case with a continuum of assets.

We are interested in how a market economy allocates consumption across individuals. For the remainder of the paper, we refer to the assets as "trees" and the consumption good as "fruit." The timing of events within period t is as follows:

1. each individual i owns a vector $\{k_{i,j}\}_{j=1}^J$ of trees which produce fruit;
2. each individual's discount factor between periods t and $t+1$ is realized;
3. individuals trade trees for fruit in a competitive market;
4. individuals consume the fruit that they hold.

We require that each individual's consumption is nonnegative in every period and we do not allow any other trades, e.g. contingent claims against shocks to the discount factor. In addition, we assume that only the owner of a tree can observe its quality, creating an adverse selection problem. Key to our equilibrium concept, which we discuss below, is that the buyer of a tree may be able to infer its quality from the price at which it is sold. Finally, we impose that only individuals with low discount factors may sell trees and henceforth call them "sellers." For symmetry, we refer to individuals with high discount factors as buyers. This configuration is reasonable in the sense that, absent an adverse selection problem, individuals with high discount factors would buy trees from individuals with low discount factors, transferring consumption from those with a high intertemporal marginal rate of substitution to those with a low one.

We now describe the competitive fruit market more precisely. After trees have borne fruit, a continuum of markets distinguished by their positive price $p \in \mathbb{R}_+$ open up. Each buyer may take his fruit to any market (or combination of markets), attempting to purchase trees in that market. Each seller may take his trees to any market (or combination of markets) attempting to sell trees in that market.

All individuals have rational beliefs about the ratio of buyers to sellers in all markets. Let $\Theta(p)$ denote the ratio of the amount of fruit brought by buyers to a market p, relative to the cost of purchasing all the trees in that market at a price p. In other words, if $\Theta(p) < 1$, there is not enough fruit to purchase all the trees offered for sale in the market, while if $\Theta(p) > 1$, there is more than enough. A seller believes that if he brings a tree to a market p, it will sell with probability $\min\{\Theta(p), 1\}$. That is, if there are excess trees in the market, the seller believes that he will succeed in selling it only probabilistically. Likewise, a buyer who brings p units of fruit to market p believes that he will buy a tree with probability $\min\{\Theta(p)^{-1}, 1\}$.

If there is excess fruit in the market, he will be rationed. A seller who is rationed keeps his tree until the following period, while a buyer who is rationed must eat his fruit.

Individuals also have rational beliefs about the types of tree sold in each market. Let $\Gamma(p) \equiv \{\gamma_j(p)\}_{j=1}^J \in \Delta^J$ denote the probability distribution over trees available for sale in a market p, where Δ^J is the J-dimensional unit simplex.[2] Buyers expect that, conditional on buying a tree at a price p, it will be a type j tree with probability $\gamma_j(p)$. Buyers only learn the quality of the tree that they have purchased after giving up their fruit. They have no recourse if unsatisfied with the quality.

Although trade does not happen at every price p, the functions Θ and Γ are not arbitrary. Instead, if $\Theta(p) < \infty$ (the buyer-seller ratio is finite) and $\gamma_j(p) > 0$ (a positive fraction of the trees for sale are of type j), sellers must find it weakly optimal to sell type j trees at price p. Without this restriction on beliefs, there would be equilibria in which, for example, no one pays a high price for a tree because everyone believes that they will only purchase low quality trees at that price.

3 Individual's Problem

We start by writing down the problem faced by an individual in an environment with arbitrary functions Θ and Γ. The individual recognizes that he can in principle trade at any positive price p. However, he may be rationed at some prices and the price may affect the quality of tree that he buys.

For any period t, history s^{t-1}, and type $j \in \{1, \dots, J\}$, let $k_{i,j,t}(s^{t-1})$ denote individual i's beginning-of-period t holdings of type j trees. For any period t, history s^t, type $j \in \{1, \dots, J\}$, and set $P \subset \mathbb{R}_+$, let $q_{i,j,t}(P; s^t)$ denote his net purchase in period t of type j trees at a price $p \in P$. The individual chooses a history-contingent sequence for consumption $c_{i,t}(s^t)$ and measures of tree holdings $k_{i,j,t+1}(s^t)$ and net tree purchases $q_{i,j,t}(P; s^t)$ to maximize his expected lifetime utility

$$\sum_{t=0}^{\infty} \sum_{s^t} \left(\prod_{\tau=0}^{t-1} \pi_{s_\tau} \beta_{s_\tau} \right) \pi_{s_t} c_{i,t}(s^t).$$

This simply states that the individual maximizes the expected discounted value of consumption, given the stochastic process for the discount factor. The individual faces a standard

[2] That is, $\gamma_j(p) \geq 0$ for all j and $\sum_{j=1}^J \gamma_j(p) = 1$.

budget constraint,

$$\sum_{j=1}^{J} \delta_j k_{i,j,t}(s^{t-1}) = c_{i,t}(s^t) + \int_0^\infty p \left(\sum_{j=1}^{J} q_{i,j,t}(\{p\}; s^t) \right) dp,$$

for all t and s^t. The left hand side is the fruit produced by the trees he owns at the start of period t. The right hand side is consumption plus the net purchase of trees at nonnegative prices p. He also faces a law of motion for his tree holdings,

$$k_{i,j,t+1}(s^t) = k_{i,j,t}(s^{t-1}) + q_{i,j,t}(\mathbb{R}_+; s^t),$$

for all $j \in \{1, \ldots, J\}$. This states that the increase in his tree holdings is given by his net purchase of that type of tree. Finally, the individual faces a set of constraints that depends on whether his discount factor is high or low.

If the individual has a high discount factor, $s_t = h$, he is a buyer, which implies $q_{i,j,t}(P; s^t)$ is nonnegative for all $j \in \{1, \ldots, J\}$ and $P \subset \mathbb{R}_+$. In addition, he must have enough fruit to purchase trees,

$$\sum_{j=1}^{J} \delta_j k_{i,j,t}(s^{t-1}) \geq \int_0^\infty \max\{\Theta(p), 1\} p \left(\sum_{j=1}^{J} q_{i,j,t}(\{p\}; s^t) \right) dp.$$

If the individual wishes to purchase q trees at a price p and $\Theta(p) > 1$, he will be rationed and so must bring $\Theta(p)pq$ fruit to the market to make this purchase. This constrains his ability to buy trees in markets with excess demand. Together with the budget constraint, this also ensures consumption is nonnegative. Finally, he can only purchase type j trees at a price p if individuals are selling them at that price, that is

$$q_{i,j,t}(P; s^t) = \int_P \gamma_j(p) \left(\sum_{j'=1}^{J} q_{i,j',t}(\{p\}; s^t) \right) dp$$

for all $j \in \{1, \ldots, J\}$ and $P \subset \mathbb{R}_+$. The left hand side is the quantity of type j trees purchased at a price $p \in P$. The integrand on the right hand side is the product of quantity of trees purchased at price p and the share of those trees that are of type j.

If the individual has a low discount factor, $s_t = l$, he is a seller, which implies $q_{i,j,t}(P; s^t)$ is nonpositive for all $j \in \{1, \ldots, J\}$ and $P \subset \mathbb{R}_+$. In addition, he may not try to sell more trees than he owns:

$$k_{i,j,t}(s^{t-1}) \geq -\int_0^\infty \max\{\Theta(p)^{-1}, 1\} q_{i,j,t}(\{p\}; s^t) dp,$$

for all $j \in \{1, \ldots, J\}$. Each tree only sells with probability $\min\{\Theta(p), 1\}$ at price p, so if $\Theta(p) < 1$, an individual must bring $\Theta(p)^{-1}$ trees to the market to sell one of them. Sellers are not restricted from selling trees in the wrong market. Instead, in equilibrium they will be induced not to do so.

Let $\bar{V}^*(\{k_j\})$ be the supremum of the individuals' expected lifetime utility over feasible policies, given initial tree holding vector $\{k_j\}$. We prove in Proposition 1 that the function $\bar{V}^*$ satisfies the following functional equation:

$$\bar{V}(\{k_j\}) = \pi_h V_h(\{k_j\}) + \pi_l V_l(\{k_j\}), \tag{1}$$

where

$$V_h(\{k_j\}) = \max_{\{q_j, k_j'\}} \left(\sum_{j=1}^{J} \delta_j k_j - \int_0^\infty p \left(\sum_{j=1}^{J} q_j(\{p\}) \right) dp + \beta_h \bar{V}(\{k_j'\}) \right) \tag{2}$$

$$\begin{aligned}
\text{subject to } & k_j' = k_j + q_j(\mathbb{R}_+) \text{ for all } j \in \{1, \ldots, J\} \\
& \sum_{j=1}^{J} \delta_j k_j \geq \int_0^\infty \max\{\Theta(p), 1\} p \left(\sum_{j=1}^{J} q_j(\{p\}) \right) dp, \\
& q_j(P) = \int_P \gamma_j(p) \left(\sum_{j=1}^{J} q_j(\{p\}) \right) dp \text{ for all } j \in \{1, \ldots, J\} \text{ and } P \subset \mathbb{R}_+ \\
& q_j(P) \geq 0 \text{ for all } j \in \{1, \ldots, J\} \text{ and } P \subset \mathbb{R}_+,
\end{aligned}$$

and

$$V_l(\{k_j\}) = \max_{\{q_j, k_j'\}} \left(\sum_{j=1}^{J} \delta_j k_j - \int_0^\infty p \left(\sum_{j=1}^{J} q_j(\{p\}) \right) dp + \beta_l \bar{V}(\{k_j'\}) \right) \tag{3}$$

$$\begin{aligned}
\text{subject to } & k_j' = k_j + q_j(\mathbb{R}_+) \text{ for all } j \in \{1, \ldots, J\} \\
& k_j \geq - \int_0^\infty \max\{\Theta(p)^{-1}, 1\} q_j(\{p\}) dp \text{ for all } j \in \{1, \ldots, J\}, \\
& q_j(P) \leq 0 \text{ for all } j \in \{1, \ldots, J\} \text{ and } P \subset \mathbb{R}_+,
\end{aligned}$$

From now on, we work with the recursive version of the individuals' problem.

Given the linearity of the problem, it is not surprising that the value functions are linear in tree holdings. That is, $V_s(\{k_j\}) = \sum_{j=1}^{J} v_{s,j} k_j$ for $s \in \{l, h\}$ and $\bar{V}(\{k_j\}) = \sum_{j=1}^{J} \bar{v}_{l,j} k_j$, where $v_{s,j}$ denotes the expected value of a tree of type j tree for an agent with discount factor β_s and $\bar{v} \equiv \pi_h v_{h,j} + \pi_l v_{l,j}$ is the expected value of a type j tree before the agent learns her discount factor.

These marginal value functions satisfy relatively simple recursive problems. A seller solves

$$v_{l,j} = \delta_j + \max_p \big(\min\{\Theta(p), 1\} p + (1 - \min\{\Theta(p), 1\}) \beta_l \bar{v}_j \big). \tag{4}$$

The individual earns a dividend δ_j from the tree and also gets p units of fruit if he manages to sell the tree at the chosen price p. Otherwise he keeps the tree until the following period. Note that an individual with a low discount factor never strictly prefers to hold onto a tree rather than try to sell it, since he can always offer it at a high price $p > \beta_l \bar{v}_j$. Of course, at such a high price, he may be unable to sell it, $\Theta(p) = 0$, in which case the outcome is the same as holding onto the tree.

For a buyer, a type j tree delivers δ_j units of fruit, each of which may either be consumed or used to purchase trees. A key result is that the value of fruit to a buyer is independent of the type of tree that produced that fruit. Define $1 + \lambda$ to be the value of a unit of fruit to a buyer. If $\lambda = 0$, the individual finds it weakly optimal to eat the fruit, while if $\lambda > 0$ he strictly prefers to purchase trees. Then

$$v_{h,j} = \delta_j (1 + \lambda) + \beta_h \bar{v}_j \tag{5}$$

In addition, the value of a unit of fruit in excess of its consumption utility satisfies

$$\lambda \equiv \max_{p \geq 0} \left(\min\{\Theta(p)^{-1}, 1\} \left(\frac{\beta_h \sum_{j=1}^{J} \gamma_j(p) \bar{v}_j}{p} - 1 \right) \right), \tag{6}$$

with $\lambda = 0$ if the maximum value is negative. The buyer uses the fruit to attempt to purchase $1/p$ trees at an optimally chosen price p. If he succeeds, with probability $\Theta(p)^{-1}$ if $\Theta(p) > 1$ and probability 1 otherwise, he enjoys the expected value of the tree next period but gives up a unit of fruit.

Proposition 1 *Let $\{v_{s,j}\}$ and λ be positive-valued numbers that solve the Bellman equations* (4), (5), *and* (6) *for $s = l, h$. Then $\bar{V}^*(\{k_j\}) \equiv \sum_{j=1}^{J} \bar{v}_j k_j$ for all k, where $\bar{v}_j = \pi_h v_{h,j} + \pi_l v_{l,j}$.*

All proofs are in the appendix. Note that for some choices of of the functions Θ and Γ, there is no positive-valued solution to the Bellman equation. In this case, the price of trees is so low that it is possible for an individual to obtain unbounded utility and there is no solution to the individual's problem. Not surprisingly, this cannot be the case in equilibrium.

4 Equilibrium

4.1 Partial Equilibrium

We are now ready to define equilibrium. We do so in two steps, first focusing on a partial equilibrium where the buyer's value of fruit λ is fixed:

Definition 1 *A partial equilibrium with adverse selection for fixed $\lambda \geq 0$ is a pair of vectors $\{v_{h,j}\} \in \mathbb{R}_+^J$ and $\{v_{l,j}\} \in \mathbb{R}_+^J$, functions $\Theta : \mathbb{R}_+ \mapsto [0, \infty]$ and $\Gamma : \mathbb{R}_+ \mapsto \Delta^J$, a set of prices $\mathbb{P} \subset \mathbb{R}_+$, and a measure μ defined on subsets of $\mathbb{P}$ satisfying the following conditions:*

1. *consistency of the value functions: for all $j \in \{1, \ldots, J\}$,*

$$
\begin{aligned}
v_{h,j} &= \delta_j(1+\lambda) + \beta_h \bar{v}_j, \\
v_{l,j} &= \delta_j + \max\left\{0, \max_{p \in \mathbb{P}} \left(\min\{\Theta(p), 1\}(p - \beta_l \bar{v}_j)\right)\right\} + \beta_l \bar{v}_j,
\end{aligned}
$$

where $\bar{v}_j \equiv \pi_h v_{h,j} + \pi_l v_{l,j}$.

2. *sellers' optimality: for all $p \in \mathbb{R}_+$ and $j \in \{1, \ldots, J\}$,*

$$
v_{l,j} \geq \delta_j + \min\{\Theta(p), 1\}(p - \beta_l \bar{v}_j) + \beta_l \bar{v}_j,
$$

with equality if $\Theta(p) < \infty$ and $\gamma_j(p) > 0$. Moreover, if $p < \beta_l \bar{v}_j$, either $\Theta(p) = \infty$ or $\gamma_j(p) = 0$ or both.

3. *buyers' optimality: for all $p \in \mathbb{R}_+$,*

$$
\lambda \geq \min\{\Theta(p)^{-1}, 1\} \left(\frac{\beta_h \sum_{j=1}^J \gamma_j(p) \bar{v}_j}{p} - 1 \right),
$$

with equality if $p \in \mathbb{P}$.

4. *all trees are offered for sale at some price $p \in \mathbb{P}$:*

$$
\pi_l K_j = \int_{\mathbb{P}} \gamma_j(p) \mu(\{p\}) dp \text{ for all } j \in \{1, \ldots, J\}.
$$

The first condition simply repeats the value functions (4) and (5), but recognizes that sellers can only sell trees at a price p if there is trade at that price, $p \in \mathbb{P}$. The second and third conditions mimic the definition of equilibrium in Guerrieri, Shimer and Wright (2010). Sellers' optimality requires that for every type of tree j, the value of selling it at an arbitrary price p (possibly with $p \notin \mathbb{P}$) cannot exceed $v_{l,j}$. Moreover, if the buyer-seller ratio at p is finite and buyers expect to be able to purchase type j trees with positive probability, then this must be a best price for selling that type of tree. Buyers' optimality requires that the value of a unit of fruit in excess of its consumption value cannot exceed λ and must be equal to λ if there is trade at that price in equilibrium. The fourth condition imposes that the quantity of type j trees held by sellers—the left hand side of the equation—is equal to the quantity of those trees offered for sale at some price $p \in \mathbb{P}$—the right hand side.

For fixed λ, we find the partial equilibrium as the solution to a sequence of optimization problems:

$$\begin{aligned} v_{l,j} = \delta_j + \max_{p,\theta} \big(&\min\{\theta,1\} p + (1-\min\{\theta,1\})\beta_l \bar{v}_j \big) \\ \text{s.t. } &\lambda \le \min\{\theta^{-1},1\} \left(\frac{\beta_h \bar{v}_j}{p} - 1 \right), \\ &v_{l,j'} \ge \delta_{j'} + \min\{\theta,1\}p + (1-\min\{\theta,1\})\beta_l \bar{v}_{j'} \text{ for all } j' < j \end{aligned} \tag{P-j}$$

where

$$\bar{v}_j = \frac{\pi_h \delta_j (1+\lambda) + \pi_l v_{l,j}}{1-\pi_h\beta_h}.$$

To solve these problems, start with type 1 trees. The last constraint disappears from Problem (P-1), and so we can solve directly for $v_{l,1}$ and $\bar{v}_1$, as well as the optimal policy p_1 and θ_1. Standard arguments ensure that the solution is unique if $\lambda \ge 0$. In general, for Problem (P-j), the first constraint and the constraint of excluding type $j-1$ trees binds, which determines p_j and θ_j as well as $v_{l,j}$ and $v_{h,j}$. The following Lemma states this claim formally, focusing on the determination of the expected marginal value of a type j tree, $\bar{v}_j$.

Lemma 1 *For fixed $\lambda \in [0, \beta_h/\beta_l - 1]$, the solution to the sequence of Problems (P-j) is as follows: If $\lambda = 0$, $\theta_1 \ge 1$. If $\lambda = \beta_h/\beta_l - 1$, $\theta_1 \in [0,1]$. Otherwise $\theta_1 = 1$. In any case,*

$$p_1 = \frac{\delta_1\beta_h(1+\pi_h\lambda)}{1+\lambda-\beta_h(1+\pi_h\lambda)} \text{ and } \bar{v}_1 = \frac{\delta_1(1+\lambda)(1+\pi_h\lambda)}{1+\lambda-\beta_h(1+\pi_h\lambda)}.$$

For $j \in \{2,\dots,J\}$, $\theta_j \le \theta_{j-1}$, $p_j > p_{j-1}$, and $\bar{v}_j > \bar{v}_{j-1}$ are uniquely defined by the following recursive system of equations:

$$\begin{aligned} p_j &= \frac{\beta_h \bar{v}_j}{1+\lambda}, \\ \bar{v}_j &= \frac{\delta_j(1+\pi_h\lambda) + \pi_l\theta_j p_j}{1-\pi_h\beta_h - \pi_l\beta_l(1-\theta_j)}, \\ \theta_j(p_j - \beta_l\bar{v}_{j-1}) &= \min\{\theta_{j-1},1\}(p_{j-1}-\beta_l\bar{v}_{j-1}). \end{aligned}$$

We focus on values of λ between 0 and $\beta_h/\beta_l - 1$ because these are the relevant ones for equilibrium. One could, however, also characterize the partial equilibrium for $\lambda > \beta_h/\beta_l - 1$.

Proposition 2 *Fix $\lambda \in [0, \beta_h/\beta_l - 1]$. There exists a partial equilibrium and any partial equilibrium is given by the solution to the Problems (P-j). More precisely:*

- *Existence: Take any $\{p_j\}$, $\{\theta_j\}$, $\{v_{h,j}\}$, and $\{v_{l,j}\}$ that solve the set of problems $\{$(P-j)$\}$. Then there exists a partial equilibrium $(v_h, v_l, \Theta, \Gamma, \mathbb{P}, \mu)$ where $\Theta(p_j) = \theta_j$, $\gamma_j(p_j) = 1$, $v_h = \{v_{h,j}\}$, $v_l = \{v_{l,j}\}$, $\mathbb{P} = \{p_j\}$, and $\mu(\{p_j\}) = \pi_l K_j$.*

- *Uniqueness: Take any partial equilibrium $(v_h, v_l, \Theta, \Gamma, \mathbb{P}, \mu)$. If $p_j \in \mathbb{P}$ and $\gamma_j(p_j) > 0$, then $(p_j, \Theta(p_j))$ solves Problem (P-j).*

The proof gives a complete characterization of the partial equilibrium and proves that any allocation that does not solve Problem (P-j) is not a partial equilibrium. Since we proved in Lemma 1 that the solution to problems (P-j) is unique, except possibly for the value of θ_1, this essentially proves uniqueness of the partial equilibrium.

4.2 Competitive Equilibrium

We now turn to a full competitive equilibrium in which λ is endogenous:

Definition 2 *A competitive equilibrium with adverse selection is a number $\lambda \in [0, \beta_h/\beta_l - 1]$, a pair of vectors $\{v_{h,j}\} \in \mathbb{R}_+^J$ and $\{v_{l,j}\} \in \mathbb{R}_+^J$, functions $\Theta : \mathbb{R}_+ \mapsto [0, \infty]$ and $\Gamma : \mathbb{R}_+ \mapsto \Delta^J$, a set of prices $\mathbb{P} \subset \mathbb{R}_+$, and a measure μ defined on subsets of $\mathbb{P}$ satisfying the following conditions:*

1. *$(\{v_{h,j}\}, \{v_{l,j}\}, \Theta, \Gamma, \mathbb{P}, \mu)$ is a partial equilibrium with fixed λ; and*
2. *markets clear: $\pi_h \sum_{j=1}^{J} \delta_j K_j = \int_{\mathbb{P}} \Theta(p) p \mu(\{p\}) dp$.*

A competitive equilibrium is a partial equilibrium plus the market clearing condition that states that the fruit brought to market by buyers is equal to the value of trees brought to the market by sellers. Recall from Proposition 2 that $\mu(\{p_j\}) = \pi_l K_j$ in partial equilibrium, where p_j is the equilibrium price of type j trees. Then the market clearing condition reduces to

$$\pi_h \sum_{j=1}^{J} \delta_j K_j = \pi_l \sum_{j=1}^{J} \Theta(p_j) p_j K_j.$$

The left hand side is the fruit held by buyers at the start of the period, while each term in the right hand side is the equilibrium cost of purchasing a particular type of tree multiplied by the buyer-seller ratio for that tree.

Proposition 3 *A competitive equilibrium* $(\lambda, v_h, v_l, \Theta, \Gamma, \mathbb{P}, \mu)$ *exists and is unique.*

The proof (to be completed)[3] shows that an increase in the value of fruit to a buyer λ drives down the price of type j trees, i.e. p_j such that $\mu(p_j) > 0$, $\theta(p_j) < \infty$, and $\gamma_j(p_j) > 0$. In addition, it makes it more difficult for sellers of good quality trees to separate themselves from those selling bad trees, reducing $\Theta(p_j)$ as well. Indeed, in the limit when $\lambda = \beta_h/\beta_l - 1$, $\Theta(p_j) = 0$ for all $j > 1$, and so trade breaks down in all but the worst type of tree. At the opposite limit of $\lambda = 0$, buyers are indifferent about purchasing trees and so $\Theta(p_1) > 1$ and buyers are rationed. By varying λ, we find the unique value at which the market clear.

5 Continuous Types of Trees

We have assumed for notational convenience that there are only a finite number of types of trees. It is conceptually straightforward to extend our analysis to an environment with a continuum of trees. This is useful because it shows that the behavior of the economy is not particularly sensitive to the number of types of trees, but rather it depends on the support of the tree distribution.

Instead of redoing all our work, we take the limit as the tree distribution becomes dense but atomless on some interval of the real line $[\underline{\delta}, \bar{\delta}]$. We let $\kappa(\delta)$ denote the density of trees on this support. The key to our analysis is that for a fixed value of λ, the partial equilibrium prices and values characterized in Proposition 2 depend only on the support of the tree distribution. In particular, the price and expected value of the lowest quality tree is

$$P(\underline{\delta}) = \frac{\underline{\delta}\beta_h(1+\pi_h\lambda)}{1+\lambda-\beta_h(1+\pi_h\lambda)} \text{ and } \bar{v}(\underline{\delta}) = \frac{\underline{\delta}(1+\lambda)(1+\pi_h\lambda)}{1+\lambda-\beta_h(1+\pi_h\lambda)}.$$

In addition, since the distribution of trees is atomless, we may simply assume $\Theta(P(\underline{\delta})) = 1$.[4] For $p < P(\underline{\delta})$, $\Theta(p) = \infty$ and $\Gamma(p)$ is defined arbitrarily. These results are unchanged from the model with a finite number of types of trees.

To analyze higher values of p, start with the condition that the seller of a type $j-1$ tree must be indifferent about representing it as a type j tree:

$$\Theta(p_j)(p_j - \beta_l\bar{v}_{j-1}) = \min\{\Theta(p_{j-1}), 1\}(p_{j-1} - \beta_l\bar{v}_{j-1}).$$

[3] We have a proof of this result for the economy with a continuum of types of trees in Section 5, available upon request.

[4] One important difference between the economy with a continuum of trees and the economy with finitely many types of trees is the market clearing condition when $\lambda = 0$. In the economy with finitely many types of trees, we used $\Theta(p_1) \geq 1$ to ensure that buyers brought all their trees to the market even when $\lambda = 0$. Here it is easier to allow buyers to consume some of their fruit and impose $\Theta(P(\underline{\delta})) = 1$.

When the types of trees are dense, we can rewrite this as a differential equation. That is, differentiate the right hand side with respect to p_{j-1} and evaluate at $p_{j-1} = p_j$ and $\bar{v}_{j-1} = \bar{v}_j$:

$$\Theta'(p_j)(p_j - \beta_l \bar{v}_j) + \Theta(p_j) = 0.$$

Also eliminate $\bar{v}_j$ using the buyer's indifference condition $\bar{v}_j = p_j(1+\lambda)/\beta_h$. This gives

$$\Theta'(p)p\left(\frac{\beta_h - \beta_l(1+\lambda)}{\beta_h}\right) + \Theta(p) = 0.$$

If $\lambda = \beta_h/\beta_l - 1$, this implies $\Theta(p) = 0$ for all $p > P(\underline{\delta})$. Otherwise, solve this differential equation using the terminal condition $\Theta(P(\underline{\delta})) = 1$ to get

$$\Theta(p) = \left(\frac{P(\underline{\delta})}{p}\right)^{\frac{\beta_h}{\beta_h - \beta_l(1+\lambda)}} \tag{7}$$

for $p > P(\underline{\delta})$. Finally, we compute the type of tree offered at a price p. We do this by eliminating $\bar{v}_j$ from the Bellman equation again using the buyer's indifference condition $\bar{v}_j = p_j(1+\lambda)/\beta_h$. This gives $\gamma_j(p) = 1$ if and only if $\delta_j = D(p)$ where

$$D(p) = p\left(\frac{1+\lambda+(\beta_h-(1+\lambda)\beta_l)(1-\Theta(p))(1-\pi_h)}{\beta_h(1+\lambda\pi_h)} - 1\right), \tag{8}$$

so buyers anticipate buying type $D(p)$ trees (and only type $D(p)$ trees) at a price p.

These equations hold as long as $\bar{\delta} \geq D(p)$. For higher prices, $\Theta(p)$ is pinned down by the indifference curve of the seller of the best type of tree.

One can of course also define a competitive equilibrium with adverse selection in this environment, so market clearing determines λ. One can prove that there is never an equilibrium with $\lambda = \beta_h/\beta_l - 1$. In such an equilibrium, buyers would only purchase the worst type of tree, but since the tree distribution is atomless, this would not use any of their fruit. Equilibrium then imposes

$$\pi_h \int_{\underline{\delta}}^{\bar{\delta}} \delta\kappa(\delta)d\delta \geq \pi_l \int_{\underline{\delta}}^{\bar{\delta}} \Theta(P(\delta))P(\delta)\kappa(\delta)d\delta,$$

with equality if $\lambda > 0$. Using the functional forms of Θ and P, one can prove directly that an increase in λ reduces the right hand side of this inequality, ensuring that the equilibrium is unique. The proof is available from the authors upon request.

6 Persistent Shocks and Continuous Time

Our model explains how adverse selection can generate liquidity frictions, in the sense that a tree only sells with a certain probability each period. But suppose that the time between periods is negligible. Will the trading frictions become negligible as well? We argue in this section that they will not. Instead, a separating equilibrium requires a real amount of calendar time before a high quality tree is sold.

To address this concern, we consider the limiting behavior of the economy when the number of periods per unit of calendar time becomes very large. That is, we take the limit as the discount factors converge to 1, holding fixed the ratio of discount rates $(1-\beta_h)/(1-\beta_l)$. But as we take this limit, we also want to avoid changing the stochastic process of shocks. With i.i.d. shocks and very short time periods, there is almost no difference in preferences between high and low types of individuals and so the gains from trade become negligible. We therefore also introduce persistent shocks into the basic model. We prove that as the period length shortens, the probability of sale in a given period falls to zero, while the probability of sale per unit of calendar time converges to a well-behaved number.

We start by introducing persistent shocks into the discrete time model. Assume that $s_t \in \{l, h\}$ follows a first order stochastic Markov process; let $\pi_{ss'}$ denote the probability that the state next period is s' given that the current state is s. A partial equilibrium with a fixed value of $\lambda \geq 0$ is still characterized by a pair of value functions $\{v_{s,j}\} \in \mathbb{R}_+^J$ that represent the value of a type s individual holding a type j tree, a function $\Theta : \mathbb{R}_+ \mapsto [0, \infty]$, a function $\Gamma : \mathbb{R}_+ \mapsto \Delta^J$, a set of prices $\mathbb{P} \subset \mathbb{R}_+$, and a measure μ defined on subsets of $\mathbb{P}$. In partial equilibrium, buyers and sellers optimize and beliefs are rational. A competitive equilibrium fixes a value of λ such that markets clear.

To simplify the exposition, we focus on parameters such that in equilibrium $\lambda = 0$, so buyers are indifferent between buying trees and consuming. In this case, problem (P-j) becomes

$$
\begin{aligned}
v_{l,j} = & \delta_j + \max_{p,\theta} \big(\min\{\theta, 1\}p + (1 - \min\{\theta, 1\})\beta_l(\pi_{ll}v_{l,j} + \pi_{lh}v_{h,j})) \big) \\
& \text{s.t. } p \leq \beta_h(\pi_{hl}v_{l,j} + \pi_{hh}v_{h,j}), \\
& \quad v_{l,j'} \geq \delta_{j'} + \min\{\theta, 1\}p + (1 - \min\{\theta, 1\})\beta_l(\pi_{ll}v_{l,j'} + \pi_{lh}v_{h,j'})) \text{ for all } j' < j
\end{aligned}
$$

and $v_{h,j} = \delta_j + \beta_h(\pi_{hl}v_{l,j} + \pi_{hh}v_{h,j})$.

As before, the worst type of tree can be sold with probability 1 at a price that leaves buyers' indifferent about purchasing the tree. We focus for analytical convenience on the case where the trees are dense and denote the worst type of tree by its dividend $\underline{\delta}$. Then

$$P(\underline{\delta}) = \frac{\beta_h \underline{\delta}}{1-\beta_h} \text{ and } v_{l,1} = v_{h,1} = \frac{\underline{\delta}}{1-\beta_h}.$$

For $p < P(\underline{\delta})$, $\Theta(p) = \infty$ and $\Gamma(p)$ is defined arbitrarily. For higher types of trees, the price and sale probabilities are pinned down by the relevant constraints and Bellman equations:

$$\begin{gathered} v_{l,j} = \delta_j + \Theta(p_j)p_j + (1-\Theta(p_j))\beta_l(\pi_{ll}v_{l,j} + \pi_{lh}v_{h,j}))) \\ v_{h,j} = \delta_j + \beta_h(\pi_{hl}v_{l,j} + \pi_{hh}v_{h,j}) \\ p_j = \beta_h(\pi_{hl}v_{l,j} + \pi_{hh}v_{h,j}), \\ \Theta(p_j)(p_j - \beta_l(\pi_{ll}v_{l,j-1} + \pi_{lh}v_{h,j-1})) = \Theta(p_{j-1})(p_{j-1} - \beta_l(\pi_{ll}v_{l,j-1} + \pi_{lh}v_{h,j-1})) \end{gathered}$$

Focusing on the limit where trees are dense, the last constraint reduces to

$$\Theta'(p_j)(p_j - \beta_l(\pi_{ll}v_{l,j} + \pi_{lh}v_{h,j})) + \Theta(p_j) = 0$$

Solve the first three constraints for $v_{l,j}$, $v_{h,j}$, and δ_j to get an expression for $\pi_{ll}v_{l,j} + \pi_{lh}v_{h,j}$. Then solving the differential equation gives

$$\Theta(p) = \frac{1-\beta_l(1-\pi_{lh}-\pi_{hl})}{\left(\frac{p}{P(\underline{\delta})}\right)^{\frac{\beta_h(1-\beta_l(1-\pi_{lh}-\pi_{hl}))}{\beta_h-\beta_l}} - \beta_l(1-\pi_{lh}-\pi_{hl})} \in [0,1]$$

for $p > P(\underline{\delta})$. Also solve those same equations for $D(p)$, the type of tree sold at price $p > P(\underline{\delta})$:

$$D(p) = p\left(\frac{1-\big(\beta_l(1-\pi_{lh}) - \beta_h\pi_{hl}\big)(1-\Theta(p))}{\beta_h(1-\beta_l(1-\Theta(p))(1-\pi_{lh}-\pi_{hl}))} - 1\right).$$

These are natural generalizations of equations (7) and (8), which obtains when $\pi_{lh} = \pi_{hh} = 1-\pi_{hl}$.

We now take the continuous time limit of this model. Define discount rates ρ_s and transition rates q_{hl} and q_{lh} as

$$\rho_s = \frac{1-\beta_s}{\Delta},\ q_{hl} = \frac{\pi_{hl}}{\Delta},\ \text{and } q_{lh} = \frac{\pi_{lh}}{\Delta}.$$

We think of $1/\Delta$ as the number of periods within a unit of calendar time. Also assume a type δ tree produces δ fruit per unit of time, or $\delta\Delta$ fruit per period. With fixed values of ρ_s, q_{hl}, and q_{lh}, the limit as $\Delta \to 0$ (and so $\beta_s \to 1$ and π_{hl} and $\pi_{lh} \to 0$) then corresponds to the continuous time limit of the model. We find that in this limit, $\Theta(p) \to 0$ but the sale rate per unit of time does not:

$$\alpha(p) \equiv \lim_{\Delta \to 0} \frac{\Theta(p)}{\Delta} = \frac{q_{hl} + q_{lh} + \rho_l}{\left(\frac{p}{P(\underline{\delta})}\right)^{\frac{q_{hl}+q_{lh}+\rho_l}{\rho_l - \rho_h}} - 1} \geq 0$$

for all $p \geq P(\underline{\delta}) \equiv \underline{\delta}/(\Delta \rho_s)$, where $\underline{\delta}/\Delta$ is the minimum dividend per unit of time. From the perspective of a seller, $\alpha(p)$ is the arrival rate of a Poisson process that permits her to sell at a price p. Equivalently, the probability that she fails to sell at a price $p > P(\underline{\delta})$ during a unit of elapsed time is $\exp(-\alpha(p))$, an increasing function of p that converges to 1 as p converges to infinity and is well-behaved in the limiting economy.

We can also simplify the expression for the type of tree sold at price p. In the continuous time limit,

$$\frac{D(p)}{p} = \rho_h + \frac{q_{hl}(\rho_l - \rho_h)\left(1 - \left(\frac{p}{P(\underline{\delta})}\right)^{-\frac{q_{hl}+q_{lh}+\rho_l}{\rho_l - \rho_h}}\right)}{q_{hl} + q_{lh} + \rho_l}.$$

For the lowest type of tree, this confirms that the price-dividend per unit of time ratio is $1/\rho_h$, while it is lower for higher priced trees, reflecting their illiquidity. Only in the special case where $q_{hl} = 0$, so buyers never anticipate needing to sell their trees, is the price-dividend ratio constant at $1/\rho_h$. We can also define the inverse of D, the price of a type δ tree, $P(\delta)$.

To close the model, we can compute the share of type j trees held by individuals with discount factor β_s, $\omega_s(\delta)$. Equating inflows and outflows, we obtain that in steady state

$$\big(q_{lh} + \alpha(P(\delta))\big)\omega_l(\delta) = q_{hl}\omega_h(\delta). \tag{9}$$

The left had side is the rate at which type δ trees get transferred from low to high types, either because of a preference shock or because of a sale. The right hand side is the rate that they are transferred in the other direction. Finally, using these steady state values, we can check the market clearing condition. In the case with $\lambda = 0$, this is an inequality constraint:

$$\int_{\underline{\delta}}^{\bar{\delta}} \omega_h(\delta)\delta\kappa(\delta)d\delta \geq \int_{\underline{\delta}}^{\bar{\delta}} \omega_l(\delta)\alpha(P(\delta))P(\delta)\kappa(\delta)d\delta.$$

The left hand side is the fruit available to individuals with a high discount factor. The right hand side is the fruit required to purchase the trees sold by individuals with low discount factors. Eliminating $\omega_h(\delta)$ using $\omega_h(\delta) + \omega_l(\delta) = 1$ and equation (9), this reduces to

$$\int_{\underline{\delta}}^{\bar{\delta}} \frac{(q_{lh} + \alpha(P(\delta)))\delta}{q_{hl} + q_{lh} + \alpha(P(\delta))}\kappa(\delta)d\delta \geq \int_{\underline{\delta}}^{\bar{\delta}} \frac{q_{hl}\alpha(P(\delta))P(\delta)}{q_{hl} + q_{lh} + \alpha(P(\delta))}\kappa(\delta)d\delta.$$

If this inequality is violated, we would instead look for an equilibrium with $\lambda \in (0, \beta_h/\beta_l - 1)$, with little conceptual change in the outcome.

Superficially, the economy in the continuous time limit looks different than the discrete time model. We can again imagine a continuum of marketplaces, each distinguished by its price p. Sellers try to sell their trees in the appropriate market, while buyers bring some of their fruit to all of the markets and consume the rest (since we are focusing on the partial equilibrium with $\lambda = 0$). In all but the worst market, there is always too little fruit to purchase all of the trees. That is, a stock of trees always remains in the market to be purchased by the gradual inflow of new fruit from buyers. Buyers are able to purchase trees immediately, but sellers are rationed and get rid of their trees only at a Poisson rate. Of course, a seller could immediately sell her trees for the low price $P(\underline{\delta})$, but she chooses not to do so.

More broadly, the point of this section is that the frictions generated by adverse selection do not disappear when the period length is short. Intuitively, it must take a real amount of calendar time to sell a tree at a high price or the owners of low quality trees would misrepresent them as being of high quality. This is in contrast to models where trading is slow because of search frictions.[5] In such a framework, the extent of search frictions governs the speed of trading and as the number of trading opportunities per unit of calendar time increases, the relevant frictions naturally disappear.

7 Discussion

Our model shows how prices and illiquidity can be used to separate trees with different qualities. High quality trees trade at a higher price but the market is less liquid. A seller could always choose to sell them at a lower price, but in equilibrium she prefers not to do so. In closing, we consider how our model can be used to understand a financial crisis characterized by widespread illiquidity and how outside intervention may increase liquidity in such markets.

Imagine that we start from a situation in which everybody thinks that all the trees are valued δ_0. Still focusing on the case with lots of buyers, $\lambda = 0$, the price is $p_0 = \delta_0\beta_h/(1-\beta_h)$. Suddenly everyone learns that there is dispersion in the quality of trees. To be concrete, suppose that the average quality of trees is still δ_0, but the support of the quality distribution is now some interval $[\underline{\delta}, \bar{\delta}]$. Information is valuable in this environment, and so assuming it is costless, all individuals would choose to learn the quality of their trees. But once sellers have

[5] See, for example, Duffie, Gârleanu and Pedersen (2005), Weill (2008), or Lagos and Rocheteau (2009) for models where trees are illiquid because of search frictions.

learned this, there is an adverse selection problem. Naturally the price of trees with $\delta < \delta_0$ falls; these trees are known to be of lower quality than before. Interestingly, the price of better trees may fall as well. This happens because in the new equilibrium, trees with $\delta > \underline{\delta}$ only sell probabilistically, reducing their liquidity and driving down the price that buyers are willing to pay for those trees. Thus the revelation of information generates an event that is qualitatively consistent with a fire sale. The price of all trees can decline and the liquidity of the market dries up. Meanwhile, buyers wait on the sidelines, possibly consuming their fruit, despite the decline in price of all types of trees.

Our model also suggests that asset purchase programs, such as the original vision of the Troubled Asset Relief Program in 2008, may alleviate adverse selection problems. Suppose a player with substantial fruit holdings, the "government" to be concrete, announces that it is willing to purchase any asset at a price $\hat{p}$. All sellers with trees that would sell for less that $\hat{p}$, i.e. with $\delta < \hat{\delta} \equiv \hat{p}(1 - \beta_h)/\beta_h$, take them to the government, which then destroys the trees. Assuming $\underline{\delta} < \hat{\delta}$, so the program is not irrelevant, the government suffers a loss, but it has the desired effect on prices and liquidity. After the government intervention, it is common knowledge that the worst type of tree produces dividend $\hat{\delta} > \underline{\delta}$. This tree sells for sure at price $\hat{p}$, while the liquidity and price of all better trees both jump up. Indeed, there are trees that sold for less than $\hat{p}$ before the government announced its program which are not sold to the government but instead become more liquid and experience an increase in price to something in excess of $\hat{p}$. This was exactly the original intent of the TARP; however, the program was never implemented as planned and so we do not know whether it would have successfully moderated adverse selection problems in the market for asset-backed securities.

8 Pooling Environment

Much of the literature on adverse selection in financial markets assumes that all trades occur at a common price p, so the equilibrium is *pooling* (see, for example, Eisfeldt, 2004; Kurlat, 2009; Daley and Green, 2010; Chari, Shourideh and Zetlin-Jones, 2010). In contrast, we find that different types of trees never trade at the same price, so the equilibrium is separating. The source of the difference in results lies in the definitions of equilibrium. In models with pooling, the environment is set up in such a way that a seller cannot even consider selling his trees at a price different than p. In contrast, this thought experiment is central to our definition of equilibrium. In this section, we consider an alternative definition of equilibrium where we restrict all trades to occur at a common price p. The rest of the environment is exactly as in our benchmark model, but the equilibrium is necessarily pooling.[6] We show that this significantly affects several important outcomes, including the cross-sectional behavior of prices, dividends, and liquidity; the nature of fire sales; and the efficacy of asset purchase programs. Thus our notion of equilibrium is central to our results.

We look for an equilibrium in which all trades occur at a price p. Sellers (individuals with low discount factors) are able to choose whether to sell their trees at that price and buyers (individuals with high discount factors) are able to choose whether to buy trees at that price. This implies that the price must leave the marginal seller indifferent about selling his tree and it must leave the buyer indifferent about buying the average tree offered for sale. Our definition of a pooling equilibrium embodies these requirements.

We look at a simple version of our model with i.i.d. preference shocks and a continuum of types of trees.[7] As before, we let $v_s(\delta)$ denote the marginal value of a type δ tree to an individual with discount factor β_s and λ denote the value of a unit of fruit to a buyer in excess of its consumption value. We assume that all trades occur at a common price p. Let $\zeta(\delta)$ denote the fraction of type δ trees that sellers attempt to sell at price p. This is equal to 0 if $p < \beta_l \bar{v}(\delta)$ and 1 if $p > \beta_l \bar{v}(\delta)$. Buyers purchase trees only if the expected value of a purchased tree is equal to the value of the foregone fruit, $p(1+\lambda)$. More formally,

Definition 3 *A pooling equilibrium with adverse selection is a triple of functions $v_h : [\underline{\delta}, \bar{\delta}] \mapsto \mathbb{R}_+$, $v_l : [\underline{\delta}, \bar{\delta}] \mapsto \mathbb{R}_+$, and $\zeta : [\underline{\delta}, \bar{\delta}] \mapsto [0,1]$, a price $p \in \mathbb{R}_+$, and a number $\lambda \in [0, \beta_h/\beta_l - 1]$ satisfying the following conditions:*

1. *consistency of the value functions: for all $\delta \in [\underline{\delta}, \bar{\delta}]$,*

$$v_h(\delta) = \delta(1+\lambda) + \beta_h \bar{v}(\delta) \text{ and } v_l(\delta) = \delta + \max\{p, \beta_l \bar{v}(\delta)\},$$

   where $\bar{v}(\delta) \equiv \pi_h v_h(\delta) + \pi_l v_l(\delta)$.

2. *sellers' optimality: for all $\delta \in [\underline{\delta}, \bar{\delta}]$, $\zeta(\delta) = \begin{cases} 1 & \text{if } p > \beta_l \bar{v}(\delta) \\ 0 & \text{if } p < \beta_l \bar{v}(\delta). \end{cases}$*

3. *buyers' optimality: $p(1+\lambda) = \beta_h \dfrac{\int_{\underline{\delta}}^{\bar{\delta}} \zeta(\delta)\bar{v}(\delta)\kappa(\delta)d\delta}{\int_{\underline{\delta}}^{\bar{\delta}} \zeta(\delta)\kappa(\delta)d\delta}$.*

4. *market clearing: $\pi_h \displaystyle\int_{\underline{\delta}}^{\bar{\delta}} \delta\kappa(\delta)d\delta \gtreqless \pi_l p \int_{\underline{\delta}}^{\bar{\delta}} \zeta(\delta)\kappa(\delta)d\delta \Leftrightarrow \lambda = \begin{cases} 0 \\ \beta_h/\beta_l - 1. \end{cases}$*

[6] We stress that the model we have analyzed thus far in the paper has a unique equilibrium. There is no pooling equilibrium in our model.

[7] Working directly with a continuum of trees allows us to avoid a tedious treatment of tie-breaking.

The definition of a pooling equilibrium consists of four parts. First is the value functions, which state that an individual with a high discount factor values his trees based on the possibility of using the fruit to purchase more trees, while an individual with a low discount factor values them both for their fruit and potentially for their resale value. This immediately implies that trees that produce more fruit are more valuable, $\bar{v}(\delta)$ is increasing.

The second part of the definition of equilibrium states that a seller will sell a tree for sure if the price exceeds the discounted value of the tree and won't sell it if the inequality is reversed. If $p = \beta_l \bar{v}(\delta)$, the sale probability is some arbitrary $\zeta(\delta)$. This implies that all trees below some threshold are sold whenever they are held by a consumer with a low discount factor.

The third part of the definition states that in equilibrium, buyers pay a fair price for trees, given their valuation of a unit of fruit at $1+\lambda$. The left hand side is the value of the fruit used to purchase a tree, while the right hand side is the expected discounted value of the tree that the buyer obtains.

Finally, the market clearing condition states that if the amount of fruit held by buyers at the beginning of the period exceeds the cost of purchase the trees sold by sellers, then the value of fruit must be equal to its consumption value (since some buyers eat fruit). If it is smaller, then the value of fruit must be so high that sellers must keep some of their worst trees and the value of fruit is driven up to β_h/β_l. Otherwise, $1+\lambda$ takes on an intermediate value and buyers do not consume any fruit.

One can prove the existence of equilibrium in this environment. We are more interested in how liquid markets are in this environment. The following proposition is key:

Proposition 4 *In any pooling equilibrium, only trees with $\delta \leq \delta^*$ are sold in the market, where*

$$\delta^* = \frac{\beta_h(1-\bar{\beta})}{\beta_l\big(1-\beta_h+\lambda(1-\pi_h\beta_h)\big)} \frac{\int_{\underline{\delta}}^{\delta^*} \delta\kappa(\delta)d\delta}{\int_{\underline{\delta}}^{\delta^*} \kappa(\delta)d\delta}. \tag{10}$$

Note that in general there is no guarantee that the pooling equilibrium is unique. Formally, there can be multiple solutions to equation (10). This is because both the left and right hand sides of equation (10) are increasing in δ^* for fixed λ, and the slope of the right hand side may be arbitrarily large, for example when the density κ is large in a neighborhood of δ^*. In that case, there can be an equilibrium with a low price in which sellers are only willing to sell bad trees and buyers pay a low price anticipating that they will purchase only bad trees. There can be another equilibrium in which more trees sell and so buyers are willing to pay more for a tree. Chari, Shourideh and Zetlin-Jones (2010) propose a slightly modified definition of equilibrium which selects the outcome with the highest price, and so we do not view this nonuniqueness as an essential feature of a pooling environment.[8]

Like the competitive equilibrium with adverse selection, some trees are illiquid in the pooling equilibrium, namely those with $\delta > \delta^*$. In fact, the distinction between liquid and illiquid trees is dichotomous in this environment. Trees that are more productive than the critical value are never sold, while trees that are less productive sell whenever they are held by an individual with a low discount factor. Moreover, a seller could always sell an illiquid asset for the market price p, but he chooses not to do so. In this sense the nature of illiquidity is similar in the two models, although it is more extreme in the pooling environment.

Still, there are important qualitative differences between the two environments. First, in the competitive equilibrium with adverse selection, we predict that higher quality trees will sell at a higher price but take longer to sell. This prediction can be tested empirically. For example, we predict that within a class of securities that look outwardly similar, those that sell for a higher price will take longer to sell but will generate higher dividends on average.[9] In the pooling equilibrium, any two trees within the same class should sell at the same time and should sell as soon as they are offered in the market. The model therefore predicts no correlation between price, time to sell, and dividend.

Second, consider a fire sale. Again, suppose the pooling economy starts from an initial condition in which everyone believes that all trees produce dividend δ_0 and there are lots of buyers, so $\lambda = 0$. A small amount of dispersion in tree quality will not affect the equilibrium, since even a seller with the best tree $\bar{\delta}$ would be willing to sell it for the price of the average tree δ_0. But if the dispersion in tree quality continues to grow, adverse selection will be a problem and so the average quality of trees sold and the equilibrium price will fall. This again looks different than in the competitive equilibrium with adverse selection. In that case, the price of a high quality tree may be higher or lower in the presence of adverse selection than in the initial condition where everyone views the trees as homogeneous. In any case, the market for high quality trees will continue to exist, although it may be thin. In the pooling equilibrium, the price of a high quality tree is unchanged by a small amount of adverse selection, while sellers withdraw it completely from the market when the adverse selection problem is too severe.

Finally, we consider the impact of an asset purchase program. Suppose the "government" stands ready to purchase as many trees as people want to sell at price $\bar{p} > p^*$. In the pooling environment, if there is any more trade in the private market, the private market price has

[8] In addition, modest regularity conditions on κ like log-concavity are enough to eliminate this source of multiple equilibrium.

[9] We include the caveat "on average" because dividends may follow a stochastic process in reality. In that case, assets are distinguished based on the expected present value of future dividends.

to equal $\bar{p}$. Sellers' indifference condition $\bar{p} = \beta_l \bar{v}(\delta^*)$ then pins down the quality of the marginal tree in the market, while buyers' indifference condition $\bar{p}(1+\lambda) = \beta_l \mathbb{E}_{\delta \leq \delta^*} \bar{v}(\delta)$ pins down the average quality. But there is no condition to pin down the amount of trees left in the private market. That is, if there is an equilibrium of the model in which the density of assets with $\delta \leq \delta^*$ is given by $\tilde{\kappa}$ after the intervention, there is another equilibrium in which it is given by $\eta\tilde{\kappa}$ for any $\eta < 1$. In particular, there is always a solution in which arbitrarily few assets are left in the private market.

This might not be a desirable outcome, and so one can imagine the government attempting to avoid this by capping the amount of assets it is willing to purchase for $\bar{p}$. In this case, the asset purchase program will be oversubscribed and the government will have to ration its purchases, presumably without knowing the quality of the assets it is buying. It follows that the private market price after the intervention is simply bounded above by $\bar{p}$. If, for example, the government is equally likely to buy any asset that sellers value at less than $\bar{p}$, one can prove that this intervention will not affect the private market price p^* but will instead simply reduce the volume of assets in circulation. If the government is somehow able to screen out the worst assets from the purchase program, perhaps by requiring sellers to hold onto the asset for some time before announcing which assets it will purchase, then the intervention will lower the private market price. In contrast, the asset purchase program appears to be a much more promising intervention if our notion of competitive equilibrium with adverse selection is the relevant one.

A Proofs

Proof of Proposition 1. Throughout this proof, let $\bar{\Theta}(p) \equiv \max\{\Theta(p), 1\}$ and $\underline{\Theta}(p) = \min\{\Theta(p), 1\}$. Fix Θ and Γ and take any positive-valued numbers $\{v_{s,j}\}$, and λ that solve the Bellman equations (4), (5), and (6) for $s = l, h$. Let p_h be an optimal price for buying trees,

$$p_h \in \arg\max_p \left(\bar{\Theta}(p)^{-1} \left(\frac{\beta_h \sum_{j=1}^J \gamma_j(p) \bar{v}_j}{p} - 1 \right) \right).$$

Similarly let $p_{l,j}$ be an optimal price for selling type j trees,

$$p_{l,j} = \arg\max_p \underline{\Theta}(p) \big(p - \beta_l \bar{v}_j\big)$$

for all δ. We seek to prove that $\bar{V}^*(\{k_j\}) \equiv \sum_{j=1}^J \bar{v}_j k_j$ where $\bar{v}_j = \pi_h v_{h,j} + \pi_l v_{l,j}$.

If $\lambda = 0$, equations (4) and (5) imply

$$\bar{v}_j = \pi_h\big(\delta_j + \beta_h \bar{v}_j\big) + \pi_l\big(\delta_j + \underline{\Theta}(p_{l,j})p_{l,j} + (1 - \underline{\Theta}(p_{l,j}))\beta_l \bar{v}_j\big).$$

for all δ. Equivalently,

$$\bar{v}_j = \frac{\delta_j + \pi_l \underline{\Theta}(p_{l,j})p_{l,j}}{1 - \pi_h\beta_h - \pi_l\beta_l(1 - \underline{\Theta}(p_{l,j}))} > 0.$$

Alternatively, if $\lambda > 0$, the same equations imply

$$\bar{v}_j = \pi_h \left(\delta_j \left(\left(1 - \bar{\Theta}(p_h)^{-1}\right) + \bar{\Theta}(p_h)^{-1} \frac{\beta_h \sum_{j'=1}^{J} \gamma_{j'}(p_h)\bar{v}_{j'}}{p_h} \right) + \beta_h \bar{v}_j \right) \\ + \pi_l\big(\delta_j + \underline{\Theta}(p_{l,j})p_{l,j} + (1 - \underline{\Theta}(p_{l,j}))\beta_l \bar{v}_j\big)$$

for all δ. Since $v_{l,j}$ and $v_{h,j}$ are positive by assumption so is $\bar{v}_j$, and equivalently we can write

$$\bar{v}_j \left(1 - \pi_h\beta_h - \pi_l\beta_l(1 - \underline{\Theta}(p_{l,j})) - \pi_h\beta_h\bar{\Theta}(p_h)^{-1} \frac{\delta_j \sum_{j'=1}^{J} \gamma_{j'}(p_h)\bar{v}_{j'}}{p_h \bar{v}_j} \right) \\ = \pi_h\delta_j \left(1 - \bar{\Theta}(p_h)^{-1}\right) + \pi_l\big(\delta_j + \underline{\Theta}(p_{l,j})p_{l,j}\big).$$

The right hand side of this expression is positive for all j. Once again since $\bar{v}_j > 0$, with $\lambda > 0$, this holds if and only if

$$1 - \pi_h\beta_h - \pi_l\beta_l(1 - \underline{\Theta}(p_{l,j})) > \pi_h\beta_h\bar{\Theta}(p_h)^{-1} \frac{\delta_j \sum_{j'=1}^{J} \gamma_{j'}(p_h)\bar{v}_{j'}}{p_h \bar{v}_j}. \tag{11}$$

If this restriction fails at any prices p_h and $p_{l,j}$, it is possible for an individual to obtain unbounded expected utility by buying and selling trees at the appropriate prices. We are interested in cases in which it is satisfied.

Next, let $\bar{V}(\{k_j\}) = \sum_{j=1}^{J} \bar{v}_j k_j$ and $V_s(\{k_j\}) \equiv \sum_{j=1}^{J} v_{s,j} k_j$ for $s = l, h$. It is straightforward to prove that $\bar{V}$ and $\bar{V}_s$ solve equations (1), (2), and (3) and that the same policy is optimal. (Include proof?)

Finally, we adapt Theorem 4.3 from Werning (2009), which states the following: suppose $\bar{V}(k)$ for all k satisfies the recursive equations (1), (2), and (3) and there exists a plan that is optimal given this value function which gives rise to a sequence of tree holdings $\{k^*_{i,j,t}(s^{t-1})\}$

satisfying

$$\lim_{t\to\infty}\sum_{s^t}\left(\prod_{\tau=0}^{t-1}\pi_{s_\tau}\beta_{s_\tau}\right)\bar{V}(\{k^*_{i,j,t}(s^{t-1})\})=0. \tag{12}$$

Then, $\bar{V}^*=\bar{V}$.

If $\lambda=0$, an optimal plan is to sell type j trees at price $p_{l,j}$ when impatient and not to purchase trees when patient. This gives rise to a non-increasing sequence for tree holdings. Given the linearity of $\bar{V}$, condition (12) holds trivially.

If $\lambda>0$, it is still optimal to sell type j trees at price $p_{l,j}$ when impatient, but patient individuals purchase trees at price p_h and do not consume. Thus

$$\begin{aligned}k'_{h,j}&=k_j+\bar{\Theta}(p_h)^{-1}\gamma_j(p_h)\frac{\sum_{j'=1}^{J}\delta_{j'}k_{j'}}{p_h}\\ k'_{l,j}&=\big(1-\underline{\Theta}(p_{l,j})\big)k_j.\end{aligned}$$

Using linearity of the value function, the expected discounted value next period of an individual with tree holdings $\{k_j\}$ this period is

$$\begin{aligned}&\sum_{j=1}^{J}\bar{v}_j\big(\pi_h\beta_h k'_{h,j}+\pi_l\beta_l k'_{l,j}\big)\\ &\qquad=\sum_{j=1}^{J}\bar{v}_j\left(\pi_h\beta_h\left(k_j+\bar{\Theta}(p_h)^{-1}\gamma_j(p_h)\frac{\sum_{j'=1}^{J}\delta_{j'}k_{j'}}{p_h}\right)+\pi_l\beta_l\big(1-\underline{\Theta}(p_{l,j})\big)k_j\right)\\ &\qquad\qquad=\sum_{j=1}^{J}\bar{v}_jk_j\left(\pi_h\beta_h+\pi_l\beta_l\big(1-\underline{\Theta}(p_{l,j})\big)+\pi_h\beta_h\bar{\Theta}(p_h)^{-1}\frac{\delta_j\sum_{j'=1}^{J}\gamma_{j'}(p_h)\bar{v}_{j'}}{p_h\bar{v}_j}\right),\end{aligned}$$

where the second equality simply rearranges terms in the summation. Equation (11) implies that each term of this sum is strictly smaller than $\bar{v}_jk_j$. This implies that there exists an $\eta<1$ such that

$$\eta>\frac{\sum_{j=1}^{J}\bar{v}_j\big(\pi_h\beta_h k'_{h,j}+\pi_l\beta_l k_{l,j}\big)}{\sum_{j=1}^{J}\bar{v}_jk_j}=\frac{\pi_h\beta_h\bar{V}(\{k'_{h,j}\})+\pi_l\beta_l\bar{V}(\{k'_{l,j}\})}{\bar{V}(\{k_j\})},$$

and so condition (12) holds. ■

Proof of Lemma 1. Write Problem (P-1) as

$$v_{l,1} = \delta_1 + \max_{p,\theta}\big(\min\{\theta,1\}p + (1-\min\{\theta,1\})\beta_l\bar{v}_1\big)$$
$$\text{s.t. } \lambda \le \min\{\theta^{-1},1\}\left(\frac{\beta_h\bar{v}_1}{p}-1\right).$$

Raising p increases the objective function and tightens the constraint, which ensures the constraint binds. Substituting the binding constraint into the objective function and eliminating the price gives

$$v_{l,1} = \delta_1 + \beta_l\bar{v}_1 + \max_{\theta\ge 0}\left(\min\{\theta,1\}\left(\frac{\beta_h\min\{\theta^{-1},1\}}{\lambda+\min\{\theta^{-1},1\}}-\beta_l\right)\right)\bar{v}_1$$

If $\lambda = 0$, any $\theta_1 \ge 1$ attains the maximum. If $\lambda = \beta_h/\beta_l - 1$, any $\theta_1 \in [0,1]$ attains the maximum. For intermediate values of λ, the unique maximizer is $\theta_1 = 1$. In any case, combining this equation with equation (5) and the definition $\bar{v}_1 = \pi_h v_{h,1} + \pi_l v_{l,1}$ gives

$$p_1 = \frac{\delta_1\beta_h(1+\pi_h\lambda)}{1+\lambda-\beta_h(1+\pi_h\lambda)} \text{ and } \bar{v}_1 = \frac{\delta_1(1+\lambda)(1+\pi_h\lambda)}{1+\lambda-\beta_h(1+\pi_h\lambda)}.$$

This is the unique solution to Problem (P-1).

We now proceed by induction. Fix $j \ge 2$ and assume for all $j' \in \{2,\dots,j-1\}$, we have established the characterization of $p_{j'}$, $\theta_{j'}$, and $\bar{v}_{j'}$ in the statement of the lemma. We establish the result for j. Setting $(\theta,p) = (\theta_{j-1},p_{j-1})$ is feasible but not generally optimal. Indeed, it delivers a value $\bar{v}_j > \bar{v}_{j-1}$ and leaves the constraint $\lambda \le \min\{\theta^{-1},1\}(\beta_h\bar{v}_j/p-1)$ slack. So consider reducing θ and increasing p while keeping $\min\{\theta,1\}(p-\beta_l\bar{v}_{j-1})$ constant. This raises the value of the objective function, leaves the constraints for $j' < j-1$ slack, and tightens the constraint $\lambda \le \min\{\theta^{-1},1\}(\beta_h\bar{v}_j/p-1)$. The optimal policy is therefore achieved when $\lambda = \min\{\theta_j^{-1},1\}(\beta_h\bar{v}_j/p_j-1)$ and $\min\{\theta_{j-1},1\}(p_{j-1}-\beta_l\bar{v}_{j-1}) = \min\{\theta_j,1\}(p_j-\beta_l\bar{v}_{j-1})$. Moreover, even if $\theta_{j-1} > 1$, it is always the case that $\theta_j < 1$ at this solution, allowing a further simplification of these expressions.

Finally, we need to prove that there is a unique value of $\bar{v}_j > \bar{v}_{j-1}$ that solves these equations. Eliminate p_j and θ_j from the Bellman equation for $\bar{v}_j$ with the binding constraints:

$$(1-\pi_h\beta_h-\pi_l\beta_l)\bar{v}_j = \delta_j(1+\pi_h\lambda) + \pi_l\min\{\theta_{j-1},1\}\frac{\big(\beta_h-\beta_l(1+\lambda)\big)^2\bar{v}_{j-1}\bar{v}_j}{\big(\beta_h\bar{v}_j-\beta_l(1+\lambda)\bar{v}_{j-1}\big)(1+\lambda)}. \quad (13)$$

If $\lambda = \beta_h/\beta_l - 1$, the last term is zero and so this pins down $\bar{v}_j$ uniquely. Otherwise we prove that there is a unique solution to equation (13) with $\bar{v}_j > \bar{v}_{j-1}$. In particular, the left hand side is a linearly increasing function of $\bar{v}_j$, while the right hand side is an increasing, concave function, and so there are at most two solutions to the equation. As $\bar{v}_j \to \infty$, the left hand side exceeds the right hand side, and so we simply need to prove that as $\bar{v}_j \to \bar{v}_{j-1}$, the right hand side exceeds the left hand side.

First assume $j = 2$ so $\theta_{j-1} = \theta_1 \geq 1$. Then we seek to prove that

$$(1 - \pi_h\beta_h - \pi_l\beta_l)\bar{v}_1 < \delta_2(1 + \pi_h\lambda) + \pi_l \frac{\big(\beta_h - \beta_l(1+\lambda)\big)\bar{v}_1}{1+\lambda}.$$

Since $\bar{v}_1 = \frac{\delta_1(1+\lambda)(1+\pi_h\lambda)}{1+\lambda-\beta_h(1+\pi_h\lambda)}$ and $\delta_1 < \delta_2$, we can confirm this directly. Next take $j \geq 3$. In this case, in the limit with $\bar{v}_j \to \bar{v}_{j-1}$, the right hand side of (13) converges to

$$\delta_j(1+\pi_h\lambda) + \pi_l \min\{\theta_{j-1}, 1\} \frac{\big(\beta_h - \beta_l(1+\lambda)\big)\bar{v}_{j-1}}{(1+\lambda)}$$
$$> \delta_{j-1}(1+\pi_h\lambda) + \pi_l \min\{\theta_{j-2}, 1\} \frac{\big(\beta_h - \beta_l(1+\lambda)\big)^2 \bar{v}_{j-2}\bar{v}_{j-1}}{\big(\beta_h\bar{v}_{j-1} - \beta_l(1+\lambda)\bar{v}_{j-2}\big)(1+\lambda)},$$

where the inequality uses the indifference condition

$$\min\{\theta_{j-2}, 1\}(p_{j-2} - \beta_l\bar{v}_{j-2}) = \min\{\theta_{j-1}, 1\}(p_{j-1} - \beta_l\bar{v}_{j-2})$$

and the assumption $\delta_{j-1} < \delta_j$. Rewriting equation (13) for type $j-1$,

$$(1 - \pi_h\beta_h - \pi_l\beta_l)\bar{v}_{j-1} = \delta_{j-1}(1+\pi_h\lambda) + \pi_l \min\{\theta_{j-2}, 1\} \frac{\big(\beta_h - \beta_l(1+\lambda)\big)^2 \bar{v}_{j-2}\bar{v}_{j-1}}{\big(\beta_h\bar{v}_{j-1} - \beta_l(1+\lambda)\bar{v}_{j-2}\big)(1+\lambda)},$$

it follows that

$$(1 - \pi_h\beta_h - \pi_l\beta_l)\bar{v}_{j-1} < \delta_j(1+\pi_h\lambda) + \pi_l \min\{\theta_{j-1}, 1\} \frac{\big(\beta_h - \beta_l(1+\lambda)\big)\bar{v}_{j-1}}{(1+\lambda)},$$

which completes the step.

Finally, set $p_j = \beta_h\bar{v}_j/(1+\lambda) > p_{j-1}$ and $\theta_j = \min\{\theta_{j-1}, 1\}(p_{j-1} - \beta_l\bar{v}_{j-1})/(p_j - \beta_l\bar{v}_{j-1}) < \theta_{j-1}$, completing the proof. ■

Proof of Proposition 2.

We first prove that the solution to Problems (P-j) describe a partial equilibrium and then prove that there is no other equilibrium.

Existence. As described in the statement of the proposition, we look for a partial equilibrium where $\mathbb{P} = \{p_j\}$, $\Theta(p_j) = \theta_j$, $\gamma_j(p_j) = 1$, $\mu(\{p_j\}) = \pi_l K_j$, and $v_{s,j}$ solves Problem (P-j). Also for notational convenience define $p_{J+1} = \infty$. To complete the characterization, we define Θ and Γ on their full support $\mathbb{R}_+$. For $p < p_1$, $\Theta(p) = \infty$ and $\Gamma(p)$ can be chosen arbitrarily, for example $\gamma_1(p) = 1$. For $j \in \{1, \dots, J\}$ and $p \in (p_j, p_{j+1})$, $\Theta(p)$ satisfies sellers' indifference condition $v_{l,j} = \delta_j + \beta_l \bar{v}_j + \Theta(p)(p - \beta_l \bar{v}_j)$ and $\gamma_j(p) = 1$. To prove that this is an equilibrium, we simply verify that under these beliefs, the four conditions in the definition of partial equilibrium hold.

The first condition, consistency of the value functions, holds by construction.

The second condition is sellers' optimality. By construction, for all $j \in \{1, \dots, J\}$ and $p \geq p_1$, $v_{l,j} = \delta_j + \beta_l \bar{v}_j + \Theta(p)(p - \beta_l \bar{v}_j)$ if $\gamma_j(p) = 1$. We must only show that $v_{l,j} \geq \delta_j + \beta_l \bar{v}_j + \Theta(p)(p - \beta_l \bar{v}_j)$ for all other p.

To prove this, first take any $j \in \{2, \dots, J\}$, $j' < j$, and $p \in (p_j, p_{j+1})$. By construction, $\Theta(p_j)(p_j - \beta_l \bar{v}_j) = \Theta(p)(p - \beta_l \bar{v}_j)$ which implies that $\Theta(p) < \Theta(p_j)$, given that $p > p_j$. Since $\bar{v}$ is increasing in j, it follows that $\Theta(p_j)(p_j - \beta_l \bar{v}_{j'}) > \Theta(p)(p - \beta_l \bar{v}_{j'})$. Also by the construction in Problem (P-j), $\Theta(p_{j'})(p_{j'} - \beta_l \bar{v}_{j'}) \geq \Theta(p_j)(p_j - \beta_l \bar{v}_{j'})$. Combining inequalities gives $\Theta(p_{j'})(p_{j'} - \beta_l \bar{v}_{j'}) > \Theta(p)(p - \beta_l \bar{v}_{j'})$ for all $p \in (p_j, p_{j+1})$ and $j' < j$.

Similarly, take any $j \in \{1, \dots, J-1\}$, $j' > j$, and $p \in (p_j, p_{j+1})$. By construction, $\Theta(p_{j+1})(p_{j+1} - \beta_l \bar{v}_j) = \Theta(p)(p - \beta_l \bar{v}_j)$, since type j is indifferent about the price p_{j+1}. Since $p < p_{j+1}$, this implies that $\Theta(p) > \Theta(p_{j+1})$. Moreover, since $\bar{v}$ is increasing in j, it follows that $\Theta(p_{j+1})(p_{j+1} - \beta_l \bar{v}_{j'}) > \Theta(p)(p - \beta_l \bar{v}_{j'})$. Also by the construction in Problem (P-$(j+1)$), $\Theta(p_{j'})(p_{j'} - \beta_l \bar{v}_{j'}) \geq \Theta(p_{j+1})(p_{j+1} - \beta_l \bar{v}_{j'})$. Combining inequalities gives $\Theta(p_{j'})(p_{j'} - \beta_l \bar{v}_{j'}) > \Theta(p)(p - \beta_l \bar{v}_{j'})$ for all $p \in (p_j, p_{j+1})$ and $j' > j$.

Third we turn to buyers' optimality condition. By construction, the inequality binds at all $p \in \mathbb{P}$. For $p < p_1$, it is satisfied because $\Theta(p)^{-1} = 0$. If $\lambda = 0$, the inequality holds for all $p \in (p_j, p_{j-1})$ because $\beta_h \bar{v}_j / p_j = 1$ and so $\beta_h \bar{v}_j / p < 1$. If $\lambda > 0$, Lemma 1 implies $\min\{\Theta(p)^{-1}, 1\} = 1$ for all $p \geq p_1$ and $\beta_h \bar{v}_j / p_j > \beta_h \bar{v}_j / p$ for all $p \in (p_j, p_{j-1})$, and so the inequality again holds for all $p > p_1$.

Finally, the requirement that all trees are offered for sale at some price holds by construction.

Uniqueness. Now take any partial equilibrium $\{v_h, v_l, \Theta, \Gamma, \mathbb{P}, \mu\}$. We first claim that $\bar{v}$ is increasing in j. This follows immediately from part 1 of the definition of equilibrium: Let $p_{j'}$ denote the price offered by j'. For $j > j'$, it is feasible to offer the same price $p_{j'}$, and since

$\delta_j > \delta_{j'}$, this gives a higher value $\bar{v}_j > \bar{v}_{j'}$. Behaving optimally gives a still higher value.

Next, the fourth piece of the definition of equilibrium implies that for each $j \in \{1, \ldots, J\}$, there exists a price $p_j \in \mathbb{P}$ with $\gamma_j(p_j) > 0$.

In the remainder of the proof, we take any $j \in \{1, \ldots, J\}$ and $p_j \in \mathbb{P}$ with $\gamma_j(p_j) > 0$. Let $\theta_j = \Theta(p_j)$. First we prove that the constraint $\lambda \leq \min\{\theta_j^{-1}, 1\}(\beta_h \bar{v}_j / p_j - 1)$ is satisfied. Second we prove that the constraint $v_{l,j'} \geq \delta_{j'} + \beta_l \bar{v}_{j'} + \min\{\theta_j, 1\}(p_j - \beta_l \bar{v}_{j'})$ is satisfied for all $j' < j$. Third we prove that the pair (θ_j, p_j) delivers value $v_{l,j}$ to sellers of type j trees. Fourth we prove that (θ_j, p_j) solves (P-j).

Step 1. To derive a contradiction, assume $\lambda > \min\{\theta_j^{-1}, 1\}(\beta_h \bar{v}_j / p_j - 1)$. Equilibrium condition (iii) implies that there is a j' with $\gamma_{j'}(p_j) > 0$ and $\lambda < \min\{\theta_j^{-1}, 1\}(\beta_h \bar{v}_{j'} / p_j - 1)$. If $\theta_j = \infty$, $\min\{\theta_j^{-1}, 1\} = 0 \leq \lambda$, which is impossible; therefore $\theta_j < \infty$. Then for all $p' > p_j$ and $\theta' = \Theta(p')$,

$$\min\{\theta', 1\}(p' - \beta_l \bar{v}_{j'}) \leq \min\{\theta_j, 1\}(p_j - \beta_l \bar{v}_{j'}) < \min\{\theta_j, 1\}(p' - \beta_l \bar{v}_{j'}).$$

The weak inequality holds from type j' sellers' optimality condition, since p_j is an optimal price for type j' sellers, while the strict inequality uses $p' > p_j$. This implies $\theta' < \theta_j$. Next observe that for all $j'' < j'$,

$$\min\{\theta', 1\}(p' - \beta_l \bar{v}_{j''}) < \min\{\theta_j, 1\}(p_j - \beta_l \bar{v}_{j''}) \leq v_{j''} - \delta_{j''} - \beta_l \bar{v}_{j''},$$

where the first inequality follows because $\theta' < \theta_j$ and $\bar{v}_{j''} < \bar{v}_{j'}$ and the second follows from type j'' sellers' optimality condition. This implies that $\gamma_{j''}(p') = 0$. Thus any $p' > p_j$ only attracts type j' sellers or higher and so delivers value at least equal to $\min\{\theta'^{-1}, 1\}(\beta_h \bar{v}_{j'} / p' - 1)$ to buyers. For p' sufficiently close to p_j, this exceeds λ, contradicting buyers' optimality.

Step 2. Sellers' optimality implies $v_{l,j'} \geq \delta_{j'} + \beta_l \bar{v}_{j'} + \min\{\theta_j, 1\}(p_j - \beta \bar{v}_{j'})$ for all j', and so the second constraint is satisfied.

Step 3. Sellers' optimality implies $v_{l,j} = \delta_j + \beta_l \bar{v}_j + \min\{\theta_j, 1\}(p_j - \beta \bar{v}_j)$ for all j, and so the policy delivers value $v_{l,j}$.

Step 4. Suppose there is a policy (θ, p) that satisfies the constraints of problem (P-j) and delivers a higher payoff. That is,

$$\begin{gathered} v_{l,j} < \delta_j + \beta_l \bar{v}_j + \min\{\theta, 1\}(p - \beta_l \bar{v}_j) \\ \lambda \leq \min\{\theta^{-1}, 1\}(\beta_h \bar{v}_j / p - 1) \\ v_{l,j'} \geq \delta_{j'} + \beta_l \bar{v}_{j'} + \min\{\theta, 1\}(p - \beta_l \bar{v}_{j'}) \text{ for all } j' < j. \end{gathered}$$

If these inequalities hold with $\theta > 1$, then the same set of inequalities holds with $\theta = 1$, and so we may assume $\theta \leq 1$ without loss of generality. Choose $p' < p$ such that

$$v_{l,j} < \delta_j + \beta_l \bar{v}_j + \theta(p' - \beta_l \bar{v}_j) \tag{14}$$

$$\lambda < \beta_h \bar{v}_j / p' - 1 \tag{15}$$

$$v_{l,j'} > \delta_{j'} + \beta_l \bar{v}_{j'} + \theta(p' - \beta_l \bar{v}_{j'}) \text{ for all } j' < j. \tag{16}$$

Now sellers' optimality condition $v_{l,j} \geq \delta_j + \beta_l \bar{v}_j + \min\{\Theta(p'), 1\}(p' - \beta_l \bar{v}_j)$, together with equation (14), implies $\Theta(p') < \theta$. This together with equation (16) implies that

$$v_{l,j'} > \delta_{j'} + \beta_l \bar{v}_{j'} + \Theta(p')(p' - \beta_l \bar{v}_{j'}) \text{ for all } j' < j,$$

and so in particular $\gamma_{j'}(p') = 0$ for all $j' < j$. But then, using equation (15), we obtain

$$\lambda < \min\{\Theta(p')^{-1}, 1\} \left(\frac{\beta_h \bar{v}_j}{p'} - 1 \right) \leq \min\{\Theta(p')^{-1}, 1\} \left(\frac{\beta_h \sum_{j'=1}^{J} \gamma_{j'}(p') \bar{v}_{j'}}{p'} - 1 \right),$$

a violation of buyers' optimality condition. This completes the proof. ■

Proof of Proposition 3. We prove that there exists a unique $\lambda \in [0, \beta_h/\beta_l - 1]$ such that the market clearing condition holds. To be completed. ■

Proof of Proposition 4. The first part of the definition of a pooling equilibrium implies $\bar{v}(\delta)$ is increasing. Then the second part implies $\zeta(\delta) = 1$ if $\delta < \delta^*$ and $\zeta(\delta) = 0$ if $\delta > \delta^*$, with no restriction on $\zeta(\delta)$ if $\delta = \delta^*$.

Next, for $\delta < \delta^*$, the value functions imply $v_h(\delta) = \delta(1 + \lambda) + \beta_h \bar{v}(\delta)$ and $v_l(\delta) = \delta + p$. Summing these and solving for $\bar{v}(\delta)$ gives

$$\bar{v}(\delta) = \frac{\delta(1 + \pi_h \lambda) + \pi_l p}{1 - \pi_h \beta_h}.$$

Then $p > \beta_l \bar{v}(\delta)$ if and only if $\delta < \delta^*$ defined by

$$p = \beta_l \frac{\delta^*(1 + \pi_h \lambda)}{1 - \bar{\beta}}. \tag{17}$$

Substituting this back into the previous equation gives

$$\bar{v}(\delta) = \frac{\big((1 - \bar{\beta})\delta + \pi_l \beta_l \delta^*\big)(1 + \pi_h \lambda)}{(1 - \pi_h \beta_h)(1 - \bar{\beta})}. \tag{18}$$

This holds if $\delta \leq \delta^*$; otherwise, the value is reduced by the fact that the asset is never resold.

Next, eliminate p from the buyer's indifference condition, the third part of the definition of equilibrium, using equation (17). Also eliminate $\bar{v}(\delta)$ using equation (18). Solving for δ^* gives

$$\delta^* = \frac{\beta_h(1-\bar{\beta})}{\beta_l\big(1-\beta_h+\lambda(1-\pi_h\beta_h)\big)} \frac{\int_{\underline{\delta}}^{\bar{\delta}} \zeta(\delta)\delta\kappa(\delta)d\delta}{\int_{\underline{\delta}}^{\bar{\delta}} \zeta(\delta)\kappa(\delta)d\delta}.$$

Imposing that $\zeta(\delta) = 1$ if $\delta < \delta^*$ and $\zeta(\delta) = 0$ if $\delta > \delta^*$ gives the expression in the statement of the Proposition. ■

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