Risk-sharing and contagion in networks

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Abstract

The aim of this paper is to investigate how the capacity of an economic system to absorb shocks depends on the specific pattern of interconnections established among economic agents. The key issue we would like to analyze is the trade-off between the risk-sharing benefits to firms of becoming more highly interconnected versus the large-scale costs resulting from an increased risk exposure. We focus on two dimensions of the network structure. The size of the (disjoint) components into which the network is divided, and the "relative *density*" of connections within each component. We find that when the probability distribution of the shocks exhibits "fat tails" components should be of the minimum possible size. In the opposite case where the probability distribution places high enough mass on small-sized shocks, the best configuration involves having all firms arranged in a single component. There are, however, intermediate conditions on the shock structure for which similarly intermediate arrangements are optimal. We also find that there is typically a conflict between optimality and pairwise stability, derived from the fact that those components that attain the best size will block admitting new members from a smaller components.

1 Introduction

Recent economic events have made it clear that looking at financial entities in isolation gives an incomplete, and possibly very misleading, impression of the potential impact of shocks to the financial system. The aim of this paper is to investigate how the capacity of the system to absorb shocks depends on the specific pattern of interconnections established among economic agents – to fix ideas, we shall think of them as financial firms. The key issue we would like to analyze is the trade-off between risk-sharing and contagion. Or, to be more specific, we want to shed light on the extent to which the risk-sharing benefits to firms of becoming more highly interconnected (which provides some sort of insurance against moderate shocks) may be offset by the large-scale costs resulting from an increased risk exposure (which, for large shocks, could entail a large wave of induced bankruptcies).

We analyze a model in which there is a network consisting of N nodes, each of them interpreted as a firm For simplicity, let us postulate that all of them have the same level of assets and liabilities but a shock may individually hit a randomly selected firm. The effect of such a shock is to decrease the income generated by its assets, thus possibly leading to the default of the firm if the resulting income falls short of its liabilities. Now think of the presence of a direct link between two firms as reflecting an exchange of their originally held assets. This asset exchange generates patterns of mutual exposure between firms, the magnitude of such exposure decreasing with their network distance. Thus, when a shock hits a firm, all the firms which are directly or indirectly linked to it become affected in proportion to their exposure to that firm. In the end, therefore, the 'network' structure determines how any given shock affects different firms and what is its overall aggregate impact on the whole system.

In order to concentrate the analysis on our basic trade-off, insurance versus contagion, we focus on two dimensions of the network structure. One is the size of the (disjoint) components into which the network is divided, i.e. the degree of *segmentation* of the system. The other concerns the "relative *density*" of connections within each component, as measured by the fraction of nodes that lie at different network distances.

Network density is important because, as explained, different network distances yield different degree of exposure. In this respect, our analysis will largely focus on contrasting two polar cases: (i) completely connected components where there is a direct link between any pair of firms in it; (ii) partially connected components where firms are placed on an underlying lattice (for simplicity, a one-dimensional ring) and each of them is directly connected only to the firms that are adjacent to it. In the first case (complete components), the mutual exposure between any pair of firms in the same component is exactly the same. Instead, in the second case (lattice components), the reciprocal exposure between two firms is heterogenous, falling with lattice distance.

Once a shock hits the assets a particular firm, other firms connected to it are affected and may even default. Specifically, this will happen for any firm whose exposure to the shock (as measured by the fraction of its currently held assets that are affected by it) exceeds the sum of firm's capital and liabilities. Under these conditions, a key concern is to identify the architecture of the system that minimizes the expected number of defaults in the system. This involves finding both the best degree of segmentation as well as the optimal link density within each separate component. As explained, the objective is to strike the best compromise between risk sharing and contagion. Naturally, the maximum extent of risk sharing obtained when all firms form part of a single and fully connected network. But this configuration, of course, also yields the highest exposure to a large shock, which could lead to the extensive default across the whole system. There are two alternative (and in some cases complementary) ways of reducing such exposure. One is by segmentation, which isolates the firms in each component from any shock that might hit every other component. The second one is reducing the density of connections in each component, which buffers the network propagation along this component of any shock that hits one of its firms.

The paper proposes a stylized model to study the problem under a fairly general structure of shocks. Some of our results can be summarized as follows. First we find that when the probability distribution of the shocks exhibits "fat tails" (i.e. attributes a high mass to large shocks), the optimal configuration involves a maximum degree of segmentation – that is, components should be of the minimum possible size. This reflects a situation where the priority is to minimize contagion. Instead, in the opposite case where the probability distribution places high enough mass on small- and medium-sized shocks, the best configuration involves having all firms arranged in a single component. The aim in this case is to achieve the highest level of risk sharing. There are, however, intermediate conditions on the shock structure for which similarly intermediate arrangements are optimal, i.e. the optimal degree of segmentation involves medium-sized components.

It is interesting to note that all of the former conclusions hold irrespectively of whether components are assumed to have either of the two polar network structures under consideration, i.e. complete or a lattice structures. But, as explained, an analogous trade-off between risk-sharing and contagion can be attained by impinging on network structure (or density). Indeed, our second set of results explores this alternative route. Note that, when a certain component is completely connected, the induced symmetry of the configuration implies that once a shock hits any firm in it, either all other members of the component go broke or none does. The potential advantage of a lattice configuration is that firm exposure is not uniform but decays with network distance. Thus, if shocks are of a suitable magnitude, only a fraction of the firms may default while all would do if they were completely connected. A first immediate consequence of this observation is that the lattice structure induces and optimal degree of segmentation that is lower (or, equivalently, a component size that is larger) than in the case of the completely connected structure. But, more importantly, we also find that it renders the lattice structure optimal (i.e. better than any segmentation in complete components) when the shock structure displays certain features. This happens, for example, when the shock distribution places a high mass on a narrow range of large shocks as well as on a wide range of small and medium size shocks. In this case, some intermediate segmentation in components with a internal lattice structure attains the lowest expected number of defaults over all configurations under consideration.

The results described so far considers structures where all agents have the same size. We also explore an extension with asymmetric agents, which makes it natural to study asymmetric structures, such as a *star*, in which a large firms is connected to a set of smaller firms, for whom the large firm is their only direct connection. Some of our results, like the tradeoff between risk sharing and contagion, extend to this framework, but new phenomena appear. For example, in a *star*, the center can become a sort of *firebreak* that prevents contagion to extend to all the spokes, and this can be useful for some intermediate values of the large shock.

Finally, we also address the issue of whether the requirements for overall optimality are compatible with individual incentives to establish links. Formally, we model those incentives through the notion of pairwise stability, which is often used in the network literature. Restricting our attention to the case where agents solely determine the degree of segmentation (for they are completely connected to all members of their component), we find that there is typically a conflict between optimality and pairwise stability. This conflict derives from the fact that those components that attain the best size (and thus minimize the default probability of their members), will block admitting new members from a smaller components. But, as it turns out, overall optimality requires that all agents face the same risk situation – which, in particular, demands that all components be of identical size.

The rest of the paper is organized as follows. Section 2 summarizes the related literature. Section 3 discusses the model: first a discrete version, then its continuum idealization. Section 4 undertakes the analysis of optimal financial structures under a variety of different assumptions on the underlying structure. Section 6 addresses network formation and explores the tension between strategic stability and optimality. Finally, Section 7 concludes with a summary and an agenda for future research. For the sake of smooth discussion, all formal proofs of our results are relegated to an Appendix.

2 Related Literature

The literature on financial contagion and systemic risk is diverse and growing fast, so we shall only discuss at some length five representative papers. The reader is referred to Allen and Babus (2009) for a recent survey.

Allen and Gale (2000) pioneered the study of the stability of interconnected financial systems. They propose a model in the Diamond and Dybvig (1983) tradition, with shocks affecting the aggregate preference for liquidity of depositors. That is, some consumers may need liquidity before the maturation of the assets in which banks have invested their deposits. This creates a "liquidity" problem for banks, which try to address it by holding deposits in (i.e. establishing directed links with) each other. In general, the liquidity needs of connected banks are balanced, so that the excess demand of liquidity for a bank is matched by an equal excess supply of another. But there is a state of the world where liquidity demands cannot be balanced and, therefore, at least one bank must go bankrupt. However, the overall effect in this case depends on the precise pattern of connections. They find, specifically, that both for a completely connected structure as well for a segmented one contagion can be averted. Instead, when all banks form part of the same component but they are not completely connected through direct links, all banks go bankrupt in the state with high liquidity needs. Our model explores similar issues but has less structural detail than that of Allen and Gale (2000). We study, however, a richer shock structure and a wider set of network arrangements, which allows us to identify a significant trade-off between risk sharing and contagion. This then allows us to carry out some welfare and strategic analysis in which we find that, depending on the nature of the shocks, different degrees of segmentation and network density are required to either meet optimality criteria or agents' networking incentives.

Leitner (2005) studies a situation in which linkages are ways to create commitment to share resources in a context where individual endowments are uncertain. Agents can choose ex-ante whether to form part of a group. In any group of agents, a profitable project requires that *every* agent in it invests some given amount of resources. Thus, the required total investment grows proportionally with group size. Since there is uncertainty as to ex post level of each individual's endowment, the commitment to share resources can expand the investment opportunities. But it can also shrink them. For it may well happen that not enough aggregate resources materialize to invest on the project but a smaller group could have undertaken successfully one at a lower-scale. Some of the considerations arising in Leitner's model are analogous to those underlying ours. However, one important difference is that risk sharing in his context is undertaken at the group level. Thus, in the end, either the whole group succeeds in carrying out the investment or it fails. No role is played, therefore, by the structure of interaction, as in Allen and Gale (2000) or the present paper.

Lagunoff and Schreft (2001) study a dynamic model in which forward-looking agents invest in projects that can be hit by shocks at an initial date. Shocks are i.i.d. and when a shock hits a project, it project fails and the agents who invested in it lose their investment. Each agent invests her initial wealth in two risky projects that are shared with her neighbors (thus the network structure is modelled as a collection of one-dimensional rings). The failure of a project leads each of the two partners to discontinue the other project she is involved in, and hence the shock spreads causing further failures due to insufficient funding. If the economy were to consist of a single ring, then any shock would eventually spread and all projects would fail. But the number of rings is uncertain. The probability distribution governing the number of prevailing rings is assumed to be such that, as the population gets large, the probability that a shock hits any one of them grows. This in turn leads to an earlier date at which agents decide to discontinue their investment, thus anticipating the onset of a "financial crisis." Lagunoff and Schreft (2001) is essentially concerned with the spread of shocks throughout the economy, abstracting from some of the issues that are central to this paper, namely, what is the best network structure that minimizes the overall impact of shocks, or whether such optimal structure is consistent with agents' incentives to connect to others.

Nier *et al.* (2007) document a surge of empirical research on the importance of interbank linkages as a channel of contagion. They model the banking system as a (random) network, where each node represents a bank and each link (which has probability p) between nodes i and j represents a loan of bank i to bank j. They study the consequences of an idiosyncratic shock hitting one of the banks in the system and relate them to the structural parameters of the system. In their model, interbank connections have two opposing effects. On the one hand, they may act as channels through which shocks propagate to the whole system, but also as channels through which shocks can be shared and absorbed by other bank's net worth. They show, through numerical simulations, that for very low levels of connectivity (p close to zero), an increase in connectivity reduces system resilience. This reflects the fact that, starting from low levels, an increase in connectivity expands the channels of shock transmission. But when connectivity is already quite high, further increases in connectivity turn out to decrease contagion. The reason is that when the system displays a high connectivity, further links cannot play a major effect in stimulating an *already high* level of contagion. But, instead, additional links can strengthen the ability of banks to withstand shocks, which is the effect that dominates in this case. Although Nier *et al.* (2007) is motivated by similar concerns to ours, their methodology is very different. They conduct the analysis on randomly generated networks and rely on simulations to study the effect of different parameter changes. In our case, we focus on regular networks and study analytically both welfare and network formation issues.

Finally, we summarize the recent paper by Carletti *et al.* (2009), which is probably the closest in approach and motivation to the present paper. This paper considers a simple six-firm context where each individual firm faces the need to find funds for its respective investment. Since these investments are risky, firms can gain from risk diversification, which is achieved by exchanging shares with two other firms. Thus, in the language of networks, the situation can be described by a (regular) network where every node/firm has two links. The analysis of focuses on comparing two different arrangements: one where all six firms are arranged in a ring; another where they are divided into two equal components. The most interesting issues arise when firms finance themselves through short term debt, which has to be rolled over before the maturation of the investment project. At the intermediate point in time where such debt roll-over has to take place a signal arrives that indicates whether at least one firm will default. Investors learn that information and have to decide whether to roll over the debt or not, depending on what this signals reveals about the prospects that each firm will default. Clearly, a negative signal will not carry the same implications in each network structure. In the segmented one, (full) asset exchange introduces (perfect) correlation in the returns experienced by firms within each component. Therefore, bad news in this case means that at least three firms will default. In contrast, when the network is given by a single ring, firms' returns are not so well correlated and thus the arrival of such bad news is associated to a lower expected number of defaults. The implication is that, under some circumstances (specifically, if the costs of bankruptcy are neither too high nor low), there will be complete liquidation for the segmented case but rolling-over of debt in under the ring network. This means that the ring is a better arrangement.

Does, however, the model by Carletti *et al.* (2009) allow for a trade-off between the two structures analogous as the one we find in this paper? There indeed is this trade-off if the bankruptcy costs are high. For, in this case, the preeminent consideration must be to minimize the probability that a negative signal arrives (in which case, because the high penalty involved, investors decide to liquidate under both configurations). And, naturally, that probability is lower when the externalization of risk is highest – or, equivalently, when the idiosyncratic component of risk is lowest. This happens in the segmented case, where every firm in each component has the same portfolio. Thus it is this configuration that is also optimal in this case.

3 The model

3.1 The Environment

We consider an environment with N firms and a continuum of small investors. At any given point in time, each firm has an investment opportunity of size I. The "normal" gross return on the project is R. But the project can be subject to shocks. For simplicity, we assume that, only *one* of the N projects is hit by a shock, and every one has the same probability of being hit by the shock. The shock can be either "small" or "big". With a small shock, whose probability of hitting a particular individual is denoted π_s the gross return of the project takes a single value, R_s. Instead, if the shock is severe, its magnitude is a random variable R_b with support on $\left[\underline{R}, \overline{R}\right]$. The main idea here is that, with some probability, a shock may come that is sufficiently large to lead to the bankruptcy of all firms that have some share on the investment project in question. Summarizing, we can write the gross unit return:

$$\widetilde{R} = \begin{cases} R \text{ with prob.} 1 - \pi_s - \pi_b \\ R_s \text{ with prob.} \pi_s \\ R_b \text{ with prob.} \pi_b \end{cases}$$

The resources needed to undertake the project are financed with liabilities on which it must pay a expected return r. The investors who lend the resources to run the project receive, in case there is no default, an endogenous unit return M. Total payoffs to the manager are thus:

$$U_i = \begin{cases} (R - M)\, I \text{withprob.} 1 - \pi_s - \pi_b \\ \max\left\{(R_s - M)\, I, 0\right\} \text{ with prob.} \pi_s \\ \max\left\{(R_b - M)\, I, 0\right\} \text{ with prob.} \pi_b \end{cases}$$

We assume

1. $R_s < M$ and $R_b < M$ so If a firm can draw only on its own resources, it is unable to pay depositors (and hence it must **default)** when either a shock s or b hits it. Since M is endogenous, we are sure this is true if $R_s < r$ and $R_b < r$.

Default entails a large cost B for firm (loss of future opportunities).

In order to diversify risks, firms are assumed to keep an α fraction of their shares on their own investment project and exchange the remaining fraction for shares held by other firms (we assume $\alpha \geq 1/2$, which may be motivated by moral hazard considerations). This implies that each firm ends up owning a fraction $(1-\alpha)$ of assets tailored to the investment projects carried out by other firms. From the point of view of any given firm, this amounts to reducing the impact of a shock on its investment project to the α shares it retains, while becoming exposed to the shock that might hit other projects through the portfolio of $(1-\alpha)$ shares held from other firms. The induced pattern of asset exchange is formalized through a network, with a link between two firms reflecting that there is some exchange of shares on each other's investment project.

As we mentioned before, in order to attract investors, every firm offers to each of them a contract that pays an amount M whenever this is feasible, i.e. when the firm escapes default. (For simplicity, we assume that, in the case of default, liquidation costs leave no residual for investors.) Given the investors' outside opportunity, we must have $M \geq r$, with the inequality being strict if there is any risk of default. In order to avoid default when a small shock hits the firm it must be the case that:

$$\begin{aligned} \alpha\,(R_s - M)\,I + (1-\alpha)\,(R-M)\,I &> 0 \qquad (1) \\ \alpha &< \frac{R-M}{R-R_s} \end{aligned}$$

This implies that, if M is close enough to r (which, as we will see, amounts to saying that the probability of a severe shock is sufficiently small), any firm hit a moderate shock can avert its default. For, in this case, the amount $(1-\alpha)(R-M)$ that the firm obtains through the shares it holds on other firms' assets is enough to cover the corresponding liability $\alpha\,(R_s - M)$. With this assumption in mind, the only case in which an investor would not recover the investment would be if a shock is sufficiently large that the firm goes bankrupt. This will in general depend on the structure of connections between firms, which we denote by Γ and the probability of a large shock π_b, so we denote the probability of bankruptcy $p\,(\Gamma, \pi_b)$ (and clearly $p\,(\Gamma, 0) = 0$). Using this we can determine the equilibrium value of M as follows:

$$(1 - p\,(\Gamma, \pi_b))\,M = r$$

so that

$$\lim_{\pi_b \to 0} M = r$$

The former considerations justify the implicit assumption that firms want to connect to others by exchanging assets. For, by so doing, each of them is able to prevent default when a moderate shock (presumed to be much more frequent than severe ones) hits it. Of course, the flip side of being part of a network is that connections can also lead the firm into default if a severe enough shock hits *another* firm. This negative effect, however, is an unavoidable by-product of the risk sharing afforded by the network with respect to more frequent moderate shocks. Indeed, in a nutshell, the primary aim of this paper can be regarded as understanding what network configurations minimize those detrimental side effects of risk-sharing.

3.2 Financial Structures and payoffs

We have assumed that firms retain an α fraction of their shares in their own investment while exchanging with other firms the complementary fraction $(1 - \alpha)$. The way in which the latter is distributed among the remaining $N - 1$ firms is determined by the financial network structure. A convenient way of representing this structure is through a matrix A of the form

$$A = \begin{pmatrix} \alpha & a_{12} & \cdots & a_{1N} \\ a_{21} & \alpha & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & \alpha \end{pmatrix} \qquad (2)$$

where for each i, j $(i \neq j)$, $a_{ij} \geq 0$ denotes the fraction of shares in the investment project run by firm i that is owned by firm j. Naturally, the following adding-up constraint must be satisfied:

$$\sum_{j \neq i} a_{ij} = 1 - \alpha \qquad (i = 1, 2...., N). \qquad (3)$$

Let $R_b \equiv R - L$. When a shock L hits the project run by some firm i, the exposure to it of all firms in the system is given by Ae_iL, where e_i is the i-th unit vector $[0, .., 1, ..0]^T$. This implies that firm i will default in response to such a shock when

$$\alpha(R - L - M) + \sum_{j \neq i} a_{ij}(R - M) < 0$$

$$\alpha L > R - M.$$

While, for each $k \neq i$, firm k defaults whenever

$$\alpha(R - M) + a_{ki}(R - L - M) + \sum_{i \neq j \neq k} a_{kj}(R - M) < 0$$

or

$$a_{ki}L > R - M$$

Suppose that, without loss of generality, we normalize $R - M \equiv 1$. Then, if in view of (3) we use the notation $a_{ii} \equiv \alpha$, then the condition for firm i to default after a moderate shock hits firm j can be concisely written as

$$a_{ij} > \frac{1}{L}.$$

Notice that our assumption that $\alpha \geq 1/2$ implies that, if any firm $j \neq i$ defaults in response to a shock hitting the project run by i, so will happen to firm i itself.

In what follows we assume that the objective of firms is to minimize the probability of default, which given the symmetry we assume for all possible structures of connections is equivalent to minimizing the number of defaults. This assumption requires some justification to see when it is reasonable. To see this let us look at the expected payoffs in a given period t. Let $\Pi_s = n\pi_s$ be the probability that an s shock hits some firm and $\Pi_b = n\pi_b$ the probability a b shock hits some firm. Then

$$
\begin{aligned}
EU_i^t &= (1-\Pi_s-\Pi_b)(R-M) \\
&+\Pi_s\left(\frac{1}{n}\left(\alpha\left(R_s-M\right)+(1-\alpha)(R-M)\right)+\frac{1}{n}\sum_{j\neq i}^{n}\left(a_{ij}\left(R_s-M\right)+\alpha(R-M)\right)\right) \\
&+\Pi_b\left(\frac{1}{n}\int_{\alpha(R-L-M)+(1-\alpha)(R-M)>0}\left(\alpha(R-L-M)+(1-\alpha)(R-M)\right)dF\left(R_b\right)\right. \\
&\left.+\frac{1}{n}\sum_{j\neq i}^{n}\int_{a_{ij}(R-L-M)+\alpha(R-M)>0}\left(a_{ij}(R-L-M)+\alpha(R-M)\right)dF\left(R_b\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
EU_i^t &= (1-\Pi_s-\Pi_b)(R-M) \\
&+\pi_s\left(\left(R_s-M\right)+(1-\alpha+\alpha(n-1))(R-M)\right) \\
&+\pi_b\left(\int_{\alpha(R-L-M)+(1-\alpha)(R-M)>0}\left(\alpha(R-L-M)+(1-\alpha)(R-M)\right)dF\left(R_b\right)\right. \\
&\left.+\sum_{j\neq i}^{n}\int_{a_{ij}(R-L-M)+\alpha(R-M)>0}\left(a_{ij}(R-L-M)+\alpha(R-M)\right)dF\left(R_b\right)\right)
\end{aligned}
$$

The probability of survival for a firm i in period t is:

$$
P_{it} = (1-\Pi_b)+\pi_b\left(\Pr\left(\alpha(R-L-M)+(1-\alpha)(R-M)>0\right)+\sum_{j\neq i}^{n}\Pr\left(a_{ij}(R-L-M)+\alpha(R-\right.\right.
$$

Hence the expected value of a firm i is:

$$
EV_i = \sum_{t=0}^{\infty}\delta^t\left(\prod_{s=0}^{t}P_{is}\right)EU_i^t
$$

Given that $P_{it} \geq 1-\Pi_b$ and that $EU_i^t \geq (1-\Pi_s-\Pi_b)(R-M)$ it is clear that

$$
EV_i \geq \sum_{t=0}^{\infty}\delta^t\left(1-\Pi_b\right)^t(1-\Pi_s-\Pi_b)(R-M) = \frac{(1-\Pi_s-\Pi_b)(R-M)}{1-\delta(1-\Pi_b)}
$$

This means that:

Remark 1 *If Π_b is sufficiently small and δ sufficiently close to zero, designing the set of connections in order to maximize the probability of survival is, in our framework, equivalent to maximizing expected profits.*

Different financial structures are associated to different matrices A. In this paper we shall focus our attention on structures varying along three different dimensions: segmentation, connectivity, and securitization:

- By *segmentation* we mean the partition of the overall population into equal groups of interconnected firms – or, in network terms, into equal-sized components. The measure of such segmentation is given by the number C of equal-sized components in which the population is divided.

- By *internal connectivity*, we refer to the strength with which a typical firm, is connected with other firms within a component. Firms can be directly or indirectly connected. A direct connection can be achieved by participating in the net revenue stream of other firms. An indirect connection by participating in other firms which themselves have participations in third parties. We describe in Appendix B a concrete way to operationalize this, via a securitization process. The two main requirements we will make on this process that are important in what follows are that:

  1. All firms in a component should be directly or indirectly related, i.e. should have some extent of risk-sharing. For otherwise, unrelated firms effectively would belong to different "economic components" and their inclusion in a common component is misleading. This amounts to requiring that $a_{ij}^q > 0$ for all i, j, where a_{ij}^q stands for the ij-entry of the matrix A^q.

  2. When all direct and indirect effects are taken into account, the fraction of shares that each firm holds of its own investment project should be the same across different configurations. For, otherwise, the comparison of their performance would be vitiated

by the consideration of, effectively, very heterogenous situations – not just in terms of the network pattern but also in terms of the effective risk externalization of their own investment project. This requires that, for all i, $a^q_{ii} = 1 - \sum_{j \neq i} a^q_{ij} = \alpha$.

3.3 The Continuum Approximation

To facilitate our analysis, we rely on a continuum version of the preceding discrete setup that reproduces the main qualitative properties of the finite model. There is a measure N of firms that face the risk of default upon the arrival of a shock of uncertain magnitude. And, if a shock of magnitude L arrives, it is taken to hit uniformly a subset of firms of measure one – thus L is both the aggregate as well as the individual impact of the shock.

Two configurations are studied, which are to be conceived as the continuum counterparts of the ring and complete interaction patterns considered in the discrete setup. On the other hand, we also allow firms to be segmented into a number of independent components. And, as advanced, the aim of our analysis will be to understand what interaction structure and degree of segmentation are optimal, in the sense of minimizing the expected number of defaults (given a specific distribution governing the arrival of shocks).

Let us start by describing in detail the formalization of segmentation and network structure in the present continuum setup. First, segmentation is simply given by the number m of independent components in which the whole population is partitioned. For any such m, the induced size of each component is simply given by an equal measure of agents given by $K + 1 \equiv N/m$.

On the other hand, the pattern of connections prevailing in each component is formalized through some real function $f(\cdot)$ that specifies the exposure faced by the firms in it. To define this function, it is useful to index (or arrange) firms in each component along a one-dimensional ring of length $K + 1$. Then, the risk exposure of a firm i to the shock experienced by (the investment project of) a firm j that is at ring distance $d(i, j)$ is given by $f(d(i, j)) \in [0, 1]$, which is simply interpreted as the share of j's investment project in i's portfolio.

The complete and ring interaction structures are captured by alternative such functions, f_c, $f_r : [0, K/2] \to [0,1]$ respectively. For the complete structure, the function $f_c(\cdot)$ must embody the feature that, if shock hits the component, all firms that are not directly affected by it bear the same consequences. Thus it has to be a constant function given by

$$f_c(d) \equiv \frac{1-\alpha}{K},$$

where the constant value $\frac{1-\alpha}{K}$ is merely a consequence of the adding-up constraint

$$2\int_0^{K/2} f_c(d)\mathrm{d}x = 1-\alpha. \tag{4}$$

This adding-up constraint merely reflects our assumption that each firm retains an α share on its own investment project, while it externalizes the remaining fraction. Thus, if a unit measure of firms in a component of size $K+1$ is hit by a shock, the measure K of agents not directly affected by it must jointly absorb the fraction $1-\alpha$ of it.

On the other hand, the continuum counterpart of the ring network must capture the key feature that firms in the component are all interconnected but at a strength that decays gradually with ring distance. This motivates positing that the function $f_r(\cdot)$ satisfies the following boundary constraints:

$$f_r(0) = \alpha \tag{5}$$

$$f_r(K/2) = 0 \tag{6}$$

together with the adding-up constraint

$$2\int_0^{K/2} f_r(x)\mathrm{d}x = 1-\alpha, \tag{7}$$

that is the analogue of (4). Among the functions consistent with (5)-(6) and (7), we want to consider those of the simplest form. This motivates our restriction to piece-wise linear functions uniquely characterized by the point (H, H) that lies in the bisectrix. This amounts to the following specification:

$$\begin{aligned} f_r(d) &= \alpha - \tfrac{\alpha-H}{H}d && \text{for} d \le H \\ &= \tfrac{HK}{K-2H} - \tfrac{2H}{K-2H}d && \text{for} H \le d \le K/2 \\ &= 0 && \text{for} d \ge K/2 \end{aligned}$$

where H is to be determined as the unique value that satisfies (7). Thus, from

$$
\begin{aligned}
2\int_0^{K/2} f_r(x)\mathrm{d}x &= 2\left(\frac{(\alpha - H)\,H}{2} + H^2 + \frac{H\,(K/2 - H)}{2}\right) \\
&= 2\left(\frac{\alpha H}{2} - \frac{H^2}{2} + H^2 + \frac{HK}{4} - \frac{H^2}{2}\right)
\end{aligned}
$$

we arrive at:

$$H\left(\alpha + \frac{K}{2}\right) = 1 - \alpha$$

hence

$$H = \frac{2(1-\alpha)}{K + 2\alpha}.$$

As explained, the function $f_r(.)$ must be monotonically decreasing. In order for this requirement to hold, we need: (i) $\alpha \geq H$ and (ii) $K/2 \geq H$. Hence we arrive at the following pair of parameter restrictions:

$$
\begin{aligned}
\alpha &\geq \frac{2(1-\alpha)}{K+2\alpha} \\
\frac{K}{2} &\geq \frac{2(1-\alpha)}{K+2\alpha}
\end{aligned}
$$

which will be maintained in what follows. They require that the component size K not be too small relative to $1-\alpha$, the degree of risk externalization. Note that for the more demanding case in this respect given by $\alpha = 1/2$, this condition simply requires that $K \geq 1$, i.e. the component must involve as many firms as those (of unit measure) that are hit by a shock. Finally, also note that the function $f_r(.)$ is continuous at H, while it is concave or convex depending on whether, respectively, K is small or large relative to $1-\alpha$ – specifically, on whether K is smaller or larger than $2(1-\alpha)/\alpha$.

Given the component size K and pattern of exposure described by the function f_c and f_r, let us now determine the extent of default induced by any given shock of magnitude L in each case. Recall the assumption that, when such a shock arrives, it hits uniformly a unit mass of firms. Whether these directly affected firms default or not is independent of the underlying configuration (i.e. component size or interaction structure). For they will default in every case if, and only if,

$$L > \frac{1}{\alpha}.$$

If this inequality is not met, then neither these firms will default nor any other in the corresponding component. This simply follows from the fact that $\alpha \geq 1/2$ and $f(d) \leq \alpha$ for any $d \geq 0$. But if those directly affected do default, what happens to all the others in the component naturally depends on the size $K + 1$ of the component and on its interaction pattern – i.e. on whether f_c or f_r applies.

In the case where the interaction pattern is complete, the implied symmetry leads to the following straightforward conclusion. All the firms indirectly affected in the component (i.e. not directly hit by the shock) will jointly default if

$$L > \frac{K}{1-\alpha}$$

whereas none of those will default otherwise. Thus, if we let $g_c(L)$ stand for the number of defaults among those firms not directly hit by the shock, it is given by the following step function:

$$g_c(L) = \begin{cases} 0 & \text{if} L \leq \frac{K}{1-\alpha} \\ K & \text{if} L > \frac{K}{1-\alpha} \end{cases}$$

Of course, when the component is connected through a ring interaction structure (as captured by f_r), the conclusion is generally not so extreme. For, in this case, whether a firm in the component defaults or not depends on its ring distance to those firms that have been directly affected. The threshold that marks the relevant "default range" is given by the distance $\hat{d}$ such that

$$f_r(\hat{d})L = 1$$

so that a firm defaults if, and only if, its distance d to the set of firms directly hit by the shock is such that

$$d < f_r^{-1}(1/L).$$

Under a ring structure, therefore, the effect of shocks on the number of defaults is not extreme and discontinuous as under complete (direct) interaction. Rather, as the magnitude L of the shock increases, the number of defaults among the firms indirectly affected by it grows gradually, as determined by the function $g_r(L) \equiv 2f_r^{-1}(1/L)$. This function is easily seen to be as follows (see an illustration in Figure **??** for $K = 20$ and $\alpha = 0.5$):

$$g_r(L) = \begin{cases} 0 & \text{for } L \leq \frac{1}{\alpha} \\ \frac{2\alpha H}{\alpha - H} - \frac{2H}{\alpha - H}\frac{1}{L} & \text{for } \frac{1}{\alpha} \leq L \leq \frac{1}{H} \\ K - \frac{K-2H}{H}\frac{1}{L} & \text{for } L \geq \frac{1}{H} \end{cases} . \tag{8}$$

[[

Definition of the function g_r

$$f(L,K,\alpha) = \begin{cases} \frac{K}{2} - \frac{K-2H(K,\alpha)}{2H(K,\alpha)}\frac{1}{L} & \text{if} \quad \frac{1}{L} \leq H(K,\alpha) \\ \frac{\alpha H(K,\alpha)}{\alpha - H(K,\alpha)} - \frac{H(K,\alpha)}{\alpha - H(K,\alpha)}\frac{1}{L} & \text{if} \quad H(K,\alpha) \leq \frac{1}{L} \leq \alpha \\ 0 & \text{if} \quad \frac{1}{L} \geq \alpha \end{cases}$$

$H(K,\alpha) = \frac{2(1-\alpha)}{K+2\alpha}$

$g(L) = f(L, 20, 0.5)$

$g(L)$

]]fhFUX4.4996in3in0ptThe function $g(L)$ specifying the expected number of firms that default in a component that displays a ring structure.Function g$_r$*Plot*

4 Optimal Financial Structures

Given the continuum setup just described, our objective now is to address a design question. We want, that is, to identify the optimal financial structures that minimize the expected number of defaults. The key presumption here is that averting default is the preeminent consideration of any firm – and, for that matter, of a putative designer as well. Such "lexicographic" preferences can be based on the large direct costs associated to default (e.g. asset liquidation) or, as our preferred option, as a reflection of the long string of future returns that default irreversibly forgoes.

To find the optimal such structure amounts to determining the best extent of *segmentation*, as well as the *pattern of connections* to be had within each component. Concerning segmentation, the issue is simply to the determine the optimal size of components. Concerning the connection pattern, on the other hand, most of our discussion will focus on the comparison of the two polar cases considered so far: the ring and the completely connected structures. Later, however, we shall briefly discuss how matters can be extended to other intermediate arrangements.

We organize the analysis in three parts. First, in Subsection 4.1 we identify specially clear-cut conditions under which the optimal segmentation is one of the two polar extremes, maximal or minimal, and the interaction structure must be complete. Then, in Subsection 4.2 we extend the analysis to contexts where intermediate levels of segmentation are optimal. Finally, in Subsection 4.3 we identify scenarios where not only intermediate levels of segmentation but also incomplete connectivity is optimal for some parameter values.

4.1 Polarized segmentation

In order to get a sharp understanding of the forces at work, we shall start by assuming that the magnitude of the shocks (the severe ones) is Pareto distributed with support $[1, \infty)$ and density $\gamma/L^{\gamma+1}$. By modulating the decay parameter γ, this formulation already allows the discussion of many questions of interest – e.g. the contrast between fat or thin tails . But, as we shall see, it yields the conclusion that the optimal degree of segmentation is always extreme (i.e. maximal or minimal). Later, we shall extend our analysis to other more general setups where a wider range of issues can be studied and the conclusions for segmentation are not extreme.

Our aim here, therefore, is to find how γ affects the optimal degree of segmentation (as described by K) for the ring and the completely connected component. Then, we shall compare the two. First, if the interaction structure is that of the ring, the expected (additional)[1] mass of firms defaulting

[1] Note that the firms hit directly by the shock default or not independently of the arrangement. (This follows from our normalization constraint that has every firm holding

when the system is divided into rings of size $K+1$ is given by:

$$D_r(\alpha, K, \gamma) = 2\int_0^{\infty} g_r(L)\, \mathrm{d}L$$
$$= 2\int_{1/H}^{\infty} \left(\frac{K}{2} - \frac{K-2H}{2H}\frac{1}{L}\right)\frac{\gamma}{L^{\gamma+1}}\mathrm{d}L + 2\int_{1/\alpha}^{1/H}\left(\frac{\alpha H}{\alpha-H} - \frac{H}{\alpha-H}\frac{1}{L}\right)\frac{\gamma}{L^{\gamma+1}}\mathrm{d}L$$
$$= 2\gamma\left[-\frac{K}{2}\frac{1}{\gamma y^{\gamma}} + \frac{K-2H}{2H}\frac{1}{(\gamma+1)y^{\gamma+1}}\right]_{1/H}^{\infty} + 2\gamma\left[-\frac{\alpha H}{\alpha-H}\frac{1}{\gamma y^{\gamma}} + \frac{H}{\alpha-H}\frac{1}{(\gamma+1)y^{\gamma+1}}\right]_{1/\alpha}^{1/H}$$

which is in turn equal to

$$\begin{aligned} D_r(\alpha, K, \gamma) &= 2\gamma\left[\frac{K}{2}\frac{H^{\gamma}}{\gamma} - \frac{K-2H}{2H}\frac{H^{\gamma+1}}{(\gamma+1)}\right] \\ &+2\gamma\left[-\frac{\alpha H}{\alpha-H}\frac{H^{\gamma}}{\gamma} + \frac{H}{\alpha-H}\frac{H^{\gamma+1}}{(\gamma+1)} + \frac{\alpha H}{\alpha-H}\frac{\alpha^{\gamma}}{\gamma} - \frac{H}{\alpha-H}\frac{\alpha^{\gamma+1}}{(\gamma+1)}\right] \\ &= 2\left[\left(\frac{K}{2}\left(\frac{1}{\gamma+1}\right) + \left(\frac{2(1-\alpha)}{K+2\alpha}\frac{\gamma}{\gamma+1}\right) - \frac{2(1-\alpha)}{K\alpha+2\alpha^2-2(1-\alpha)}\left(\alpha - \frac{\gamma}{\gamma+1}\frac{2(1-\alpha)}{K+2\alpha}\right)\right)\right] \\ &\times\left(\frac{2(1-\alpha)}{K+2\alpha}\right)^{\gamma} + 2\left[\frac{2(1-\alpha)}{K\alpha+2\alpha^2-2(1-\alpha)}\frac{\alpha^{\gamma+1}}{(\gamma+1)}\right] \end{aligned} \tag{9}$$

Studying the behavior of the derivative of the above expression w.r.t. K we can derive the optimal degree of segmentation for the case of the ring. For simplicity we focus our attention on the case $\alpha = 1/2$; in this case, as recalled above the minimal admissible value of K is 1 and its maximal value is $N-1$.

Proposition 2 *Assume $\alpha = 1/2$. When the shock has a Pareto distribution, the optimal degree of segmentation for the ring structure is maximal (i.e. the optimal value of K is 1) if $\gamma < 1$ and is minimal (the optimal K is $N-1$) if $\gamma > 1$.*

Proof: See the Appendix.

Thus, for the ring structure, when the distribution of the shocks exhibits fat tails, defaults are minimized by minimizing links, that is, by breaking the system into disjoint components of minimal size. Otherwise it is optimal to have a single ring component.

---

the same share of its own investment project.) Thus, in comparing different structures, it is enough to focus on how the shock affects those firms that are indirectly affected by it.

Next, we turn to studying the analogous question for the case where the components are completely connected. In this case, the expected mass of firms defaulting when the components are of size $K+1$ is:

$$D_c(\alpha, K, \gamma) = K \ \Pr\left(L \geq \frac{K}{1-\alpha}\right) = K \ \left(\frac{1-\alpha}{K}\right)^{\gamma}. \tag{10}$$

Hence

$$\frac{\partial D_c}{\partial K} = -(\gamma - 1)\left(\frac{1-\alpha}{K}\right)^{\gamma}$$

which is monotone and readily implies that the optimal degree of segmentation is the same as for the ring. Thus we have:

Proposition 3 *When the shock has a Pareto distribution, the optimal degree of segmentation for the completely connected structure is maximal if* $\gamma < 1$ *and is minimal if* $\gamma > 1$.

Finally, given γ, it remains to compare the expected mass of defaults with the ring and the completely connected structures at the respective optimal values of K to identify the optimal structure. We will show (again by focusing our attention for simplicity on $\alpha = 1/2$) that the optimal structure is always completely connected, be it at maximal or minimum segmentation.

Proposition 4 *When the shock has a Pareto distribution, if* N *is large enough, the completely connected structure (weakly) dominates the ring structure for all values of* γ.

Proof: See the Appendix.

4.2 Intermediate Degrees of Segmentation

The findings in Propositions (2) and (3) show that there is indeed a trade-off between risk sharing and contagion. When the distribution of the shocks has the simple Pareto structure and thus it either has, or does not have, fat tails, the optimal degree of segmentation is always extreme, i.e. maximal or minimal. We show next that this is no longer true when the distribution of the shocks is less clear-cut, as for instance when it is given by the mixture of two Pareto distributions.

Proposition 5 *Suppose that the shock is distributed with probability* p *as a Pareto distribution with parameter* $\gamma > 1$ *and with probability* $1 - p$ *as a Pareto distribution with parameter* $\gamma' < 1$*. Then, there are values* p_0 *and* p_1 *with* $0 < p_0 < p_1 < 1$ *such that, if* $p \in (p_0, p_1)$*, the optimal degree of segmentation for a completely connected component is attained at an intermediate level* K^**, with*$1 < K^* < N - 1$.

Proof: See the Appendix.

The previous result establishes that an intermediate level of segmentation is optimal among completely connected structures when the shock distribution involves a mixture with positive probabilities of displaying both fat and thin tails. A similar conclusion arises when the components display a ring structure, although a close-form solution in this case is hard to obtain. We illustrate matters, therefore, through the following example.

Example 6 *Set again* $\alpha = 0.5$*, and let* $\gamma = 2$*,* $\gamma' = 0.5$ *and* $p = 0.95$*. For these values we find that for the ring structure the optimal degree of segmentation obtains at* $\hat{K} = 7.76$*, at which value the expected mass of defaults reaches its minimum given by* 0.145*. In contrast, we find that the optimal degree of segmentation among completely connected components is reached at* $\tilde{K} = 5.65$*, and the expected mass of defaults is* 0.126*. This implies that, if* N *is large enough (so that the population can be segmented in equal-size components of almost any desired size), a completely connected structure is optimal. It is interesting to observe that* $\tilde{K} < \hat{K}$*, which can be heuristically understood as a reflection of the fact that, when arranged optimally, ring components compensate for a connectivity lower than for complete ones though an increase in size.*

4.3 Sparse Connections

Let us consider now the case where the shock is the mixture of a Pareto distribution and a Dirac distribution, putting all the probability mass on a single shock realization $\bar{L}$, larger than $(N-1)/(1-\alpha)$. Thus even if K is at its maximum feasible size of $N-1$, such a shock will cause the default of all firms in the component. In this case we show that the completely connected structure component is dominated by the ring when $1 < \gamma < 2$ and $\bar{K}$ is sufficiently large.

Proposition 7 *Let $\alpha = 1/2$ and assume that, with probability p, the shock follows a Pareto distribution with parameter $2 > \gamma > 1$ and with probability $1-p$ it equals $\bar{L} = 2(N-1)+1$. In addition, suppose that*

$$\frac{(1-p)}{p} < (\gamma - 1) \ \left(\frac{1}{2(N-1)}\right)^{\gamma}. \tag{11}$$

Then, for all values of N such that

$$N > 1 + \left(\frac{1}{2^{\gamma-1}} + \frac{\frac{1}{2^{\gamma-1}}\frac{1}{(\gamma+1)}}{(\gamma-1)\ \frac{1}{2^{\gamma+1}} + \frac{1}{2^{\gamma}} - \frac{1}{\gamma+1}}\right)^{\frac{1}{2-\gamma}} \tag{12}$$

the optimal financial structure is a single ring with no segmentation.

Proof: See the Appendix.

The previous result shows that there are simple shock scenarios where the optimal arrangement includes taking advantage of the limited connectivity afforded by the ring structure. In those cases, the ring provides a suitable compromise between the extent of *risk sharing* allowed by extensive *indirect* connectivity (i.e. minimal segmentation) and the limits to wide *risk contagion* resulting from sparse *direct* connection (i.e. a low node degree).

But, in general, such a trade-off between risk sharing and risk contagion that is at the heart of our model can be captured though less polarized structures than the ring and complete ones. The next remark briefly discusses one simple possibility in this respect, obtained by interpolating between those two extremes in a way that is tractable and intuitive.

Remark 8 *Consider a piece-wise linear interpolation between the polar cases of the ring and the complete network. This interpolation can be obtained by linearly shifting the "kink"* (x, y) *along the segment joining the kinks in the two polar cases, i.e. along the segment defined by the points* (H, H) *and* $(\frac{K}{2}, \frac{1-\alpha}{K})$*. Since we want to maintain that all the firms in the component are at least indirectly connected with each other – that is, all have a positive degree of exposure to each other – the horizontal intercept for all these functions must continue to be at the point* $(K/2, 0)$*. Thus, the only degree of freedom left is the vertical intercept* $(0, z)$*, which must be determined to satisfy the adding-up constraint*

$$2 \int_0^\infty f(d; x, K)\ \mathrm{d}d = 1 - \alpha \tag{13}$$

Figure **??** *illustrates the construction. The closer* x *is to* $K/2$*, the closer is* $f(\cdot)$ *to the exposure function of the complete structure, while as it approaches* H *it tends to that of the ring structure.*

fhFU4.075in2.7717in0ptLinear interpolation between the complete and ring structures in the continuum version of the model.InterpolationFigure

5 Asymmetric structures

So far we have concentrated the discussion on groups of firms where all of them are exactly identical. Although this allows us to obtain analytical results and to gather intuition, the real world contains firms of very different sizes, so it is useful to see how our framework extends when firms are different in size.

Consider a situation with two kinds of firms. One type is exactly as the ones we have been considering so far. The othe one has size $\beta > 1$, or more precisely, its returns when there are no shock are those of the other type multiplied by β. The distribution of shocks is the same for both kinds of firms, and they have the same distribution as we have been considering so far. The only difference is that the probability of a shock hitting a large firm is β times the probability it hits the other type of firm. In a sense one can view large firms as one of the completely connected components we have considered until now, but complete integration lifts the requirement that each "unit" of the composite needs to keep responsibility of a proportion α of its own revenue stream.

Having different kinds of firms allows us to study new types of structures. We concentrate our attention on "stars" where a large firm of type β is connected directly to a number of smaller firms. The structure of connections without securitization, taking into account that the values of exchanged assets has to the same for every pair, is:

$$A = \begin{pmatrix} \delta & (1-\delta)/\beta & (1-\delta)/\beta & \cdots & (1-\delta)/\beta \\ (1-\delta) & \delta & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (1-\delta) & 0 & 0 & \cdots & \delta \end{pmatrix}$$

Once firms do a further round of securitization to guarantee that requirement 1 of section 3.2 is met we have:

$$A^2 = \begin{pmatrix} \delta^2 + (1-\delta)^2 & 2\delta(1-\delta)/\beta & (1-\delta)/\beta & \cdots & (1-\delta)/\beta \\ 2\delta(1-\delta) & \delta^2 + (1-\delta)^2/\beta & (1-\delta)^2/\beta & \cdots & (1-\delta)^2/\beta \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2\delta(1-\delta) & (1-\delta)^2/\beta & (1-\delta)^2/\beta & \cdots & \delta^2 + (1-\delta)^2/\beta \end{pmatrix}$$

and in order to guarantee that the all firms have an exposure to themselves at least equal to the minimum α we make $\alpha = \delta^2 + (1-\delta)^2/\beta$ and then the large firm a larger exposure to itself $\alpha' = \delta^2 + (1-\delta)^2 > \alpha$ so that requirement 2 of section 3.2 is met. Noting $\alpha' - \alpha = (1-\delta)^2(\beta-1)/\beta$, this means we can rewrite

$$A^2 = \begin{pmatrix} \alpha' & (1-\alpha')/\beta & (1-\alpha')/\beta & \cdots & (1-\alpha')/\beta \\ 1-\alpha' & \alpha & (\alpha'-\alpha)/(\beta-1) & \cdots & (\alpha'-\alpha)/(\beta-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1-\alpha' & (\alpha'-\alpha)/(\beta-1) & (\alpha'-\alpha)/(\beta-1) & \cdots & \alpha \end{pmatrix}$$

As in previous sections, a firm defaults (and so loses all future output) provide its exposure to the firm where the shock hits is bigger than its rents.

Formally a firm i fails when a shock hits j in the same component if $a_{ij}L > 1$ for a regular firm and if $a_{ij}L > \beta$ for a large firm.

We now compare two types of arrangements. In both of them we have 4β firms. One arrangement, which we call the *star structure*, is composed of two *stars*, each one with a β firm in the center and β spokes of unit size. The other arrangement, which we call the *symmetric structure*, is composed of two complete components. One of them has two firms of size β, and the other one has 2β firms of unit size. We can show that:

Proposition 9 *When $\delta > 1/3$, the star structure has a higher number expected number of defaults than the symmetric structures whenever*

$$\frac{\beta}{\alpha'} \leq L \leq \frac{\beta}{\alpha}, \frac{\beta}{1-\alpha'} \leq L \leq \frac{2\beta - 1}{1-\alpha}$$

the star structure has a lower number expected number of defaults than the symmetric structures whenever

$$\frac{\beta}{1-\alpha} \leq L \leq \frac{\beta}{1-\alpha'}, \frac{2\beta - 1}{1-\alpha} \leq L \leq \frac{\beta - 1}{\alpha' - \alpha}$$

When $\delta \leq 1/3$, the star structure has a higher number expected number of defaults than the symmetric structures whenever

$$\frac{\beta}{\alpha'} \leq L \leq \frac{\beta}{\alpha}, \frac{\beta - 1}{\alpha' - \alpha} \leq L \leq \frac{2\beta - 1}{1-\alpha}$$

the star structure has a lower number expected number of defaults than the symmetric structures whenever

$$\frac{\beta}{1-\alpha} \leq L \leq \frac{\beta - 1}{\alpha' - \alpha}, \frac{2\beta - 1}{1-\alpha} \leq L \leq \frac{\beta}{1-\alpha'}$$

The relative dominance of the symmetric structure in the range

$$\frac{\beta}{\alpha'} \leq L \leq \frac{\beta}{\alpha}$$

is easy to explain. The star forces the center to hold more of its own assets (α' versus α) than the symmetric structire, to exchange with the smaller spokes and keep the moral hazard constraint. This, in turn makes the risk sharing smaller, which is bad against relatively small shocks. On the other hand this protects her against large shocks, which is why it dominates in the

$$\frac{\beta}{1-\alpha} \leq L \leq \frac{\beta}{1-\alpha'}$$

range. What happens in the ranges

$$\frac{\beta}{1-\alpha'} \leq L \leq \frac{2\beta-1}{1-\alpha}, \frac{2\beta-1}{1-\alpha} \leq L \leq \frac{\beta-1}{\alpha'-\alpha}$$

is a bit more subtle. There we have situations where a shock is large enough to kill everyone in the symmetric structure, but is kept "contained" in the center when it arises at the spokes because of the larger size and internal exposure of the center, which acts as a sort of *firebreak*.

6 Stability and optimality

In this section we want to examine the relationship between overall optimality of the pattern of connections and the individual incentives to form those connections. We restrict our attention to the case where complete connections within a component are both efficient and individually optimal. In this situation the only possible issue that arises is the size of the components that firms would want to create.

Formally, we model the individual incentives through the notion of bilateral equilobrium, which is often used in the network literature (see e.g. Goyal and Vega Redondo 2007).

Definition 10 *A strategy profile s^* is a bilateral equilibrium (BE) if the following conditions hold:*

1. *For any $i \in N$ and every $s_i \in S_i$, $EU_i(s^*) \geq EU_i(s_i, s^*_{-i})$*
2. *For any pair of players $i, j \in N$ and every strategy pair (s_i, s_j),*

$$EU_i(s_i, s_j, s^*_{-i-j}) > EU_i(s^*_i, s^*_j, s^*_{-i-j}) \Longrightarrow EU_j(s_i, s_j, s^*_{-i-j}) < EU_j(s^*_i, s^*_j, s^*_{-i-j})$$

We now can state our result.

Proposition 11 *Suppose that the shock L is distributed with probability p as a Pareto distribution with parameter $\gamma > 1$ and with probability $1-p$ as a Pareto distribution with parameter $\gamma' < 1$ and p and $(1-p)$ are such that the optimal segmentation obtains at a strictly interior level $K^* \in (1, N)$. Then, the optimal structure is (generically) not a bilateral equilibrium.*

Proof: See the Appendix.

In words, we find that there is typically a conflict between group optimality and individual incentives. This conflict derives from the structure of the socially optimal configuration and an externality in the formation of the network. Social optimality requires that all components be of identical size, a size which is (generically) smaller than the group-optimal one. Then if the components were all of optimal size, individual members of each group would have an incentive to join other groups, which would thus become larger and closer to the optimal size. These actions while beneficial to the firms seceding and to the admitting groups, create a externality on those which remain behind. Once groups of optimal size (the one minimizing the default probability of their members) are formed, its members would block admitting new members from a smaller components, hence generating a socially inferior bilateral equilibrium.

7 Conclusion

We have proposed a stylized model to study the problem that arises when firms need to share resources to weather shocks that can threaten their survival, but are then exposed to risk coming from those same connections that help them in the time of need. We have found that when large shocks are likely firms would like to stay in very small groups, but that large groups are desirable when most shocks are of moderate size. Intermediate structure arise when the weight is more balanced between very large and moderate shocks. Some distributions also make it desirable some internal "detachment" within groups as well. We have additionally studied asymmetric structures where a "central" agent can have useful properties as a *firebreak*.

Finally, we find that there is typically a conflict between individual and social optimality, arising from the fact that those components that attain the best size (and thus minimize the default probability of their members), will block admitting new members from a smaller components.

There are many issues that this paper did not study in depth. Although we have explored some conditions under which asymmetries and loose internal connections are best, we do not have a general theorem that explains the likely circumstances under which different topologies will be optimal. Since our work has identified a disparity between efficiency and equilibrium outcomes, it would also be interesting to do some work with distributions of shocks that mimic those observed in reality. In that way we can have an idea for whether those disparities are severe enough to warrant policy measures that modify incentives for business group formation.

8 Appendix A

Proof of Proposition 3: It is enough to show that

$$\frac{\partial D_r(1/2, K, \gamma)}{\partial K} \gtrless 0 \Leftrightarrow \gamma \lessgtr 1 \tag{14}$$

at all $K > 1$. From the expression (9) that gives the expected number of defaults under a ring structure we obtain, making $\alpha = 1/2$, the following expression:

$$D_r(1/2, K, \gamma) = \left(K\left(\frac{1}{\gamma+1}\right) - \frac{2}{K-1}\frac{1}{\gamma+1}\right)\left(\frac{1}{K+1}\right)^{\gamma} + \frac{1}{K-1}\frac{1}{(\gamma+1)}\left(\frac{1}{2}\right)^{\gamma-1}.$$

and hence

$$\begin{aligned} &\tfrac{\partial D_r(1/2,K,\gamma)}{\partial K} = -\tfrac{1}{(K-1)^2}\tfrac{1}{(\gamma+1)}\left(\tfrac{1}{2}\right)^{\gamma-1} \\ &+\left(K\left(\tfrac{-\gamma}{\gamma+1}\right)\tfrac{1}{K+1} + \tfrac{2}{K-1}\tfrac{1}{K+1}\tfrac{\gamma}{\gamma+1} + \left(\tfrac{1}{\gamma+1}\right) + \tfrac{2}{(K-1)^2}\tfrac{1}{\gamma+1}\right)\left(\tfrac{1}{K+1}\right)^{\gamma} \end{aligned} \tag{15}$$

Now note that the inequality $\partial D_r(1/2, K, \gamma)/\partial K > 0$ is equivalent to:

$$\frac{2\,(K-1)}{K+1}\gamma + (K-1)^2 + 2 > (K+1)\left(\frac{K+1}{2}\right)^{\gamma-1} + \gamma K\frac{(K-1)^2}{K+1}$$

or

$$(K-1)^2\left(1-\frac{\gamma K}{K+1}\right)+2\left(1+\gamma\frac{K-1}{K+1}\right)>(K+1)^\gamma\,\frac{1}{2^{\gamma-1}}$$

which can be rewritten as

$$(K-1)^2+2+\gamma\,(K-1)\,(2-K)>(K+1)^\gamma\,\frac{1}{2^{\gamma-1}}. \tag{16}$$

So, using the identities

$$(K-1)^2+2+(K-1)\,(2-K)=K+1$$

and

$$(K+1)^\gamma\,\frac{1}{2^{\gamma-1}}=2\left(\frac{K+1}{2}\right)^\gamma$$

we can equivalently write (16) as follows

$$\frac{K+1}{2}-\left(\frac{K+1}{2}\right)^\gamma>\frac{1}{2}(1-\gamma)\,(K-1)\,(2-K)\,.$$

Define the function $\varphi(K,\gamma)$ by

$$\varphi(K,\gamma)=\frac{K+1}{2}-\left(\frac{K+1}{2}\right)^\gamma-\frac{1}{2}(1-\gamma)\,(K-1)\,(2-K)$$

so that the desired conclusion can be stated as follows:

$$\varphi(K,\gamma)\gtrless 0\Leftrightarrow\gamma\lessgtr 1.$$

Note that $\varphi(K,1)\equiv 0$ for all K. Therefore, it follows from the fact that, for all $K>1$,

$$\begin{aligned}\frac{\partial(K,\gamma)}{\partial\gamma} &= -\left(\frac{K+1}{2}\right)^\gamma\ln\frac{K+1}{2}+\frac{1}{2}\,(K-1)\,(2-K)\\ &< -\ln\frac{K+1}{2}+\frac{1}{2}\,(K-1)\,(2-K)<0.\end{aligned}$$

This completes the proof.

Proof of Proposition 4: From the particularization to $\alpha=1/2$ of (9) and (10) we have:

$$\begin{aligned}&D_c(1/2,K,\gamma)-D_r(1/2,K,\gamma) \qquad (17)\\ =\ &\left(\frac{1}{2}\right)^\gamma\left(\frac{1}{K}\right)^{\gamma-1}-K\left(\frac{1}{\gamma+1}\right)\left(\frac{1}{K+1}\right)^\gamma+\frac{2}{K-1}\frac{1}{\gamma+1}\left(\left(\frac{1}{K+1}\right)^\gamma-\left(\frac{1}{2}\right)^\gamma\right)\end{aligned}$$

Hence, again using l'Hôpital,

$$\begin{aligned}
&\lim_{K\to 1}\left(D_c(1/2,K,\gamma)-D_r(1/2,K,\gamma)\right)\\
=&\left(\frac{1}{2}\right)^{\gamma}\left[1-\left(\frac{1}{\gamma+1}\right)\right]+\lim_{K\to 1}\frac{1}{K-1}\frac{2}{(\gamma+1)}\left[\left(\frac{1}{K+1}\right)^{\gamma}-\left(\frac{1}{2}\right)^{\gamma}\right]\\
=&\left(\frac{1}{2}\right)^{\gamma}\left(\frac{\gamma}{\gamma+1}\right)+\frac{-\gamma 2^{\gamma}}{(\gamma+1)\,4^{\gamma}}=0.
\end{aligned}$$

This implies that the two structures are equivalently optimal when $\gamma<1$, since in this case a maximal segmentation ($K=1$) minimizes the expected number of defaults under both the ring and the completely connected structures

Consider now the case $\gamma>1$, for which minimal segmentation ($K=N-1$) is optimal. For large enough population size we can approximate (17) by

$$\lim_{K\to\infty}\left(D_c(1/2,K,\gamma)-D_r(1/2,K,\gamma)\right)=\lim_{K\to\infty}\left[\left(\frac{1}{2}\right)^{\gamma}\left(\frac{1}{K}\right)^{\gamma-1}-K\left(\frac{1}{\gamma+1}\right)\left(\frac{1}{K+1}\right)^{\gamma}\right]$$

which is negative if, and only if,

$$(\gamma+1)<2^{\gamma}\lim_{K\to\infty}\left(\frac{K}{K+1}\right)^{\gamma}$$

that indeed holds as long as $\gamma>1$. This completes the proof.

Proof of Proposition 5: The expected mass of the firms defaulting when the shock follows a mixture of a Pareto distribution with parameter γ and a Pareto distribution with parameter γ', with weights respectively p and $1-p$, is:

$$D_c(\alpha,K,\gamma,\gamma',p)=pK\,\left(\frac{1-\alpha}{K}\right)^{\gamma}+(1-p)\,K\,\left(\frac{1-\alpha}{K}\right)^{\gamma'}$$

Hence

$$\frac{\partial D_c}{\partial K}=-p\,(\gamma-1)\left(\frac{1-\alpha}{K}\right)^{\gamma}-(1-p)\left(\gamma'-1\right)\left(\frac{1-\alpha}{K}\right)^{\gamma'}.$$

Since we assume that $\gamma > 1$ and $\gamma' < 1$, we have $\frac{\partial D_c}{\partial K} > 0$ iff:

$$(1-p)\left(1-\gamma'\right)\left(\frac{1-\alpha}{K}\right)^{\gamma'} > p\left(\gamma-1\right)\left(\frac{1-\alpha}{K}\right)^{\gamma}$$

or

$$K > \left(\frac{p\left(\gamma-1\right)}{\left(1-p\right)\left(1-\gamma'\right)}\right)^{\frac{1}{\gamma-\gamma'}}\left(1-\alpha\right)$$

which implies that the function is minimized at an strictly interior point:

$$K^{*}(p) = \left(\frac{p\left(\gamma-1\right)}{\left(1-p\right)\left(1-\gamma'\right)}\right)^{\frac{1}{\gamma-\gamma'}}\left(1-\alpha\right)$$

provided this point is admissible, i.e. $K^{*}(p) \in [1, N-1]$. This condition is satisfied for some suitably chosen intermediate range $p \in [p_0, p_1]$ since $K^{*}(p) = 0$ for $p = 0$ and $K^{*}(p) \to \infty$ as $p \to 1$. This completes the proof.

Proof of Proposition 7: With $\bar{L} = 2(N-1)+1$**,** the expected mass of defaults in the case of completely connected components – now denoted by $D_c(1/2, K, \gamma, p)$ – is given by:

$$D_c(1/2, K, \gamma, p) = (1-p)K + pK\left(\frac{1}{2K}\right)^{\gamma}. \tag{18}$$

Taking the derivative with respect to K yields

$$(1-p) - (\gamma-1)\,p\left(\frac{1}{2K}\right)^{\gamma}$$

which is always negative as long as (11) is satisfied. This establishes that the optimal degree of segmentation in the case of the completely connected structure is minimal, that is obtains at the maximal value of K, given by $N-1$.

Next, noting that when $\alpha = 1/2$

$$H = \frac{1}{K+1}.$$

and

$$g(L) = \frac{K}{2} - \frac{K - 2/(K+1)}{2/\left(K+1\right)}\frac{1}{L}$$

for $K+1 \leq L$, we see that the expected mass of defaults in the case of the ring structure is:

$$D_r(1/2, K, \gamma.p) = (1-p)\,2\left(\frac{K}{2} - \frac{K-2/(K+1)}{2/(K+1)}\frac{1}{2K+1}\right)$$
$$+2p\left[\left(\frac{K}{2}\left(\frac{1}{\gamma+1}\right) + \left(\frac{1}{K+1}\frac{\gamma}{\gamma+1}\right) - \frac{2}{K-1}\left(\frac{1}{2} - \frac{\gamma}{\gamma+1}\frac{1}{K+1}\right)\right)\left(\frac{1}{K+1}\right)^{\gamma}\right]$$
$$+2p\left[\frac{2}{K-1}\frac{1}{2^{\gamma+1}}\frac{1}{(\gamma+1)}\right]$$

It suffices then to show that, when $K = N-1$ (the optimal value for the completely connected structure), the expected defaults are smaller for the ring than for the completely connected structure. Hence we set $K = N-1$. The expected defaults are larger for the complete than for the ring when the following inequality is satisfied:

$$D_c(1/2, N-1, \gamma.p) > D_r(1/2, N-1, \gamma.p) \Leftrightarrow$$
$$(1-p)(N-1) + p(N-1)\ \left(\frac{1}{2K}\right)^{\gamma} > (1-p)\,2\left(\frac{N-1}{2} - \frac{(N-1)-2/N}{2/N}\frac{1}{2(N-1)+1}\right) +$$
$$p2\left[\left(\frac{(N-1)}{2}\left(\frac{1}{\gamma+1}\right) + \left(\frac{1}{N}\frac{\gamma}{\gamma+1}\right) - \frac{2}{N-2}\left(\frac{1}{2} - \frac{\gamma}{\gamma+1}\frac{1}{N}\right)\right)\left(\frac{1}{N}\right)^{\gamma}\right] + p2\left[\frac{2}{N-2}\frac{1}{2^{\gamma+1}}\frac{1}{(\gamma+1)}\right]$$

or, using (11),

$$(\gamma-1)\ \left(\frac{1}{2(N-1)}\right)^{\gamma} 2\left[\frac{(N-1)^2+N-1-2}{2}\frac{1}{2(N-1)+1}\right] = \frac{(1-p)}{p}2\left[\frac{(N-1)^2+(N-1)-2}{2}\frac{1}{2(N-1)+1}\right]$$
$$> -(N-1)\ \left(\frac{1}{2(N-1)}\right)^{\gamma} + 2\left[\left(\frac{(N-1)}{2}\left(\frac{1}{\gamma+1}\right) - \frac{1}{\gamma+1}\left(\frac{1}{N-2}\right)\right)\left(\frac{1}{N}\right)^{\gamma}\right] + 2\left[\frac{2}{N-2}\frac{1}{2^{\gamma+1}}\frac{1}{(\gamma+1)}\right]$$

or equivalently

$$(\gamma-1)\ \left(\frac{1}{2(N-1)}\right)^{\gamma} 2\left(\frac{(N-1)^2+(N-1)-2}{2}\frac{1}{2(N-1)+1}\right) + (N-1)\ \left(\frac{1}{2(N-1)}\right)^{\gamma}$$
$$-2\left(\left(\frac{(N-1)}{2}\left(\frac{1}{\gamma+1}\right) - \frac{1}{\gamma+1}\left(\frac{1}{N-2}\right)\right)\left(\frac{1}{N}\right)^{\gamma}\right)$$
$$-\left(\frac{1}{N-2}\frac{1}{2^{\gamma-1}}\frac{1}{(\gamma+1)}\right) > 0$$

Notice that for $N > 5$

$$\begin{aligned}
& (\gamma - 1) \ \left(\frac{1}{2(N-1)}\right)^{\gamma} 2\left(\frac{(N-1)^2 + N - 3}{2} \frac{1}{2(N-1)+1}\right) + (N-1) \ \left(\frac{1}{2(N-1)}\right)^{\gamma} \\
& -2\left(\frac{(N-1)}{2}\left(\frac{1}{\gamma+1}\right) - \frac{1}{\gamma+1}\left(\frac{1}{N-2}\right)\right)\left(\frac{1}{N}\right)^{\gamma} - \left(\frac{1}{N-2}\frac{1}{2^{\gamma-1}}\frac{1}{(\gamma+1)}\right) \\
> \ & (\gamma - 1) \ \left(\frac{1}{2(N-1)}\right)^{\gamma} 2\left(\frac{(N-1)^2}{4(N-1)}\right) \\
& + (N-1) \ \left(\frac{1}{2(N-1)}\right)^{\gamma} - \left(\frac{N-1}{\gamma+1}\right)\left(\frac{1}{N-1}\right)^{\gamma} - \left(\frac{1}{N-3}\frac{1}{2^{\gamma-1}}\frac{1}{(\gamma+1)}\right) \\
= \ & \frac{1}{(N-1)^{\gamma-1}}\left((\gamma-1) \ \frac{1}{2^{\gamma+1}} + \frac{1}{2^{\gamma}} - \frac{1}{\gamma+1}\right) - \frac{1}{N-2}\left(\frac{1}{2^{\gamma-1}}\frac{1}{(\gamma+1)}\right),
\end{aligned}$$

which is bigger than zero for N large enough so that

$$\frac{N-2}{(N-1)^{\gamma-1}} > (N-1)^{2-\gamma} - \frac{1}{2^{\gamma-1}} > \frac{\frac{1}{2^{\gamma-1}}\frac{1}{(\gamma+1)}}{(\gamma-1) \ \frac{1}{2^{\gamma+1}} + \frac{1}{2^{\gamma}} - \frac{1}{\gamma+1}}$$

A sufficient condition for this to hold is that (12) holds, which establishes the desired conclusion and completes the proof.

Proof of Proposition 9 In order to prove the result we need to describe the losses for different kinds of shocks. First, we look at the losses when a shock hits the center. But before we do that we show

Lemma 12 *In this ranking we use*

$$\frac{\beta-1}{\alpha' - \alpha} > \frac{\beta}{1-\alpha'} \Longleftrightarrow \delta > 1/3 \tag{19}$$

Proof.

$$\begin{aligned}
\frac{\beta-1}{\alpha'-\alpha} &> \frac{\beta}{1-\alpha'} \\
(\beta-1)\left(1-\alpha'\right) &> \beta\left(\alpha'-\alpha\right) = \beta\frac{\beta-1}{\beta}(1-\delta)^2 \\
2(1-\delta)\delta &> (1-\delta)^2 \\
2\delta &> 1-\delta \Longleftrightarrow \delta > 1/3
\end{aligned}$$

Now, when $\delta > 1/3$

$$E_c(L) = \begin{cases} 0 & \text{for } L \leq \frac{1}{\alpha} \\ 1 & \text{for } \frac{1}{\alpha} \leq L \leq \frac{\beta}{1-\alpha'} \\ 1+\beta & \text{for } \frac{\beta}{1-\alpha'} \leq L \leq \frac{\beta-1}{\alpha'-\alpha} \\ 2\beta & \text{for } L \geq \frac{\beta-1}{\alpha'-\alpha} \end{cases}$$

when $\delta \leq 1/3$

$$E'_c(L) = \begin{cases} 0 & \text{for } L \leq \frac{1}{\alpha} \\ 1 & \text{for } \frac{1}{\alpha} \leq L \leq \frac{\beta-1}{\alpha'-\alpha} \\ \beta & \text{for } \frac{\beta-1}{\alpha'-\alpha} \leq L \leq \frac{\beta}{1-\alpha'} \\ 2\beta & \text{for } L \geq \frac{\beta}{1-\alpha'} \end{cases}$$

the difference lying in whether the center or the rest of the periphery dies earlier ■

Now when a shock hits a peripheric firm

$$E_p(L) = \begin{cases} 0 & \text{for } L \leq \frac{1}{\alpha} \\ \beta & \text{for } \frac{\beta}{\alpha'} \leq L \leq \frac{\beta}{1-\alpha'} \\ 2\beta & \text{for } L \geq \frac{\beta}{1-\alpha'} \end{cases}$$

So overall

$$E_{star}(L) = \begin{cases} 0 & \text{for } L \leq \frac{1}{\alpha} \\ \frac{1}{2} \times 1 + \frac{1}{2} \times 0 & \text{for } \frac{1}{\alpha} \leq L \leq \frac{\beta}{\alpha'} \\ \frac{1}{2} \times 1 + \frac{1}{2} \times \beta & \text{for } \frac{\beta}{\alpha'} \leq L \leq \frac{\beta}{1-\alpha'} \\ \frac{1}{2} \times (1+\beta) + \frac{1}{2} \times 2\beta & \text{for } \frac{\beta}{1-\alpha'} \leq L \leq \frac{\beta-1}{\alpha'-\alpha} \\ 2\beta & \text{for } L \geq \frac{\beta-1}{\alpha'-\alpha} \end{cases} \tag{20}$$

$$E'_{star}(L) = \begin{cases} 0 & \text{for } L \leq \frac{1}{\alpha} \\ \frac{1}{2} \times 1 + \frac{1}{2} \times 0 & \text{for } \frac{1}{\alpha} \leq L \leq \frac{\beta}{\alpha'} \\ \frac{1}{2} \times 1 + \frac{1}{2} \times \beta & \text{for } \frac{\beta}{\alpha'} \leq L \leq \frac{\beta}{1-\alpha'} \\ \frac{1}{2} \times \beta + \frac{1}{2} \times 2\beta & \text{for } \frac{\beta}{1-\alpha'} \leq L \leq \frac{\beta-1}{\alpha'-\alpha} \\ 2\beta & \text{for } L \geq \frac{\beta-1}{\alpha'-\alpha} \end{cases} \tag{21}$$

An alternative to a star arrangement is to place two β players together and 2β regular players together in complete components. In that case, since there is no asymmetry within component, every player keeps α of his revenue stream. The shock reaches with probability 1/2 each of the components, and hence the expected losses are

$$E_{symm}(L) = \begin{cases} 0 & \text{for } L \leq \frac{1}{\alpha} \\ \frac{1}{2} \times 1 + \frac{1}{2} \times 0 & \text{for } \frac{1}{\alpha} \leq L \leq \frac{\beta}{\alpha} \\ \frac{1}{2} \times 1 + \frac{1}{2} \times \beta & \text{for } \frac{\beta}{\alpha} \leq L \leq \frac{\beta}{1-\alpha} \\ \frac{1}{2} \times 1 + \frac{1}{2} \times 2\beta & \text{for } \frac{\beta}{1-\alpha} \leq L \leq \frac{2\beta-1}{1-\alpha} \\ 2\beta & \text{for } L \geq \frac{2\beta-1}{1-\alpha} \end{cases} \tag{22}$$

This allows us to compare the two kinds of structures. Notice that

Lemma 13

$$\frac{2\beta - 1}{1 - \alpha} > \frac{\beta}{1 - \alpha'} \Longleftrightarrow \delta > 1/3 \tag{23}$$

Proof.

$$\begin{aligned} \frac{2\beta - 1}{1 - \alpha} &> \frac{\beta}{1 - \alpha'} \\ (\beta - 1)\left(1 - \alpha'\right) + \beta\left(1 - \alpha'\right) &> \beta(1 - \alpha) \\ (\beta - 1)\left(1 - \alpha'\right) &> \beta\left(\alpha' - \alpha\right) \\ \frac{\beta - 1}{\alpha' - \alpha} &> \frac{\beta}{1 - \alpha'} \end{aligned}$$

so the result is then true by (19). ■

The result now follows by comparing the ranges and values of the symmetric structures in equation (22) with those of the star in equations (20) and (21).

Proof of Proposition 11: In the proof of proposition (5) we have established that the function $D_c(.)$ is convex in K, and under our assumptions it has a unique interior maximum.. Let then $K^* \in (1, N-1)$ be the solution to the problem

$$\min_K D_c(\alpha, K, \gamma, \gamma', p).$$

That is, the level of segmentation minimizing defaults within a complete component. We now establish

A A bilateral equilibrium cannot have a completely connected cluster C with $K > K^*$. This is true because for any $i \in C$ its payoff from deleting any link (selling any participation in another firm) will lead to an increased payoff since by convexity

$$D_c(\alpha, K-1, \gamma, \gamma', p) > D_c(\alpha, K, \gamma, \gamma', p)$$

and this would violate part 1 of the definition of bilateral equilibrium.

B A bilateral equilibrium cannot have two clusters C_1 and C_2 with $K_1 < K^*$ and $K_2 < K^*$. This is true because a pair i, j where $i \in C_1$ and $j \in C_2$ can buy a participation in each other (and simultaneously reduce the participations in all other members of their respective components). In this way, and using again convexity their respective payoff satisfy

$$\begin{aligned} D_c(\alpha, K_1+1, \gamma, \gamma', p) &> D_c(\alpha, K_1, \gamma, \gamma', p) \\ D_c(\alpha, K_2+1, \gamma, \gamma', p) &> D_c(\alpha, K_2, \gamma, \gamma', p) \end{aligned}$$

and this would violate part 2 of the definition of bilateral equilibrium.

Let $Q = int\left(\frac{N}{K^*+1}.\right)$. By A and B above, the only candidate for a bilateral equilibrium then has: Q (complete) clusters of size K^*+1 and one (complete) cluster of size $N - Q(K^*+1)$ (the remainder). Such a structure is in fact a bilateral equilibrium as members of the clusters of size K^* do not want to change. And members of the remainder will not be accepted in any other component.

In contrast, the socially optimal structure solves

$$\begin{gathered} \min_{K_i,n} \textstyle\sum_{i=1}^{n} \frac{K_i+1}{N} D_c(\alpha, K_i, \gamma, \gamma', p) \\ \text{s.t.} \textstyle\sum_{i=1}^{n} \frac{K_i+1}{N} = 1 \end{gathered}$$

The first order conditions for this problem (which given the concavity of D_c are sufficient for optimality) require that for any pair of clusters with $K_i \neq K_j$:

$$\frac{1}{N} D_c(\alpha, K_i, \gamma, \gamma', p) + \frac{\partial D_c(\alpha, K_i, \gamma, \gamma', p)}{\partial K_i} \frac{K_i+1}{N} = \frac{1}{N} D_c(\alpha, K_j, \gamma, \gamma', p) + \frac{\partial D_c(\alpha, K_j, \gamma, \gamma', p)}{\partial K_j} \frac{K_j+1}{N}$$

Since $\left.\frac{\partial D_c(\alpha, K_i, \gamma, \gamma', p)}{\partial K_i}\right|_{K_i = K^*} = 0$, a necessary condition for efficiency of the stable structure is:

$$\frac{1}{N} D_c(\alpha, K^*, \gamma, \gamma', p) = \frac{1}{N} D_c(\alpha, N - Q(K^*+1), \gamma, \gamma', p) + \left.\frac{\partial D_c(\alpha, K_j, \gamma, \gamma', p)}{\partial K_j}\right|_{K_j = N - Q(K^*+1)} \frac{N - Q(}{}$$

which cannot be true by the strict convexity of $D_c(.)$ to the left of K^*. This completes the proof.

9 Appendix B

As mentioned in section 3.2, we can generate different financial structures within a component via a *securitization* process, by which we indicate the (possibly iterative) procedure through which connected firms exchange shares on their whole array of *prevailing* asset holdings. By subsequent rounds of securitization, firms' portfolios may include assets of distant firms on the underlying network.

More precisely, the extent of segmentation can be identified with the number C of equal-sized components in which the population is divided. And, then, given any such C, if $K+1 \equiv N/C$ stands for the number of firms in each component (for simplicity, we assume C divides N), two polar cases will be considered: the ring component where $z=2$; the complete component where $z=K$. These embody the extreme cases of partial and full connectivity consistent with our framework and therefore are the starkest manifestations of the main forces at work. In terms of the matrix representation introduced above, A would be a block matrix with identical square submatrices along the main diagonal respectively given by:

$$A_{K,c} = \begin{pmatrix} \alpha & (1-\alpha)/K & (1-\alpha)/K & \cdots & (1-\alpha)/K \\ (1-\alpha)/K & \alpha & (1-\alpha)/K & \cdots & (1-\alpha)/K \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (1-\alpha)/K & (1-\alpha)/K & (1-\alpha)/K & \cdots & \alpha \end{pmatrix}$$

for the complete components and

$$A_{K,r} = \begin{pmatrix} \alpha & (1-\alpha)/2 & 0 & \cdots & (1-\alpha)/2 \\ (1-\alpha)/2 & \alpha & (1-\alpha)/2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (1-\alpha)/2 & (1-\alpha)/2 & 0 & \cdots & \alpha \end{pmatrix}$$

for the ring components, where the dimensions of these matrices is $K+1$, the size of the components. But the above matrices implicitly presumes that there is just one round of securitization, i.e. an exchange of assets between directly connected firms. This, in effect, limits substantially the risk-sharing possibilities in the ring since the burden of every shock hitting any firm i is only shared by the two firms directly connected to it. The situation can be much improved by successive rounds of securitization. To explain precisely what they entail, let $q = 1, 2, ...$ index such securitization rounds and denote by

$$A^1 = \begin{pmatrix} a^1_{11} & a^1_{12} & \cdots & a^1_{1N} \\ a^1_{21} & a^1_{22} & \cdots & a^1_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a^1_{N1} & a^1_{N2} & \cdots & a^1_{NN} \end{pmatrix}$$

the matrix that prevails after the first round – i.e. A^1 equals either $A_{K,c}$ or $A_{K,r}$, depending on the size K of the component and the pattern of direct connections prevailing in it (complete or ring). Then, if a subsequent round of asset exchange takes place, the induced pattern of asset shares can be computed by simply composing A^1 with itself. This gives rise to the matrix $A^2 = A^1 \times A^1$ and, in general, to the the matrix $A^q = (A^1)^q$ after q rounds, where the last superindex denotes q-fold composition. It is easy to see that, starting from any A_1 (i.e. irrespectively of the initial pattern of shares), an unlimited number of securitization rounds yields the matrix

$$\lim_{q \to \infty} A^q = \begin{pmatrix} \frac{1}{K+1} & \frac{1}{K+1} & \cdots & \frac{1}{K+1} \\ \frac{1}{K+1} & \frac{1}{K+1} & \cdots & \frac{1}{K+1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{K+1} & \frac{1}{K+1} & \cdots & \frac{1}{K+1} \end{pmatrix}$$

where all firms in the component share the same conditions, both ex-ante and ex-post – i.e. they are exact "clones" and risk-sharing (but also risk-exposure) is maximal.[2]

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[2]The essence of the argument is as follows. The matrix A is

(i) stochastic, for all its rows sum up to 1;

It is worth emphasizing that the preceding limit conclusion applies independently of the initial matrix A^1 and, therefore, of the underlying network of direct connections. This underscores the need to impose certain limitations on the number q of securitization rounds – not only for the sake of being realistic, but also for the theoretical reason of escaping a trivial analysis. A natural way of anchoring the model at a suitable value of q derives from the following two-fold requirement:

1. All firms in a component should be directly or indirectly related, i.e. should have some extent of risk-sharing. For otherwise, unrelated firms effectively would belong to different "economic components" and their inclusion in a common component is misleading. This amounts to requiring that $a_{ij}^q > 0$ for all i, j, where a_{ij}^q stands for the ij-entry of the matrix A^q.

2. When all direct and indirect effects are taken into account, the fraction of shares that each firm holds of its own investment project should be the same across different configurations. For, otherwise, the comparison of their performance would be vitiated by the consideration of, effectively, very heterogenous situations – not just in terms of the network pattern but also in terms of the effective risk externalization of their own investment project. This requires that, for all i, $a_{ii}^q = 1 - \sum_{j \neq i} a_{ij}^q = \alpha$.

In light of 1. and 2., when we compare components of a certain size $K+1$ that have different network architectures (i.e. complete and ring networks), the value of q must be set as follows. In the case of a complete component (i.e. $A^1 = A_{K,c}$), condition (i) is non-binding, so we can choose any value of q such that $a_{11}^q = \alpha$. Thus, for simplicity, we choose $q = 1$ and assume – in

(ii) aperiodic, since the main-diagonal entries are all positive;

(iii) and strongly connected since the entries of A^N are all positive.

Then, it is well known by standard theory of Markov chains that $\lim_{q \to \infty} A^q = A^*$ and that there exists some $\mu^* \in \Delta^{N-1}$ such that $\mu A^* = \mu^*$ for all $\mu \in \Delta^{N-1}$. This obviously implies that all rows of A^* must be identical, which in view of the symmetry of all nodes in the underlying network implies that all entries in A^* must be identical.

order to meet (ii) – that $a^1_{11} = \alpha$. On the other hand, for a ring component, condition (i) requires that $q \geq K/2$. So, again for simplicity, we choose the lowest value $q = K/2$ (we assume K is even) that meets that constraint. This then implies that $A^1 = A_{K,r}$ must be such that the diagonal entries of $A^{K/2} \equiv (A^1)^{K/2}$ satisfy $a^{K/2}_{ii} = 1 - \sum_{j \neq i} a^{K/2}_{ij} = \alpha$ for each $i = 1, 2, ..., K+1$. Also note, for future reference, that, after any finite rounds of securitization, the entries $a^{K/2}_{ij}$ are decreasing in the distance $d(i,j)$ between firms i and j in the underlying ring, i.e. $d(i,j') > d(i,j) \Rightarrow a^{K/2}_{ij'} < a^{K/2}_{ij}$.

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