DEMAND CREATES ITS OWN SUPPLY

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Abstract

As recent events attest, modern economies may have trouble enforcing Say's Law. An economy with decentralized markets and trades between goods and a liquid asset, money, has two equilibria. In full-employment, output is determined by supply. But a higher demand for liquidity is self-fulfilling and precipitates the economy to an equilibrium where output is determined by demand: the increase of the private uncertainty about the ability to sell goods generates the higher demand for liquidity. That state may be a trap: a reverse shift from pessimism to optimism may not be sufficient, without policy intervention, to restore full-employment.

Keywords: Say's law, money, liquidity trap, paradox of thrift, multiple equilibria, strategic complementarity.

JEL codes: E00, E10, E21, E41

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1 Introduction

In the current crisis, the demand for precautionary saving and the reduction of consumption is playing an important contribution. Uncertainty about employment and income raises the motive for saving and the lower demand for goods feeds into the uncertainty. The mechanism, which has some relation with the paradox of thrift, is analyzed here in a model of general equilibrium with money as the medium of exchange. These issues have been raised by Leijonhufvud (1968), following Keynes (1936), who argued that in an economy where goods are traded with money, Say's law may not hold. The present paper proves the point in the context of general equilibrium, rational decision making and expectations formations.

In any contemporary economy, exchanges are between goods and money. Money is liquid as it can be used to trade any good. Agents increase their money balances through sales and use these balances to buy consumption goods. Both inflows and outflows of money are subject to random micro-shocks but in a "standard" regime of economic activity, these shocks can assumed to be relatively small and agents can afford to keep a relatively low level of money inventories. Such a regime depends on individual expectations. If agents expect that opportunities for sales are subject to a larger uncertainty, they reduce their consumption to accumulate more money as a precaution. But the reduction of consumption by some agents may increase the sale uncertainty of others and raise the demand for money. The higher demand for money (liquidity) may be self-fulfilling.

In this paper, the sudden increase of the demand for money shifts the economy from an equilibrium with a regime of high consumption to another equilibrium with a regime of low consumption where agents attempt to accumulate higher money balances and there is insufficient aggregate demand and output.

Money is valuable because agents are spatially separated. The spatial separation of agents has been the foundation of models of money since Samuelson (1958) where agents cannot enter in bilateral credit agreements because they are temporally separated in different generations and meet only once. That important property has been embodied by Townsend (1980) in a setting with infinitely lived agents who are paired along the two opposite lanes of a "turnpike", each selling and consuming with his *vis à vis* on alternate days and carrying money from a day of production to the next day for consumption. Any two agents are paired at most once. That property is preserved here in an adaptation of the Townsend model in which agents are in a continuum of mass one and matched pairwise in each period with two other agents, one customer for the produced good and one supplier of the consumption good.

As shown in these models, the essential property of money is to enable transactions in separate, non-centralized markets. Recent literature has linked the holding of money with search, beginning with the moneyless model with search and exchange of "coconuts" by Diamond (1982), and continuing with the money model of Diamond (1984), and Diamond and Yellin (1990). There is no search in the present model. In a search model, a higher quantity of money holding facilitates the search and is positively related to the level of activity. In the present model, a high level of activity reduces the need to save for future transactions because agents expect a high inflow of cash in the future. When business is expected to be slow, agents attempt to save more in money and the reduction of demand slows down the business. The structure of the model intuitively generates multiple equilibria.

In the full-employment equilibrium, there is not more than one period between a sale and a consumption. Money is necessary for consumption but because a stable and high inflow of cash is expected, a relatively low level of money balance is sufficient to maintain a high level of consumption. No agent is cash constrained for consumption and producers can always sell.

In an equilibrium with unemployment, a producing agent cannot sell when he is matched with an agent who has no money. Because of the probability of no sale, agents attempt to accumulate money. But the higher balances for some agents must result in a smaller or no balance for others because the endogenous money price of goods is the same in the two equilibria and the total quantity of money is not affected by the regime of activity.

When the economy is in a full-employment stationary equilibrium, a negative shock of expectations is sufficient to push the economy to an equilibrium with unemployment: the fear of smaller opportunities for sale induces agent to keep money: if they do not have an urgent need to consume they choose to save, but this act of saving reduces the opportunity of another agent to sell his production. The two equilibria with and without full employment are not symmetric: in the stationary equilibrium with unemployment a jump of optimism may not be sufficient to nudge the economy into a recovery: in full-employment, all agents who don't have a high need for consumption can shift to saving. In the economy with unemployment, agents who are liquidity-constrained cannot jump to consumption even if they become optimistic about the future.

The paper owes an intellectual debt to Green and Zhou (1998, 2002) and Zhou (1999). These papers focus on the existence of an equilibrium where agents with identical preferences meet according to an exogenous Poisson process. They emphasize that there is a continuum of the equilibrium value of the price level. In the present paper, the emphasis is on the multiple equilibria with and without full-employment and on the dynamics that are generated by the heterogeneity of preferences.

The paper differs from Guerrieri and Lorenzoni (2009) who analyze the unique equilibrium of a model similar to Lagos and Wright (2005) where a centralized market in the last part of each period enables agents to reconstitute their money balances. Because of the linear utility at the end of a period, the equilibrium is effectively a sequence of three periods equilibria. An illiquid asset is introduced that cannot be traded in the decentralized market of the second part of a period during which agents need money to buy goods. In the present paper, there is no centralized market and the equilibrium in any period depends on all the future periods.

The model is presented in Section 2. In order to simplify the analysis, agents are constrained in the maximum of cash they can hold, by assumption. Since the higher demand for cash is what generates an inefficient equilibrium, the restriction should not limit the validity of the properties. The assumption is relaxed later in the paper.

The stationary equilibrium with full employment and low demand for money is presented in Section 3. The dynamics of the two regimes of high and low consumption are first analyzed in Section 4. In the following section, these two regimes are shown to be equilibria under suitable parameter conditions. Section 6 shows that the steady state of the low regime with unemployment may be a trap out of which no optimism about the future can lift the economy. Section 7 shows that the properties are robust when there is no exogenous upper-bound on money holding, and when the consumption indivisibility is replaced by a utility function that is continuous for any positive consumption.

2 The model

There is a continuum of infinitely lived agents, indexed by $i \in [0, 1)$. The utility of agent i in period t is $u(x_{it}, \theta_{it})$, where $\theta_{it} \in \{0, 1\}$ are i.i.d. random variables that represent shocks to the utility of consumption. When $\theta_{it} = 1$, agent i has a higher need to consume in period t than when $\theta_{it} = 0$. If $\theta_{it} = 0$, the agent is, in period t, of the *low type*, and if $\theta_{it} = 1$, the agent is of the *high type*. The probability of the high type is exogenous and equal to α, $(0 < \alpha < 1)$, which is known by all agents.

To simplify the exposition, we assume that

$$u(x, \theta) = \begin{cases} 1, & \text{if } x \geq 1, \\ -\theta c, & \text{if } x < 1. \end{cases} \tag{1}$$

The assumption of indivisibility that is embodied in (1) is taken as a first step and will be relaxed in Section 7.3. The welfare of agent i in any period, say period 0, is the discounted expected sum of the utilities of consumption in the future periods:

$$U_i = E\Big[\sum_{t \geq 0} \beta^t u(x_{it}, \theta_{it})\Big], \qquad \text{with} \qquad \beta = \frac{1}{1+\rho} < 1. \tag{2}$$

Agents produce goods which they sell, and consume goods produced by others. Goods are not storable. In each period, an agent meets two other agents, one buyer and one seller, according to a process of random matching that is defined by the function

$$\phi_t(i) = \begin{cases} i + \xi_t, & \text{if } i + \xi_t < 1, \\ i + \xi_t - 1, & \text{if } i + \xi_t \geq 1. \end{cases} \tag{3}$$

Any agent i can sell his product to agent $j = \phi_t(i)$ and consume the good produced by agent $\phi_t^{-1}(i)$.

The variables $\xi_t \in (0, 1)$ are random, i.i.d., with a uniform density. The process embodies the absence of a double coincidence of wants and implies that an agent has a zero probability to find the same match in a future period[1]. The present setup with no centralized market is similar to Townsend's turnpike that fits a circle of infinite

[1] One could use other matching functions ϕ_t provided that they satisfy the property that for any subset $\mathcal{H}$ of $[0, 1)$, $\mu(\mathcal{H}) = \mu\Big(\phi_t(\mathcal{H})\Big)$, where μ is the Lebesgue-measure on $[0, 1)$. The property is required for a uniform random matching of all agents.

diameter with random pairing of atomistic agents between the two lanes (Townsend, 1980).

In order to simplify the demand for money, it is assumed that each agent is like a two-headed household: at the beginning of period t, say a day, one head of household i can go out with some cash (if there is any in the household) to buy a consumption good from a randomly matched supplier $\phi_t^{-1}(i)$. The second head stays at home to service the customer $\phi_t(i)$: if that customer buys, he produces and sells one unit of the good, at no cost. The two heads meet at the end of the day to consume if a purchase has been made. A setup with a single person who buys and sells with a random order during the day would probably generate the same essential properties but the analysis would be more complicated.

To summarize, in each period t, events proceed in the following sequence:

1. Each agent i first learns his type, *i.e.* the value of $\theta_{i,t}$. The probability of the high type ($\theta_{i,t} = 1$) is equal to α.

2. Each agent decides to carry a quantity of money m "to the market" (which is not centralized). The decision about m has to take place at the beginning of the day, before the eventual production and sale during the day.

3. For each agent, the seller produces either 0 or 1 (since agents demand 0 or 1 in the utility function). The production is cost-free. The seller posts a price p and there is no bargaining. One can assume that the seller produces instantly after he knows whether he has a buyer.

We will consider symmetric equilibria where in any period, all sellers post the same price, p. That price does not have to be constant over time, but in equilibrium it will be constant. The price p is publicly known in a rational expectation equilibrium. Suppose an agent decides to consume in a given period: he carries an amount of money $\tilde{m}$ to the market. To carry $\tilde{m} < p$ would mean no consumption. Hence, for a posted seller's price p, buying agents bring each $\tilde{m} \geq p$ to the market. Since all sellers sell at the same price p, the agent does not gain a strict benefit by bringing a quantity $\tilde{m} > p$ to the market.

Given the price p posted by all sellers and known to buyers who bring a quantity of money $\tilde{m} \geq p$, a seller will post a price $\tilde{p}$. Since any buying customer brings $\tilde{m} \geq p$, a value $\tilde{p} < p$ is sub-optimal for the seller. If the seller deviates and posts a price $\tilde{p} > p$ he will sell only to the customers who bring $\tilde{m} > p$. The profitability for the seller of posting a price $\tilde{p} > p$ depends on the distribution of customers who bring to the market $\tilde{m} > p$. But remember that a customer is indifferent between $\tilde{m} = p$ and $\tilde{m} > p$. (The customer has *ex ante* a probability zero of meeting the seller who deviates by posting $\tilde{p} > p$). In this context, as no buyer has a strict incentive to deviate from $\tilde{m} = p$ and no seller has a strict incentive to deviate from $\tilde{p} = p$, it is reasonable to assume that agents will coordinate on the outcome where all sellers post the price p and agents bring the quantity of money $m = p$ to the market. Any other equilibrium would imply a distribution of prices for the sellers and a distribution of the money balances carried to the market. Such an equilibrium, if it exists, would require remarkable coordination power from the agents, to put it mildly, and its stability properties would not be straightforward. In the present context, such an equilibrium, if feasible, is not realistic[2].

Lemma 1

In an equilibrium where all sellers post a price p in a given period, all agents who consume in that period carry to the market an amount of money $m = p$ to buy goods. No buyer has a strict incentive to deviate by bringing a different quantity of money to the market. A seller who deviates from posting p gets a strictly smaller payoff.

3 First properties

3.1 Steady state equilibrium with full-employment

In a full-employment equilibrium by definition, all agents produce and sell, (and hence consume). When some agents do not sell (and others do not consume), there is unemployment.

Assume that each agent has a quantity of money at least equal to $\bar{m}$ at the beginning of the period and that sellers post a price $p < \bar{m}$. If all agents carry an amount of money $\tilde{m} = p$ to their matched seller, all agents consume and sell in the period. Each agent has the same amount of cash at the end of the period. No consumer has an incentive to deviate by hoarding: he would just miss the utility of consumption in the period. From Lemma 1, no seller has an incentive to deviate. We have the following result.

[2]That outcome could also be reinforced by the assumption of a vanishing cost of carrying money for buyers, in which case a deviation from $\tilde{m} = p$ is strictly sub-optimal. However, such an assumption of a small cost turns out to be superfluous.

Proposition 1 *Assume that the quantity of money of each agent is bounded below by $\bar{m} > 0$. Then any price $p \leq \bar{m}$ that is constant over time determines a steady state equilibrium with full employment.*

There is a continuum of equilibrium price levels. The continuum of price equilibria is of no special interest here. The main property is that for all these prices there is full employment. All the full-employment equilibria with different p have the same real allocation of resources which is the socially optimal allocation.

In a full employment equilibrium, money is indispensable for transactions, but there is no precautionary motive since agents are sure that they will be able to sell and replenish their cash at the end of each period.

Using Lemma 1, we will consider only equilibria with a price that is constant over time. That price will be normalized to 1. All money holdings in the interval $I_k = [k, k+1)$, with k a non-negative integer, generate the same opportunities for trade. An agent with an amount of money $m \in I_k$ at the beginning of a period will be defined to be in *state* k. In state 0, an agent is liquidity constrained and cannot consume. We first make the technical assumption that money holdings are bounded: there is N arbitrary such that the quantity of money held by any agent is strictly smaller than $N + 1$.

Lemma 2

There exists N such that in any equilibrium and in any period, any agent with money greater than N at the beginning of any period consumes one unit.

Lemma 3

Assume that the support of the initial distribution of money is bounded by N_0. Then in any equilibrium, the support of the distribution of money is bounded by $Max(N_0, N)$.

Definition

- Let π_t by the probability of not making a sale.
- Deterministic path: $\tilde{\pi}_t = \{\pi_s, ...\}_{s \geq t}$.

- Evolution of the cash balance of agent i

$$\begin{cases} m_{i,t+1} = m_{i,t} + 1 - x_{i,t} \text{ with probability } 1 - \pi_t, \\ \\ m_{i,t+1} = m_{i,t} - x_{i,t} \text{ with probability } \pi_t. \end{cases} \tag{4}$$

$$1 - \pi_t = \int x_{i,t} di.$$

- $x_{i,t} = x^*(m_{i,t}, \theta_{i,t}, \tilde{\pi}_t)$ maximizes

Equilibrium

An equilibrium is defined by an initial distribution of money, a path $\tilde{\pi}_0$ with perfect foresight and a consumption function $x_t = x^*(m_t, \theta_t, \tilde{\pi}_t)$ that maximizes the utility

$$E[\sum_{t \geq 0} \beta^t u(x_t, \theta_t)],$$

and a sequence of Γ_t of distribution of money that satisfies (1)

$$A(M_t) = Max E(u(x_t, \theta_t) + \beta A(M_{t+1})$$

Assumption 1 *The aggregate quantity of money M is such that $M < N$.*

The assumption eliminates the degenerate case where all agents hold the maximum quantity of money that is allowed.

Let $\Gamma(t)$ be the vector of the distribution of agents at the beginning of period t across states

$$\Gamma(t) = (\gamma_0(t), \gamma_1(t), \dots, \gamma_N(t))',$$

where $\gamma_k(t)$ is the mass of agents in state k. Any distribution of money must have a total mass of 1 and a mean equal to the aggregate money supply M:

$$\sum_0^N \gamma_k(t) = 1, \qquad \sum_{k=1}^N k\gamma_k(t) = M. \tag{5}$$

3.2 The consumption of the high type agents

In general, the demand for money should depend on the type of the agent (high or low) and the opportunities of future sales as determined by the path of the probabilities of making a sale in future periods. The behavior of the high type agents is not ambiguous however. As shown in the next result, they consume whenever they can.

Proposition 2

In an equilibrium, any high-type agent consumes unless he is liquidity-constrained.

The property is intuitive. For a high-type agent, saving today entails a penalty c. The best use of the additional money is to consume it in the future when he is also a high type. Because of the discounting, the value of avoiding that future penalty is smaller than the penalty today. The agent is better off by not saving. The formal argument follows the intuition and is presented in the Appendix.

4 Dynamics in two regimes

Given Proposition 2, we can focus for the rest of the paper, on the consumption function of the low-type agents. One can anticipate that these agents save if and only if their money balances is below some target level. We are therefore led to consider two types of consumption functions. We will show in Section 5 that each of this functions is optimal for an individual in the relevant equilibrium. Each of the two consumption functions determine the evolution of the distribution of money balances in the economy and therefore a regime of the economy. In the *high regime*, low-type agents consume whenever they can, that is when they have at least one unit of money. In the *low regime*, they save unless they reach a maximum (or targeted) level of money. That value is fixed and equal to N here, but will be endogenized later[3].

4.1 The high regime

At the beginning of the first period, period 0, the distribution of money, $\Gamma(0)$, is given. Since all agents except those in state 0 consume, and the matching is independent of the money balance, each agent faces the same probability $\pi(t)$ of not making a sale in period t and being unemployed. The probability $\pi(t)$ is equal to the fraction of agents in state 0, $\gamma_0(t)$. The evolution of the distribution of money is given by

$$\Gamma(t+1) = H(\pi_t).\Gamma(t), \qquad \text{with} \qquad \pi_t = \gamma_0(t), \tag{6}$$

[3]When there is no exogenous upper-bound on individual money holding, if the low type have a consumption function such that in a steady state, they consume when their balance reaches N, that value of N is also the (endogenous) upper-bound of the distribution of money.

and the transition matrix

$$H(\pi) = \begin{pmatrix} \pi & \pi & 0 & 0 & \dots & 0 \\ 1-\pi & 1-\pi & \pi & 0 & \dots & 0 \\ 0 & 0 & 1-\pi & \pi & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \dots & 0 & 0 & 1-\pi & \pi \\ 0 & \dots & 0 & 0 & 0 & 1-\pi \end{pmatrix}. \tag{7}$$

For example, in the first line, the agents who are in state 0 at the end of period t are in that state either (i) because they met another agent in state 0, with probability π, and that determines H_{11}, or (ii) because they were in state 1 at the beginning of period t, consumed but met with a buyer with no cash and therefore got no inflow of cash which determines $A_{12} = \pi$. Likewise for the other elements of the matrix H.

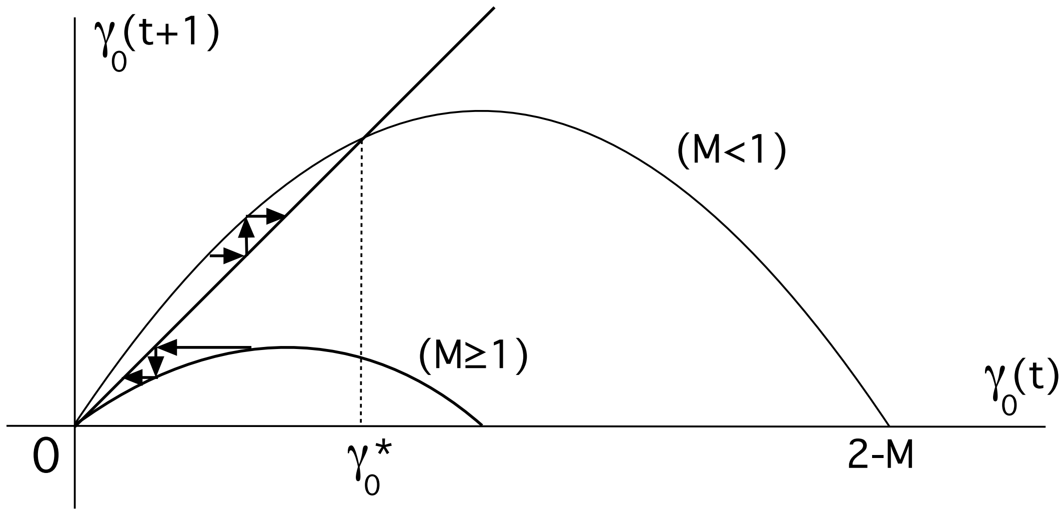
Given the normalized price $p = 1$, Proposition 1 showed that if individual money balances are bounded below by 1, the high regime has a steady state with full-employment that is also an equilibrium. The next result (proven in the Appendix) shows that under the condition that the aggregate quantity of money, M, is at least equal to one, the high regime converges to that steady state with full-employment.

Proposition 3

In the high regime (where all agents who are not liquidity-constrained consume), the distribution of money $\Gamma(t)$ converges to a limit. Let M be the aggregate quantity of money.

- *If $M \geq 1$, for any initial distribution of money, the rate of unemployment converges to zero. At the limit, no agent is liquidity-constrained.*
- *If $M < 1$, the rate of unemployment converges to $\pi^* = 1 - M$. The distribution of money converges such that $\gamma_0^* = \pi^*$, $\gamma_1^* = 1 - \gamma_0^*$, and $\gamma_k^* = 0$ for $k \geq 2$.*

In the case where $M \geq 1$, the economy may have liquidity-constrained agents at the beginning of time. But a diffusion process of money takes places that reduces gradually the mass of constrained agents to zero. The real economy at the limit is invariant to the initial distribution of money: it has full employment. The limit distribution of money, which has no incidence on the real economy, does depend on the initial distribution of money. As a particular case, any distribution with full employment and $\gamma_0 = 0$ is invariant through time.



When $M < 1$, the economy converges to a steady state with unemployment and the fraction γ_0 of liquidity constrained agents tends to a positive value. When $M > 1$, the economy converges to full-employment with no liquidity constrained agents in the limit.

Figure 1: Dynamics of the liquidity constrained agent in the regime of high consumption (three states).

The case with three states

When there are three states, the quantity of money is bounded by 2. Using the equations of the quantity of agents and of money, $\sum_{k=1}^{2} \gamma_k = 1$ and $\sum_{k=1}^{2} k\gamma_k = M$, to eliminate $\gamma_1(t)$ and $\gamma_2(t)$, the distribution of money has one degree of freedom. Its dynamics can be expressed in function of $\gamma_0(t)$:

$$\gamma_0(t+1) = \gamma_0(t)(\gamma_0(t) + \gamma_1(t)),$$

which is equivalent to

$$\gamma_0(t+1) = \gamma_0(t)\Big(2 - M - \gamma_0(t)\Big). \tag{8}$$

The evolution of $\gamma_0(t)$ is represented in Figure 1 for the cases $M \geq 1$ and $M < 1$. When $1 \leq M < 2$, at the limit, $\gamma_0^* = \pi^* = 0$ and the distribution of money in states 1 and 2, (γ_1^*, γ_2^*) is determined uniquely by the unit mass of agents and the quantity of money. When M increases, γ_1^* decreases and γ_2^* increases.

4.2 The low regime

In the low regime, consumption is generated by the fraction α of the high-type agents in states 1 to $N-1$, and all agents in state N. As the fraction of agents who consume is $1-\pi_t$,

$$\pi(t) = 1 - \alpha \sum_{k=1}^{N-1} \gamma_k(t) - \gamma_N(t), \tag{9}$$

which can be written as

$$\pi(t) = 1 - \alpha(1 - \gamma_0(t)) - (1-\alpha)\gamma_N(t). \tag{10}$$

The value of $\pi(t)$ is equal to zero if and only if $\gamma_0(t) = 0$ and $\gamma_N(t) = 1$ which is possible only if $\gamma_N(t) = 1$ and $M = N$. (All individuals are in the highest state). We have the following result.

Proposition 4

In the low regime, with $M < N$, there is unemployment in all periods: $\pi(t) > 0$ for all $t \geq 1$.

The evolution of the distribution of money is now given by

$$\Gamma(t+1) = L(\pi_t).\Gamma(t), \qquad \text{with} \qquad \pi_t \text{ given in (9)}, \tag{11}$$

where the transition matrix $L(\pi)$ takes a form that depends on N.

For $N = 2$,

$$L(\pi) = \begin{pmatrix} \pi & \alpha\pi & 0 \\ 1-\pi & (1-\alpha)\pi + \alpha(1-\pi) & \pi \\ 0 & (1-\pi)(1-\alpha) & 1-\pi \end{pmatrix}. \tag{12}$$

For example in the first line, the mass of liquidity constrained agents $\gamma_0(t+1)$ comes from the constrained agents (not consuming) who do not make a sale, with probability π, and that defines L_{11}, and the agents in state 1 of the high type who do not make a sale, with probability $\alpha\pi$, which defines L_{12}. Likewise for the other elements of the matrix L.

For $N \geq 3$,

$$L(\pi) = \begin{pmatrix} \pi & \alpha\pi & 0 & 0 & \dots & 0 \\ 1-\pi & a & \alpha\pi & 0 & \dots & 0 \\ 0 & b & a & \alpha\pi & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & b & a & \alpha\pi & 0 \\ 0 & \dots & 0 & b & a & \pi \\ 0 & 0 & 0 & \dots & b & 1-\pi \end{pmatrix}, \tag{13}$$

with

$$\begin{cases} a = (1-\alpha)\pi + \alpha(1-\pi), \\ \\ b = (1-\pi)(1-\alpha), \end{cases}$$

and where the middle lines are omitted for $N = 3$.

The dynamics of the economy are completely specified by equation (10) where the matrix $L(\pi)$ is given in (11) or (12), and π_t in (8).

The stationary economy

Let e be the row-vector with $N+1$ components equal to 1. One verifies that $e.L(\pi) = e$ for any π. (For any distribution of money, the transition matrix L keeps the total quantity of money invariant). Fix a value of π. The matrix $L(\pi)$ has an eigenvalue equal to 1 and that eigenvalue is of order 1 (Lemma 2 in the Appendix). There is a unique vector Γ^* such that $B(\pi).\Gamma^* = \Gamma^*$ and $e.\Gamma^* = 1$. The vector Γ^* defines a stationary distribution of money holdings that depends on the probability π and can be written as a function of π: $\Gamma^* = (\gamma_0^*(\pi), \ldots, \gamma_N^*(\pi))$. We can then compute the total amount of money of the distribution $\Gamma^*(\pi)$, which by an abuse of notation is equal to

$$M(\pi) = \sum_{k=1}^{N} k\gamma_k^*(\pi).$$

The next result shows that this function is invertible and that the level of aggregate money M determines, in a decreasing function, the unemployment rate in the steady state. (The lenghty proof is given in the Appendix).

Proposition 5

The low regime with money balances bounded by N has a unique steady state. In that steady state, the rate of unemployment is a function of the aggregate quantity of money M that is strictly decreasing from 1 to 0 when M increases from 0 to N.

Proposition 5 is illustrated in Figure 2 for two values of N, $N = 2$ and $N = 3$, and in each case for two values of the share of high-types, $\alpha = 0.25$ and $\alpha = 0.75$. One can observe that for a given quantity of money, the rate of unemployment rises with N. When N rises, low-type agents save more (up to a higher value of money balances). That depresses the demand and raises the unemployment rate. The points A and B correspond to the same quantity of money for $N = 2$ and $N = 3$. The unemployment rate at B is higher. It will be shown in Section 7 that both the stationary distributions

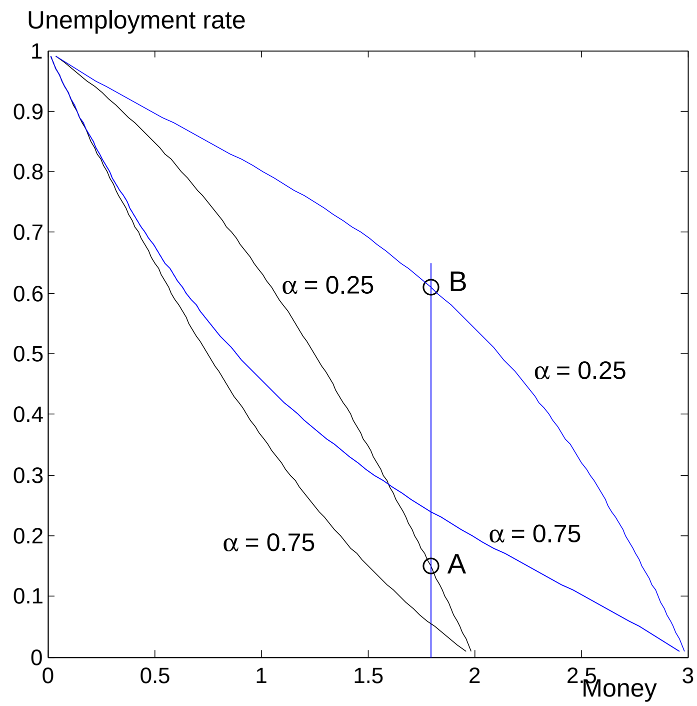


Figure 2: Money and unemployment in the stationary economy under the low regime for $N = 2$ and $N = 3$. (If $M \geq N$, the unemployment rate is 0). For each value of N, two cases are presented with low ($\alpha = 0.25$) and high ($\alpha = 0.75$) consumption. For a quantity of money equal to 1.7944, the unemployment rate is equal to 0.15 when $N = 2$ and to 0.6105 when $N = 3$. The impact of money on unemployment is smaller in the state B with the higher unemployment rate than in state A.

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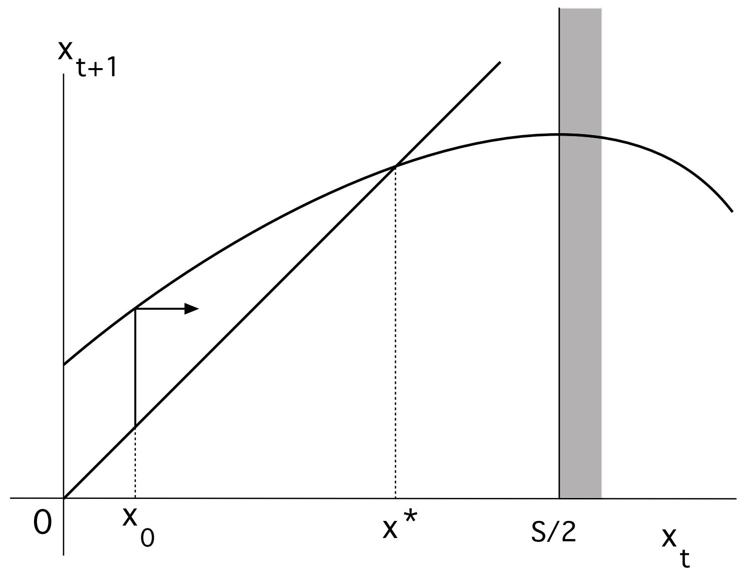
associated at the points A and B are equilibria, for the same quantity of money. At the point B for example, the unemployment rate is higher because agents save more. But the higher saving is the optimal strategy when the unemployment rate is higher.

The case with three states

As for the high regime, we consider the case of three states with $N = 2$ that has only one degree of freedom and we characterize the evolution of $\gamma_0(t)$. Let $S = 2 - M$. From the constraints on the distribution of money $\Gamma(t) = (\gamma_0(t), \gamma_1(t), \gamma_2(t))$, we have $-2\gamma_0(t) + S = \gamma_1(t) \geq 0$. Hence, any initial value of the fraction of agents in state 0, $\gamma_0(0)$, must satisfy the condition

$$\gamma_0(0) \leq S/2. \tag{14}$$



$\gamma_0(t) = x_t$, $S = 2 - M$.

Figure 3: Dynamics of the fraction of liquidity-constrained agents in the low regime

The analysis which are presented in the Appendix, shows that for $t \geq 0$,

$$\begin{aligned} \gamma_0(t+1) &= P(\gamma_0(t)), \qquad \text{with} \\ P(x) &= -(1-2\alpha)^2 x^2 + (1-2\alpha)^2 Sx + \alpha(1-\alpha)S^2. \end{aligned} \tag{15}$$

The polynomial $P(x)$ has its maximum at $x = S/2$. One verifies[4] that $P(S/2) < S/2$. Since $P(x)$ is increasing on the interval $[0, S/2]$, there is for $x > 0$ a unique value x^* such that $P(x^*) = x^*$ and $x^* < S/2$. For any admissible value of x_0 which must be in the interval $[0, S/2]$ by (13), the sequence $x_{t+1} = P(x_t)$ converges to x^* monotonically. The evolution of x_t is represented in Figure 3.

During the transition, the mass of liquidity-constrained agents, $\gamma_0(t)$, varies monotonically towards its steady state. The unemployment rate is a linear function of $\gamma_0(t)$:

$$\pi(t) = -(1-2\alpha)\gamma_0(t) + (1-\alpha)S, \tag{16}$$

and converges to the limit π^* that is determined by the equation

$$M = \psi(\pi^*) = 2 - \frac{\pi^*\Big(1 - \pi^*(1-2\alpha)\Big)}{1 - \alpha - \pi^*(1-2\alpha)}. \tag{17}$$

[4] $P(S/2) = \left(\frac{(1-2\alpha)^2}{4} + \alpha(1-\alpha)\right) S^2$. Since $S < 1$, this expression is strictly smaller than $S/2$.

The function $\psi(\pi)$ has a negative derivative and is strictly decreasing, as shown already in Proposition 5. Let ϕ be its inverse function with $\pi = \phi(M)$. One verifies that

$$\phi(0) = 1, \quad \phi(2) = 0, \quad \phi(1.5 - \alpha) = 0.5. \tag{18}$$

Proposition 6

In a low regime, with $N = 2$ and $M < 2$, the distribution of money converges to a stationary distribution and the unemployment rate converges to a limit π^* that is a decreasing function $\phi(M)$ such that $\phi(0) = 1$, $\phi(1) = \pi_1^* > 0$, and $\phi(2) = 0$.

(i) On the dynamic path, the mass of liquidity-constrained agents, $\gamma_0(t)$, is a monotone function of time. The unemployment rate, $\pi(t)$, is an increasing (decreasing) function of $\gamma_0(t)$ when $\alpha > 1/2$, ($\alpha < 1/2$).

(ii) In the special case where $\alpha = 1/2$, the distribution of money and the unemployment rate are constant for all periods with $\gamma_0 = (S/2)^2$.

The property in (i) has an intuitive interpretation. For example, assume that the economy is initially at full-employment with $\gamma_0(0) = 0$. In the low regime, $\gamma_0(t)$ increases over time. The evolution of the unemployment rate depends on the share of high-type agents, α. If α is small, the shift to the low regime makes most agents shift to saving: demand drops by a large amount and the unemployment rate shoots up. Gradually, as more agents accumulate large money balances (at N), they consume and the unemployment rate decreases. The direction of the evolution is inverse when α is large.

From Proposition 2, if $N = 2$, there are only two possible consumption functions: agents in state 1 and of the low type either consume (and that is the high regime), or save (which generates the low regime). For simplicity, the restriction $N = 2$ is maintained in the next two sections. It is removed in Section 7.

5 Optimal consumption functions

So far, we have considered how the distribution of money and the unemployment rate depends on the consumption function. We now determine which consumption function is optimal. We assume that there are 3 states ($N = 2$). Let $U_k(t)$ be the expected utility of an agent in period t from future consumption when he holds an amount of money k, after his consumption decision in period t. That utility does not depend on his consumption in period t. Since an agent with money holding 2 at the beginning of a period consumes 1, we define $U_k(t)$ for $k = 0, 1$. The values of $U_0(t)$ and $U_1(t)$ are determined by straightforward backward induction and depend on the regime. For $N = 2$, each of the two regimes is determined by the behavior of a low-type in state 1. We begin with the low regime.

5.1 Equilibrium in the low regime

The consumption function in the low regime is defined such that an agent saves if and only of the low-type and in state 1. If the agent saves, his utility is $U_1(t)$. If he consumes, his utility is $1 + U_0(t)$. The necessary and sufficient condition for the low regime to be an equilibrium is that for any t,

$$U_1(t) - U_0(t) \geq 1. \tag{19}$$

An agent in period t with a balance 1 after his consumption decision will make a sale in period t with probability $1 - \pi_t$, in which case he will have a balance of 2 at the beginning of next period. He then consumes in that period, get a utility of consumption of 1 and be left with a balance 1 that provides a utility $U_1(t+1)$ from the consumptions after period $t+1$. So summarize, the expected utility in period t from making a sale is that period is $\beta(1 - \pi_t)(1 + U_1(t+1))$.

If the agent does not make a sale, his balance at the beginning of period $t+1$ is equal to 1. In that period, if he is of the high type, with probability α, he consumes that balance and has a utility $1 + U_0(t+1)$. If is of the low type, he does not consume and keeps his balance of 1 that provides him with a utility $U_1(t+1)$. The expected utility of not making a sale is therefore $\beta\pi_t(\alpha(1 + U_0(t+1) + (1-\alpha)U_1(t+1))$.

Combining the two possible sale outcomes,

$$U_1(t) = \beta\Big(\pi_t\alpha U_0(t) + (1 - \pi_t\alpha)U_1(t) + 1 - \pi_t(1-\alpha)\Big).$$

Using the same argument for an agent with no money after the consumption decision, and the notation $U(t)' = (U_0(t), U_1(t))$, we have in matricial form

$$U(t) = \beta A_t U(t+1) + \beta B_t, \quad \text{with}$$

$$A_t = \begin{pmatrix} \pi_t + \alpha(1-\pi_t) & (1-\pi_t)(1-\alpha) \\ \\ \pi_t\alpha & 1 - \pi_t\alpha \end{pmatrix}, \quad B = \begin{pmatrix} \alpha(1 - \pi_t(1+c)) \\ \\ 1 - \pi_t(1-\alpha) \end{pmatrix}. \tag{20}$$

Define $X_t = U_1(t) - U_0(t)$. From the previous equations, using $a_t = \alpha(1-\pi_t)+\pi_t(1-\alpha)$ in (12),

$$X_t = \beta a_t X_{t+1} + \beta(1 - a_t + \pi_t \alpha c). \tag{21}$$

The stationary solution X^* of this equation is

$$X^* = \beta \frac{1 + \pi^* \alpha c - a^*}{1 - \beta a^*},$$

where π^* and a^* are the steady state value of π and a. Simple algebra shows that the inequality $X^* \geq 1$ is equivalent to $\pi^* \alpha c / \rho \geq 1$ which leads to the next result.

Proposition 7

When $N = 2$, the low regime steady state is an equilibrium if and only if $\pi^* \alpha c / \rho \geq 1$, where π^* is the rate of unemployment as described in Proposition 6.

The result has a simple interpretation: the discounted expected value of the cost of unemployment measured as the product of the probability of the high type and the penalty for not consuming in the high type must be greater than one.

For the dynamic path, we have sufficient conditions that are established in the Appendix using the difference equation (20).

Proposition 8

When $N = 2$, the low regime is an equilibrium for any period t under either the following sufficient conditions:

- $\pi^* \alpha c > \rho$ *with $\alpha \leq \frac{1}{2}$ and $\gamma_0(0) < \gamma_0^*$, or $\alpha > 1/2$ and $\gamma_0(0) > \gamma_0^*$, where γ_0 is the initial mass of agents in state 0 and π^*, γ_0^* are values in the steady state of the low regime.*
- *for values of α and $\gamma_0(0)$ that do not satisfy one of the previous conditions, $\pi(0)\alpha c > \rho$, with $\pi(0) = -(1 - 2\alpha)\gamma_0(0) + (1 - \alpha)(2 - M)$.*

In the second item of the proposition, the unemployment rate rises on the dynamic path and the condition $\pi(0)\alpha c > \rho$ implies the inequality of Proposition 7 in the steady state.

The most interesting case to consider is when the economy is initially at full employment with $\gamma_0(0) = 0$. The previous sufficient conditions are simpler

$$\begin{cases} \text{if } \alpha \leq \dfrac{1}{2}, & \pi^* \alpha c > \rho, \\ \text{if } \alpha > \dfrac{1}{2}, & \pi(0)\alpha c > \rho, \text{ with } \pi(0) = (1-\alpha)(2-M). \end{cases} \tag{22}$$

5.2 Equilibrium in the high regime

We assume that $M > 1$ which is the most interesting case. The consumption function of the high regime is optimal in period t if an agent in state 1 and of the low type prefers to consume rather than save, that is if $X_t = U_2(t) - U_1(t) \leq 1$. The analysis of the low regime can be used here if we replace α by 1 in the previous equations. The difference equation (20) takes now the form

$$X_t = \beta(1-\pi_t)X_{t+1} + \beta\pi_t(1+c). \tag{23}$$

The stationary solution is equal to $\beta\pi^*(1+c)/(1-\beta(1-\pi^*))$, and since $\pi^* = 0$ (Proposition 3), it is equal to 0. The consumption function is trivially optimal near the steady state. Since the high regime converges to the steady state with full employment (Proposition 3), there is T such that if $t > T$, $X_t < 1$. Using (22), if $X_{t+1} < 1$, a sufficient condition for $X_t < 1$ is that $c < \rho$. By induction, for any t, $X_t < 1$.

Proposition 9

When $N = 2$, *if* $M > 1$ *and* $c < \rho$, *for any initial distribution of money, the high regime is an equilibrium.*

Note the condition $c < \rho$ in the proposition is sufficient and the result holds for any distribution of money. The high regime could be an equilibrium under a weaker assumption for particular distributions of money in the first period. For example, if $M \geq 1$ with a uniform distribution, the full employment stationary state is an equilibrium, for any value of c (Proposition 3). By continuity, if the initial money distribution is not too different from a full-employment distribution, the high regime defines an equilibrium that converges to full employment.

6 Unemployment Trap

Suppose that the economy is a stationary equilibrium with full employment. There exists a value $\bar{c}_1$ such that if $c > \bar{c}_1$, one of the inequalities in (21) is satisfied and by Proposition 8, the low regime path is also an equilibrium: an exogenous shift of (perfect foresight) expectations towards pessimism can push the economy on the path with an employment rate that converges to a positive value. The switch to the low regime is self-fulfilling.

Suppose now that the economy is in the stationary equilibrium of the low regime with $\pi^*\alpha c > \rho$ (Proposition 7). Can an exogenous change of animal spirits lift the economy out of that state and set the economy on a path back to full employment?

Let period 0 be the first period in which some low-type in state 1 switch to consumption. In that period, we must have $X_0 \leq 1$. We now show that if c is sufficiently large, this inequality cannot be satisfied.

Using (20) with $\alpha = 1$ and $X_1 \geq 0$, (more money is better), $X_0 \geq \beta\pi(0)(1+c)$, where $\pi(0)$ is the unemployment rate in period 0. If $\alpha = 1$, the only agents who do not consume are the liquidity-constrained in state 0. Hence $\pi(0) = \gamma_0^*$, where γ_0^* is the mass of liquidity-constrained agents in the steady state of the low regime and depends only on M, (Proposition 6). X_0 cannot be smaller than 1 if $\beta\gamma_0^*(1+c) > 1$, which is equivalent to

$$\gamma_0^*(1+c) > 1+\rho. \tag{24}$$

If c is sufficiently large (to satisfy $\pi^*\alpha c > \rho$ and (23)), then $X_1 > 1$ and there is no first period in which agents in state 1 with a low type shift to consumption instead of saving. The stationary equilibrium in the low regime is the only equilibrium. We have proven the next result.

Proposition 10

When $N = 2$, there exists a value $\bar{c}$ such that if $c > \bar{c}$,

(i) if the economy is at or near the full-employment stationary equilibrium in the high regime, in any period a shift of expectations can push the economy to a low regime path with an unemployment rate that converges to a strictly positive value;

(ii) if the economy is at the stationary equilibrium of the low regime with positive unemployment, that is the unique equilibrium.

The previous result shows the existence of a liquidity trap equilibrium. In that equilibrium, agents attempt to accumulate money balances because of the uncertainty of future exchanges. There is an asymmetry between the high regime with full employment and the low regime that leads to a liquidity trap. In any period, a switch from the high to the low regime can occur, but if the economy has been sufficiently long in a low regime, the economy may not switch back to a path toward full employment and the low regime with a permanent positive unemployment rate may be the only equilibrium. The non-symmetry arises because individuals are never constrained on their saving, but liquidity-constrained individuals are, by definition, constrained on their consumption.

7 Properties

This section examines the robustness of the previous properties. We separately remove the exogenous upper-bound on money balances and the indivisibilities in consumption.

7.1 Endogenously bounded distributions and multiple unemployment equilibria

We begin by the intuitive result that the accumulation of money for any agent is finite. When all agents consume above some level of money, the distribution of money will be bounded.

Proposition 11

In any period and any equilibrium, there exists a finite N_t such that any agent with a balance $m \geq N_t$ consumes.

A steady state with a high regime and $M > 1$ has full-employment (Proposition 3). In that steady state, the value N_t in Proposition 11 is equal to 1. All agents consume. The distribution of money matters only because it is bounded *below* by 1. That distribution may be bounded or unbounded. The interesting case will be the low regime that generates unemployment.

To simplify the argument, and without loss of generality, we assume that the initial distribution of money is bounded. There is no restriction after the first period. The following result follows immediately from the mechanics of the evolution of the distribution of money.

Proposition 12

Consider an arbitrary equilibrium and let n_t be the maximum of the upper-bound of the support of the distribution of money and the integer N_t of Proposition 11. Assume that the initial distribution is bounded: n_0 is finite. Then in that equilibrium, for any t, $n_{t+1} \leq n_t$.

From the previous Lemma, if the initial distribution of money is bounded and the economy tends in equilibrium to a steady state, the support of the distribution is bounded in that steady state.

In a low regime steady state, if an agent has a consumption function such that he consumes in state k, then the upper-bound of his balance in the steady state will be not strictly greater than k: because of the probability of not making a sale while being of the high-type, the set $\{0, \dots, k\}$ is a sink for evolution of the distribution of the agent's balance. Hence in the steady state, the consumption function for states strictly higher than k is irrelevant. We can therefore restrict an optimal consumption to be monotone in the following sense: We will say that a low-type agent has a k-*consumption* function if he consumes only when his balance is at least equal to k.

Proposition 13

In the steady state of an equilibrium with a low regime, low-type agents consume if and only if their balance is equal to some finite N, where N is the upper-bound of the support of the distribution of money. Furthermore,

$$N \leq \bar{N} = 1 + \frac{1 + \alpha c}{\rho}.$$

From the previous discussion, we can now characterize all low regime equilibria in a steady state. The distribution of money with upper-bound N is the eigenvector of the matrix $L(\pi)$ in equation (12) or dimension N, where π is determined by (9), and the total quantity of money is equal to M.

The utility of holding money in such a steady state is determined as in Section 5. Extending (19) to the vector of dimension N: $U^N(t) = (U_0^N(t), \dots, U_{N-1}^N(t))'$, by the same recursive argument as in Section 5,

$$U^N(t) = \beta A^N(\pi_t) U(t+1) + \beta B^N(\pi_t), \tag{25}$$

where the square matrix A^N of dimension N and the $N \times 1$ matrix B^N are defined by

$$A^N(\pi) = \begin{pmatrix} \alpha(1-\pi)+\pi & b & 0 & \dots & \dots & 0 \\ \pi\alpha & a & b & 0 & \dots & 0 \\ 0 & \pi\alpha & a & b & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \pi\alpha & a & b \\ 0 & \dots & 0 & 0 & \pi\alpha & 1-\pi\alpha \end{pmatrix}, \quad B^N(\pi) = \begin{pmatrix} \alpha - \pi\alpha(1+c) \\ \alpha \\ \vdots \\ \alpha \\ 1-\pi(1-\alpha) \end{pmatrix}. \tag{26}$$

The steady state with a low regime is an equilibrium if and only if the N-consumption function is optimal and the $N+1$ consumption function is not. We have therefore the necessary conditions

$$\begin{cases} U_k^N - U_{k-1}^N \geq 1 \quad \text{for} \quad 1 \leq k \leq N-1, \\ \\ U_N^{N+1} - U_{N-1}^{N+1} \leq 1. \end{cases} \tag{27}$$

Note that in the second equation, all the values U_k^{N+1} have to be computed by (24) using the matrices $A^{N+1}(\pi)$ and $B^{N+1}(\pi)$ that are associated to the $N+1$-consumption function and with the same unemployment rate that is used for the matrices $A^N(\pi)$ and $B^N(\pi)$.

Proposition 14

For a given quantity of money, a steady state of an equilibrium with a low regime is characterized by the following necessary and sufficient conditions:

(i) The upper-bound of the distribution of money and the consumption function of the low-type are described by Proposition 13 for some value N.

(ii) The distribution of money and the unemployment rate are unique as characterized by Proposition 5.

(iii) The levels of utility of an agent after his consumption decision are given by (24) and (25). They satisfy the conditions (26).

(iv) There is no other value of $N \leq \bar{N}$ such that the conditions (26) are satisfied.

The result indicates that all steady state equilibria in the low regime can be found after a finite number of computations. The last condition (iv) is introduced because we could not rely in this paper on the standard concavity property of the utility of money balances as the optimization problem is over integers. However, as a practical method, using intuition, if there is a steady state equilibrium with an N-consumption function, it is sufficient to verify that (26) does not hold for $N+1$ and that the $N+1$-consumption function is not optimal. The next section provides an illustration.

7.2 Multiple equilibrium unemployment rates

Recall that in Figure 2, for a given quantity of money (equal to 1.7944), there are two steady states of the low regime with N-consumption functions where $N=2$ and $N=3$, unemployment rates $\pi_2=0.15$ and $\pi_3=0.6105$, respectively (points A and B). We now apply Proposition 14 to show that each steady state is an equilibrium. The cost of not consuming for a high type is set at $c=4$ and the discount factor is equal to $\beta=0.9$. The intuition for the multiplicity is straightforward. If agents attempt to accumulate more balances, demand is lower, the unemployment rate is higher which provides the incentive for more accumulation.

1	2	3	4	5	6
		Max m= 2		Max m=3	
Balance	Distribution of balances	Utility "mid-day"	Difference	Utility "mid-day"	Difference
0	0.0083	6.5838		5.9111	
1	0.1889	7.6333	**1.0495**	6.8434	0.9322
2	0.8028			7.6988	**0.8554**

Table 1: Utilities for two consumption functions in the stationary equilibrium with unemployment rate $\pi=0.15$. The equilibrium is represented by the point A in Figure 2. Additional structural parameters: $\beta=0.9$, $c=4$.

The equilibrium with the low unemployment rate $\pi_2=0.15$ is described in Table 1. The distribution of money is reported in Column 2. Less than 1% of the agents are liquidity-constrained while the unemployment rate is 15%. The lack of demand is generated by the fear of being liquidity constrained with a high type. Hence only 85% of the agents consume whereas more than 99% could consume.

The vector U as determined by (24) and (25) is reported in Column 3. The first part of condition (26) is shown to be true in Column 4, with a difference U_1-U_0 that is greater than 1. Column 4 presents the vector of utilities (for the same unemployment rate π_2) of a low-type individual who choose the 3-consumption function, that is consumes if and only if his balance is equal to 3. The second part of the condition (26) is verified in Column 6 with $U_2-U_1<1$. Note that in this case, the value of U_1-U_0 is no longer greater than one.

1	2	3	4	5	6
		Max m= 3		Max m=4	
Balance	Distribution of balances	Utility "mid-day"	Difference	Utility "mid-day"	Difference
0	0.0928	0.4486		0.2599	
1	0.2369	2.3757	1.9271	2.1152	1.8554
2	0.4534	3.4304	**1.0547**	3.0333	0.9181
3	0.2169			3.8110	**0.777**

Table 2: Utilities for two consumption functions in the stationary equilibrium with unemployment rate $\pi = 0.6105$. The equilibrium is represented by the point B in Figure 2. Same structural parameters as in Table 1.

The second steady state equilibrium with the high unemployment rate $\pi_3 = 0.6105$ is represented in Table 2. Agents now accumulate balances up to 3. The fraction of liquidity constrained agents, 0.093 is also much lower than the unemployment rate. The conditions (26) are verified in Columns 4 and 6.

7.3 Continuous utility function

Assume that the utility function in (1) is replaced by

$$u(x_t, \theta_t) = Max(x_t + \theta_t c(x_t - 1), 1), \tag{28}$$

where x is a positive real number. The utility functions of the two types, high and low, are represented in Figure (4). Both are stylized expressions of a concave functions with declining marginal utilities. The "saturation point" is the same for both types with a consumption of 1, and for a consumption smaller than 1, the marginal utility is higher for the high type. The rest of the model is unchanged. Producers can sell any quantity of their produced good with a zero cost of production, but given the utility function of the buyer, they will never sell more than one unit.

We now show that the main property of the discrete model, multiple equilibrium steady states with an without unemployment, is preserved when the utility function with indivisibilities in (1) is replaced by a utility of the type in (27). Under that utility function, there is a general equilibrium with full-employment. The distribution of money is not unique under full-employment but this non uniqueness does not matter for the real allocation and we have seen it already in the discrete model. Furthermore, the discrete distribution of money (over integers) that supports the steady state equilibrium with unemployment is stable: after a perturbation of that distribution to a

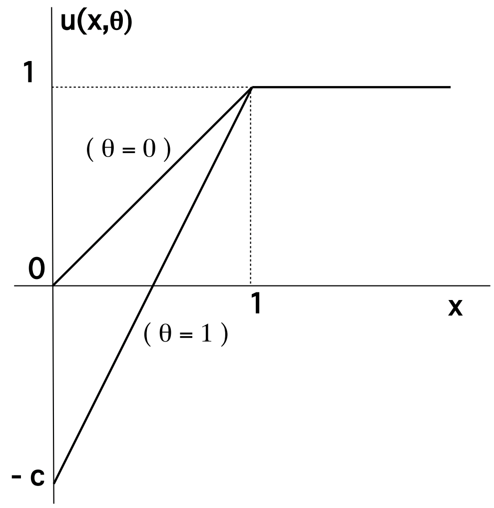


Figure 4: Utility functions for the high type ($\theta = 1$) and the low type ($\theta = 0$).

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continuous distribution, the distribution of money in the general equilibrium reverts over time to the initial discrete distribution. A full algebraic analysis is beyond the scope of this paper, and some arguments will be intuitive. Given the stability of the discreteness property of the distribution of money (which is due to the kink in the utility function), one may presume that the dynamic properties are also maintained for the continuous utility function (27).

Lemma 1 will hold in this extension of the model, as will be discussed later. We therefore assume that the price of real goods in money is equal to 1. We consider the following consumption function

$$c(m, \theta) = \begin{cases} Min(m, 1) & \text{if } \theta = 1, \\ \\ Min(m-k, 1) & \text{if } \theta = 0, \quad \text{for some integer } k. \end{cases} \tag{29}$$

We take $k = 1$ but the method presented here could be used for any k: a high-type consumes as much as possible up to 1 and a low-type consumes any "surplus" over 1 up to a maximum of 1. We also assume that the aggregate quantity of money is strictly smaller than 2. (Otherwise, for $k = 1$, there is no equilibrium with unemployment).

Recall that the state of an agent is defined by the interval $I_i = [i, i + 1)$, with i an integer $i \geq 0$. Under the present consumption function, no sale is greater than 1 and therefore in a stationary state, the balance of an agent must be in one of the intervals I_i, $0 \leq i \leq 2$.

We consider only a steady state equilibrium. Assume first that the distribution of money is the same as the stationary distribution of Proposition 6: its support is discrete with atoms at the points $(0, 1, 2)$. That assumption will be re-examined below. This distribution is also stationary under the continuous consumption function in (28) and it generates the same unemployment rate. It remains to show that the consumption function (28) is optimal in that environment.

The main task is to prove that a low type agent in state I_1 does not have a balance strictly smaller than 1 after his consumption. Let v_0 and v_1 the marginal utilities of money for an agent with a balance in $I_i = [i, i+1)$, net of his consumption in the period. By induction, we have the equations

$$\begin{cases} v_1 = \beta\Big((1-\pi)v_1 + \pi((1-\alpha)v_1 + \alpha v_0)\Big), \\ v_0 = \beta\Big((1-\pi)(((1-\alpha)v_1 + \alpha v_0) + \pi((1-\alpha)v_0 + \alpha(1+c))\Big). \end{cases} \tag{30}$$

For example, on the first line consider an agent who is in I_1 after his consumption and has a small increment of money dm. His utility increase is $v_1 dm$. He makes a sale with probability $1-\pi$ in which case he is in I_2 in the next period, consumes 1 and is back to I_1 after his consumption in that period. His money dm is carried over to the next period with a utility gain $v_1 dm$ in that period. If he makes no sale, with probability π, his post-consumption position is in I_1 if he is of the low type, with probability $1-\alpha$, or in the interval I_0 if he is of the high type and consumes one unit. By backward induction, we get the first equation. Note that the only time when agent consumes the "extra" dm is when he is liquidity constrained and of the high type. In that case, the marginal utility of consumption is $1 + c$.

The solution (v_0, v_1) in the linear system (29) depends on the parameters β, α, π and on c.

$$v = \beta G v + \beta H.$$

with

$$G = \begin{pmatrix} \alpha(1-\pi) + \pi(1-\alpha) & (1-\pi)(1-\alpha) \\ \pi\alpha & 1-\pi\alpha \end{pmatrix}, \quad H = \begin{pmatrix} \pi\alpha(1+c) \\ 0 \end{pmatrix}.$$

It follows that

$$v_0 - v_1 = \frac{\pi\alpha}{1-a}(1 + c - v_0), \qquad \text{with} \qquad a = \alpha(1-\pi) + \pi(1-\alpha) < 1.$$

The marginal utility of money when the agent is in I_0 after his consumption, v_0, depends on the future consumption when the agent is of the high type and has a marginal utility of consumption $1+c$. Because of the discounting and the probability of a high type is less than 1, $v_0 < 1+c$. Hence

$$v_0 > v_1. \tag{31}$$

We have verified the intuitive property that for an individual, the marginal utility of money decreases with the level of his balance.

For given parameters β, α and π, the value of v_0 is a linear function of c, and increasing. If $c = 0$, $v_0(0) < 1$ because of the discounting argument that was used in the previous paragraph. Now increase the value of c gradually. There is c^* such that $v_0(c^*) = 1$. For that value c^*, because of (30), $v_1(c^*) < 1$. By continuity, there exists a value c^{**} such that if $c \in (c^*, c^{**})$,

$$v_1(c) < 1 < v_0(c). \tag{32}$$

Under these inequalities, the consumption function (28) is optimal.

Establishing the equations (29) required that an agent sales are equal to 0 or 1 and therefore that the support of the distribution of money is discrete at integer points. We now argue intuitively that the equilibrium is robust to some perturbation of the money distribution. Assume some value of c such that $v_1(c) < 1 < v_0(c)$ and consider a perturbation of the money distribution, from an initial distribution over integers. The marginal utility of the money in a given interval I_i may not be constant anymore. By an abuse of notation, let $v_0(m)$ and $v_1(m)$ the marginal utility of money (after a possible consumption) in the intervals I_0 and I_1. By continuity, when the perturbation is sufficiently small, for any $m \in I_0$ and $m' \in I_1$,

$$v_1(m) < 1 < v_0(m).$$

The consumption function (28) is still optimal.

Take any perturbation in the distribution of money, if agents have the consumption function (28), then the distribution of money converges to a discrete distribution on the support $\{0, 1, 2\}$. We provide here only an intuitive argument. Project the distribution of money on the interval $[0, 1)$ by the congruence operator with modulus 1.

Assume that agent A has a balance 0 (mod. 1) at the beginning of the period. After his consumption decision, his balance is still 0 (mod. 1) and his consumption does not change the balance of the agent he buys from (mod. 1). He then meets a customer B in the second part of the period who has m. An examination of the various possible types and states shows that at the end of the period, either the two agents have switched their balance (mod. 1) or they have the same balance. The match of these agents does not change the distribution of money. To summarize, there is no attrition of the mass of agent with balance 0 at the beginning of a period. (The name attached on that balance may change).

We now show that the mass of agents with balance 0 increases if the mass of balances in $(0, 1)$ is strictly positive. Consider an agent with balance $m \in (0, 1)$. If he consumes 0 (mod. 1), his purchase does not change the distribution of money, as shown in the previous paragraph). The other possible case is that he consumes m. Let m' be the balance of his seller (after the consumption decision of the seller). If $m' = 0$ we are back to the previous case and there is no impact on the distribution of balances. If $m' \in (0, 1)$, then the balance of his supplier increases to $m + m'$ (mod. 1). The interesting case is $m + m' = 0$. In that case, neither of the two agents had a balance equal to 0 before the match and both of them have a balance equal to 0 after the exchange. That match increases the mass of agents with a balance 0. Such a match occurs with a strictly positive probability as long as the mass of agents with a balance $m \in (0, 1)$ is strictly positive[5]

Although there is no indivisibility in consumption, the distribution of money in the steady state is still discrete (because of the strong curvature of the consumption at the point 1). The driving force toward the discrete distribution is the (infinitely) strong concavity of the continuous utility function at the point 1.

To conclude, reconsider the pricing policy of sellers and assume that the distribution of money holdings is discrete. If all sellers have a price p, no seller has an incentive to deviate from that price and the argument that let to Lemma 1 still applies. If the distribution of money holding is a perturbation of a discrete distribution, the argument also applies.

8 Conclusion and policy

The multiple equilibria are not due to the properties of the production technology but originate exclusively in the self-fulfilling expectations of consumers' expectations

[5] This is a point where one relies on intuition. To be more formal, one can either discretize the interval $[0, 1)$ in an arbitrarily large number K of sub-intervals, or one can assume a density function $f_t(m)$ of the distribution of m and show that after an exchange, the density at the point 0 increases by $\int_0^1 f_t(m_t) f_t(1 - m_t) dt$, which is strictly positive as long as $f_t(m)$ is not identical to 0.

about their future opportunities to sell. The multiplicity of equilibrium output is thus driven by the demand. The properties of the model refute Say's law.

The most effective policy to get an economy out of an unemployment trap may be a subsidy of consumption that changes the return to saving. Such a policy may induce all agents who are not liquidity-constrained to consume and thus generate the high regime, which converges to full-employment (Proposition 3, assuming the non-trivial case $M > 1$). Note that under such a policy, the high regime is the only equilibrium. Such a subsidy could be financed by issuing money.

A uniform lump-sum distribution of money can make the switch to a high regime possible, but it cannot force the economy into the high regime. If expectations remain of a low regime, that regime may still be an equilibrium. The policy has some effect however because the rate of unemployment in the stationary equilibrium of the low regime is inversely related to the money supply (Proposition 5). Only a very large increase of the quantity of money can eliminate the unemployment.

The reduction of the price level by policy to a new value that is still an equilibrium value has the same effect as an expansion of money in this model. That effect is similar to the Pigou effect. It is well known however, that such a deflation may have a negative impact on demand when agents have nominal debts (Fisher, 1933). In the model presented here there is no financial intermediation. In a model with financial intermediation, agents would accumulate some debt. The possibility of negative balances would not alter the properties of the model: agents would still face an (negative) lower-bound on their balances and agents at that lower-bound would be the liquidity constrained agents.

Financial intermediation may actually make matters worse because the maximum of the credit line they grant is endogenous. In a high regime, financial intermediation is useless (in the present model[6]). If the economy switches to a low regime, financial institutions probably would toughen the credit requirement and thus amplify the shock. Such an effect seems to have been observed in the most recent financial crisis and recession.

[6]The only risk that agent face here is that of demand since we focus on this issue. Hence, there is no risk in the high regime. In general, there should also be a production shocks in which case financial intermediation would be useful in the high regime. But the issue of the endogenous upper-bound on individual debts remains when a shock to aggregate demand occurs.

APPENDIX: Proofs

Proposition 2

For an argument more formal than in the text, assume that an agent, called agent A, has at least one unit of money in period 0 and is of the high type. If he consumes, he has m_1 money holding at the end of the period (which depends on the random sale in the period). Let $\{m_t\}$, $t \geq 2$, his optimal path of money holding from period $t = 2$ on. That random path depends on the consumption decisions and the trading path of the agent (which is independent of the path of types).

Assume now that the agent saves in period 0. We call him agent B. That agent has the same random sequence of trading opportunities and types as agent A. His holding of money at the end of period 1, is therefore $m'_1 = m_1 + 1$. For the same random events at agent A, agent B cannot have for all $t \geq 1$, $m'_t \geq m_T + 1$: that strategy would generate a welfare smaller by $-c$ compared to that of agent A. (Remember that the agent has a high-type in period 0). Let T be the smallest value such that $m'_{T+1} = m_{T+1}$. Assume that $T \geq 2$. Since an agent consumes at most one unit, by definition of T, in periods t with $1 \leq t \leq T-1$, agent B and A have the same consumption, in period T agent A does not consume and agent B consumes. At the end of period T, the two agents have the same money holding m_{T+1} and therefore the same expected utility for the future. Given the realizations of trade opportunities and types until period T, the utilities of agents A and B for the given history of trades and types in periods t with $0 \leq t \leq T$, $\tilde{U}_A$ and $\tilde{U}_B$, satisfy the inequality

$$\tilde{U}_B \leq \tilde{U}_A - c + \beta^T c = -c(1 - \beta^T).$$

Since $T \geq 1$,

$$\tilde{U}_B \leq \tilde{U}_A - c(1 - \beta).$$

Adding up over the disjoints events $\{m'_{T+1} = m_{T+1}\}$ for some T, with total probabilities not greater than 1, the expected utilities of agents A and B in period 0 satisfy the inequality

$$U_B \leq U_A - c(1 - \beta).$$

Saving for an agent of the high type who can consume is not optimal. □

Proposition 3

Let $S_k(t) = \sum_{j=k}^{N} \gamma_j(t)$. Given the matrix $H(\pi)$, for any n with $2 \le n \le N$,

$$S_k(t+1) = S_k(t) - \pi(t)\gamma_k(t) \le S_k(t).$$

Any such sequence $S_k(t)$ is monotone decreasing, bounded below by 0 and therefore converges. (The distribution of money evolves over time by increasing first-order stochastic dominance).

Since, $\gamma_N(t) = S_N(t)$ and for $2 \le k \le N-1$, $\gamma_k(t) = S_k(t) - S_{k+1}(t)$, $\gamma_k(t)$ converges for any $2 \le k \le N$.

Because $\gamma_1(t) = M - \sum_{k=2}^{N} k\gamma_k(t)$ and $\gamma_0(t) = 1 - \sum_{k=1}^{N} \gamma_k(t)$, $\gamma_1(t)$ and $\gamma_0(t)$ also converge. Let γ_0^* be the limit of $\gamma_0(t)$.

If $\gamma_0^* > 0$, since $\pi(t) = \gamma_0(t)$, we must have, by induction from $k = N$, and using the expression of the matrix H, $\gamma_k^* = 0$ for $k \ge 2$. In this case, $M = \gamma_1^* = 1 - \gamma_0^* < 1$. If $M \ge 1$, then $\gamma_0^* = 0$. $\square$

To prove Proposition 5, we first establish some Lemmata.

Lemma 2

For any $\pi \in (0,1)$, the matrix $L(\pi)$, defined in (12) has an eigenvalue equal to 1 that is of order 1.

Recall that the matrix $L(\pi)$ is square and of dimension $N+1$. Call e the row-vector of ones and of dimension $N+1$. Any distribution Γ has the sum of its components equal to 1. Since the matrix $L(\pi)$ is a function in the set of distributions of dimension $N+1$, $e.L(\pi) = 1$ (which can also be verified directly). Hence, 1 is an eigenvector of the matrix $L(\pi)$. To show that it is of order 1, consider the matrix

$$L(\pi) - I = \begin{pmatrix} \pi - 1 & \alpha\pi & 0 & 0 & \dots & 0 \\ 1-\pi & a-1 & \alpha\pi & 0 & \dots & 0 \\ 0 & b & a-1 & \alpha\pi & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & b & a-1 & \alpha\pi & 0 \\ 0 & \dots & 0 & b & a-1 & \pi \\ 0 & 0 & 0 & \dots & b & -\pi \end{pmatrix}, \tag{33}$$

where I is the identity matrix of dimension $N+1$. Let Δ_j be the determinant for the first j rows and columns of this matrix. Replacing the last row of Δ_N by the sum of all rows of Δ_N and using $a+b-1=-\alpha\pi$, $\Delta_N=-b\Delta_{N-1}$ with $b=(1-\alpha)(1-\pi)$. By induction, $\Delta_N=(-b)^{N-1}(\pi-1)\neq 0$. The unit eigenvalue is of order 1.

Lemma 3

For any $\pi\in(0,1)$, and Γ such that $\sum_k \gamma_k=1$, $L^t(\pi)\Gamma$ tends to the eigenvector with sum of the components equal to one that is associated to the unit eigenvalue of $L(\pi)$ as t tends to infinity.

We will replace $L(\pi)$ by L in the proof. Since the sum of the elements of L in each column is equal to 1, for any vector v, $|Lv|=|\sum_{ij}L_{ij}v_j|=|\sum_j v_j|\leq\sum_j|v_j|$. The matrix L is contracting. Furthermore, if there are two non identical distributions $v\neq w$ with $\sum v_j=\sum w_j=1$, then $|L.(v-w)|=|\sum_j(v_j-w_j)|<\sum_j|v_j-w_j|$.

The sequence $\Gamma^t=L^t\Gamma$ has at least one accumulation point, $\tilde{\Gamma}$, because it belongs to a compact. Let Γ^* be the eigenvector associated to the unit eigenvalue of $L(\pi)$ such that $\sum_k\gamma_k^*=1$. Using the definition of $\tilde{\Gamma}$, the difference $\tilde{\Gamma}-\Gamma^*$ can be approximated arbitrarily closely by $L^k\tilde{\Gamma}-L^k\Gamma^*=L^{k-1}L.(\tilde{\Gamma}-\Gamma^*)$. If $\tilde{\Gamma}\neq\Gamma^*$, by the previous paragraph, $|\tilde{\Gamma}-\Gamma^*|\leq|L.(\tilde{\Gamma}-\Gamma^*)|$, a contradiction. Therefore, the sequence $\Gamma^t=L^t\Gamma$ has the limit Γ^*. $\square$

We denote by $\succeq$ the ordering according to first-order stochastic dominance. Comparing to distributions Γ and $\tilde{\Gamma}$,

$$\Gamma'\succeq\Gamma\quad\text{if and only if for any } K<N,\quad\sum_{k=0}^{k=K}\gamma'_k\geq\sum_{k=0}^{k=K}\gamma_k.$$

Lemma 4

For any distribution of money Γ, let $\tilde{\Gamma}(\pi)=L(\pi)\Gamma$, where $L(\pi)$ is the transition matrix given in (12). If $\pi'\geq\pi$, then $\tilde{\Gamma}(\pi')\succeq\tilde{\Gamma}(\pi)$.

The case $N=2$ is trivial. Assume $N\geq 3$. For $K=1$, the inequality is verified because $\gamma_0>0$. For $K=2$, using the expression of $L(\pi)$ in (12),

$$\tilde{\gamma}_0(\pi)+\tilde{\gamma}_1(\pi)=\gamma_0+\Big(\pi(1-\alpha)+\alpha\Big)\gamma_1+\alpha\pi\gamma_2,$$

which is non-decreasing in π.

Likewise, for $K \leq N-1$,

$$\sum_{k=0}^{K} \tilde{\gamma}_k(\pi) = \sum_{k=0}^{K-1} \gamma_k(\pi) + \Big(\pi(1-\alpha)+\alpha\Big)\gamma_K + \zeta_K \pi \gamma_{K+1}, \tag{34}$$

with $\zeta_K = \alpha$ if $K \leq N-2$, and $\zeta_K = 1$ if $K = N-1$.

The right-hand side in (33) is non-decreasing in π. $\square$

The result has an intuitive interpretation. When π increases, the aggregate demand falls which shifts distribution of money "to the left".

Lemma 5

Let there be two distributions Γ^2 and Γ^1 such that $\Gamma^2 \succeq \Gamma^1$. For any $\pi \in (0,1)$,

$$L(\pi)\Gamma^2 \succeq L(\pi)\Gamma^1.$$

To prove that the first component of $L(\pi)\Gamma^2$ is at least equal to that of $L(\pi)\Gamma^1$, use $\gamma_0^2 + \gamma_1^2 \geq \gamma_0^1 + \gamma_1^1$ and take the difference

$$\begin{aligned} \pi\gamma_0^2 + \alpha\pi\gamma_1^2 - \pi\gamma_0^1 - \alpha\pi\gamma_1^2 &= \pi(\gamma_0^2 - \gamma_0^1) + \alpha\pi(\gamma_1^2 - \gamma_1^1) \\ &\geq (1-\pi\alpha)(\gamma_0^2 - \gamma_0^1), \end{aligned}$$

which is non negative because $\gamma_0^2 \geq \gamma_0^1$.

A similar argument is applied for the sum of the first two components of $L(\pi)\Gamma^j$, j=1,2. In general, the difference between the sum of the first k components, $3 \leq k \leq N-1$ is

$$\sum_{j=0}^{k-1}(\gamma_j^2 - \gamma_j^1) + ((1-\alpha)\pi + \alpha)(\gamma_k^2 - \gamma_k^1) + \alpha\pi(\gamma_{k+1}^2 - \gamma_{k+1}^1)$$

$$\geq (1-\alpha\pi)\sum_{j=0}^{k-1}(\gamma_j^2 - \gamma_j^1) + ((1-\alpha)\pi + \alpha - \alpha\pi)(\gamma_k^2 - \gamma_k^1)$$

$$\geq (1-\alpha-(1-\alpha)\pi)\sum_{j=0}^{k-1}(\gamma_j^2 - \gamma_j^1) = (1-\alpha)(1-\pi)\sum_{j=0}^{k-1}(\gamma_j^2 - \gamma_j^1),$$

which is non negative. $\square$

The proof of the following result is left as an exercise.

Lemma 6

If $\Gamma^2 \succeq \Gamma^1$, then the quantity of money in the distribution Γ^2 is not strictly greater than in the distribution Γ^1.

We can now prove Proposition 5.

Proposition 5

Define $\Gamma^*(\pi)$ as the steady state distribution associated to π. For any t, $L(\pi)^t\Gamma^* = \Gamma^*$. Take $\pi' > \pi$ and define the sequence $\Gamma^t = L(\pi')^t\Gamma^*$. By Lemma 4,

$$\Gamma^1 = L(\pi')\Gamma^* \succeq L(\pi)\Gamma^* = \Gamma^* = \Gamma^0.$$

By Lemma 5, if $\Gamma^t \succeq \Gamma^{t-1}$,

$$\Gamma^{t+1} = L(\pi')\Gamma^t \succeq L(\pi')\Gamma^{t-1} = \Gamma^t.$$

The sequence Γ^t defines an increasing sequence of distribution in the sense of first-order stochastic dominance. By Lemma 3, it converges to the eigenvector of $L(\pi')$ and defines the distribution of money in the steady state associated to $L(\pi')$. By Lemma 6, the quantity of money in that distribution is not strictly smaller than in the distribution $\Gamma^*(\pi)$. The quantity of money in the distribution $\Gamma^*(\pi)$ is a continuous function of π because $\Gamma^*(\pi)$ is continuous. The proof is concluded by taking the limits for $\pi \to 0$ and $\pi \to 1$. $\square$

The low regime with $N = 2$

For any t, the quantity of money is $M = 2(1 - \gamma_0(t) - \gamma_1(t)) + \gamma_1$,

$$\gamma_1(t) = -2\gamma_0(t) + S, \qquad \text{with} \qquad S = 2 - M,$$

and the rate of unemployment, $\pi(t)$, is equal to $\gamma_0(t) + (1-\alpha)\gamma_1(t)$. Hence,

$$\pi(t) = -(1-2\alpha)\gamma_0(t) + (1-\alpha)S.$$

Using the transition matrix L for the case $N = 2$ in (11),

$$\begin{aligned}
\gamma_0(t+1) &= \pi(t)\Big(\gamma_0(t) + \alpha(-2\gamma_0(t) + S\Big), \\
&= -(1-2\alpha)^2\gamma_0(t)^2 + (1-2\alpha)^2 S\gamma_0(t) + \alpha(1-\alpha)S^2 = P(\gamma_0(t)).
\end{aligned}$$

Proposition 8

Assume that $\pi^*\alpha c > \rho$ which is equivalent to $X^* > 1$. On the path of the low regime that converges to the steady state, $X_t > 1$ for t sufficiently large.

Suppose that $X_{t+1} > 1$. Then using the equation of backward induction (20), a sufficient condition for $X_t > 1$ is

$$\pi_t \alpha c > \rho. \tag{35}$$

From Proposition 6, π_t varies monotonically on the transition path. It decreases with time if $\alpha < 1/2$ and $\gamma_0(0) < \gamma_0^*$, or $\alpha > 1/2$ and $\gamma_0(0) > \gamma_0^*$. If π_t is decreasing, then $X_t > 1$ for all t.

In the other parametric cases, π_t is increasing and if $\pi_0 \alpha c > \rho$, then (34) is satisfied for all t and since for some T, $X_t > 1$ for $t > T$, by backward induction, $X_t > 1$ for all t. In this case, the steady state condition $X^* > 1$ holds.

If $\alpha = 1/2$, the economy is stationary for all periods and the necessary and sufficient condition for the optimality of consumption is $\pi^* \alpha c > \rho$. $\square$

Proposition 11

Let us extend the definition of U_k that was given at the beginning of Section 5. We consider an equilibrium that may not be a steady state and define U_k as the utility of an agent after his consumption holding a balance k, $k \geq 0$ after his consumption decision, excluding the utility of any consumption in the period. We omit the time subscript which is not necessary for the argument.

Let n be an integer, $n \geq 1$. The value of U_n is bounded above by $\beta/(1-\beta)$ which is the utility of an agent who consumes in every period. Likewise, U_0 is bounded below by $-\alpha c \beta/(1-\beta)$ which is the utility of an agent who never consumes. Hence,

$$\sum_{k=1}^{n} (U_k - U_{k-1}) \leq \frac{1+\alpha c}{\rho}. \tag{36}$$

Recall that a low-type agent saves if and only his balance k is such that $U_k - U_{k-1} \geq 1$. Since for any $k \geq 0$, $U_k \geq U_{k-1} \geq 0$, there are at most n_1 values of k such that $U_k - U_{k-1} \geq 1$ and $n_1 \leq (1+\alpha c)/\rho$. Hence there exists N such that $U_k - U_{k-1} < 1$ for any $k \geq N$. $\square$

Proposition 12

Let N be the upper-bound of the money distribution. We prove by contradiction that in a low regime, if a low type agent holds a balance $m < N$, he saves. (Recall that all high-type agents consume when possible). Suppose that for some $n < N$ all agents holding n consume. Then the set $\{0, 1, \ldots, m\}$ forms a sink in the evolution of the distribution of money. By definition of the upper-bound N, we would have $N \leq n$ which is a contradiction. We have then for any k with $1 \leq k \leq N-1$,

$$U_k - U_{k-1} \geq 1.$$

Using the upper-bound in (35),

$$N - 1 \leq \frac{1 + \alpha c}{\rho}.$$

□

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