Do Aggregate Fluctuations Depend on the Network Structure of Firms and Sectors?*

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Abstract

We construct a theoretical model that encompasses both firms' and sectors' network structure by considering a lower-dimension economic unit, that is, sector-specific establishments of multi-sectoral firms. The model suggests a reduced-form relation where aggregate production is a function of all the establishment-specific idiosyncratic shocks filtered by the network structure of the economy. We show that aggregate fluctuations depend on the geometry and magnitude of cross-effects across establishments, which is measured by the eigenvalues and eigenvectors of the network matrix. Moreover, the equilibrium levels and their dispersion depend on the Bonacich centrality of establishments within the network structure of the economy. Different network structures entail different aggregate volatilities due to the fact that the presence of direct relations averages out the idiosyncrasies across establishments.

Keywords: Granular hypothesis, Aggregate fluctuations, Networks, Conglomerates, Intersectoral linkages

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1 Introduction

The standard diversification argument claims that independent sectoral shocks tend to average out as the level of disaggregation increases, as well as idiosyncratic shocks to firms average out as the number of firms increases.[1] In this perspective, the existence of aggregate fluctuations is only or at least mainly due to shocks that affect contemporaneously all the *grains* of the economy, such as fluctuations in intrinsically macroeconomic variables, e.g., the inflation rate, financial turmoils, or policy shocks.

A recent stream of literature tries to provide a microfoundation for the existence of aggregate shocks. We will refer to this stream as the granular hypothesis (GH) literature. The GH suggests two possible types of "grains" from which aggregate fluctuations might originate, that is, firms or sectors. For example, Gabaix [16] notices that the empirical size distribution of firms is fat-tailed. Hence, the baseline assumption of the diversification argument, that is, the finite variance of the distribution from which firms are drawn, is not supported empirically. For example, a power law for the size-distribution of firms allows for a relevant impact of idiosyncratic shocks on aggregate fluctuations. As a consequence, the variability in sales of the 100 top US firms can explain as much as 1/3 of aggregate variability. Other studies that look at firms to at least partially explain aggregate fluctuations are, among others, Jovanovic [24], Durlauf [15], Bak et al. [2], and Nirei [31]. In the case of sectoral shocks, the seminal paper of Long and Plosser [26] has been followed by several works, namely, Horvath [21] and [22], Conley and Dupor [9], Dupor [14], Shea [33], the same Bak et al. [2], Scheinkman and Woodford [32], Carvalho [7], Acemoglu, Ozdaglar, and Tahbaz-Salehi [1], and Carvalho and Gabaix [8]. In particular, Carvalho [7] applies the tools of network theory to the input-output tables, linking aggregate variability to the network structure of intersectoral trade. The presence of sectors that work as hubs to the economy make idiosyncratic shocks that would normally be irrelevant propagate to the aggregate level.

These two explanations may be overlapping. On the one hand, aggregate fluctuations can be originated through idiosyncratic shocks to sectors. On the other hand, the existence of big firms allows for the transmission of micro-level shocks to macro-level variables. Aggregate fluctuations are therefore either alternatively or jointly facilitated by idiosyncratic shocks to sectors and firms. In this paper we consider the two features, that is, sector- and firm-specific variability, as the two faces of the same phenomenon, which jointly contribute to the propagation of idiosyncratic shocks to the aggregate.

[1] See, e.g., Lucas [27] and, more recently, the irrelevance theorem of Dupor [14].

The baseline intuition of our model is that big firms are not sector-specific. In other words, we can view firms as an intersectoral network of sector-specific business units. From now on, we will refer to the networks of business units with the term "conglomerates," and we will call each atomistic business unit a "firm." Each firm produces a sector-specific commodity, and it can be part of a conglomerate. We make shocks originate at the firm level, so that conglomerate- or sector-wide fluctuations result as aggregations of multiple firm-specific shocks.

We present a static economy with multiple sectors. A continuum of households consumes a combination of good types according to their respective complementarity. Each good type is produced in a sector, where sector-specific firms compete à la Cournot among them. Each firm can have links with other firms in other sectors. If a link exists, then the two linked firms are part of the same conglomerate. The marginal cost of each firm depends on the production of the other firms linked to it, that is, on the production of the conglomerate. In network theory terminology, firms represent the vertices of a graph where we can note different components, that is, distinct path-connected subnetworks of firms. The network structure of conglomerates overlaps independently with the distribution of firms among sectors. We take the network structure as given and we explore the equilibrium output given the network.

We model peer effects among firms in a way similar to Ballester et al. [3], where the profit of each agent-firm is concave with respect to its own production and linear with respect to other firms' actions. This leads to a very tractable linear structure of the equilibrium solution and therefore permits a matrix representation. The basic result of this literature is that each agent's action is a function of its network centrality, a concept originally borrowed from the sociological literature. For example, see Bonacich [5]. The novelty of our model with respect to this stream of literature is the presence of multiple interrelated markets.[2] This permits the existence of intersectoral spillovers and transmission mechanisms that can be reduced to their peer-effects nature. A negative shock to firm i in sector s transmits not only as a positive shock to other competitor firms in the same sector but also as a negative shock to firms in other sectors both directly, if they are part of the same multisectoral conglomerate, and indirectly, through the complementarities on the demand side across commodities. Thus, the diffusion of idiosyncratic shocks from one firm to another depends on the existence of a

[2] İlkılıç [23] shares with our model the same logic and assumes profit maximization to be taken at the conglomerate level. This is game-theoretically more sophisticated but less tractable once we analyze aggregate volatility and we try to bring the model to the data.

transmission path that connects the two firms. This transmission path does not coincide with the network path of a conglomerate. It is instead a mix of network components and existence of markets for more or less related goods.

The multisectoral models à la Long and Plosser permit the transmission to the aggregate level of idiosyncratic sectoral shocks. The use of input-output tables permits to track the transmission mechanism of shocks across sectors. This explains partially the micro-foundation of aggregate fluctuations and intersectoral comovement as a product of sector-specific technological fluctuations in the style of RBC models. Nevertheless, it is not clear what a sector-specific shock is, in the sense that a sectoral shock is likely to be the aggregation of lower-dimensional "granular" shocks. Similarly, Gabaix [16] considers firm-specific shocks, even accounting for the industrial specialization of the core business of each firm. Nevertheless, it is not clear in this case either what a firm-specific shock is, being firm-specific production the complex aggregation of contract relations, internal organization, and so on. We partially overcome this ambiguities by considering establishment-specific shocks. Given the high level of disaggregation, we are more likely to characterize the idiosyncratic shocks as a product of chance, like mistakes in accounting, strikes, mismatches in the logistics procedures, or simply temporary bad luck in production. In our framework, sector- and firm-specific shocks result as aggregations at the sectoral and firm level of establishment-specific volatility. We would like to contribute to this literature by showing how the transmission of idiosyncratic shocks can account for a greater part of aggregate fluctuations once we take into account not only shocks to sectors or to firms separately, but considering the joint structure of intersectoral linkages and conglomerate relationships.

The paper is organized as follows. Section 2 maps the data on intersectoral linkages and conglomerates into graph-theoretic language. Section 3 presents the set-up for the model and derives the equilibrium solution. Section 4 expresses the equilibrium solution in terms of Bonacich centrality measures. Section 5 shows the effects of the network structure on aggregate volatility and performs some counterfactual exercises using both data from US Census Bureau and BEA and simulated random networks. Section 6 draws the final conclusions and suggests future lines of research. Proofs, figures, and tables are provided in the Appendix.

2 The Network Structure of the Economy

In this section we use Detailed Benchmark Input-Output data compiled by the Bureau of Economic Analysis to describe intersectoral linkages and the

County Business Patterns by the US Census Bureau to describe some features of the conglomerate relations. We consider 2002 data.

2.1 The Network Structure of Intersectoral Linkages

Let $\mathscr{S} \equiv \{1, \cdots, S\}$ be the set of sectors ordered by the N.A.I.C.S. code at a certain digit level, where S is the total number of sectors. In the commodity-by-commodity direct requirements tables the typical (s, s') entry gives the input share evaluated at producers' prices of (row) commodity s as an intermediate input in the production of (column) commodity s'. In order to interpret the data, we assume that each commodity s is produced only in sector s. This is an approximation of what we can extract from the input-output tables, where each commodity can be produced by several sectors. Nevertheless, we can always assign for each commodity a typical sector that produces the great majority of the quantity of a specific commodity. This simplification does not entail qualitative problems and can be seen more as an abuse of notation. We use the requirements tables as a proxy for (aggregate) complementarity across sectors. If commodity s is used as an intermediate input in the production of commodity s' with a share $\tilde{\beta}_{ss'}$, then we say that commodity s is complementary to commodity s' and $\tilde{\beta}_{ss'}$ parameterizes the degree of complementarity. From a theoretical point of view, the complementarity between sectors can be technological or related to preferences, and can relate to both intermediate and final products. In this sense, the data in the direct requirements tables are an equilibrium product of the interaction between technological and consumption complementarities across sectors and at different levels of intermediation. From an aggregative perspective, if we treat the direct requirements as an exogenous endowment of the economy, then interpreting them as consumption or production complementarities does not yield different implications qualitatively and the problem becomes a matter of analytical tractability, as we see in the model below. According to this interpretation, the requirements tables tell us that, while commodity s is a complement of commodity s' with a certain degree $\tilde{\beta}_{ss'}$, commodity s' is a complement of commodity s with degree $\tilde{\beta}_{s's}$, where $\tilde{\beta}_{ss}$ is generally not equal to $\tilde{\beta}_{s's}$. Shocks to the production of a certain commodity can affect the production of other commodities either downstream, that is, from a sector that supplies the commodity to other sectors that demand the commodity, or upstream, that is, from a sector that demands the commodity to other sectors that supply the commodity. In order to analyze the transmission of shocks both downstream and upstream, we can construct a symmetric version of the input-output matrix that accounts for both downstream and upstream complementarity across sectors. Hence, we map the asymmetric input-output matrix into a symmetric complementarity matrix by adding the two corresponding entries for each pair of sectors (s, s') and normalizing them in the interval $[0, 1/(S-1)]$, that is,

$$\beta_{ss'} = \beta_{s's} = \frac{1}{S-1} \frac{\tilde{\beta}_{ss'} + \tilde{\beta}_{s's}}{\max\{\tilde{\beta}_{ss'} + \tilde{\beta}_{s's}\}},$$

for every $s \neq s'$ in $\mathscr{S}$. This transformation dismisses part the information contained in the input-output table but it is necessary for analytical tractability later on. Moreover, we ignore for the moment the complementarity of each commodity with respect to itself.

We use the elements of the set $\mathscr{S}$ of sectors as labels for the vertex set $V(\mathscr{S})$, that is, $V(\mathscr{S}) \equiv \{v_1, \cdots, v_S\}$.

Definition 1. The edge set of intersectoral linkages The edge set $E(\mathscr{S})$ of intersectoral linkages is a subset of $[V(\mathscr{S})]^2$ such that

$$E(\mathscr{S}) \equiv \{\{v_s, v_{s'}\} \in V(\mathscr{S})^2 \, | s \text{ is a complement of } s', \text{ with } s \neq s'\}.$$

In other words, $E(\mathscr{S})$ is defined by the link between elements of the set $V(\mathscr{S})$ of all nodes-sectors. Note that according to our definition of $E(\mathscr{S})$ a sector cannot be complementary to itself. The direct requirement of sector s of intermediate inputs from sector s itself does not enter the edge set, so that there are no self-links. The maximal consumption of commodity s, which is given by α_s/β_{ss}, captures this feature of the input-output tables. The link that defines $E(\mathscr{S})$ is undirected, that is, if $\{v_s, v_{s'}\}$ belongs to E, then also $\{v_{s'}, v_s\}$ belongs to E. We define also the weight function W as a real-valued function from $E(\mathscr{S})$ to $[0, 1/(S-1)]$ that assigns to each element $\{v_s, v_{s'}\}$ in $E(\mathscr{S})$ a weight equal to $W_S(\{v_s, v_{s'}\}) = \beta_{ss'}$.[3]

Definition 2 (The network of intersectoral linkages)**.** The network $\mathbf{g}(\mathscr{S}) \equiv (V(\mathscr{S}), E(\mathscr{S}), W_S)$ of intersectoral linkages is a network of vertex set $V(\mathscr{S})$, edge set $E(\mathscr{S})$, and weight function W_S, where every element of $E(\mathscr{S})$ is an undirected link between two distinct elements v_s and $v_{s'}$ of $V(\mathscr{S})$ with associated weight $W_S(\{v_s, v_{s'}\}) = \beta_{ss'}$.

For example, we represent graphically the network structure of intersectoral linkages in Figure 3.[4] The relative central position of the nodes reflects

[3] We could use the term vertex instead of the term node. Similarly, we could use the terms adjacency relation, edge, or arc instead of the term link, the term link strength instead of weight, and the term graph instead of network. The literature uses this terminology interchangeably.

[4] For the representations of graphs we use the software UCINET 6 by Borgatti, Everett, and Freeman, distributed by Analytic Technologies.

the weights associated to the links. Central sectors have strong links with the other sectors, that is, they are more complementary to the other sectors, while peripheral nodes have weak links. The central sectors for the US economy are "manufacturing" (node 5), "professional and business services" (node 10), and "educational services, health care, and social assistance" (node 11). The centrality of these sectors is clearer in Figure 4, where we dichotomize the network by constructing a new edge set where there is a link between two nodes only if the original link is greater than the threshold level of 0.01, and there is no link otherwise. The alternative weight function assigns the weight 1 to all links in the new edge set.

The adjacency matrix is a particular representation of weighted networks, which reproduces which pairs of nodes are linked together and with which weight.

Definition 3 (The network matrix of intersectoral linkages)**.** The network matrix $\mathbf{B}$ of intersectoral linkages is the adjacency matrix of the network $\mathbf{g}(\mathscr{S})$. In other words, $\mathbf{B}$ is a real-valued symmetric $S \times S$ matrix with typical element $\mathbf{B}_{ss'} = \beta_{ss'} \in [0, 1/(S-1)]$ if $\{v_s, v_s'\}$ belong to $E(\mathscr{S})$ and $W_S(\{v_s, v_{s'}\}) = \beta_{ss'}$, and $\mathbf{B}_{ss'} = 0$ if $\{v_s, v_s'\}$ does not belong to $E(\mathscr{S})$.

We reproduce in Table 3 in the Appendix the matrix of intersectoral linkages for the case $S = 17$, derived from BEA's commodity-by-commodity direct requirements tables for 2002.

2.2 The Network Structure of Conglomerates

An *establishment* is a single physical location at which business is conducted or services are provided. Each establishment is classified on the basis of its major activity, that is, each establishment is characterized by a sector of specialization. An establishment is not necessarily identical with a company or an enterprise, which may consist of one establishment or more. It constitutes the very "grains" of organized economic activity, in the sense that no matter what conceptual framework we employ we can always express any corporation, division, subsidiary, company, and so on, as a differently organized combination of a certain number of establishments. For example, a conglomerate is a collection of corporations involving a parent company and one or more subsidiaries. Each subsidiary can have its own subsidiaries itself. Each company of the conglomerate, be it a parent company or a subsidiary, can be decomposed into one or more establishments, connected to one another by one or more legal, accounting, or economic ties. From now on, we call each establishment a *firm* and any collection of establishments a *conglomerate*.

Let $\mathscr{F}$ be the set of establishments/firms and F be the total number of firms. Each firm is sector-specific, so the set of sectors defines a partition P_S of the set of firms. Let $P_S \equiv \{\mathscr{F}_1, \cdots, \mathscr{F}_S\}$ be a set of subsets of $\mathscr{F}$ such that $\bigcup_{s \in \mathscr{S}} \mathscr{F}_s = \mathscr{F}$ and $\mathscr{F}_s \cap \mathscr{F}_{s'} = \emptyset$ if $\mathscr{F}_s$ and $\mathscr{F}_{s'}$ belong to P_S, for every $s \neq s'$ in $\mathscr{S}$. For every sector s in $\mathscr{S}$, we call n_s the cardinality of $\mathscr{F}_s$, that is, $n_s \equiv \#(\mathscr{F}_s)$. By construction, $\sum_{s \in \mathscr{S}} n_s = F$. We order the firms in $\mathscr{F}$ by their sector of activity, that is,

$$\mathscr{F} = \{1, \cdots, n_1, n_1 + 1, \cdots, n_1 + n_2, \cdots, F\}, \tag{1}$$

where n_s is the number of establishments in sector s, for all s in $\mathscr{S}$.[5]

We use the elements of the set $\mathscr{S}$ of firms as labels for the vertex set $V(\mathscr{F})$, that is, $V(\mathscr{F}) \equiv \{v_1, \cdots, v_F\}$.

Definition 4 (The edge set of linked firms)**.** The edge set $\tilde{E}(\mathscr{F})$ of linked firms is a subset of $[V(\mathscr{F})]^2$ such that

$$\tilde{E}(\mathscr{F}) \equiv \{\{v_i, v_j\} \in V(\mathscr{F})^2 \,|i\text{'s output depends on } j\text{'s output, with } i \neq j\}.$$

In other words, $\tilde{E}(\mathscr{F})$ is defined by the link between elements of the set $V(\mathscr{F})$ of all nodes-firms. Note that there are no self-links, that is, a firm's performance cannot depend on its own performance. The links that define $E(\mathscr{F})$ are undirected, that is, if $\{v_i, v_j\}$ belongs to E, then also $\{v_j, v_i\}$ belongs to E.

Definition 5 (The network of linked firms)**.** The network $\tilde{\mathbf{g}}(\mathscr{F}) \equiv (V(\mathscr{F}), \tilde{E}(\mathscr{F}))$ of linked firms is an undirected network of vertex set $V(\mathscr{F})$ and edge set $\tilde{E}(\mathscr{F})$, where every element of $\tilde{E}(\mathscr{F})$ is an undirected link between two distinct elements v_i and v_j of $V(\mathscr{F})$.

A path from i to j of length l is a sequence of $l \in \mathbb{N}$ links that indirectly connects node v_i to node v_j. For example, there may be no direct link between i and j, that is, $\{v_i, v_j\}$ may not belong to the edge set $\tilde{E}(\mathscr{F})$. Nevertheless, if there exists a node v_k such that both $\{v_i, v_k\}$ and $\{v_k, v_j\}$ belong to $\tilde{E}(\mathscr{F})$, then there is path of length 2 from i to j. A component $C(\tilde{E})$ of vertex set $V(\mathscr{F})$ and edge set $\tilde{E}(\mathscr{F})$ is a subset of the vertex set $V(\mathscr{F})$ such that for each pair of nodes v_i and v_j in the component C it is possible to find a path of some length l in $\tilde{E}(\mathscr{F})$ between v_i and v_j. There may be one or more

[5]According to the County Business Patterns of the US Census Bureau, there were around 5.5 million employer establishments in the US in 2002. The number of establishments and their dimension, measured in terms of average number of employees per establishment, are distributed unevenly across sectors and change over time.

components in the vertex set. The set of components constitutes a partition of the vertex set $V(\mathscr{F})$.

In the model we assume that a firm does not compete with firms within the same conglomerate. This is not necessarily true in practice but it is a necessary assumption for analytical tractability.[6] Hence, we construct a network that respects this limitation. Let $V(\mathscr{F}_s)$ be the vertex set of the firms that belong to sector s in $\mathscr{S}$. By construction, $v_i \in V(\mathscr{F}_s)$ if and only if $i \in \mathscr{F}_s$. Moreover, if $v_i \in V(\mathscr{F}_s)$ and $v_j \in V(\mathscr{F}_{s'})$ with $s \neq s'$, then necessarily $i \neq j$.

Definition 6 (The edge set of conglomerates)**.** The edge set $E(\mathscr{S})$ of conglomerates is a subset of $\tilde{E}(\mathscr{F})$ such that

$$E(\mathscr{F}) \equiv \{\{v_i, v_j\} \in V(\mathscr{F}_s) \times V(\mathscr{F}_{s'}) \,| \text{if } v_i, v_j \in C(\tilde{E}), \\ \text{then } \nexists v_k \in C(\tilde{E}) \setminus \{v_i, v_j\} \text{ such that } v_k \in \mathscr{F}_s \cup \mathscr{F}_{s'}\}.$$

In other words, the edge set of conglomerates is a subset of the edge set where each path can only include nodes that belong to different sectors.

For every component $C(E)$ in $V(\mathscr{F})$ that we obtain from the edge set $E(\mathscr{F})$ and for every sector s in $\mathscr{S}$, we can find at most one node v_i in $C(E)$ such that firm i belongs to $\mathscr{F}_s$. This leads us to the definition of conglomerate.

Definition 7 (Conglomerate)**.** A conglomerate $C(E)$ is a component of the vertex set $V(\mathscr{F})$ and edge set $E(\mathscr{F})$.

The conglomerates are multisectoral components of $V(\mathscr{F})$. For each firm i whose node v_i belongs to the component $C(E)$, we say that firm i is part of conglomerate $C(E)$. A conglomerate cannot have more than one firm in each sector, and each sector can have at most one firm that belong to a certain conglomerate. We call $\mathscr{C}(E)$ the set of all components $C(E)$ of the vertex set $V(\mathscr{F})$ through the edge set $E(\mathscr{F})$. By construction, $\mathscr{C}(E)$ is a partition of the vertex set $V(\mathscr{F})$, and for every $C(E)$ in $\mathscr{C}(E)$ and every s in $\mathscr{S}$ we have that $\#(C(E) \cap V(\mathscr{F}_s)) \leq 1$. The cardinality of each conglomerate measures its sectoral diversification.

We define also the weight function W_F as a real-valued function from $E(\mathscr{F})$ to $[0, 1]$ that assigns to each element $\{v_i, v_j\}$ in $E(\mathscr{F})$ a weight, or link strength, equal to $W_F(\{v_i, v_j\}) = \epsilon$, where $\epsilon \in [0, 1]$.

[6] In reality a parent company and a subsidiary can be competitors in the same market and this may be an important source - or lack thereof - of comovement across firms.

Definition 8 (The network of conglomerates)**.** The network $\mathbf{g}(\mathscr{F}) \equiv (V(\mathscr{F}), E(\mathscr{F}), W_F)$ of conglomerates is an undirected network of vertex set $V(\mathscr{F})$, edge set $E(\mathscr{F})$, and weight function W_F, where every element of $E(\mathscr{F})$ is an undirected link between two distinct elements v_i and v_j of $V(\mathscr{F})$ with associated weight $W(\{v_i, v_j\}) = \epsilon \in [0, 1]$.

We present in Figure 5 a simulated random network of conglomerations with $F = 100$ and $S = 20$. Each node is a firm, and each group of linked nodes is a conglomerate. Each conglomerate is characterized by a different shape of its nodes. On the upper left side there is a group of one-firm conglomerates, while the rest of the firms are distributed among conglomerates of different sizes. Since each firm belongs to a different sector within the same component, the size of the components reflects the diversification of each conglomerate. A random network that satisfies the properties of Definition 8 tends generate a size distribution of conglomerates, as we can see in Figure 6. This resembles the empirical sectoral diversification of large conglomerates, which we reproduce in Figure 7.

We can define the adjacency matrix of the conglomerations, that reproduces which of nodes of the vertex set of firms $V(\mathscr{F})$ are linked according to $E(\mathscr{F})$ and with which weight.

Definition 9 (The network matrix of conglomerations)**.** The network matrix Γ of conglomerations is the adjacency matrix of the network $\mathbf{g}(\mathscr{F})$. In other words, Γ is a real-valued symmetric $F \times F$ matrix with typical element $\gamma_{ij} = \epsilon \in [0, 1]$ if $\{v_i, v_j\}$ belong to $E(\mathscr{F})$ and $W(\{v_i, v_j\}) = \epsilon$, and $\gamma_{ij} = 0$ if $\{v_i, v_j\}$ does not belong to $E(\mathscr{F})$.

Given the ordering of $\mathscr{F}$ in (1) and Definition 8, if the number F of firms is enough greater than the number S of sectors, then the matrix Γ of conglomerations is a rather sparse matrix, with several zero entries and just a few non-nil entries.

3 The Model

Consider a multisector economy. There is a continuum of mass 1 of identical households whose utility depends on the consumption c_s of different commodities. There are $S \in \mathbb{N}$ commodities, where $s \in \mathscr{S} = \{1, \cdots, S\}$. There are S productive sectors, each of them producing a different sector-specific commodity.[7] Each sector s is populated by n_s firms, and each firm i within

[7] We use the term commodity and sector interchangeably because we assume that only sector s produces commodity s.

sector s produces an undifferentiated quantity q_i of good s competing à la Cournot with the other firms within the same sector. Each firm may have a link with firms in other sectors, and the more productive the linked firms, the lower the marginal production cost.

On the preference side, each household owns a symmetric share of the profits realized on the production side, and employs these resources to consume different good types. We omit an index for the generic household, though we bear in mind that each control variable on the preference side should have it. The utility function of the household is linear-quadratic, that is,

$$U(c_1, \cdots, c_S) = \sum_{s \in \mathscr{S}} \alpha_s c_s - \frac{1}{2} \sum_{s \in \mathscr{S}} \beta_{ss} c_s^2 + \sum_{s \in \mathscr{S}} \sum_{s' \neq s} \beta_{ss'} c_s c_{s'},$$

with $\alpha_s \geq 0$, $\beta_{ss} > \beta_{ss'}$, and $\beta_{ss'} \in [0, 1/(S-1)]$ for all s and s' in $\mathscr{S}$.

The value of $\sum_{s=1}^{S} \alpha_s$ measures the absolute size of the economy, that is, α_s is high enough so that the utility function is strictly increasing and strictly concave separately with respect to the consumption of each commodity c_s. The intuition is the following: suppose that $\alpha_s = \sum_{s'=1}^{S} \alpha_{s'}$, that is, the household only consumes the good of type s. The marginal utility is always decreasing but positive only up to $c_s < \alpha_s/\beta_{ss}$. Hence, α_s represents the upper bound for the values of c_s such that the utility function shows the standard properties of strictly positive, strictly decreasing marginal utility. Below this level, consumption is compatible with the standard properties of utility functions. The parameter β_{ss} measures the concavity of the utility function separately with respect to each good, and $\beta_{ss'}$ parameterizes instead the degree of complementarity between different consumption goods. If $\beta_{ss'} = 0$, the goods are independent. If $0 < \beta_{ss'} \leq 1/(S-1)$, the goods are complements at different degrees. The upper bound on the possible values of $\beta_{ss'}$ avoids increasing returns in the model. For a similar version of this utility function, see Bloch [4]. In order to rule out the possibility of nil demand of any good type, we may assume that $\alpha_s > 0$ for every s. For simplicity, we also assume that $\beta_{ss'} = \beta_{s's}$, that is, the relation of complementarity between two good types is reciprocal.[8]

[8] Since the model does not incorporate any input-use mechanism on the production side, this is a fair approximation of the aggregate technological relation between two sectors. An alternative modeling strategy is to consider a unique final good consumed by the households and produced using the S intermediate commodities through an aggregative equation that takes into account the technological complementarities among commodities. Since this would not change the structure of the aggregate equilibrium, we choose the simplest formulation.

The household maximizes its utility subject to the budget constraint

$$\sum_{s=1}^{S} p_s c_s = \sum_{s=1}^{S} \Pi_s,$$

where p_s is the price of each good s and

$$\Pi_s \equiv \sum_{i \in \mathscr{F}_s} \pi_i^s \tag{2}$$

represents the share of the household in the profits realized by each firm i operating in sector s. The set $\mathscr{F}_s$ is the set of firms that operate in sector s and π_i^s is the profit of the i-th firm in $\mathscr{F}_s$. Since there is a continuum of households of mass 1 and shares are equal across households, each dividend coincides analytically with total profits. The maximization problem is

$$\begin{aligned} \max_{\{c_s\}_{s \in \mathscr{S}}} & \sum_{s \in \mathscr{S}} \alpha_s c_s - \frac{1}{2} \sum_{s \in \mathscr{S}} \beta_{ss} c_s^2 + \sum_{s \in \mathscr{S}} \sum_{s' \neq s} \beta_{ss'} c_s c_{s'} \\ \text{subject to } & \sum_{s \in \mathscr{S}} p_s c_s = \sum_{s=1}^{S} \Pi_s. \end{aligned} \tag{3}$$

The first order condition (FOC) yields a linear inverse demand function for each commodity,

$$p_s = \alpha_s - \beta_{ss} c_s + \sum_{s' \neq s} \beta_{ss'} c_{s'}. \tag{4}$$

For a discussion of the parameter values for which we have positive prices and quantities, see Bloch [4].

On the production side, there are n_s firms in each sector s. They compete à la Cournot and share the sector-specific demand expressed by the households. The maximization problem for firm i in sector s is

$$\max_{q_i} \quad \pi_i^s \equiv p_s q_i - m_i q_i, \tag{5}$$

where m_i is the marginal cost of producing one unit of good s. We assume that the marginal cost is invariant across good types. Moreover, we suppose that the marginal cost is linearly increasing in the firm i's own production and decreasing in the production of any other firm that has a link with firm i. Definition 8 implies that the maximum degree of any firm i is $S-1$, that is, the number of sectors other than its own. The marginal cost m_i of firm i is

$$m_i = \frac{\delta}{2} q_i - \sum_{\substack{j \in \mathscr{F} \\ j \neq i}} \gamma_{ij} q_j - \xi_i, \tag{6}$$

where $\delta > 0$ parameterizes the concavity of each firm's profits in own production. The idiosyncratic component ξ_i is an iid random variable with mean μ and finite variance σ^2. The element $-\sum_{\substack{j\in\mathscr{F}\\ j\neq i}} \gamma_{ij} q_j$ represents how the marginal cost of a firm decreases with the production of the firms that are linked to it. According to Definition 8, γ_{ij} can be either 0 if i does not share a link with j or ϵ if i does share a link with j. The conjecture behind lies on a similar way of thinking as Goyal and Joshi [17], where the marginal cost is linearly decreasing in the number of links that a firm has. We provide hereafter two examples in which the marginal cost of production is connected to the production of the linked firms. For a similar argument, see Bloch [4].

Example 1 (Common use of a facility.)**.** Firms in the same conglomerate have to pay a fixed amount ε for every produced unit in order to construct a common warehouse, independently of the good type produced. We can treat this portion as a proportional taxation that every firm in the conglomerate agrees to pay. The rationale for this commitment is that the cost of each produced unit decreases with the size of the common warehouse, a sort of a public good for the firms that are part of the conglomerate. Firm i in conglomerate κ_i has therefore a unitary cost that is composed of $\tilde{\delta} q_i$ for contracting its own workers, $-\tilde{\xi}_i$ for some idiosyncratic technological endowment, ε as the compulsory proportional contribution to the common warehouse, and $-\varepsilon \sum_{j\in\kappa_i} q_j$ as the unit cost of operating the warehouse which decreases with the size of the warehouse. Hence, the marginal cost is

$$m_i = \tilde{\delta} q_i - \tilde{\xi}_i + \varepsilon - \varepsilon \sum_{j\in\kappa_i} q_j = (\tilde{\delta} - \varepsilon) q_i - \sum_{\substack{j\in\kappa_i\\ j\neq i}} \varepsilon q_j + \varepsilon - \tilde{\xi}_i,$$

which has the same form as (6), once we impose $\delta \equiv 2(\tilde{\delta} - \varepsilon)$, $\tilde{\xi}_i$ distributed iid with mean ε and finite variance σ, and $\gamma_{ij} \equiv \mathbb{I}_i^j \varepsilon$, where $\mathbb{I}_i^j = 1$ if $j \in \kappa_i$ and $\mathbb{I}_i^j = 0$ otherwise.

Example 2 (Common R&D production.)**.** Every firm within a conglomerate produces a technology which is nonrival for the members of the conglomerate. As in Goyal and Moraga [18] for example, we characterize production as an innovation effort. Technology produced in one sector by a firm of the conglomerate can be applied in another sector with an adaptation cost, $(1-\varepsilon)$, that we can treat as an iceberg cost on the original technology. In order to be sold in the market, that is, outside the borders of the conglomerate, the product must be patented and therefore the complementarity or the substitutability on the demand side depend on the legal rather than technological compatibility between one good and another. The costs of each firms are convex in the firm's own innovation effort and depend on the effort exerted by the other firms in the same conglomerate, whereas the marginal cost is equal across firms of the same conglomerate up to a scaling constant ξ_i. The resulting expression for m_i is the same as in (6).

Remember that Definition 8 implies that $\gamma_{ij} = 0$ if i and j produce the same good type s, that is, two firms that belong to the same conglomerate cannot compete with one another. Moreover, $\gamma_{ij} = 0$ also if there exists a path between i and j in $\mathbf{g}(\mathscr{F})$ such that a firm k on that path belongs to either the sector of firm i or the sector of firm j. The structure of the marginal costs we assume basically implies that the correlation of produced quantities between two firms that share a link should be higher than the correlation between two firms that do not share a link, which is at the origin of our concept of linkage between two firms. The share of each unit i within sector s depends on the conglomerate to which i belongs, and more specifically on the conglomerate's dimension and the dimension of the sectors in which the conglomerate has a firm.

There is a market clearing condition for each sector that guarantees that at equilibrium the demand for good s expressed on the preference side is equal to the supply provided by the production side, i.e.,

$$c_s \leq \sum_{i \in \mathscr{F}_s} q_i, \tag{7}$$

for every s in $\mathscr{S}$. This means that the households can consume only up to the total production provided by all the firms that operate in sector s. Another market clearing condition refers to the financial market, that is, all the profits realized by the firms are distributed among the households in equal shares, as (2) states.

Definition 10 (Equilibrium)**.** An equilibrium for the economy is a set of household decisions $\{c_s\}_{s \in \mathscr{S}}$, firms decisions $\{q_i\}_{i \in \mathscr{F}}$, dividends $\{\Pi_s\}_{s \in \mathscr{S}}$, profits $\{\pi_i\}_{i \in \mathscr{F}}$, and prices $\{p_s\}_{s \in \mathscr{S}}$ such that $\{c_s\}_{s \in \mathscr{S}}$ solves (3) given $\{p_s\}_{s \in \mathscr{S}}$ and $\{\Pi_s\}_{s \in \mathscr{S}}$, $\{q_i\}_{i \in \mathscr{F}}$ solves (5) given (4), and the market clearing conditions (7) and (2) hold for every $s \in \mathscr{S}$.

Let us consider the FOC of (5) and the market clearing condition (7). If we substitute for $c_{s'} = \sum_{j \in F_{s'}} q_j$ in (4), we obtain

$$p_s = \alpha_s - \beta_{ss} \sum_{j \in F_s} q_j + \sum_{s' \neq s} \beta_{ss'} \left(\sum_{j \in F_{s'}} q_j \right),$$

for every s in $\mathscr{S}$. We can plug this inverse demand function and the expression for the marginal cost (6) inside the i-th firm's problem (5). The resulting firm i's problem is

$$\max_{q_i} \quad \left[\alpha_s - \beta_{ss} \sum_{j \in F_s} q_j + \sum_{s' \neq s} \beta_{ss'} \left(\sum_{j \in F_{s'}} q_j \right) \right] q_i - \left[\frac{\delta}{2} q_i - \sum_{\substack{j \in \mathscr{F} \\ j \neq i}} \gamma_{ij} q_j - \xi_i \right] q_i,$$

whose FOC with respect to q_i yields

$$\alpha_s - \beta_{ss} q_i - \beta_{ss} \sum_{j \in F_s} q_j + \sum_{s' \neq s} \beta_{ss'} \left(\sum_{j \in F_{s'}} q_j \right) - \delta q_i + \sum_{\substack{j \in \mathscr{F} \\ j \neq i}} \gamma_{ij} q_j + \xi_i = 0.$$

Thus, the equilibrium solution satisfies

$$(\delta + \beta_{ss}) q_i + \beta_{ss} \sum_{j \in F_s} q_j - \sum_{s' \neq s} \beta_{ss'} \left(\sum_{j \in F_{s'}} q_j \right) - \sum_{\substack{j \in \mathscr{F} \\ j \neq i}} \gamma_{ij} q_j = \alpha_s + \xi_i, \qquad (8)$$

for every s in $\mathscr{S}$ and every i in $\mathscr{F}_s$. The FOC (8) identifies three aspects that characterize a firm's optimal production choice. First, the width of the sector in which it operates, parameterized by α_s. Second, its idiosyncratic level of technology, identified by ξ_i, which affects the cost-efficiency of production. Third, the position of the firm within the network structure of production, that is, some centrality measure relative to the network of conglomerates Γ and its interaction with the sectoral composition of the economy $\mathbf{B}$. We assume for simplicity that the concavity of the utility function with respect to the consumption of each commodity alone is the same across good types. This does not affect relevantly the results.

Assumption 1. The concavity of the utility function with respect to each good's consumption is the same across all good types, that is, $\beta_{ss} = \beta$ for all s in $\mathscr{S}$.

We use Definition 3 and Definition 9 to express the equilibrium solution in matrix form. Let $\bar{\xi} \equiv [\xi_1, \cdots, \xi_N]'$ be the vector of length F of the idiosyncratic shocks to all firms grouped by sector. Similarly, let $\bar{q} \equiv [q_1, \cdots, q_N]'$ be the vector of optimal quantities. We define $\bar{\alpha} \equiv [\alpha_1, \cdots, \alpha_1, \cdots, \alpha_S, \cdots, \alpha_S]'$ as the vector of length F of the sectors' shares within the overall size of the economy, where each sector share α_s is repeated n_s times, where n_s is the number of firms within sector s. Let $\mathbf{I}_S$ be the $S \times S$ identity matrix and $\mathbf{U}_s$ be the $n_s \times n_s$ matrix of ones. If we compute $\mathbf{B} - \beta * \mathbf{I}_S$, we can construct the $F \times F$ block matrix $\bar{\mathbf{B}}$, where each block is the expansion of $\mathbf{B} - \beta * \mathbf{I}_S$ by $\mathbf{U}_s$. In order to represent the equilibrium in matrix form, we need a matrix that accounts for both the network of intersectoral linkages and the network of conglomerates.

Definition 11 (The interaction matrix of the economy)**.** The interaction matrix Θ of the economy is the real-valued symmetric $F \times F$ matrix given by $\Theta \equiv \bar{\mathbf{B}} + \Gamma$.

The matrix Θ summarizes both the network of intersectoral linkages and the network of conglomerates. The typical element θ_{ij} of Θ is either $\theta_{ij} = \beta_{ss'} + \gamma_{ij}$ if $s \neq s'$, or $\theta_{ij} = -\beta$ if $s = s'$, where firm i operates in sector s and firm j operates in sector s'. Let $\mathbf{I}_F$ be the $F \times F$ identity matrix. Hence, the necessary condition (8) for the equilibrium solution consists of the matrix equation

$$\Psi \bar{q} = \bar{\alpha} + \bar{\xi}, \tag{9}$$

where Ψ is

$$\Psi \equiv \left[(\delta + \beta)\mathbf{I}_F - \Theta \right]. \tag{10}$$

We call $\bar{q}^*$ the solution of the matrix equation (9).

Proposition 1. *The matrix equation (9) has a unique generic solution $\bar{q}^*$, and*

$$\bar{q}^* = \Psi^{-1} \left[\bar{\alpha} + \bar{\xi} \right], \tag{11}$$

where Ψ^{-1} is the inverse of Ψ.

The equilibrium production is a function of the relative dimensions of the sectors composing the economy and the idiosyncratic productivities of all the firms in the economy filtered by the network matrix of the economy. The existence and uniqueness of the equilibrium solution does not mean that $q_i^* \geq 0$ for every i in $\mathscr{F}$. This depends on the interaction between Ψ^{-1} and $\bar{\alpha} + \bar{\xi}$, which may contain negative elements.

4 Equilibrium and Bonacich Centrality

We need to characterize the equilibrium solution to obtain sufficient conditions for which $\bar{q}^* \geqq 0$. Let $\underline{\theta} \equiv \min\{\theta_{ij} | i \neq j\}$ and $\overline{\theta} \equiv \max\{\theta_{ij} | i \neq j\}$. Since $\gamma_{ij} \geq 0$ for every i and j in $\mathscr{F}$ and $\beta_{ss} > \beta_{ss'} \geq 0$ for every $s \neq s'$ in $\mathscr{S}$, then $\underline{\theta} = -\beta < 0$ and $\overline{\theta} \geq 0$. In order to obtain sufficient conditions for which $\vec{q}^* \geqq 0$ we need to decompose Ψ. Since $\Psi = [(\delta+\beta)\mathbf{I}_F - \Theta]$ according to (10), we have to reformulate Θ. First, we isolate the positive cross-effects of Θ and we normalize them by the highest cross-effect $\overline{\theta}$.

Definition 12 (The network matrix of the economy)**.** The network matrix $\mathbf{G}$ of the economy is the $F \times F$ real-valued matrix where the typical element is

$$g_{ij} \equiv \frac{\theta_{ij} + \beta}{\overline{\theta} + \beta}$$

for $i \neq j$ and $g_{ii} = 0$ otherwise.

Note that $\overline{\theta} + \beta > 0$ since $\beta > 0$. By construction, $g_{ij} \in [0, 1]$ for all i and j in $\mathscr{F}$.

We can decompose the matrix Ψ into three components of cross-effects between firms,

$$\Psi = -\left[-(\delta+\beta)\mathbf{I}_F - \beta\mathbf{U}_F + \left(\overline{\theta} + \beta\right)\mathbf{G}\right], \tag{12}$$

where $\mathbf{U}_F$ is the $F \times F$ matrix of ones. The component $-(\delta+\beta)\mathbf{I}_F$ reflects the concavity of firms' profits in own production, the component $-\beta\mathbf{U}_F$ mirrors the competition in homogeneous quantities within the same sector and uses this uniform substitutability as a benchmark value for all the firms, and the component $\left(\overline{\theta} + \beta\right)\mathbf{G}$ represents the complementarity of production decisions between firms within the same conglomerate, measured from the benchmark value of the competition à la Cournot.

Note that we can treat the network matrix of the economy as the adjacency matrix of a weighted network, which we call network of the economy, that accounts for both the network of intersectoral linkages and the network of conglomerates. In order to define the network of the economy, we need a vertex set, an edge set, and a weight function.

Definition 13. The edge set of the economy The edge set E of the economy is a subset of $[V(\mathscr{F})]^2$ such that

$$E \equiv \{\{v_i, v_j\} \in [V(\mathscr{F})]^2 \; | \{v_i, v_j\} \in E(\mathscr{F}) \text{ or } \{v_s, v_{s'}\} \in E(\mathscr{S})\},$$

where $i \in \mathscr{F}_s$ and $j \in \mathscr{F}_{s'}$.

The weight function W is a real-valued function from $V(\mathscr{F})^2$ to $[0, 1]$.

Definition 14 (The network of the economy)**.** The network $\mathbf{g} \equiv (V(\mathscr{F}), E, W)$ of the economy is an undirected network of vertex set $V(\mathscr{F})$ and edge set E, where every element of E is an undirected link between two distinct elements v_i and v_j of $V(\mathscr{F})$, with associated weight $W(\{v_i, v_j\}) = g_{ij} \in [0, 1]$.

As an example, we report in Figure 8 the representation of the network matrix $\mathbf{G}$ for the US economy, where we can see the blocks of intersectoral linkages at different complementarity levels and the conglomerate relations across the sectors.

Let $\lambda_{max}(\mathbf{G})$ be the highest eigenvalue of $\mathbf{G}$. This value is crucial for the characterization of the equilibrium solution.

Assumption 2. The network matrix $\mathbf{G}$ satisfies

$$\lambda_{max}(\mathbf{G}) < \frac{\delta+\beta}{\overline{\theta}+\beta}.$$

The intuition behind Assumption 2 is that the concavity of own production, measured by $\delta+\beta$, must be higher than the maximal complementarity between own production and other firms' choices, measured by $(\overline{\theta}+\beta)\lambda_{max}(\mathbf{G})$.

Proposition 2 (Debreu and Herstein, 1953). $\left[\mathbf{I}_F - (\overline{\theta}+\beta)/(\delta+\beta)\mathbf{G}\right]^{-1} \geqq 0$ *if and only if Assumption 2 holds.*

Let $\lambda_{min}(\mathbf{G})$ be the lowest eigenvalue of $\mathbf{G}$. Moreover, we call $\rho(\mathbf{G}) \equiv \max\left(|\lambda_{min}(\mathbf{G})|, |\lambda_{max}(\mathbf{G})|\right)$ the spectral radius of $\mathbf{G}$. Since $\mathbf{G}$ is square and symmetric, $\lambda_{max}(\mathbf{G}) \geq 0$ and $\lambda_{min}(\mathbf{G}) \leq 0$. Moreover, $\lambda_{max}(\mathbf{G}) \geq -\lambda_{min}(\mathbf{G})$.[9] Hence, Assumption (2) implies that

$$\lambda_{min}(\mathbf{G}) > -\frac{\delta+\beta}{\overline{\theta}+\beta}, \tag{13}$$

and therefore the spectral radius $\rho(\Theta) \equiv \max\{|\lambda_{max}(\mathbf{G})|, |\lambda_{min}(\mathbf{G})|\} < (\delta+\beta)/(\overline{\theta}+\beta)$. The weaker condition in (13) is sufficient for $\Psi^{-1} \geqq 0$, and it is necessary if $\mathbf{G}$ describes a regular network, that is, a network without loops and multiple edges where each node has the same number of neighbors. See Bramoullé, Kranton, and D'Amours [6, Corollary 1].[10] As the highest eigenvalue represents the population-wide pattern and level of positive cross-effects, the lowest eigenvalue represents another important feature of the network. The lower it is, the higher the number of links that connect distinct sets of firms, and the greater the impact of each firm's production decision on other firms' production.

[9] See Cvetković et al. [10, Theorem 0.13].

[10] In our framework, the network structure of the economy would be a regular network if all the conglomerates had the same number of firms distributed in the same collection of sectors and each firm within a conglomerate was connected to each other.

Note that

$$\mathbf{U}_F \bar{q}^* = q^* \bar{1}_F,$$

where $q^* \equiv \sum_{i \in \mathscr{F}} q_i$. Using (12), we can rewrite (9) as

$$\left[(\delta+\beta)\mathbf{I}_F - \left(\overline{\theta}+\beta\right)\mathbf{G}\right]\bar{q}^* = -\beta q^* \bar{1}_F + \bar{\alpha} + \bar{\xi},$$

that is,

$$(\delta+\beta)\left[\mathbf{I}_F - \frac{\overline{\theta}+\beta}{\delta+\beta}\mathbf{G}\right]\bar{q}^* = -\beta q^* \bar{1}_F + \bar{\alpha} + \bar{\xi}.$$

If Assumption 2 holds, the matrix $\left[\mathbf{I}_F - (\overline{\theta}+\beta)/(\delta+\beta)\mathbf{G}\right]$ is invertible.[11] If it is invertible and $(\overline{\theta}+\beta)/(\delta+\beta)$ is small enough, then we can express $\left[\mathbf{I}_F - (\overline{\theta}+\beta)/(\delta+\beta)\mathbf{G}\right]$ by a Newman series, that is,

$$\left[\mathbf{I}_F - \frac{\overline{\theta}+\beta}{\delta+\beta}\mathbf{G}\right]^{-1} = \sum_{k=0}^{+\infty}\left(\frac{\overline{\theta}+\beta}{\delta+\beta}\right)^k \mathbf{G}^k,$$

where $\mathbf{G}^k$ as the k-th power of $\mathbf{G}$. We call $g_{ij}^{[k]}$ the typical element of $\mathbf{G}^k$, where k is a positive integer. The iterations of $\mathbf{G}$ for $k \in \{2, 3, \cdots\}$ keep track of the indirect connections in the network of both conglomerates and sectors. The entry $g_{ij}^{[k]}$ yields the number of paths of length k necessary to pass from i to j in the network $\mathbf{g}$ of Definition 14. Hence, the typical element of

$$\left(\frac{\overline{\theta}+\beta}{\delta+\beta}\right)^k \mathbf{G}^k$$

measures the number of paths number of paths of length k necessary to pass from any firm to any other firm in $\mathbf{g}$ weighted by $(\overline{\theta}+\beta)/(\delta+\beta)$. Thus, from (9) we derive

$$(\delta+\beta)\bar{q}^* = \sum_{k=0}^{+\infty}\left(\frac{\overline{\theta}+\beta}{\delta+\beta}\right)^k \mathbf{G}^k\left[-\beta q^* \bar{1}_F + \bar{\alpha} + \bar{\xi}\right].$$

Our framework of establishment-level production mixes the network of conglomerates with the network of intersectoral linkages. A change in one firm's production does not only affect the production of the firms that are connected to it through the conglomerate network, but also the production in the sector in which the firm operates. This increases or decreases the demand for other good types, depending on whether these other goods are complementary to

[11] The Frobenius theory of nonnegative matrices implies that Assumption 2 holds if $(\overline{\theta}+\beta)/(\delta+\beta)$ is bounded above by the largest column sum of $\mathbf{G}$.

the firm's good. Hence, the production in other sectors is affected as well. For example, let us take the first iteration of $\mathbf{G}$. If we want to see the direct effect of a shock from i to j, we can look at $g_{ij}^{[1]} = (\gamma_{ij} + \beta_{ss'} + \beta)/(\overline{\theta} + \beta)$, where i belongs to s and j belongs to s'. The conglomerate effect is measured by γ_{ij}, while the intersectoral relation through the demand side is measured by $\beta_{ss'}$. The rest of the elements simply rescale the effect with respect to the substitutability among homogeneous goods β. Even if the firms are not part of the same conglomerate, a shock to i has an effect on the demand for the good type produced by j. Let us now take the second iteration of $\mathbf{G}$, so as to see the indirect effect of a shock from i to j. By indirect effect we mean the effect of a shock to i on j through the impact on all the firms m that are directly connected both to i and to j. We can look at $g_{ij}^{[2]} = \sum_{m=1}^{F}(\gamma_{im} + \beta_{ss''})(\gamma_{mj} + \beta_{s''s'})$, where $i \in F_s$, $j \in F_{s'}$, and $m \in F_{s''}$, for every m and every s''. The element $g_{ij}^{[2]}$ sums up all the possible firms m's and sectors s'''s that could be the conduit of propagation of a shock from i to j. Higher powers of $\mathbf{G}$ yield weaker levels of intermediation, and the sum of $\mathbf{G}^k$ for every $k \geq 0$ accounts for the whole stream of intermediate degrees, that is, for any path length k from 0 to ∞. The decay factor $(\overline{\theta} + \beta)/(\delta + \beta)$ assures that there exists always a level of intermediation such that the impact is negligible.

Definition 15 (Weighted Bonacich Centrality)**.** Consider a network $\mathbf{g}$ with adjacency matrix $\mathbf{G}$, a scalar a such that $[\mathbf{I}_F - a\mathbf{G}]^{-1}$ is well defined and non-negative, and a vector $\bar{x} \in \mathbb{R}^F$. The vector of *weighted Bonacich centralities* of parameter a in $\mathbf{g}$ and weights $\bar{x}$ is

$$\bar{b}(\mathbf{g}, a, \bar{x}) \equiv [\mathbf{I}_F - a\mathbf{G}]^{-1}\,\bar{x}.$$

The weighted Bonacich centrality of firm i is $b_i(\mathbf{g}, a, \bar{x}) = \sum_{j \in \mathscr{F}} x_j m_{ij}(\mathbf{g}, a)$, where x_j is the j-th element of the vector $\bar{x}$ of weights and $m_{ij}(\mathbf{g}, a)$ is the typical element of $[\mathbf{I}_F - a\mathbf{G}]^{-1}$. This centrality measure is the sum of all the weighted loops $x_i m_{ii}(\mathbf{g}, a, \bar{x})$ from firm i to itself and of all the other paths $\sum_{j \neq i} x_j m_{ij}(\mathbf{g}, a, \bar{x})$ from i to every other firm j. In our framework, we set $a = (\overline{\theta}+\beta)/(\delta+\beta)$ and we refer to $\bar{b}(\mathbf{g}, (\overline{\theta}+\beta)/(\delta+\beta), \bar{x})$ as $\bar{b}(\bar{x})$ for simplicity. We report in Figure 1 the unweighted Bonacich centrality for the 558 firms that mirrors the distribution across sectors of the 5.6 million establishments of the County Business Patterns. In other words, on the horizontal axis there are all the firms in $\mathscr{F}$, ordered by sector as in (1), while on the vertical axis there are the Bonacich centralities $b_i(\bar{1}_F)$ of each firm i in $\mathscr{F}$. The different segments of Bonacich centrality correspond to firms that belong to different sectors.

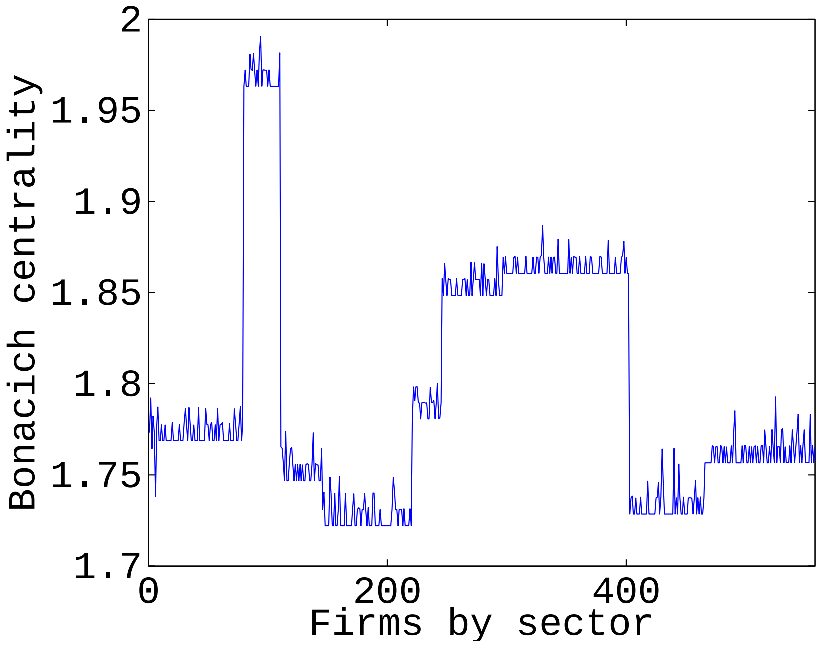

Figure 1: Unweighted Bonacich centrality for 558 firms in 14 sectors. The different segments corresponds to different sectors.

Proposition 3 (The Equilibrium Solution)**.** *Suppose Assumption 2 holds. Then, the unique interior equilibrium solution is*

$$\bar{q}^* = \frac{1}{\delta+\beta}\left[\bar{b}(\bar{\alpha}) + \bar{b}(\bar{\xi})\right] - \frac{\beta\left(b(\bar{\alpha}) + b(\bar{\xi})\right)}{(\delta+\beta)(\delta+\beta+\beta b(\bar{1}_F))}\bar{b}(\bar{1}_F). \tag{14}$$

Note that if $\alpha_s = \tilde{\alpha}$ for all s in $\mathscr{S}$ and $\xi_i = 0$ for all i in $\mathscr{F}$, then $\bar{b}(\bar{\alpha}) = \alpha\bar{b}(\bar{1}_F)$, $\bar{b}(\bar{\xi}) = \bar{0}_F$, and the solution boils down to

$$\bar{q}^* = \frac{\tilde{\alpha}}{\delta+\beta+\beta b(\bar{1}_F)}\bar{b}(\bar{1}_F),$$

which is the equilibrium solution of Ballester et al. [3, Theorem 1]. Hence, our model can be seen as a generalization of that result to case of S sectors and idiosyncratic shocks. Our general result shares with the special case the dependence of each firm's production on the - in our case weighted - centrality degree of the firm. The weights are the dimensions of the sectors in which the firms operate and the idiosyncratic productivities. The source of heterogeneity for equilibrium production is the position of each firm with respect to the complex nexus of intersectoral relations, conglomerates among sectors, and the whole list of technological idiosyncrasies of all the firms in the economy. The production of each firm depends positively on how path-central that firm is in the network structure $\mathbf{g}$.

Let us call $\alpha_{min} \equiv \min_s\{\alpha_s\}$, $\xi_{min} \equiv \min_i\{\xi_i\}$, and $\xi_{max} \equiv \max_i\{\xi_i\}$. Note that since $m_{ij} > 0$ for every i and j in $\mathscr{F}$, $b_i(\bar{x}) > 0$ for every i in $\mathscr{F}$ and every vector $\bar{x}$ of strictly positive weights.

Proposition 4. *If the condition*

$$\alpha_{min} > \frac{\beta}{\delta+\beta} b(\bar{1}_F)(\xi_{max} - \xi_{min}) - \xi_{min}$$

holds, then $q_i^* > 0$ for every i in $\mathscr{F}$.

If the smallest sector of the economy is large enough with respect to the magnitude of the idiosyncratic shocks, then the production of any firm is always positive. If the fundamental uncertainty coming from the idiosyncratic shocks is too wide, that is, if the firm-specific productivities vary too much across firms, then there is the possibility that some firms do not produce positive quantities. Take the case where ξ_i follows a uniform distribution over the interval $[0, 1]$. Then, $\xi_{max} \leq 1$, $\xi_{min} \geq 0$, and $\xi_{max} - \xi_{min} \leq 1$. Hence, the condition of Proposition 4 holds if

$$\alpha_{min} > \frac{\beta}{\delta+\beta} b(\bar{1}_F).$$

Remark 1. The symmetry of $\mathbf{G}$ is not relevant for Proposition 3. In case of asymmetry of $\mathbf{G}$, we can substitute Assumption 2 with the condition $(\overline{\theta} + \beta)\rho(\mathbf{G}) < \delta + \beta$, where $\rho(\mathbf{G})$ is the spectral radius of $\mathbf{G}$, and the rest of the results follow.

Remark 2. We impose $n_s = n$ for all s in $\mathscr{S}$ without any loss of generality. Letting α_s to vary across sectors simply complicates the notation but does not affect the conditions behind Proposition 3.

5 Network structure and aggregate volatility

Now we will study how aggregate volatility, that is, the variance of the production choices vector, depends on the network structure of the economy. The standard diversification argument maintains that a series of idiosyncratic shocks to the *grains* of the economy, which in our case are the single firms, average out at the aggregate level due to the law of large numbers.

In particular, we can apply a variation of the Central Limit Theorem (CLT) to argue that the variance of aggregate production, that is, σ_{GDP}, decays at rate $1/\sqrt{F}$ as the number of grains pertubed by iid shocks, F, increases.[12] In our model, we expect the variance of our aggregate production measure to decrease at a lower rate then $1/\sqrt{F}$ as F increases. This is due to the fact that equilibrium production choices are connected through the network structure of the economy. A similar argument was presented in Jovanovic [24], where the author easily provides examples where the endogenous data do not obey the law of large numbers, albeit the exogenous data do. For even earlier contributions, see Diamond [13] and Mortensen [29].

We can express $\bar{b}(\bar{x})$ as

$$\bar{b}(\bar{x}) = \left[\mathbf{I}_F - \frac{\overline{\theta}+\beta}{\delta+\beta}\mathbf{G}\right]^{-1} \bar{x} = \sum_{k=0}^{+\infty} \left(\frac{\overline{\theta}+\beta}{\delta+\beta}\right)^k \mathbf{G}^k \bar{x}.$$

Proposition 5 (Finite dimensional spectral theorem)**.** *Let G in $\mathbb{R}$ be symmetric. Then, G has F linearly independent real eigenvectors. Moreover, these eigenvectors can be chosen such that they are orthogonal to each other and have norm one.*

According to Theorem 5 and given that G is symmetric by construction, there exists a real orthonormal matrix $\mathbf{V}$ such that

$$\Lambda = \mathbf{V}^T \mathbf{G} \mathbf{V}$$

is a diagonal matrix with elements $\Lambda_{ii} = \lambda_i(\mathbf{G})$, where $\lambda_i(\mathbf{G})$ is the i-th eigenvalue of $\mathbf{G}$. Remember that a matrix $\mathbf{V}$ is orthonormal if its transpose is equal to its inverse, that is, $\mathbf{V}^{-1} = \mathbf{V}^T$.

Proposition 6. *Suppose Assumption 2 holds. Then,*

$$\frac{1}{\delta+\beta}\left[\mathbf{I}_F - \frac{\overline{\theta}+\beta}{\delta+\beta}\mathbf{G}\right]^{-1} = \mathbf{V}\tilde{\Lambda}\mathbf{V}^{-1},$$

where $\tilde{\Lambda}$ is a diagonal matrix whose generic diagonal element is

$$\tilde{\lambda}_i(\mathbf{G}) \equiv \frac{1}{\delta+\beta-(\overline{\theta}+\beta)\lambda_i(\mathbf{G})}, \tag{15}$$

and $\mathbf{V}$ is an orthonormal matrix of eigenvectors of $\mathbf{G}$.

[12] See for example Gabaix [16] for the aggregation of firms' production into a unique final good, and Dupor [14] for the aggregation of sector-specific productions through a standard dynamic multisector model à la Long and Plosser [26].

Proposition 6 tells us that the equilibrium production levels depend on the eigenvalues and the eigenvectors of the network matrix $\mathbf{G}$, where $\tilde{\Lambda}$ contains the eigenvalues and $\mathbf{V}$ contains the eigenvectors. The eigenvalues measure the cross-effects of each firm, while the eigenvectors account for the particular position of each firm within the network. The more the positive cross-effects across firms, the higher the eigenvalues and therefore the higher the production. Let v_{im} be the row i-column m element of $\mathbf{V}$ and v_{mj}^{-1} be the row m-column j element of $\mathbf{V}^{-1}$.

In order to study the volatility of our economy, we can look at the variance-covariance matrix of the equilibrium production. Since we deal with a static model, we will use the concepts of variance and volatility interchangeably.

Definition 16 (The variance-covariance matrix)**.** The variance-covariance matrix $\Sigma(\mathbf{G})$ is the $F \times F$ real-valued matrix given by

$$\Sigma(\mathbf{G}) \equiv E\left[\left(\bar{q}^* - E\left[\bar{q}^*\right]\right)\left(\bar{q}^* - E\left[\bar{q}^*\right]\right)^T\right],$$

where the expectation operator $E\left[\cdot\right]$ is defined over the probability distribution of $\bar{\xi}$.

The diagonal entries of variance-covariance matrix account for the volatility of each firm, while the off-diagonal entries account for the comovement between different firms. Since there exist links between firms, both in the network of intersectoral linkages and in the network of conglomerates, there exists covariance in equilibrium production across firms although the idiosyncratic shocks $\bar{\xi}$ are independently distributed. Hence, the variance-covariance matrix depends on the network structure of the economy.

Proposition 7. *The variance-covariance matrix of the equilibrium production can be decomposed into three components, that is,*

$$\Sigma(\mathbf{G}) = \sigma^2\left[\mathbf{I}_F + \Sigma_U + \Sigma_G\right], \tag{16}$$

where

$$\Sigma_U \equiv D^2 F\left(\sum_{i\in\mathscr{F}}\sum_{m\in\mathscr{F}}\sum_{j\in\mathscr{F}} v_{im}\tilde{\lambda}_m(\mathbf{G})v_{mj}^{-1}\right)^2 \sum_{i\in\mathscr{F}}\sum_{m\in\mathscr{F}}\sum_{j\in\mathscr{F}}\left(v_{im}\tilde{\lambda}_m(\mathbf{G})v_{mj}^{-1}\right)^2 \mathbf{U}_F,$$

$$\Sigma_G \equiv D\mathbf{V}\tilde{\Lambda}\mathbf{V}^{-1}\mathscr{V}\mathbf{V}\tilde{\Lambda}\mathbf{V}^{-1},$$

$$D \equiv \frac{-\beta(\delta+\beta)}{\delta+\beta+\beta b(\bar{1}_F)},$$

and the typical element of $\mathscr{V}$ is

$$\mathscr{V}_{jk} = \sum_{i \in \mathscr{F}} \sum_{m \in \mathscr{F}} \left(v_{im} \tilde{\lambda}_m(\mathbf{G}) \right)^2 \left((v_{mj}^{-1})^2 + (v_{mk}^{-1})^2 \right).$$

We can also derive a measure of aggregate volatility that is comparable with the one used in Horvath [21], Dupor [14], and Carvalho [7]. This formulation is a simplified measure of the actual aggregate variance, whose properties are best described by the variance-covariance matrix $\Sigma(\mathbf{G})$.

Definition 17 (Aggregate volatility)**.** The aggregate volatility $\sigma_Y^2(\mathbf{G})$ is a scalar given by

$$\sigma_Y^2(\mathbf{G}) \equiv E\left[\left(\frac{1}{F} \sum_{i=1}^{F} (q_i - E[q_i]) \right)^2 \right].$$

The advantage of $\sigma_Y^2(\mathbf{G})$ is that it shuts down the covariance between firms and focuses on the aggregate variance. In fact, $\sigma_Y^2(\mathbf{G})$ corresponds to a transformation of the diagonal elements of $\Sigma(\mathbf{G})$. Consequently, it depends on the network structure of the economy.

Proposition 8. *Suppose Assumption 2 holds. Then, the aggregate volatility is a function of the eigenvalues of the network matrix, that is,*

$$\sigma_Y^2(\mathbf{G}) = \frac{\sigma^2}{F^2} \left(\frac{\delta + \beta}{\delta + \beta + \beta b(\bar{1}_F)} \right)^2 \sum_{i \in \mathscr{F}} \sum_{j \in \mathscr{F}} \sum_{m \in \mathscr{F}} \left(v_{im} \tilde{\lambda}_m(\mathbf{G}) v_{mj}^{-1} \right)^2, \tag{17}$$

where v_{im} is the row i-column m element of $\mathbf{V}$, v_{mj} is the row m-column j element of $\mathbf{V}^{-1}$, and $\tilde{\lambda}_m(\mathbf{G})$ is defined in (15).

The elements v_{im} and v_{mj} account for the presence of a path from i to j that involves m as an intermediary node, so as the number of links grows their values will become non-nil. Hence, if the number of links grows at the same rate as F, then the aggregate volatility will decline at a rate lower than F, and if the link rate is even higher than F, then aggregate volatility may not decline at all. See Acemoglu, Ozdaglar, and Tahbaz-Salehi [1] for the limiting distributions of aggregate volatility in presence of cross-effects.

Equations (16) and (17) relate the aggregate variance and covariance to the absolute value of the network matrix' eigenvalues. Hence, the typical element of the variance-covariance matrix $\Sigma(\mathbf{G})$ increases if the absolute value of any eigenvalue of $\mathbf{G}$ increases, that is, if the high and positive eigenvalues increase and the low and negative eigenvalues decrease. In other words, the variance-covariance matrix shrinks if the most path-central elements of the economy lose some weight and the least path-central gain some weight.[13] The heterogeneity in the path-centrality degrees of firms is directly related to the heterogeneity of the equilibrium production levels and, since firms are path-connected, to the covariance of equilibrium quantities. The higher the difference in the path-centrality degrees the higher the dispersion of the production levels.

We run some simulations of our model to understand the relationship between the network structure $\mathbf{g}$ of the economy and aggregate volatility. We derive the matrix $\mathbf{B}$ of intersectoral linkages from the direct requirements tables of the BEA for 2002, as in Table 3. We assign to each sector s a certain number of firms n_s and a certain dimension α_s. We take the n_s's from the County Business Patterns of the US Census Bureau for 2002 and the α_s's from the Gross Ouput by Industry accounts of the BEA for 2002. We report the summary of the data in Table 2. We construct a matrix $\hat{\Gamma}$ of conglomerations using an algorithm that generates a random network that satisfies Definition 9. We use $\mathbf{B}$ and $\hat{\Gamma}$ to derive the network matrix $\hat{\mathbf{G}}$ of the economy as in Definition 12. Moreover, we consider $T = 100$ periods. For each period t, we make a different draw $\bar{\xi}_t$ of $\bar{\xi}$. For each draw, we derive $\bar{q}_t$ holding the rest of the parameters and the network structure $\hat{\mathbf{G}}$ constant. We define real GDP at time t as follows,

$$\mathrm{GDP}_t \equiv \sum_{i \in \mathscr{F}} q_{it}. \tag{18}$$

Figure 2 represents how the real GDP fluctuates across time due to idiosyncratic shocks, given a fixed network matrix $\hat{\mathbf{G}}$ determining aggregate volatility $\sigma_Y^2(\hat{\mathbf{G}})$.

We make different counterfactual exercises to analyze the effect of the network structure on aggregate volatility. We keep the number of firms across sectors taken from the County Business Patterns constant across cases. In Case 1, we suppose that there are no idiosyncrasies across sectors and firms, except that the number of firms across sectors is different. The sectors have the same size $\alpha_s = \alpha/S$ for every s in $\mathscr{S}$ and the firms have the same productivity $\xi_i = 0$ for every i in $\mathscr{F}$. In Case 2, we introduce idiosyncratic productivities across firms that follow a normal distribution of mean 0 and standard deviation 1. In Case 3, we introduce different sector shares that mirror the distribution of industry shares in the US economy. In Case 4,

[13] See Bramoullé, Kranton, and D'Amours [6] for an interpretation of the extreme eigenvalues of a network matrix.

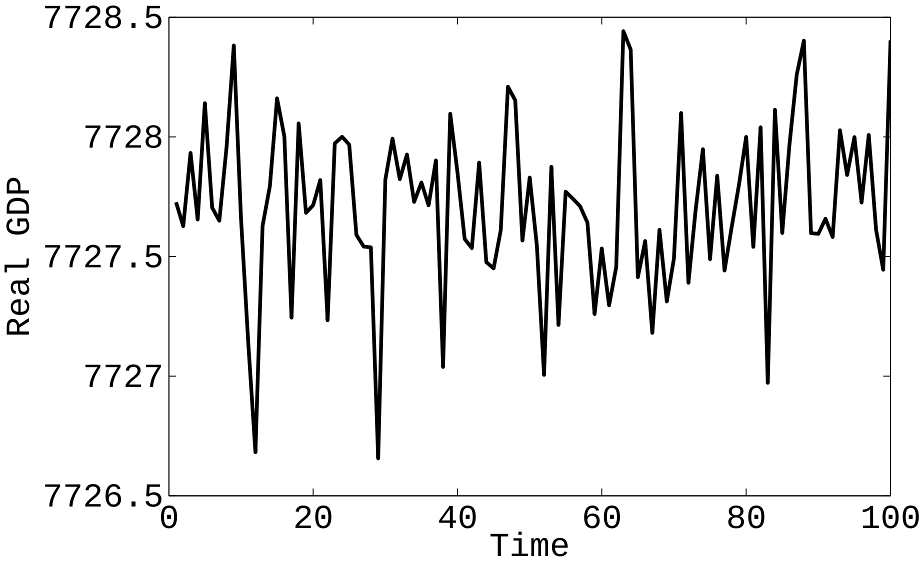

Figure 2: Aggregate fluctuations over time given a fixed network structure.

we introduce intersectoral linkages that mirror the complementarity across sectors derived from the inpu-output tables in the US economy. In Case 5, we introduce conglomerate relations simulated through a random algorithm that respects Definition 8. In Case 6, we increase by 20% the number of links in the conglomerate network. Table 1 reports the results about different aggregate statistics, that is, the aggregate volatility according to Definition 17 normalized to the value of Case 1, the standard deviation of output levels (STD), and the GDP as defined in (18).[14]

Cases	$\sigma^2_Y(\mathbf{G})$	STD(q)	GDP/F
1) No idiosyncrasies	1	0.1099	6.8628
2) Shocks to firms	1	0.1103	6.8622
3) Shocks to sectors	1	5.6490	8.4396
4) Network of sectors	0.9087	5.7793	9.3927
5) Network of firms	0.9053	5.7771	9.4222
6) Bigger firms	0.9046	5.7766	9.4278

Table 1: Numerical exercises. Sources: BEA. Year: 2002. Simulation algorithms available upon request.

[14]We include the standard deviation of outputs in order to make our results comparable with Acemoglu, Ozdaglar, Tahbaz-Salehi [1].

Cases 1, 2, and 3 confirm the theory of Proposition 7 and Proposition 17, that is, aggregate volatility changes only if we change the network structure $\hat{\mathbf{G}}$. The aggregate volatility does not decrease because we introduce changes in $\hat{\mathbf{G}}$ only from Case 4. The standard deviation instead increases with the introduction of the idiosyncrasies in firm-specific productivities and sector shares of Cases 2 and 3. The inclusion of the network of intersectoral linkages in Case 4 decreases the volatility by around 10%. The reason is that the idiosyncrasies introduced by the different sector shares in Case 3 are smoothed out partially by the intersectoral linkages, which make the production in a sector have positive feedbacks into the production of another sector. The standard deviation increases in Case 4 simply because of the different number of firms across sectors. The existence of the conglomerate relations in Case 5 reduces even further aggregate volatility. The rationale behind is the same: the idiosyncratic shocks to the single firms are smoothed out across firms within the same conglomerate. The magnitude of the drop in volatility from Case 4 to Case 5 is smaller than from Case 3 to Case 4 because of the sparseness of the network matrix of conglomerate relations. For example, in the numerical exercise of Table 1, Case 5 introduces 171 links to the overall network matrix $\mathbf{g}$ out of the $558 * 557/2$ possible links. The standard deviation decreases as well from Case 4 to Case 5 for the same reason as the aggregate volatility. As a control, in Case 6 we introduce around 20% more links with respect to the benchmark of Case 5. The volatility and the standard deviation decrease accordingly. The new links are randomly assigned to the network of conglomerates in Case 5. Note that the aggregate product, measured by the average of our definition of GDP in (18), increases along the different cases. This is due to the positive spillovers on the production side across firms and on the demand side across sectors/commodities. In fact, the most relevant shifts in aggregate production are due to the introduction of the intersectoral linkages in Case 4 and of the conglomerate relations in Case 5. We can measure these increases by the increases in the (average) Bonacich centralities of the network.[15] These centrality measures crucially depend on the eigenvalues and eigenvectors of the network matrix, as Proposition 6 suggests.

6 Conclusion

We explain the transmission of idiosyncratic shocks to the aggregate level by considering intersectoral linkages and conglomerate relations across dif-

[15] See Ballester et al. [3, Theorem 2] for an early characterization of the dependence of equilibrium outcome on the Bonacich centrality.

ferent firms. We express aggregate output as a reduced-form function of the idiosyncratic shocks filtered by the network structure of the economy. The aggregate volatility depends on the network structure of the economy. In particular, we can express the dispersion in production levels as a function of the eigenvalues and eigenvectors of the network matrix.

The model helps us to understand how the network structure influences aggregate volatility. We show that the more connected the economy, the more the law of large numbers smooths out the idiosyncrasies across sectors and firms. Our counterfactual exercises in Section 6 show that the introduction of intersectoral linkages and conglomerates decreases aggregate volatility. There are three extensions that future research might explore. First, it remains to understand which types of network structure favor aggregate volatility and which others temper it, other things equal. We could compare two extreme cases. On the one hand, a certain set of conglomerates with a certain size distribution consists of complete components of firms, that is, components where each node is connected to each other. On the other hand, the same conglomerates with the same size distribution consist of star-like components, with one central node and two or more peripheral nodes. The aggregate volatility might change if we pass from one case to the other. Moreover, we could simulate the transition between the two extreme cases by parameterizing the intermediate network structures with some key variables. Following Newman, Strogatz, and Watts [30], we could derive the average component size in a random network from the moments of the degree distribution. Hence, different average component sizes correspond, other things equal, to different levels of, say, the average degree. We could generate a family of random networks for each chosen average component size. For example, a high average component size would mean an economy characterized by big firms. With the same logic, we could derive families of network structures that would adhere to some stylized facts that we may want to examine, for example, the intersectoral diffusion of conglomerates, the concentration ratios within each sector, and so on. If we analyze the intermediate cases, the volatility might not change monotonically from one extreme to the other.

Second, future research could explore a framework of network formation using the payoff structure introduced in this paper. We could set up a two-stage game where the model presented so far would represent the second stage, being the first stage devoted to network formation. There would be a trade-off between the cost of forming a link and the benefits of belonging to a component. Firms would act strategically and decide with whom to share a link depending on the potential equilibrium outcomes in the second stage. Another possibility is dynamic network formation model as in

König, Tessone, and Zenou [25], where the timing of the two stages is inverted. First, the agents realize their equilibrium production given previous period's network. Second, given the equilibrium result they choose which other agents to share a link with. Different network structures arise and it is possible to identify stationary network structures that follow the properties of nested split graphs.[16] These networks are also known as "interlink stars" in Goyal and Joshi [17] and Goyal et al. [19]. The main property of these networks is the core-periphery structure, which reminds the stylized structure of the conglomerates, that is, central parent companies that specialize the conglomerate into a core business and peripheral subsidiaries that diversify the production to smooth out sector-specific fluctuations.

Third, the model has also several policy implications. For instance, future work might analyze the application of our framework to model discretionary policy interventions. In times of crisis, there may be interventions aimed at stabilizing aggregate output. Each intervention entails a public cost, so a key question is which economic agent we should stabilize first in order to decrease aggregate volatility the most with the least public cost. Suppose that a given set of firms is subject to idiosyncratic shocks such as demand shifts, strikes, productivity fluctuations, or simply bad luck. The law of large numbers reduces the scope of public intervention since the random idiosyncratic happenings in different directions would compensate each other if the set of firms is large enough. Nevertheless, this argument does not hold if there exist direct connections between firms, for example financial liability relations. Hence, idiosyncratic shocks can transmit to the aggregate level and can generate aggregate fluctuations in presence of a network structure of inter-firm relations. Discretionary policy in this sense plays a key role in stabilizing output: as much as - negative - idiosyncratic shocks transmit from one agent to the other up to the aggregate level, so does the stabilization policy of the public authority. If the public authority bails out a troubled firm, it stabilizes the performance of all the firms directly or indirectly connected to it by a path of inter-firm financial ties. My model suggests that the firm that should be stabilized in order to obtain the most substantial aggregate effect is the most path-central firm of the economy, considering both intersectoral linkages and conglomerate relations. Moreover, the reduced-form of the model suggests a practical way of identifying the key firm(s), that is, by deriving the eigenvalues of the network matrix, isolating the highest, and deducing which is the most path-central firm. This method has important analogies with the static Principal Component Analysis (PCA), where my model identifies the common components with the bundled up cross-effects among different

[16]See, for example, Mahadev and Peled [28, Chapter 5].

agents. Future research could explore further the link between centrality measures and PCA in networks.

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Appendix A: Proofs

Proof of Proposition 1. The set of parameters β, δ, and ε for which $\det(\Psi) = 0$ has Lebesgue measure zero in $\mathbb{R}^3$. Hence, the matrix of cross-effects Ψ is generally nonsingular. Hence, Ψ is generically nonsingular, so it is invertible and we call its inverse Ψ^{-1}. □

Proof of Proposition 2. See Theorem III* in Debreu and Herstein [12, p. 601]. □

Proof of Proposition 3. We can express $\bar{q}^*$ in terms of weighted Bonacich centralities, that is,

$$(\delta + \beta)\bar{q}^* = -\beta q^* \bar{b}(\bar{1}_F) + \bar{b}(\bar{\alpha}) + \bar{b}(\bar{\xi}). \tag{19}$$

The individual firm's production is

$$(\delta + \beta)q_i^* = -\beta q^* b_i(\bar{1}_F) + b_i(\bar{\alpha}) + b_i(\bar{\xi}),$$

which we can sum up over all i's in $\mathscr{F}$ and obtain

$$(\delta + \beta)q^* = (\delta + \beta)\sum_{i \in \mathscr{F}} q_i^* = -\beta q^* b(\bar{1}_F) + b(\bar{\alpha}) + b(\bar{\xi}),$$

where $b(\bar{x}) \equiv \sum_{i \in \mathscr{F}} b_i(\bar{x})$ for any $\bar{x} \in \mathbb{R}^F$. Thus,

$$q^* = \frac{b(\bar{\alpha}) + b(\bar{\xi})}{\delta + \beta + \beta b(\bar{1}_F)}, \tag{20}$$

which we substitute in (19). □

Proof of Proposition 4. If $\alpha_s = \alpha_{min}$ for every s in $\mathscr{S}$, then $\bar{b}(\bar{\alpha}) = \alpha_{min}\bar{b}(\bar{1}_F)$. Since $\alpha_{min} \equiv \min_s\{\alpha_s\} > 0$ by construction and $b_i(\bar{x}) > 0$ for every i in $\mathscr{F}$ and every vector $\bar{x}$ of strictly positive weights, we have that $\bar{b}(\bar{\alpha}) \geq \alpha_{min}\bar{b}(\bar{1}_F)$. Similarly, $\bar{b}(\bar{\xi}) \geq \xi_{min}\bar{b}(\bar{1}_F)$ and $\bar{b}(\bar{\xi}) \leq \xi_{max}\bar{b}(\bar{1}_F)$. Substituting these values in the equilibrium expression of Proposition 3 we obtain the condition for $\bar{q}^*$ being a vector of strictly positive entries of Proposition 4. □

Proof of Proposition 5. See, for example, Halmos [20, Chapter 79]. □

Proof of Proposition 6. We perform the eigendecomposition of $\mathbf{G}$ and obtain

$$(\delta + \beta)\bar{q}^* = \sum_{k=0}^{+\infty} \left(\frac{\overline{\theta} + \beta}{\delta + \beta}\right)^k (\mathbf{V}\Lambda\mathbf{V}^{-1})^k \left[-\beta q^* \bar{b}(\bar{1}_F) + \bar{\alpha} + \bar{\xi}\right],$$

that is,

$$(\delta+\beta)\bar{q}^* = \mathbf{V}\sum_{k=0}^{+\infty}\left(\frac{\overline{\theta}+\beta}{\delta+\beta}\right)^k \Lambda^k \mathbf{V}^{-1}\left[-\beta q^* \bar{b}(\bar{1}_F)+\bar{\alpha}+\bar{\xi}\right].$$

If Assumption 2 holds, then $(\delta+\beta)/(\overline{\theta}+\beta) > \lambda_{max}(\mathbf{G}) \geq \lambda_i(\mathbf{G})$ for every i in $\mathscr{F}$, that is,

$$\frac{\overline{\theta}+\beta}{\delta+\beta}\lambda_i(\mathbf{G}) < 1,$$

for every i in $\mathscr{F}$. Hence,

$$\sum_{k=0}^{+\infty}\left(\frac{\overline{\theta}+\beta}{\delta+\beta}\lambda_i(\mathbf{G})\right)^k = \frac{\delta+\beta}{\delta+\beta-(\overline{\theta}+\beta)\lambda_i(\mathbf{G})}$$

for every i in $\mathscr{F}$. □

Proof of Proposition 8. According to Proposition 6, if Assumption 2 holds we can express the equilibrium production as

$$\bar{q}^* = \mathbf{V}\tilde{\Lambda}\mathbf{V}^{-1}\left[-\beta q^* \bar{1}_F+\bar{\alpha}+\bar{\xi}\right].$$

According to Definition 15,

$$\bar{b}(\bar{x}) = \left[\mathbf{I}_F - \frac{\overline{\theta}+\beta}{\delta+\beta}\mathbf{G}\right]^{-1}\bar{x} = (\delta+\beta)\mathbf{V}\tilde{\Lambda}\mathbf{V}^{-1}\bar{x},$$

and

$$b(\bar{x}) = (\delta+\beta)\bar{1}_F^T\mathbf{V}\tilde{\Lambda}\mathbf{V}^{-1}\bar{x}. \tag{21}$$

Hence,

$$\bar{q}^* = -\frac{\beta}{\delta+\beta}q^*\bar{b}(\bar{1}_F)+\mathbf{V}\tilde{\Lambda}\mathbf{V}^{-1}\left[\bar{\alpha}+\bar{\xi}\right],$$

which given (20) is equivalent to

$$\bar{q}^* = -\frac{\beta}{\delta+\beta}\frac{b(\bar{\alpha})+b(\bar{\xi})}{\delta+\beta+\beta b(\bar{1}_F)}\bar{b}(\bar{1}_F)+\mathbf{V}\tilde{\Lambda}\mathbf{V}^{-1}\left[\bar{\alpha}+\bar{\xi}\right],$$

which given (21) gives us

$$\bar{q}^* = -\frac{\beta\bar{1}_F^T\mathbf{V}\tilde{\Lambda}\mathbf{V}^{-1}\left[\bar{\alpha}+\bar{\xi}\right]}{\delta+\beta+\beta b(\bar{1}_F)}\bar{b}(\bar{1}_F)+\mathbf{V}\tilde{\Lambda}\mathbf{V}^{-1}\left[\bar{\alpha}+\bar{\xi}\right]. \tag{22}$$

Since there is a source of uncertainty in (22) given by the stochastic ξ, let us compute the expected value of $\bar{q}^*$,

$$E\left[\bar{q}^*\right] = -\frac{\beta \bar{1}_F^T \mathbf{V}\tilde{\Lambda}\mathbf{V}^{-1}\bar{\alpha}}{\delta + \beta + \beta b(\bar{1}_F)}\bar{b}(\bar{1}_F) + \mathbf{V}\tilde{\Lambda}\mathbf{V}^{-1}\bar{\alpha} \tag{23}$$

We look at the deviation from the expected production of firm i,

$$q_i^* - E\left[q_i^*\right] = -\frac{\beta \bar{1}_F^T \mathbf{V}\tilde{\Lambda}\mathbf{V}^{-1}\bar{\xi}}{\delta + \beta + \beta b(\bar{1}_F)}b_i(\bar{1}_F) + \left[\mathbf{V}\tilde{\Lambda}\mathbf{V}^{-1}\bar{\xi}\right]_i,$$

that is,

$$q_i^* - E\left[q_i^*\right] = -\frac{\beta \sum_{i\in\mathscr{F}}\sum_{m\in\mathscr{F}}\sum_{j\in\mathscr{F}} v_{im}\tilde{\lambda}_m v_{mj}^{-1}\xi_j}{\delta + \beta + \beta b(\bar{1}_F)}b_i(\bar{1}_F) + \sum_{m\in\mathscr{F}}\sum_{j\in\mathscr{F}} v_{im}\tilde{\lambda}_m v_{mj}^{-1}\xi_j,$$

where v_{im} is the row i-column m element of $\mathbf{V}$ and v_{mj}^{-1} is the row m-column j element of $\mathbf{V}^{-1}$. Thus,

$$\frac{1}{F}\sum_{i=1}^{F}\left(q_i^* - E\left[q_i^*\right]\right) = \frac{1}{F}\left(1 - \frac{\beta b(\bar{1}_F)}{\delta + \beta + \beta b(\bar{1}_F)}\right)\sum_{i\in\mathscr{F}}\sum_{m\in\mathscr{F}}\sum_{j\in\mathscr{F}} v_{im}\tilde{\lambda}_m v_{mj}^{-1}\xi_j,$$

from which we obtain

$$\sigma_Y^2 = \frac{1}{F^2}\left(\frac{\delta + \beta}{\delta + \beta + \beta b(\bar{1}_F)}\right)^2 E\left[\left(\sum_{i\in\mathscr{F}}\sum_{m\in\mathscr{F}}\sum_{j\in\mathscr{F}} v_{im}\tilde{\lambda}_m v_{mj}^{-1}\xi_j\right)^2\right].$$

The idiosyncratic shocks are independently and identically distributed across firms, so

$$E\left[\left(\sum_{i\in\mathscr{F}}\sum_{m\in\mathscr{F}}\sum_{j\in\mathscr{F}} v_{im}\tilde{\lambda}_m v_{mj}^{-1}\xi_j\right)^2\right] = \sum_{i\in\mathscr{F}}\sum_{m\in\mathscr{F}}\sum_{j\in\mathscr{F}}\left(v_{im}\tilde{\lambda}_m v_{mj}^{-1}\right)^2 E\left[\xi_j^2\right],$$

where $E\left[\xi_j^2\right] = \sigma^2$ for every j in $\mathscr{F}$.

$\square$

Proof of Proposition 7. Given (23) and (22), we have that

$$\bar{q}^* - E\left[\bar{q}^*\right] = -\frac{\beta \bar{1}_F^T \mathbf{V}\tilde{\Lambda}\mathbf{V}^{-1}\bar{\xi}}{\delta + \beta + \beta b(\bar{1}_F)}\bar{b}(\bar{1}_F) + \mathbf{V}\tilde{\Lambda}\mathbf{V}^{-1}\bar{\xi}.$$

Moreover, given (21), we know that

$$\bar{b}(\bar{1}_F) = (\delta + \beta)\mathbf{V}\tilde{\Lambda}\mathbf{V}^{-1}\bar{1}_F.$$

Hence, we obtain that

$$\bar{q}^* - E\left[\bar{q}^*\right] = \mathbf{V}\tilde{\Lambda}\mathbf{V}^{-1}\left[\frac{-\beta(\delta+\beta)}{\delta+\beta+\beta b(\bar{1}_F)}\bar{1}_F^T\mathbf{V}\tilde{\Lambda}\mathbf{V}^{-1}\bar{\xi}\bar{1}_F + \bar{\xi}\right].$$

Since ξ_i is identically and independently distributed for every i in $\mathscr{F}$ with mean 0 and variance σ^2, we can decompose the variance-covariance matrix of equilibrium production into the three components $\mathbf{I}_F$, Σ_U, and Σ_G. $\square$

Appendix B: Figures and Tables

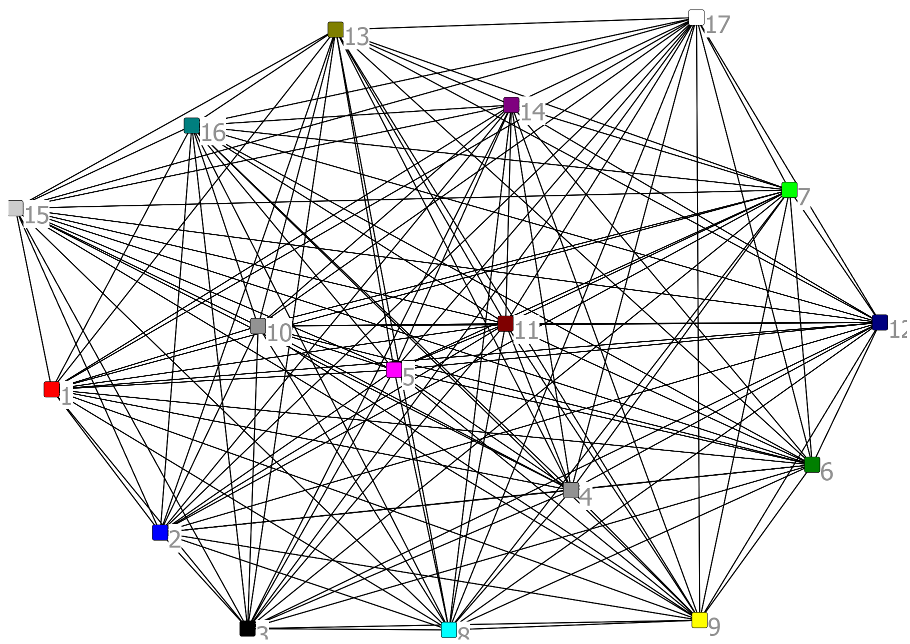

Figure 3: The network structure of intersectoral linkages.

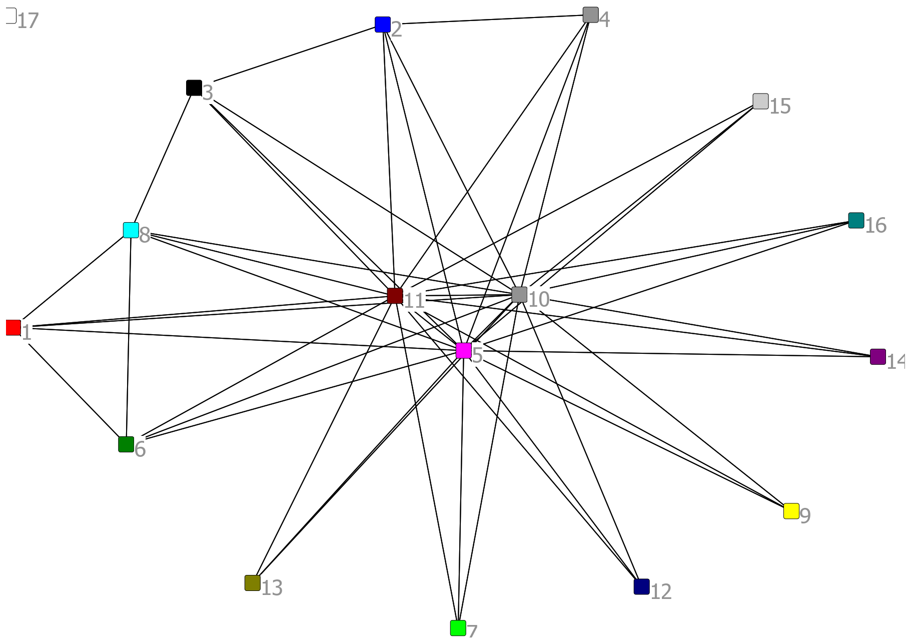

Figure 4: The dichotomized network structure of intersectoral linkages.

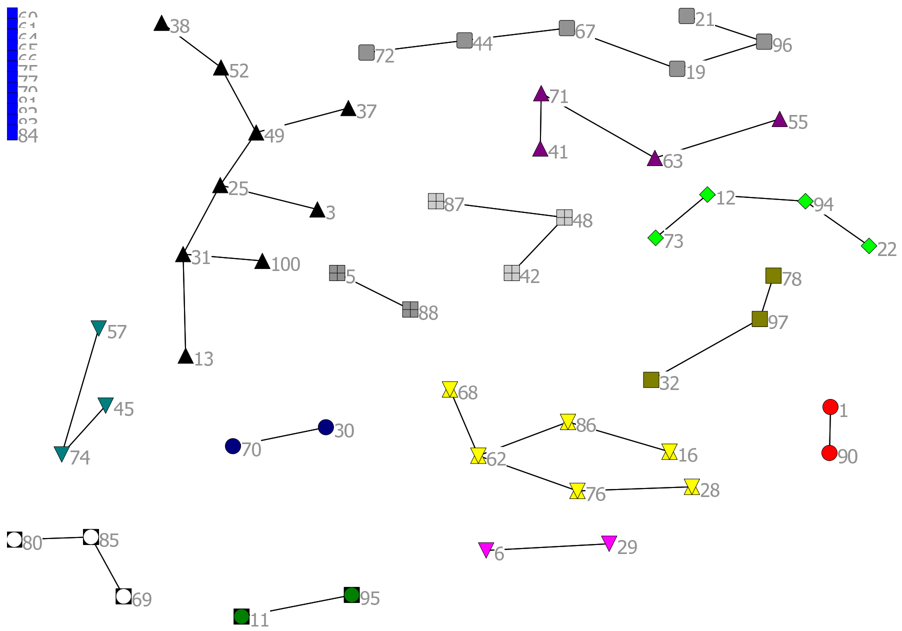

Figure 5: An example of the network structure of conglomerations.

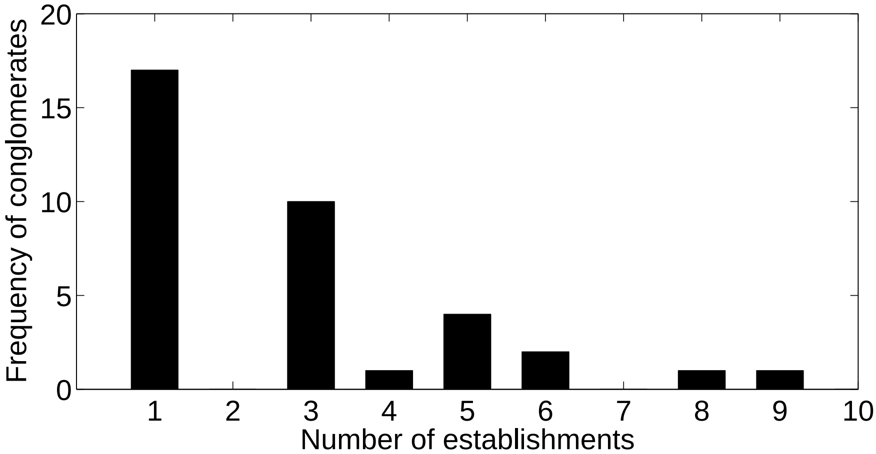

Figure 6: An example of diversification distribution of conglomerates generated according to Definition 7.

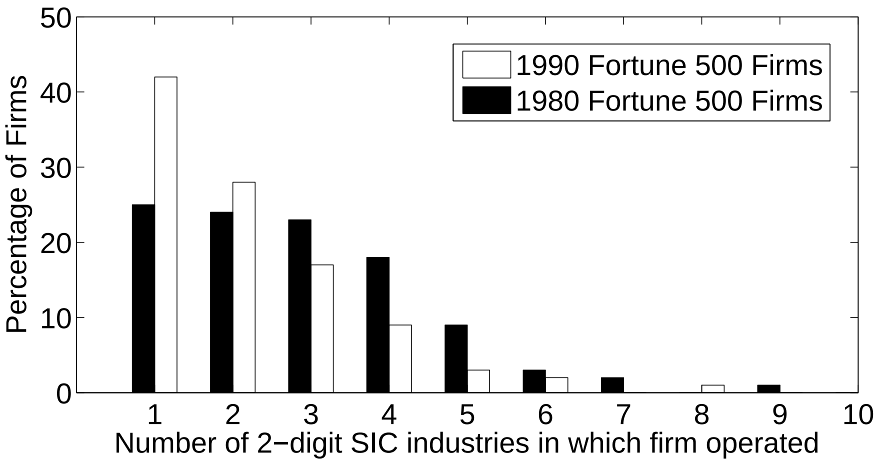

Figure 7: Frequency distribution of diversification: *Fortune* 500. Source: Davis, Diekman, and Tinsley [11, Figure 2].

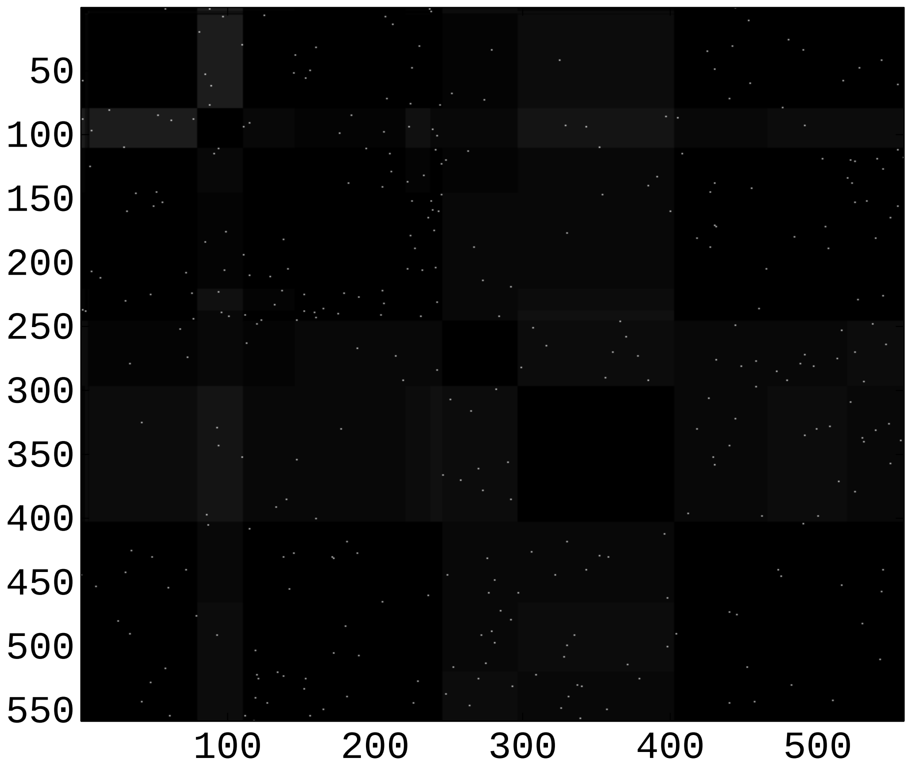

Figure 8: A representation of the network matrix $\mathbf{G}$ of the economy. Data sources: BEA's direct requirements tables for intersectoral linkages, number of establishments per sector from the US Census Bureau's County Business Patterns. Year: 2002. On the horizontal and vertical axis there are 558 firms ordered by sector of activity. These represent the 5524784 establishments distributed across the 14 sectors of the US economy, expressed in tens of thousands and rounded up within each sector. Elements in the matrix represent whether there is a connection or not. The different shading represent the different degree of complementarity, from low complementarity (darker) to high complementarity (lighter). The blocks represent the 14 different sectors. A dark block means that between block s's sector and block s''s sector there is low complementarity, a light block that there is high complementarity. The white dots correspond to the existence of a conglomerate relation between column i's firm and row j's firm. These dots are almost white because the intensity of the link between two firms within the same conglomerate is much higher with respect to any other pair of firms that do not share a link. The most important feature of this representation is that it highlights the sparseness of the matrix of conglomerations Γ with respect to the matrix of intersectoral linkages $\bar{\mathbf{B}}$.

IO code	**Sector**	**Gross Output**	**Employer establishments**	**Companies**
1	Agriculture, forestry, fishing, and hunting	240.8	29250	249290
2	Mining	188.7	19324	102029
3	Utilities	320.4	6223	18896
4	Construction	970.6	729842	2780323
5	Manufacturing	3848.3	310821	601181
6	Wholesale trade	894	347319	711083
7	Retail trade	1030.9	745872	2584689
8	Transportation and warehousing	579.2	167865	976826
9	Information	959.6	76443	309117
10	Finance, insurance, real estate, rental, and leasing	3438.4	507281	3047522
11	Professional and business services	1780.6	1061706	4877023
12	Educational services, health care, and social assistance	1295.7	629550	2430839
13	Arts, entertainment, recreation, accommodation, and food services	704.9	538265	1645857
14	Other services, except government	464	392656	2677613
	Total	16716.1	5524784	22974655

Table 2: Gross output (in billions of dollars), number of employer establishments, and number of nonemployer companies by industry. Year: 2002. Sources: BEA (accounts), US Census Bureau (County Business Patterns and Survey of Business Owners). We report the number of nonemployer companies only for illustrative purposes. In fact, these companies constitute three quarters of all establishments in the economy but account for only around 3% of total sales and receipts data. Hence, we consider only employer establishments in the analysis. We do not consider the residual category “Industries not classified” in the list of industries because it is not present in the list of industries used by the BEA.

IO code	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	0	560	722	567	6250	1156	137	1008	336	3279	1716	223	444	360	219	612	096
2	560	0	3032	1033	3891	519	180	759	379	2953	3018	166	358	302	263	546	147
3	722	3032	0	706	2427	430	317	1399	454	1486	2114	328	721	402	375	403	116
4	567	1033	706	0	6187	867	840	788	518	1734	2760	143	313	476	396	276	082
5	6250	3891	2427	6187	0	2529	1243	4127	2447	2230	4787	2492	3481	3239	2832	5245	173
6	1156	519	430	867	2529	0	309	1084	575	1403	2557	409	685	617	559	779	170
7	137	180	317	840	1243	309	0	662	352	2179	2078	081	228	317	163	059	060
8	1008	759	1399	788	4127	1084	662	0	790	2297	2995	335	658	725	850	642	485
9	336	379	454	518	2447	575	352	790	0	1843	3658	451	851	706	736	552	248
10	3279	2953	1486	1734	2230	1403	2179	2297	1843	0	3257	2288	2122	3076	1170	1378	126
11	1716	3018	2114	2760	4787	2557	2078	2995	3658	3257	0	2413	3118	2530	2804	3028	116
12	223	166	328	143	2492	409	081	335	451	2288	2413	0	276	274	237	077	071
13	444	358	721	313	3481	685	228	658	851	2122	3118	276	0	536	449	281	074
14	360	302	402	476	3239	617	317	725	706	3076	2530	274	536	0	374	305	070
15	219	263	375	396	2832	559	163	850	736	1170	2804	237	449	374	0	094	173
16	612	546	403	276	5245	779	059	642	552	1378	3028	077	281	305	094	0	166
17	096	147	116	082	173	170	060	485	248	126	116	071	074	070	173	166	0
Mean	1040	1065	908	1040	3152	862	542	1153	876	1931	2526	604	859	842	688	850	140
Std	1557	1273	878	1486	1815	724	677	1046	931	961	1155	866	1033	1032	854	1341	106

Table 3: Complementarity matrix. All the entries are in 10^{-5}. Year: 2002. Source: BEA commodity-by-commodity direct requirements tables. We do not consider in the simulation exercises the following categories: "Government" (15), "Scrap, used and secondhand goods" (16), and "Other inputs" (17). The reason is that they are not reported in the list of industries of US Census Bureau's County Business Patterns.