Decentralized trading with private information*

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Abstract

We contribute to the recently developed theory of asset pricing in decentralized markets. We extend this literature to characterize an environment in which some agents have superior private information. In our model, agents have an additional incentive to trade assets to learn information that other agents have. First, we show that uninformed agents can learn all the useful information the long run, and that the long-run allocations are Pareto efficient. In the long run, therefore, the allocations coincide with those of the standard centralized market equilibrium such as in Grossman-Stiglitz. Second, we show that agents with private information receive rents, and the value of information is positive. This is in contrast with the centralized markets in which prices fully reveal information and the value of information is zero. Finally, we provide characterization of the dynamics of the trades.

1 Introduction

This paper provides a theory of trading and information in environments which are *informationally decentralized*. These markets have three key frictions: (1) trading is decentralized (bilateral), (2) information about transactions is known only to the parties of the

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transaction, and (3) some agents have private information. Duffie, Garleanu, and Pedersen (2005) started a research agenda of providing a theory of asset pricing in decentralized environments with public information.[1] They note that many important markets are decentralized such as over-the-counter markets and private-auction markets. Examples of such markets include mortgage-backed securities, swaps and many other derivatives, and real estate markets to name a few. Many of these markets feature informational frictions as well which are a focus of our paper – prices for transactions are not publicly observable and some agents are better informed than others.

We are motivated by three interconnected sets of issues. The first is whether trading in informationally decentralized markets leads to an efficient outcome. This is an important open issue because the presence of any of the three key frictions may lead to highly inefficient outcomes. Our second focus is on the value of information and the evolution of such value in our environment. Are the informed agents better off than uninformed? Can and should the uninformed agents learn private information? This is one of the classis issues of the asset pricing literature in environments with private information and *centralized* tradingsuch as Grossman and Stiglitz (1980). They conclude that prices are fully revealing (in the absence of noise traders), and private information has no value. Our analysis answers these questions in our decentralized environment. Finally, our interest is in the dynamics of trades. In this regard, we are motivated by the classic analysis of Glosten and Milgrom (1985) and Kyle (1985). The difference with this literature is that there markets are informationally centralized in the sense that all agents observe all transactions. In contrast, in our environment information about transactions is private.

Specifically, our environment is as follows. Agents start with different endowments of two assets, match randomly, and trade in bilateral meetings. Any information about their trade is private to the parties of the transaction. A proportion of agents are informed and have superior information over the uninformed agents. This information is about the probability that an asset will pay off in a given state of the world and determines how valuable the asset is. In each period, the game can end with some probability, and the agents have to consume their endowments of assets, or the game continues to the next period. This formulation is one way to introduce discounting in the model. The only information that the agents observe are the history of their matches, but not the

[1] There is a large literature now studying such markets – see, e.g., Duffie, Garleanu and Pedersen 2007, Duffie and Manso 2007, Lagos 2007, Lagos and Rochetau 2007, Lagos, Rochetau, and Weill 2007, Vayanos 1998, Vayanos and Weill 2007, and Weill 2007, among others.

endowments of other agents or their trades. Uninformed agents form beliefs about the value of an asset based on a history of trades they conduct. This environment is technically and conceptually challenging to analyze because the distribution of beliefs about the value of the asset is endogenous and changes over time. An uninformed agent not only has to form a belief about the state of the world but also to form a belief about other agents' beliefs as they influence the future opportunities of trading.

We derive two sets of results. The first set of results are a theoretical examination of efficiency of equilibrium and of the value of information and its evolution. We first show that the long-run allocations are Pareto efficient, and our decentralized environment converges to allocations achieved in Grossman-Stiglitz' perfectly revealing equilibrium. The argument is by contradiction. If an uniformed agent does not converge to an efficient allocations there is a profitable deviation on his part where he constructs a trade thal allows him to learn the state of the world and then take advantage of this information. We show that the losses of such experimentation can be made smaller than the gains of learning the state of the world.

If the initial allocations are not Pareto-efficient, i.e., if there are gains from trade[2], the informed agents receive a higher lifetime utility than uninformed agents. In other words, private information has a positive value. The intuition is that the uninformed agents will learn the true state of the world only in the long run and additionally have to conduct potentially unprofitable trades in the short run to learn the state of the world. That is why in the short run there are profitable trading opportunities for the informed agents. This result is in contrast with the Grossman-Stiglitz analysis where prices fully reveal all the private information, and information has no value. Yet, the uninformed agents can learn all the usefull information, and, in the long run, the value of information converges to zero.

The second set of results is on the theoretical and numerical analysis of the dynamics of trades. We first consider a static example in which there is only one round of trading that is useful to illustrate the inutiition about the trades and strategies of the agents. We show that the static allocations are inefficient. We then develop a method to numerically compute a specific equilibrium of the game. Our simulations and examples show how the behavior of informed agents differ depending on their endowment of the valuable asset. The asset position of the agent who starts with a low endowment of the valuable asset follows a hump-shaped profile. This agent accumulates the valuable asset above his long-run position

[2]If the initial allocations are already Pareto optimal, we show that a version of no trade theorem (similar to, e.g., Brunnermeier 2001 for a detailed exposition of this topic) holds.

before the information is revealed. To do so, he mimics the behavior of the uninformed agents and takes advantage of the fact that uninformed agents do not know which asset is more valuable. Upon accumulating a sufficient amount, this agent sells some of the valuable asset at more advantagous terms of trade as information dissipates across agents. The strategy of the informed agent with a large initial endowment of the valuable asset is different. He decumulates his endowment of the valuable asset. His strategy is determined by considerations of signalling that his asset is valuable – to do so he exchanges small amounts of assets for large amounts of the other asset. Finally, we show in the examples that it takes longer to converge to efficient allocations in our environment with private information than if all information is public.

Our paper is related to several other strands of the literature. The most closely related is Duffie and Manso (2007) and Duffie, Giroux, and Manso (2007) who also consider a private information trading setup with decentralized markets and focus on information percolation in these environments. They derive important closed form solutions for the dynamics of the trade in an environment similar to ours while we have a more general setup and derive strong results about the long-run allocations and general dynamics. Amador and Weill (2007, 2008) is an interesting study of information dispersion in an environments with private and public information. The difference in this paper is that ours is a model of trade rather than solely of information transmission.

Our work is also related and extends papers by Wolinsky (1990) and Blouin and Serrano (2001) who consider a version of Gale (1987) economy with indivisible good and heterogenous information about its value. They show that the information is not fully revealed and allocations are not ex-post efficient. The difference of our paper is that we allow for endogenously determined prices rather than assuming fixed terms of trade. Dubey, Geanakoplos, and Shubik (1987) and Glosten and Milgrom (1985) is related but they consider a model where there are commonly observed signals ("prices") through which uninformed agents learn. In our environment all prices are determined as a part of equilibrium.

The paper is structured as follows. Section 2 describes the environment. Section 3 defines an equilibrium of the game. Section 4 provides characterization of the equilibrium. Section 5 is a static example. Section 6 is a numerical solution of the game. Section 7 concludes. The Appendix contains most of the formal proofs which are sketched in the body of the paper.

2 Setup and trading game

This section describes the setup of our model and defines the decentralized trading game.

2.1 Environment

There are two states of the world $S \in \{S_1, S_2\}$ and two assets. Asset $j \in \{1, 2\}$ pays one unit of consumption if and only if state S_j is realized. There is a continuum of agents with von-Neumann-Morgenstern expected utility $E[u(c)]$, where E is the expectation operator.

At date 0, each agent is randomly assigned a type i, which determines his initial endowment of the two assets, denoted by the vector $x_{i,0} \equiv \left(x^1_{i,0}, x^2_{i,0}\right)$. There is a finite set of types N and each type $i \in N$ is assigned to a fraction f_i of agents. The aggregate endowment of both assets is 1:

$$\sum_i f_i x^j_{i,0} = 1 \text{ for } j = 1, 2. \tag{1}$$

We make the following assumptions on preferences and endowments. The first assumption is symmetry insuring that the endowments of assets are mirror images of each other.

Assumption 1. *(Symmetry) For each type $i \in N$ there exists a type $j \in N$ such that $f_i = f_j$ and $(x^1_{i,0}, x^2_{i,0}) = (x^2_{j,0}, x^1_{j,0})$.*

The second assumption imposes usual properties on the utility function, as well as boundedness and Inada conditions.

Assumption 2. *The utility function $u(\cdot)$ is twice continuously differentiable on R^2_{++}, increasing, strictly concave, bounded above, and satisfies $\lim_{x \to 0} u(x) = -\infty$.*

Finally, we assume that the initial endowments are interior.

Assumption 3. *The initial endowment $x_{i,0}$ is in the interior of R^2_+ for all types $i \in N$.*

The uncertainty about the state of the world is realized in two stages. First, nature draws a binary signal $s \in \{s_1, s_2\}$, with equal probabilities. Second, given the signal s, nature selects the state S_1 with probability $\phi(s)$. We assume that the signal s_1 is favorable to state S_1 and that the signals are symmetric, that is, $\phi(s_1) > 1/2$ and $\phi(s_2) = 1 - \phi(s_1)$. After the signal s is realized, an exogenous random fraction α of agents of each type privately observes the realization of s. The agents who observe s are the informed agents, denoted by I, those who do not observe it are the uninformed agents, denoted by U. The informed agents know s exactly and assign probability $\phi(s)$ to the state S, while the uninformed agents, prior to trading, assign probability 1/2 to both values of the signal s and, hence, to both values of the state S. Throughout the paper, we assume that agents do not observe the information of their counterparts and that endowments are privately observable.

2.2 Trading

After the realization of the signal s, but before the realization of the true state S, all agents engage in a trading game, set in discrete time.

At the beginning of each period $t \geq 1$, the game ends with probability $1 - \gamma$ and continues with probability γ. When the game ends, the state S is publicly revealed and agents consume the payoffs of their assets.[3] The possibility that the game ends is a reason for the agents to trade assets, as they want to acquire insurance prior to the revelation of S.

If the game does not end, all the agents are randomly matched in pairs and a round of trading takes place. With probability 1/2 one of the two agents is selected to be the proposer. In that case, he can make a take it or leave it offer $z = \left(z^1, z^2\right) \in R^2$ to the other agent, that is, he offers to deliver z^1 of asset 1 in exchange for $-z^2$ of asset 2. The other agent can either accept or reject the offer. If an agent with endowment x offers z to an agent with $\tilde{x}$ and the offer is accepted, the endowment of the agent who made the offer becomes $x - z$, and the endowment of the agent who accepted the offer becomes $\tilde{x} + z$. The proposer can only make feasible offers, i.e., offers such that $x - z \geq 0$, and the responder can only accept if $\tilde{x} + z \geq 0$. If the offer is rejected, both agents keep their endowments x and $\tilde{x}$. This concludes the trading round.

Notice that an agent does not observe the endowment of his opponent and does not know whether his opponent is informed or not. Moreover, an agent only observes the trading round he is involved in. Therefore, both trading and information revelation take place through decentralized, bilateral meetings. Notice also that, apart from the presence of asymmetric information, this game follows closely the bargaining game in Gale (1987).

[3] We make this assumption to simplify the exposition. Alternatively we could have assumed that with probability $1 - \gamma$ state S becomes public information, or, with slight modifications, that the signal s becomes public information and trading continues after that.

3 Equilibrium definition

We now define a perfect Bayesian equilibrium of our trading game. Let us first describe individual histories. The event $h_0 \in N \times \{U, I_1, I_2\}$ denotes the agent's initial endowment-type, whether the agent is uninformed or informed, and, in the latter case, whether he observes s_1 or s_2. For each period $t \geq 1$, the event h_t is a vector which includes the indicator variable $\iota_t \in \{0, 1\}$, indicating whether the agent was selected as the proposer, the offer made $z_t \in R^2$, and the indicator variable $r_t \in \{0, 1\}$, indicating whether the offer was accepted. The sequence $h^t = \{h_0, h_1, ..., h_t\}$ denotes the history of play up to period t for an individual agent. Let H^t denote the space of all possibly histories terminating at time t, and H^∞ the space of infinite histories.

Next, we describe the strategy and beliefs of a given agent. Let $\mathcal{C}$ be the space of probability distributions over R^2, corresponding to the space of all possible offers z.[4] If the agent is selected as the proposer at time t, his behavior is described by the map:

$$\sigma_t^p : H^{t-1} \to \mathcal{C},$$

which assigns a probability distribution $\sigma_t^p \left(\cdot | h^{t-1}\right)$ to each history h^{t-1}. If he is selected as the responder, his behavior is described by:

$$\sigma_t^r : H^{t-1} \times R^2 \to [0, 1],$$

which denotes the probability that the agent accepts the offer $z \in R^2$ for each history h^{t-1}. The strategies are restricted to be feasible for both players. The agent's strategy is then described by $\boldsymbol{\sigma} = \{\sigma_t^p, \sigma_t^r\}_{t=0}^{\infty}$. The agent's beliefs are described by two functions:

$$\begin{aligned} \delta_t &: H^{t-1} \to [0, 1], \\ \delta_t^r &: H^{t-1} \times R^2 \to [0, 1], \end{aligned}$$

which describe, respectively, the probability assigned to the signal s_1 after each history h^{t-1}, at the beginning of the trading round, and the probability assigned to s_1 after each each history h^{t-1}, if the agent is selected as the responder and receives offer z. The agent's

[4] To simplify the measure-theoretic apparatus we restrict $\mathcal{C}$ to be the space of discrete, finite distributions, so that only a finite subset of H^t will be reached with positive probability in a symmetric equilibrium. None of the arguments in our proofs require this restrictions.

beliefs are denoted compactly by $\boldsymbol{\delta} = \{\delta_t, \delta_t^r\}$.

We focus on symmetric equilibria where all agents play the same strategy $\boldsymbol{\sigma}$. Given the strategy $\boldsymbol{\sigma}$, let us construct the probability space $(\Omega, \mathcal{F}, P)$, which will be used both to represent the ex ante uncertainty from the point of view of a single agent *and* to capture the evolution of the cross sectional distribution of individual histories in the economy. Let $\Omega = \{s_1, s_2\} \times H^\infty$. A point $\omega = (s, h^\infty)$ in Ω describes the whole potential history of play for a single agent, if the game continues for infinitely many periods. The probability $P(\omega)$ is constructed in detail below. With a slight abuse of notation, we can use $\left(s, h^t\right)$ to denote the associated event in the space Ω, i.e., to denote the subset of Ω which includes all the $\omega = (s, h^\infty)$ such that the truncation of h^∞ at time t is equal to h^t. From the point of view of an individual agent at date 0, the probability that the signal is s and the game ends in period t at history h^{t-1} is then equal to $(1-\gamma)\gamma^{t-1} P\left(\left(s, h^{t-1}\right)\right)$.[5] The filtration $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 ... \subset \mathcal{F}$ is generated by the information sets of the agent at the beginning of each period t.

Before constructing the probability measure P explicitly, it is useful to highlight the crucial relation between P and the cross sectional distribution of individual histories in the economy: at the beginning of time t, the mass of agents with history h^{t-1} is given by $P\left(\left(s, h^{t-1}\right)|s\right)$.

We can then construct the probability $P(\omega)$ recursively, in the following manner. The probability of the event $\left(s, h^0\right)$ for each $h^0 \in H^0$ is determined by the exogenous assignment of endowments and information at date 0. For example, $P\left(\left(s_1, h^0\right)\right) = (1/2)\,\alpha f_i$ if $h_0 = (i, I_1)$, given that the probability of s_1 is $1/2$, the agent receives the information I_1 with probability α and the endowment i with probability f_i. Given $P\left(\left(s, h^{t-1}\right)\right)$ for all $h^{t-1} \in H^{t-1}$, the probability $P\left(\left(s, h^{t-1}\right)\right)$ is defined by iteration, as follows. Given that agents are randomly matched the probability of receiving offer z in period t, for an agent who is not selected as the proposer, is equal to[6]

$$\psi_t(z|s) = \int \sigma_t^p\left(z|h^{t-1}\right) dP(\omega|s).$$

[5] An alternative, equivalent representation of individual uncertainty, can be obtained defining the state space $\tilde{\Omega} = \{s_1, s_2\} \times H$, where $H = \cup_{t=0}^\infty H^t$, and the probability $\tilde{P}\left(\left(s, h^t\right)\right) = P\left(\left(s, h^t\right)\right)$. The probability $\tilde{P}\left(\left(s, h^t\right)\right)$ is the unconditional probability of the outcome $\left(s, h^t\right)$, while $P\left(\left(s, h^t\right)\right)$ is the probability of that outcome *conditional on the game ending at time* t. The advantage of using the probability space in the text will be clear when we introduce cross sectional distributions.

[6] Given that $\sigma_t^p\left(z|h^{t-1}\right)$ and $\sigma_t^r\left(z|h^{t-1}\right)$ are constant for all $\omega \in \left[s, h^{t-1}\right]$, the integrals in this equation and in the next one can be computed only using our knowledge of $P\left(\left[s, h^{t-1}\right]\right)$.

Next, we can construct the probability that offer z is accepted in the period t, which is

$$\chi_t(z|s) = \int \sigma_t^r\left(h^{t-1}, z\right) dP(\omega|s).$$

Given ψ_t and χ_t, it is then possible to construct $P\left(\left(s, h^t\right)\right)$. Let $h^t = \left(h^{t-1}, h^t\right)$ and suppose $h_t = (0, z, 0)$, then

$$P\left(\left(s, h^t\right)\right) = \frac{1}{2}\left(1 - \sigma^r\left(z, h^{t-1}\right)\right)\psi_t(z|s) P\left(\left(s, h^{t-1}\right)\right),$$

given that $P\left(\left(s, h^{t-1}\right)\right)$ is the probability of reaching history h^{t-1} for a given agent, 1/2 is the probability that the agent is selected as the receiver, $\psi_t(z|s)$ is the probability that he receives offer z, and $1 - \sigma^r\left(z, h^{t-1}\right)$ is the probability that he rejects the offer. In a similar way, we have

$$\begin{aligned}
P\left(\left(s, h^t\right)\right) &= \frac{1}{2}\sigma^r\left(h^{t-1}, z\right)\psi_t(z|s) P\left(\left(s, h^{t-1}\right)\right) \text{ if } h_t = (0, z, 1), \\
P\left(\left(s, h^t\right)\right) &= \frac{1}{2}\left(1 - \chi_t(z|s)\right)\sigma^p\left(z|h^{t-1}\right) P\left(\left(s, h^{t-1}\right)\right) \text{ if } h_t = (1, z, 0), \\
P\left(\left(s, h^t\right)\right) &= \frac{1}{2}\chi_t(z|s)\sigma^p\left(z|h^{t-1}\right) P\left(\left(s, h^{t-1}\right)\right) \text{ if } h_t = (1, z, 1).
\end{aligned}$$

The probability measure P will play a central role in the rest of the analysis. For now, we can use it to check that the beliefs are consistent with Bayes' rule on the equilibrium path. This requires that $\delta_t\left(h^t\right)$ satisfies

$$\delta_t\left(h^t\right) = \frac{P(\left(s, h^{t-1}\right))}{\sum_s P((s, h^{t-1}))},$$

for all histories $h^t \in H^t$ such that $\sum_s P(\left(s, h^{t-1}\right)) > 0$. A similar restriction can be imposed on the beliefs δ_t^r. We assume that the beliefs of the informed agents always assign probability 1 to the signal s observed at date 0, that is,

$$\delta_t\left(h^t\right) = \delta_t^r\left(h^t, z\right) = 1 \text{ for all } z \text{ and all } h^t \text{ s.t. } h_0 = (i, I_1), \tag{2}$$

$$\delta_t\left(h^t\right) = \delta_t^r\left(h^t, z\right) = 0 \text{ for all } z \text{ and all } h^t \text{ s.t. } h_0 = (i, I_2). \tag{3}$$

That is, informed agents do not change their beliefs on the signal s, even after observing off-the-equilibrium-path behavior from their opponents.

Given the symmetry of the environment, we will focus on equilibria where strategies and beliefs are "symmetric across states." To define formally this property let $\left[h^t\right]^c$ denote the "complement" of history h^t, that is, a history where: the initial endowment is symmetric to the initial endowment in h^t; if the agent is informed I_{-j} replaces I_j; all the offers received and made at each stage are symmetric to the offers made in h^t, while the responses are the same as in h^t. In particular, if the offer $z = \left(z^1, z^2\right)$ is in h_t, then $z^c = \left(z^2, z^1\right)$ is in $\left[h_t\right]^c$. We say that strategy $\boldsymbol{\sigma}$ is symmetric across states if the agent's behavior is identical when we replace asset 1 with asset 2 and h^t with $\left[h^t\right]^c$, that is, $\sigma_t^p(z|h^{t-1}) = \sigma_t^p(z^c|\left[h^{t-1}\right]^c)$ and $\sigma_t^r(h^{t-1}, z) = \sigma_t^r(\left[h^{t-1}\right]^c, z^c)$. For beliefs, we require $\delta\left(h^t\right) = 1 - \delta\left(\left[h^t\right]^c\right)$ and $\delta^r\left(h^t, z\right) = 1 - \delta^r\left(\left[h^t\right]^c, z^c\right)$. This form of symmetry is different from the standard symmetry requirement that all agents with the same characteristics behave in the same manner which we also assume. Throughout the paper, we will use symmetry to mean symmetry across states, whenever there is no confusion.

We can now define an equilibrium formally.

Definition 1 *A perfect Bayesian equilibrium which is symmetric across states is given by a strategy $\boldsymbol{\sigma}$, beliefs $\boldsymbol{\delta}$ on the signal s, and a probability space $(\Omega, \mathcal{F}, P)$, such that:*
(i) $\boldsymbol{\sigma}$ is optimal for an individual agent at each history h^{t-1}, given his beliefs $\boldsymbol{\delta}$ on the signal s and given that he believes that his opponent, at each round t, is randomly drawn from $P\left(\left[s, h^{t-1}\right]|s\right)$ and plays $\boldsymbol{\sigma}$;
(ii) the beliefs $\boldsymbol{\delta}$ are consistent with Bayes' rule whenever possible and satisfy (2) and (2);
(iii) the probability space is consistent with all agents playing $\boldsymbol{\sigma}$;
(iv) strategies and beliefs are symmetric across states.

Notice that, given that agents are atomistic, an agent, after observing a finite number of opponents playing off-the-equilibrium-path, still holds that all other agents are following $\boldsymbol{\sigma}$. Therefore, he assumes that their future behavior is still described by the functions $\psi_t\left(z|s\right)$ and $\chi_t\left(z|s\right)$ constructed above. His only problem when forming expectations in this case, is to assign a probability to the signal s. For this, he follows his (off-the-equilibrium-path) beliefs δ_t and δ_t^r.

Finally, we construct two stochastic processes, which describe the equilibrium dynamics of an individual agent's endowments and beliefs, conditional on the game not ending. Take the probability space $(\Omega, \mathcal{F}, P)$ and let $x_t(\omega)$ and $\delta_t(\omega)$ denote the endowment and belief of the agent at ω (the definition of $\delta_t\left(\omega\right)$ involves a slight abuse of notation). Since an agent's current endowment and beliefs are, by construction, in his information set at time

t, $x_t(\omega)$ and $\delta_t(\omega)$ are $\mathcal{F}_t$-measurable stochastic processes on $(\Omega, \mathcal{F}, P)$, which describe the evolution of endowment and beliefs, conditionally on the game not ending.

4 Characterization of the equilibrium outcomes

In this section, we provide a characterization of the equilibrium in the long run, i.e., along the path when the game has not ended. We first consider the behavior of informed agents and show that they equalize their marginal rates of substitution in the long run. Then we show that the uninformed agents have the same marginal rate of substitution as the informed agents. This result implies that the uninformed agents can either construct a trade that allows them to learn the state arbitrarily well or that there are no gains from trade from the beginning of the game. This result implies that the value of information is zero in the long run, similar to the full revelation in Grossman and Stiglitz (1980). We then show that if the allocation is not Pareto efficient initially, then there are informational rents, in contrast with the classical results of Grossman and Stiglitz (1980). We finish the section with characterization of the dynamics of trade. The difficulty with analyzing this problem that we overcome is that the cross sectional distribution of beliefs is changing along the equilibrium path and is endogenous to trades that agents make or can potentially make. The agent when deciding to trade needs to know not only his belief of the state of the world, but also the beliefs of other agents, as well as forecast how they will evolve. Our arguments are essentially by contradiction, as we show that if the above results fail to hold then agents can construct deviating trades which increase their expected utility.

4.1 Rational Expectations Equilibrium and Pareto Efficiency

We start first by considering rational expectations equilibrium of an economy with the same initial conditions and information structure as in the economy described in Section 2.1. This equilibrium will provide a useful benchmark for our long run results described below. We keep the discussion rational expectations equilibria brief as this exposition is quite standard (see, e.g. Grossman (1989)).

Let $\delta^I(s)$ denote the belief of an informed agents after signal s, so $\delta^I(s_1) = 1$ and $\delta^I(s_2) = 0$. The *rational expectations equilibrium* of the pure endowment economy with I types of agents with endowments $x_{i,0}$ satisfying (1) and beliefs $\delta^U = 0.5$ for a fraction $1-\alpha$ of each type, and $\delta = \delta^I(s)$ for a fraction α consists of prices $p(s) \in R^2_+$ and allocations

$\{x_i^*(s,\delta)\}_{i=1}^I$ s.t. for $s \in \{s_1, s_2\}, \delta \in \{\delta^U, \delta^I(s)\}$

$$x_i^*(s,\delta) = \arg \max_{p(s)x \le p(s)x_{i,0}} E\{u(x)|p(s),\delta\} \tag{4}$$

and

$$\sum_{i=1}^I f_i x_i^{j*}(s,\delta) = 1 \text{ for } j = 1, 2. \tag{5}$$

In the fully revealing equilibrium $p(s_1) \neq p(s_2)$, which imply that the expections in (4) are the same for all δ and hence informed and uninformed get the same equilibrium allocations $x_i^*(s,\delta) = x_i^*(s,\delta^I(s))$. For (5) to be satisfied it must be true that the prices are equal to the ratio of probabilities:

$$\frac{p^1(s)}{p^2(s)} = \frac{\phi(s)}{1-\phi(s)}.$$

It is also easy to verify that any fully revealing rational expectations equilibrium is Pareto Efficient, and that all Pareto Efficient allocations are fully revealing rational expecations equilibria for some economy.

It is a well known result that there can be no rational expectations equilibria which do not reveal the information fully. For such equilibrium to be possible, it must be true that $p(s_1) = p(s_2)$. Then (4) implies that $\pi(\delta)u'(x_i^{1*}(s,\delta))/(1-\pi(\delta))u'(x_i^{2*}(s,\delta)) = 1$, where $\pi(\delta)$ is the probability that an agent with belief δ assigns to consuming good 1. In state s_1, for example, $\pi(\delta^U) = 0.5$, while $\pi(\delta^I(s_1)) > 0.5$, which implies that (5) would necessarily be violated.

4.2 Preliminary considerations

In this section we define the per period and lifetime utility of agents in a symmetric equilibrium of our trading game. We use the martingale convergence theorem to show that both lifetime utility and beliefs converge in the long run, conditionally on the game not ending.

An agent who assigns probability δ to signal s_1 assigns probability

$$\pi(\delta) \equiv \delta\phi(s_1) + (1-\delta)\phi(s_2)$$

to state S_1. For an informed agent, δ is always equal to either 0 or 1 and $\pi(\delta)$ is equal to either $\phi(s_1)$ or $\phi(s_2)$. If the game ends, an agent with the endowment-belief pair (x, δ)

receives the expected payoff

$$U(x, \delta) \equiv \pi(\delta)u(x^1) + (1 - \pi(\delta))u(x^2).$$

Using the stochastic processes x_t and δ_t, we can then define a stochastic process u_t for the equilibrium expected utility of an agent if the trading game ends on round t,

$$u_t(\omega) \equiv U(x_t(\omega), \delta_t(\omega)).$$

We can then derive a stochastic process v_t for the expected lifetime payoff of an agent at the beginning of round t,

$$v_t(\omega) \equiv (1 - \gamma)E\left\{\sum_{s=t}^{\infty} \gamma^{s-t} u_s(\omega) \mid \mathcal{F}_t\right\}. \tag{6}$$

The next two lemmas establish that both the beliefs δ_t and the values v_t are bounded martingales and converge in the long run.

Lemma 1 *Let δ_t be the equilibrium sequence of beliefs. Then there exists a random variable δ^∞ such that*

$$\lim_{t \to \infty} \delta_t(\omega) = \delta^\infty(\omega) \ \textit{a.s.}$$

Proof. The beliefs δ_t are evaluated along the equilibrium path, so they are always consistent with Bayes' rule. The law of iterated expectations then implies that δ_t is a martingale,

$$\delta_t = E\left[\delta_{t+1} | \mathcal{F}_t\right].$$

Since δ_t is bounded in $[0, 1]$, the result follows from the Martingale Convergence Theorem. ■

Lemma 2 *There exists a random variable $v^\infty(\omega)$ such that*

$$\lim_{t \to \infty} v_t(\omega) = v^\infty(\omega) \ \textit{a.s.}$$

Proof. An agent always has the option to reject any offers and offer zero trades from period t onwards, wait the end of the game, and consume x_t. This implies that

$$u_t \leq E\left[v_{t+1} \mid \mathcal{F}_t\right].$$

Equations (6) implies that

$$v_t = (1-\gamma)u_t + \gamma E\left[v_{t+1} \mid \mathcal{F}_t\right]. \tag{7}$$

Combining these results gives

$$v_t \leq E\left[v_{t+1} \mid \mathcal{F}_t\right],$$

which shows that v_t is a submartingale. It is bounded above because the utility function $u(\cdot)$ is bounded above , therefore it converges by the Martingale Convergence Theorem. ■

Note that these results hold for both informed and uninformed agents.

4.3 Long run characterization: informed agents

We now proceed to characterize the long run properties of the equilibrium. We first focus on informed agents and show that their marginal rates of subsitution converge in probability to the same constant. The intuition for this result is that if it was not true, informed agents in the long run could construct an offer that would be accepted with positive probability by some other informed agents with a different marginal rate of substitution, increasing their utilities above the equilibrium payoffs. This is a modified version of the argument used by Gale (2000) for a decentralized market with full information.

Proposition 1 ***(Convergence of MRS for informed agents)*** *There exist two positive scalars $\kappa(s_1)$ and $\kappa(s_2)$ such that, conditional on each s, the marginal rate of substitution of informed agents converges in probability to $\kappa(s)$:*

$$\lim_{t\to\infty} P\left(\left|\frac{\phi(s)u'(x_t^1(\omega))}{(1-\phi(s))\,u'(x_t^2(\omega))} - \kappa(s)\right| > \varepsilon \mid \delta_t(\omega) = \delta^I(s), s\right) = 0 \text{ for all } \varepsilon > 0. \tag{8}$$

Proof. We provide a sketch of the proof here and leave the complete proof to the appendix. Without loss of generality, suppose signal s_1 is realized. By Lemma 2 when t is sufficiently large $U(x_t(\omega), 1)$ can be made sufficiently close to $v_t(\omega)$ for almost all realizations of ω. Let $\Omega^{I,1}$ be the subset of histories of informed agents in state s_1, i.e., those ω such that $\delta_t(\omega) = 1$.

If (8) is violated it is possible to construct two sets $A_t, B_t \subset \Omega^{I,1}$, both of positive measure, such that if $\omega \in A_t$ and $\tilde{\omega} \in B_t$ the difference between the marginal rates of substitution of two informed agents with $\omega \in A_t$ and $\tilde{\omega} \in B_t$ is at least $\zeta > 0$, i.e.,

$$\frac{\phi u'(x_t^1(\omega))}{(1-\phi)u'(x_t^1(\omega))} < \frac{\phi u'(x_t^1(\tilde{\omega}))}{(1-\phi)u'(x_t^1(\tilde{\omega}))} - \zeta.$$

But then agent with a history $\omega \in A_t$ can offer a small trade $z = (-\varepsilon, p\varepsilon)$ at a price in between the two marginal rates of substitution, such as

$$p = \frac{\phi u'(x_t^1(\omega))}{(1-\phi)u'(x_t^1(\omega))} + \frac{\zeta}{2},$$

and stop trading after this offer is either accepted or rejected. The utility of the agent ω is higher if his offer is accepted since

$$\begin{aligned} U(x_t^1(\omega) - z, \phi) &= \phi u(x_t^1(\omega) + \varepsilon) + (1-\phi)u(x_t^2(\omega) - p\varepsilon) \\ &\approx U(x_t^1(\omega), \phi) + \left[\phi u'(x_t^1(\omega)) - (1-\phi)pu'(x_t^2(\omega))\right]\varepsilon \\ &\approx v_t(\omega) + (1-\phi)pu'(x_t^2(\omega))\frac{\zeta}{2}\varepsilon. \end{aligned}$$

By choosing t sufficiently large and ε sufficiently small, we can make the approximation errors in the above equation sufficiently small, so that this trade strictly improves the utility of the first type of agents, $U(x_t^1(\omega) - z, \phi) > v_t(\omega)$. The agent with $\tilde{\omega} \in B_t$ is also better off by a similar argument. Therefore, all informed agents in B would accept the offer. Since there is a positive probability for agent ω to meet an agent $\tilde{\omega} \in B$, the strategy just described gives strictly higher utility than the equilibrium strategy, and we have a contradiction.

This argument shows that there is a sequence of $\kappa_t(s_1)$, possibly varying over time, to which marginal rate of substitution of the informed agents converge. In the appendix, we show how to extend the argument above to show that $\kappa_t(s_1)$ is constant over time, completing our proof. ■

Two useful remarks on the argument above: First, there may be a mass of uninformed agents who also potentially accept the offer of z, but this only increases the probability of acceptance, which further improves the utility of the agent ω. Second, the deviation described (offering z at time t and stop trading afterwards) is not necessarily the best deviation for agent ω. Potentially, there may be better sequence of offers that the agent ω can make to improve his utility. However, since our argument is by contradiction, it is enough to focus on simple deviations of this form. We will follow a similar approach in many of the following proofs.

The following corollary shows that the allocations of informed agents converge.

Corollary 1 *For all informed agents, the process $\{x_t\}$ almost surely converges to a constant.*

Proof. Suppose this is not the case. From Lemma 1 there must exist two subsequences of $x_t(\omega)$, one converging to to x' and the other to x'', both leading to the same marginal rates of substitution:

$$\frac{\phi u'(x^{1\prime})}{(1-\phi)u'(x^{2\prime})} = \frac{\phi u'(x^{1\prime\prime})}{(1-\phi)u'(x^{2\prime\prime})}.$$

This is possible only if $x' > x''$ or $x' < x''$ which violates Lemma 2. ■

4.4 Long run characterization: uninformed agents

We now turn to the characterization of equilibria for uninformed agents. The main difficulty here is that uninformed agents, upon receiving offers or having their offers accepted or rejected, might change their beliefs. Thus agents, who might be willing to accept some offer ex ante, before updating their beliefs, might reject it after an update. An additional complexity comes from the fact that the updated beliefs are not necessarily pinned down by Bayes' rule after an arbitrary offer, since this offer may not occur in equilibrium. For these reasons, we need a strategy of proof different from the one used for informed agents.

Our arguments are based on finding strategies that allow uninformed to learn the signal s at an arbitrarily small cost in the long run. The existence of such strategies implies that either agents eventually learn the signal or the benefits of such learning goes to zero. In the next section, we show that in both cases equilibrium allocations converge to ex post Pareto efficient allocations in the long run.

The specific learning strategies of the uninformed depend on wether the marginal rates of substitution of informed agents converge to the same value in both states of the world, $\kappa(s_1) = \kappa(s_2)$ or to different values. For this reason we split this section in two parts, one analyzing the case $\kappa(s_1) \neq \kappa(s_2)$ and the other the case $\kappa(s_1) = \kappa(s_2)$.

4.4.1 The case $\kappa(s_1) \neq \kappa(s_2)$

We begin by considering the case in which the marginal rates of substitutions for informed agents converge to different values in states s_1 and s_2.

Proposition 2 ***(Convergence of MRS for uninformed agents)*** *Suppose $\kappa(s_1) \neq \kappa(s_2)$. Conditional on each s, the marginal rate of substitution of all agents, evaluated at the full information probabilities $\phi(s)$ and $1-\phi(s)$, converges in probability to $\kappa(s)$:*

$$\lim_{t\to\infty} P\left(\left|\frac{\phi(s)u'(x_t^1(\omega))}{(1-\phi(s))\,u'(x_t^2(\omega))} - \kappa(s)\right| > \varepsilon \mid s\right) = 0 \text{ for all } \varepsilon > 0. \tag{9}$$

Proof. We provide a sketch of the proof here and leave the complete proof to the appendix. In this sketch we develop arguments that apply when agents are sufficiently close to their long run values, in the appendix we make precise how, and in what order, the appropriate limits are taken.

Since symmetry implies $\kappa(s_1) = 1/\kappa(s_2)$, assume without loss of generality that $\kappa(s_1) > 1 > \kappa(s_2)$. Consider an offer $z = (\varepsilon, -\varepsilon)$.

Observation 1. For ε small enough offer z is accepted by some informed agents in the long run if $s = s_1$. By Proposition 1, in the long run there is a positive mass of informed agents that, by accepting z, would increase their expected utility above its equilibrium level:

$$\begin{aligned} U(x_t(\omega) + z, \phi(s_1)) &\approx U(x_t(\omega), \phi(s_1)) + (\kappa(s_1) - 1)(1-\phi(s_1))u'(x_t^2(\omega)^2)\varepsilon \\ &> U(x_t(\omega), \phi(s_1)) = v_t(\omega). \end{aligned}$$

Observation 2. Offer z cannot be accepted by any agent, informed or uninformed, in state s_2, except possibly by a vanishing mass of agents. Suppose to the contrary that a positive fraction of uninformed accepted z in state s_2. By an argument symmetric to the one above, informed agents in state s_2 are strictly better off making offer z, if this offer is accepted with positive probability, given that it would bring them to $x_t(\omega) - z$. But then an optimal deviation on their part is to make such an offer and strictly increase their expected utility above its equilibrium level, leading to a contradiction.

Using these two observations, we are ready to prove (9). Suppose without loss of generality that (9) is not satified for a positive mass of uninformed agents in state $s = s_1$. For concreteness, suppose there is a non-vanishing mass of uninformed agents with endowments such that $\phi(s_1)u'(x^1(\omega))/\left((1-\phi(s_1))\,u'(x^2(\omega))\right) < \kappa(s_1)$. Let us construct a profitable deviation from the equilibrium strategy, for these uninformed agents.

Suppose at period T the uninformed agent is sufficiently close to his long run expected utility. If in the match in period T he is chosen as the receiver, he rejects all trades and stops trading in all subsequent periods (e.g., he can make the offer $(0,0)$ when selected as the proposer and reject all offers when selected as the receiver). If in period T the agent is the proposer, he offers $\hat{z} = (\varepsilon, -\varepsilon)$. If the offer is rejected, he stops trading in all subsequent periods. Similarly, he stops trading if his offer is accepted in T but he is not chosen as the proposer at $T+1$. Finally, if the offer is accepted and in period $T+1$ he is chosen as the proposer, he offers $z^* = (\theta, -\zeta\theta)$, where

$$\zeta = \frac{1}{2}\left(\kappa(s) + \frac{\phi(s_1)u'(x_t^1(\omega))}{(1-\phi(s_1))\,u'(x_t^2(\omega))}\right)$$

and θ is positive and sufficiently small. In choosing θ and ε, we set ε to be small relative to θ, i.e. $\varepsilon = o(\theta)$. After period $T+1$ the agent stops trading.

With this strategy, if $s = s_1$, the uninformed agent after two rounds has an allocation $x_t(\omega) - \hat{z} - z^*$ at least with probability $(1/4)\,\alpha^2\gamma$, since with this probability he is chosen as the proposer and he is matched with an informed agent both in periods T and $T+1$. In all other contigencies, his allocation is either $x_t(\omega)$ or $x_t(\omega) - \hat{z}$, which for any δ implies $U(x_t(\omega) - \hat{z}, \delta) = U(x_t(\omega), \delta) + o(\theta)$ since $\varepsilon = o(\theta)$. This implies that the ex-ante payoff of the uninformed agent from following this strategy is

$$\begin{aligned} & U(x_t(\omega), \delta_t(\omega)) + \frac{1}{4}\alpha^2\gamma\left(\kappa(s_1) - \frac{\phi(s_1)u'(x_t^1(\omega))}{(1-\phi(s_1))\,u'(x_t^2(\omega))}\right)(1-\phi(s_1))\,u'(x_t^2(\omega))\theta + o(\theta) \\ > \; & U(x_t(\omega), \delta_t(\omega)) \approx v_t(\omega). \end{aligned}$$

This shows that this is a profitable deviation for the uninformed, leading to a contradiction.
■

The crucial piece of the proof is observation 2. Once we have established that for any small ε offer $(\varepsilon, -\varepsilon)$ is accepted by some informed agents in state s_1 and by essentially no agents in state s_2, it is easy to show how uninformed agents can improve their utility. By making offer z they can then learn that the signal is s_1 with arbitrary precision (i.e., an acceptance can be made an arbitrarily strong signal in favor of s_1). After learning s_1, they can improve their utility by trading with informed agents and equilizing their marginal rate of substitution, as informed agents did in the proof of Proposition 1. Since ε can be chosen arbitrarily small, the utility loss for all other realizations can be made small relative to the expected gain from learning and trading with the informed, leading to the desired contradition.

4.4.2 The case $\kappa(s_1) = \kappa(s_2)$

When $\kappa(s_1) = \kappa(s_2)$, learning the signal is more difficult for the uninformed than in the previous case. Informed agents in both states have the same marginal rates of substitutions and their strategies for small offers might be the same. For this reason, we pursue a different strategy of proof. Consider, any uninformed agent whose marginal rate of subsitution $m_t \equiv \pi(\delta_t)u'(x_t^1)/\left((1-\pi(\delta_t))\,u'(x_t^2)\right)$ fails to converge to $\kappa(s)$. For example suppose there is a subsequence of m_t that converges to a value $m < \kappa(s)$. Suppose once uninformed agent's marginal rate of substitution is close to m, such an agent deviates from his equilibrium strategy and makes the offer $z = (\varepsilon, -\varepsilon\zeta)$, where $\zeta = (\kappa(s)+m)/2$ when selected as the proposer. If the probabilities of acceptance of offer z are sufficiently different in the two states, the agent learns s. Once he knows s, the agent can further improve his utility by following the strategy described in the proof of Proposition 2. On the other hand, if these probabilities are sufficiently similar, his updated beliefs should be close to his beliefs δ_t, but then the trade z is constructed in such a way that it increases his expected utility when evaluated at the beliefs δ_t. Both of these cases lead to a contradiction that the agent's long run payoff converged. We formally state the proposition below and leave the proof to the appendix.

Proposition 3 *Suppose $\kappa(s_1) = \kappa(s_2) = 1$ and suppose that for some $s \in \{s_1, s_2\}$, $\varepsilon > 0$ and $\zeta > 0$, there is an infinite sequence of periods $\{t_k\}_{k=1}^{\infty}$ such that*

$$P\left(\left|\frac{\pi(\delta_{t_k}(\omega))u'(x_{t_k}^1(\omega))}{(1-\pi(\delta_{t_k}(\omega)))\,u'(x_{t_k}^2(\omega))} - \kappa(s)\right| > \zeta \mid s\right) > \varepsilon \tag{10}$$

for $k = 1, 2, \ldots$. Then

$$\lim_{k\to\infty}(t_{k+1} - t_k) = \infty. \tag{11}$$

There are two differences between Propositions 2 and 3. First, here we show convergence of agent's marginal rates of substitution evaluated at his beliefs, $\pi(\delta_t)$ rather than at the full information probabilities $\phi(s)$. Second, here we show convergence in a weaker sense than convergence in probability in the previous section. Namely, we show that marginal rates of substitution converge for all periods except for a sequence of periods $\{t_k\}_{k=1}^{\infty}$, with $t_{k+1} - t_k \to \infty$. That is, there may be periods where marginal rates of substitution differ across agents, but these periods become increasingly rare as time progresses. It turns out that this result is strong enough to show that there are no equilibria with $\kappa(s_1) = \kappa(s_2)$.

Proposition 4 *There is no symmetric equilibrium with $\kappa(s_1) = \kappa(s_2)$.*

Proof. Once again, we sketch the arguments, leaving the technical details to the appendix. Symmetry implies that $\kappa(s_1) = \kappa(s_2) = 1$ and Proposition 3 implies that for t large the marginal rates of substitution of almost all agents are close to 1. Feasibility implies that

$$\int x_t^1 d\Gamma(\omega|s) = \int x_t^2 d\Gamma(\omega|s) = 1 \tag{12}$$

Suppose $s = s_1$ and consider any agent with $\delta_t(\hat{\omega}) \in (0, 1/2]$. Interiority of beliefs implies that there is positive probability of observing the same history in both states. Symmetry of equilibrium and Bayes rule implies that in equilibrium for any history $\hat{\omega}$ s.t. $\delta_t(\hat{\omega}) \in (0,1)$ there is another history $\tilde{\omega}$ s.t. $\delta_t(\tilde{\omega}) = 1 - \delta_t(\hat{\omega})$ and that there are two agents whose asset positions are symmetric to each others in all dates: $\left\{\left(x_i^1(\hat{\omega}), x_i^2(\hat{\omega})\right)\right\}_{i=1}^{t} = \left\{\left(x_i^2(\tilde{\omega}), x_i^1(\tilde{\omega})\right)\right\}_{i=1}^{t}$. Moreover if $\delta_t(\hat{\omega}) \leq 1/2$, than then measure of agents with history $\tilde{\omega}$ is strictly greater than the measure of agents with history $\hat{\omega}$.[7] Then for any measure of agents with history $\hat{\omega}$ we can find the same measure of agents with history $\tilde{\omega}$. We can form a new distribution $\tilde{\Gamma}(\omega|s_1)$ by setting $\tilde{\Gamma}(\omega|s_1) = \Gamma(\omega|s_1) - \Gamma(\hat{\omega}|s_1)$ if $\omega \in \{\hat{\omega}, \tilde{\omega}\}$ and $\tilde{\Gamma}(\omega|s_1) = \Gamma(\omega|s_1)$ otherwise. By construction $\int x_t^1 d\tilde{\Gamma}(\omega|s_1) = \int x_t^2 d\tilde{\Gamma}(\omega|s_1)$. Repeating this procedure for all ω with $\delta_t(\omega) \in (0, 1/2]$, we construct a distribution $\Gamma^*(\omega|s_1)$ s.t.

$$\int x_t^1 d\Gamma^*(\omega|s_1) = \int x_t^2 d\Gamma^*(\omega|s_1) \tag{13}$$

and $\Gamma^*(\omega|s_1) = 0$ for all ω s.t. $\delta_t(\omega) = 0$. Since for almost all ω s.t. $\Gamma^*(\omega|s_1) > 0$, $\pi(\delta_t)u'(x_t^1)/(1-\pi(\delta_t))u'(x_t^2)$ converges to 1, and $\pi(\delta_t)/(1-\pi(\delta_t)) > 1$, it must be true that $x_t^1(\omega) > x_t^2(\omega)$, which contradicts (13). ■

[7] See the proof of Lemma 10 in the appendix for the formal proof of these properties.

4.5 Efficiency and informational rents

The characterization of the behavior of informed and uninformed agents in the previous section allows us to derive the key results about the long run efficiency and the value of information.

Theorem 1 ***(Efficiency in the long run)*** *Equilibrium allocations converge to efficient allocations in the long run, i.e.,*

$$\lim_{t\to\infty} P\left(\left|x_t^1 - x_t^2\right| > \varepsilon\right) = 0 \text{ for all } \varepsilon > 0. \tag{14}$$

For any $s \in \{s_1, s_2\}$ the long run marginal rate of substitution $\kappa(s)$ is equal to the ratio of the conditional probabilities of states S_1 and S_2,

$$\kappa(s) = \phi(s)/(1 - \phi(s)).$$

Proof. Proposition 4 rules out the case $\kappa(s_1) = \kappa(s_2)$. First suppose that $\kappa(s) > \phi(s)/(1-\phi(s))$ for some s. Then Propositions 1 and 2 imply that $\lim_{t\to\infty}\left(x_t^1 - x_t^2\right) < 0$ a.s. This, however, violates (12). We rule out the case $\kappa(s) < \phi(s)/(1-\phi(s))$ analogously. Since $\kappa(s) = \phi(s)/(1-\phi(s))$, Propositions 1 and 2 imply (14). ■

This theorem establishes that in the long run equilibrium allocations coincide with a rational expectation equilibrium defined in Section 4.1 for *some* initial allocations. It does not show whether starting with the *same* initial allocations centralize rational expectation equilibrium and the decentrilized matching environment we consider converges to the *same* long run outcomes. To futher explore whether informed agents can achieve higher payoff than uninformed agents, we define a concept *value of information.* Consider any agent in period t with a history h^t. Let $\delta_t(h^t)$ be beliefs of such agent in period t, and $v(x_t|h_t)$ his expected payoff. Let $v(x_t, t|s)$ be the payoff of an agent in period t if he had endowment x_t and assigned probability 1 to the signal s. Thus, $v(x_t, t|s)$ is a payoff of a hypothetical informed agent in period t with endowment x_t.[8] If an agent with a history h^t could costlessly learn signal s, his expected utility would increase by $\mathcal{I}(h^t) = \delta\left(h^t\right) v(x_t, t|s_1) + (1-\delta\left(h^t\right))v(x_t, t|s_2) - v(x_t|h_t)$. We call $\mathcal{I}_t$ the *value of infor-*

[8] In equilibrium there might not exist informed agents with endowment x_t in period t. However, we can formally extend our game by addition of a measure 0 of such agents in period t. Since they are of measure zero, such an extension does not change the equilibrium of the game, but the payoffs $v(x_t, t|s)$ become well defined.

mation. Since upon costlessly learning s an agent can continue to persue his equilirbium strategy, the value of information is always nonnegative, $\mathcal{I}_t \geq 0$.

The first result that follows shows that a famous no trade theorem due to Milgrom and Stokey (1982) holds in our settings, and if the initial allocations are Pareto Efficient, the value of information is zero.

Theorem 2 *Suppose $x^1_{0,i} = x^2_{0,i}$ for all i. Then there is no trade in equilibrium and $\mathcal{I}_t = 0$ for all t.*

Proof. It is a straightforward adaptation of the proof of Theorem 1 in Milgrom and Stokey (1982) once one notices that allocations are Pareto Efficient if and only if $x^1_i = x^2_i$ for all i. ■

One of the implications of this result is that informed and uninformed agents with the same initial endowments receive the same payoff if the initial allocation is efficient. Combining the insight of Theorem 2 with the result from Theorem 1, it is easy to obtain the following corrollary

Corollary 2 *The value of information is zero in the long run, i.e. for all s*

$$\lim_{t\to\infty} P(\mathcal{I}_t > \varepsilon | s) = 0 \textit{ for any } \varepsilon > 0.$$

The question remains whether informed agents can get a higher utility than uninformed if the initial allocation is not Pareto Efficient. The next theorem shows that this is indeed the case, and the the following two sections we explore which strategies informed agent can persue to increase their utility.

Theorem 3 *Suppose $x^1_{0,i} \neq x^2_{0,i}$ for some i. Then the value of information is positive in period 0, i.e. there exists some $\varepsilon > 0$ s.t.*

$$P(\mathcal{I}_0 \geq \varepsilon | s) > 0$$

for all s.

Proof. In the appendix. ■

5 A static example

The previous sections provided characterization of the equilibrium allocations in the long run. This and the following sections provide further insights on the behavior of agents along the transition. We start with a simple static example, which both will illustrate the ways in which the informed agents can achieve higher utility than the uninformed, and provide intuition for the results on the next section, in which we solve a fully dynamic version of the game numerically.

Suppose there are two types of agents with endowments (e_h, e_l) and (e_l, e_h) with $e_h > e_l > 0$ and $e_l + e_h = 1$. There is only one round of random matching, after which the game ends, the state of the world is realized, and each agent consumes his endowment. We further simplify the environment by assuming that the fraction of informed agents is negligible. This implies that their presence does not affect strategies of uninformed agents.

There are many equilibria of the static game. We describe one of them, which is robust to many refinements, including Cho-Krepts (1988) intuitive criteria for out of equilibrium beliefs. Figures 1 and 2 provide the intuition for that equilibrium construction using Edgeworth box.

Consider first an uninformed maker of the offer with endowment (e_h, e_l). We can focus only on the case if he is matched with an agent with endowment (e_l, e_h), because if he is matched with an agent with endowment (e_h, e_l) it is impossible to find a trade that improve utilities of both of them, and hence no trade occurs in equilibrium in such matches.

The uninformed proposer makes an offer $\left(z^{1*}, z^{2*}\right)$ that solves

$$\max_{z^1, z^2} 0.5u(e_h - z^1) + 0.5u(e_l - z^2)$$

s.t.

$$0.5u(e_l + z^1) + 0.5u(e_h + z^2) \geq 0.5u(e_l) + 0.5u(e_h) \tag{15}$$

The solution to this problem involves $e_h - z^{1*} = e_l - z^{2*}$ and it is represented by the red dot on the Figures 1 and 2.

There are two possibilities for informed agents. They may have received a signal that their initial endowment is higher for the good which consumption is more likely. We call such agents rich informed. The other possibility is that they have more of a good for the state which is less likely to occur and we call such agents poor informed.

Suppose informed proposer with endowment (e_h, e_l) observed signal s_1, which makes him rich informed. Such agent makes an offer $\left(z^{1**}, z^{2**}\right)$ that solves

$$\max_{z^1, z^2} \phi u(e_h - z^1) + (1-\phi)u(e_l - z^2)$$

s.t.

$$\phi u(e_l + z^1) + (1-\phi)u(e_h + z^2) \geq \phi u(e_l) + (1-\phi)u(e_h) \tag{16}$$

$$0.5u(e_h - z^1) + 0.5u(e_l - z^2) \leq 0.5u(e_h - z^{1*}) + 0.5u(e_l - z^{2*}) \tag{17}$$

A poor informed proposer mimics the uninformed proposer by making an offer $\left(z^{1*}, z^{2*}\right)$.

To complete construction of equilibrium we need to describe strategies of receivers. The uninformed agents generally accepts all the offers that satisfy (15) and reject all the other offers, with two exceptions. If they see any offer in the yellow region depicted on Figure 1 (formally, any offer that satisfies (17)) the cutoff for their accepted offers are determined by the indifference curve of the poor informed receiver of the offer (formally, by (16)). Similarly, for any offer which lies in the yellow region on Figure 1, they accept only offers to the left of the indifference curve of a rich informed agent, shown in the red dashed line.

To understand why this is the equilibrium, consider Figures 1 and 2. First, consider an uniformed agent who makes an offer. This offer must maximize his utility and also ensure that an agent who receives the offer accepts such trade, i.e., it should give a weakly higher utility than the initial endowment. Since we assumed in this static example that the fraction of informed agents is negligible, the agent who receives the offer does not update his beliefs. Both of the uninformed agents value each of the assets equally, the slopes of their indifference curves on the 45 degree line are equal to 1/2 (the red dotted line is the indifference curve for the uninformed agent who receives an offer and the blue dotted line is an indifference curve for the agent who makes an offer). Hence, the red dot on Figure 1 represents such an equilibrium point.

Next, consider a strategy of the rich informed proposer. His indifference curve is steeper (blue dashed line) that those of uninformed as he knows that good 1 is more valuable. The shaded area on the figure shows all offers that give the rich informed agent who makes it higher utility than the utility which the uninformed agent who makes it (and a higher utility than that of the poor informed agent, as we show below). For any offer that is being made in that region, the uninformed receiver must infer that the the agent who makes the offer is a rich informed agent, and therefore that the informed agent observed signal s_1. Before making a decision whether to accept or reject such an offer, the agent who receives

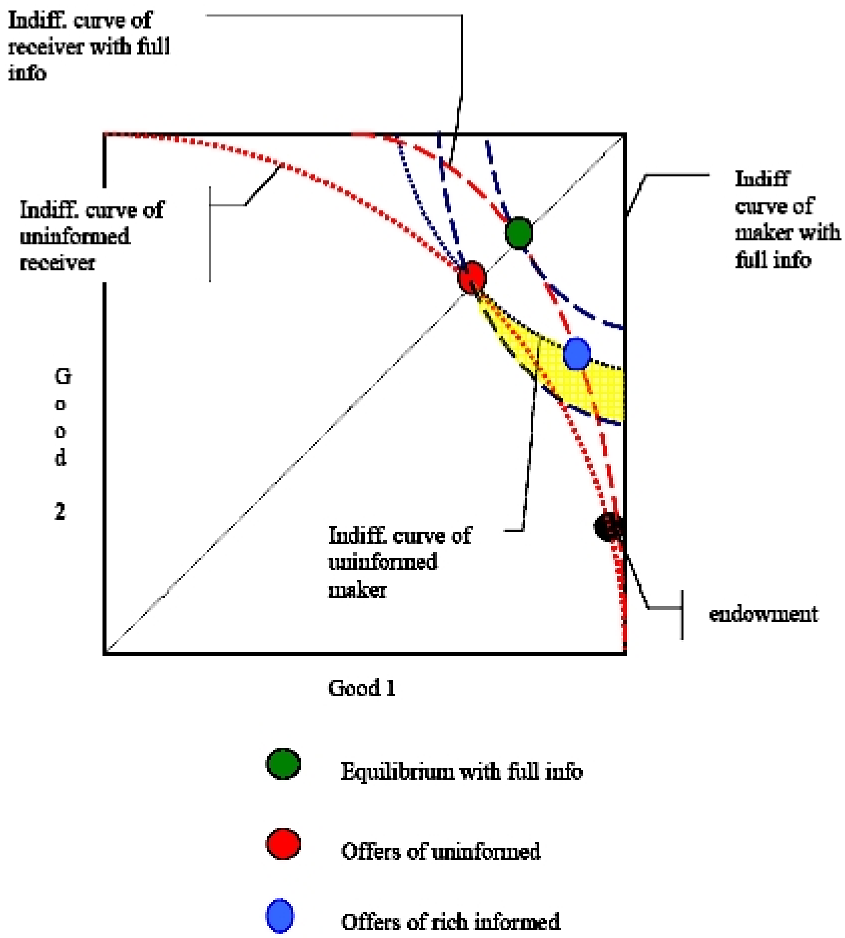

Figure 1: Rich informed agent

the offer updates his beliefs. His indifference curve then becomes steeper and is represented on the Figure 1 by a red dashed line. Therefore, the rich informed agent makes an offer that maximizes his utility in the shaded region (i.e., in the region that changes the beliefs of the counterpart) and leaves the receiver weakly better off than rejecting the offer. This is given by the blue dot on the graph. Note, that the price that an agent who receives the offer pays for good 1 (defined as a quantity of good 2 paid for a unit of good 1) is higher than: (a) the price at which uninformed agents offer good 1, and (b) than the price that informed agents would offer in an equilibrium where all agents have full information (green dot on the graph).

Why is the green dot not an equilibrium? The reason is that at that point both the agents who are informed and uninformed can make the the offer – hence, the agent who recieves such offer does not change his beliefs and does not shift the indifference curve, remains on the dotted red indifference curve, and rejects the offer.

This example shows one of the possibilities for the informed agents to receive higher payoff than uninformed agents. By offering a small amount of good 1 at a high price, the informed agent can credibly signal that they have information. The reason is that only if an agent is informed that state s_1 is more likely he is willing to retain so much of good 1. Such signalling leads to the ex-post inefficiency. One can see that inefficiency by observing that the equilibrium point (the blue dot) does not lie on the 45 degree line. In contrast, the full information equilibrium (the green dot) is efficient and lies on the 45 degree line.

Figure 2 considers a case when an informed agent receives information that state 2 is more likely. If he makes any offer that signals his type (an offer in the shaded area), the uninformed agent updates his beliefs, changes the indifference curve to the one represented by the dashed line, and rejects the offer. In other words, such an offer reveals to the uninformed agent that his reservation value is higher than an ex-ante reservation value, evaluated with equal probabilities for two states of the world (i.e., the one on the dotted red indifference curve). Therefore, such offer would not be accepted by the receiver with the updated beliefs. For this reason the poor informed agent would rather replicate the strategies of the uninformed agent than reveal his type, i.e., make offers at the red dot.

This example illustrate several important features of agents' strategies which we show will remain true in a computed equilibrium of the dynamic game. First, since rich informed agents need to signal their information, it usually takes longer to reach efficient outcomes than with full information. Second, rich informed types generally prefer to sell little of

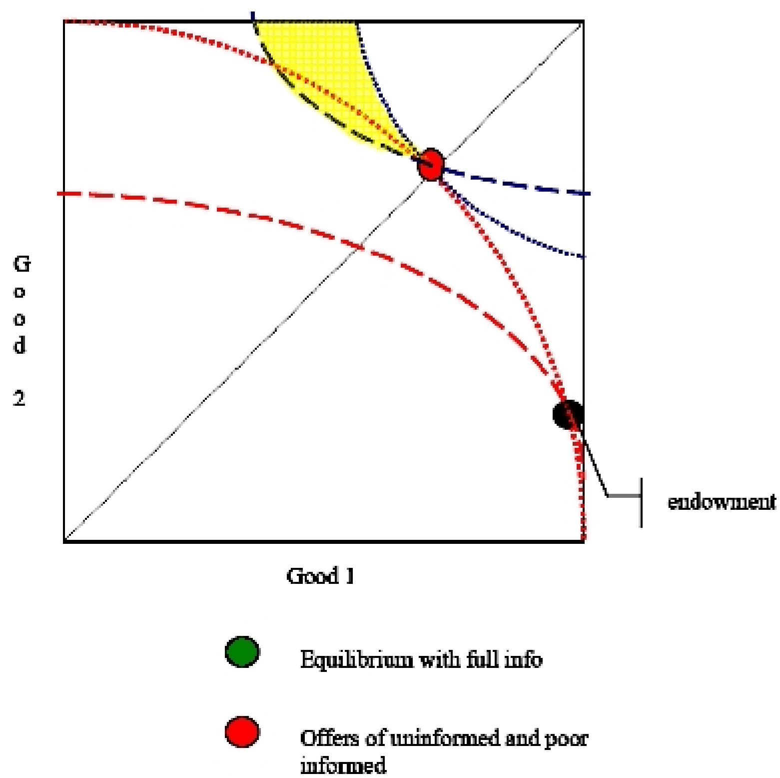

Figure 2: Poor informed agent

their endowment early in the game, and they decrease their position slowly. Finally, poor informed sell their endowment relatively fast, taking advantage of the fact that uninformed agents do not know which asset is more valuable.

6 Numerical illustration

In this section we illustrate quantitatively the theoretical results of the paper. The analysis of this section also allows to show some interesting properties of equilibria to complement our theoretical derivations. We also contrast our results with the case when all the information is public.

Let agents per period utility be $u(x) = -\exp(-x)$. In appendix, section 9.3, we show that with such utility function agents' strategies depend on the difference in their endowments $x^1 - x^2$, rather than separately on x^1 and x^2. Then behavior of an agent in period t depends only on 3 variables, $\left(x^1 - x^2, \delta, t\right)$.

Suppose there are two types of agents: half of agents starts with the endowment $(2, 0)$ and the other half starts with the initial endowment $(0, 2)$. Consider the state of the world s_1, i.e., the first good is more valuable.

Figure 3 shows how the average positions $x^1 - x^2$ of informed agents evolve over time, both when information if private, and if the signals were publicly observed.

The dotted lines in the picture show the case if the signal is public. The dashed red line describes the average position $\left(x^1 - x^2\right)$ of the poor agents. We see that as such agents meet other agents (rich agents in that case), the poor agent trades to acquire the second asset and finally converges to the efficient allocation in which $x^1 = x^2$. A similar strategy is for the rich agent who sells some of his first asset, acquires the second asset and converges to the efficient allocation. The question may arise why the convergence does not occur in one period. The reason is that some agents may meet with the agents of the same type, for example if in the first period, two "rich" agents meet, there will be no trade.

A different picture arises in the setting with private information. Consider first the poor informed agents, i.e., those who started with the endowment $(0, 2)$. Their trades are depicted by the solid red line. As was discussed in the theoretical part, such agents know that in the long run the uninformed agents will learn the true state of the world and the terms of trade will turn against the poor agents. Therefore, the poor informed agents buy the first asset as fast as possible, and the red line rises steeply. Moreover, they buy more of

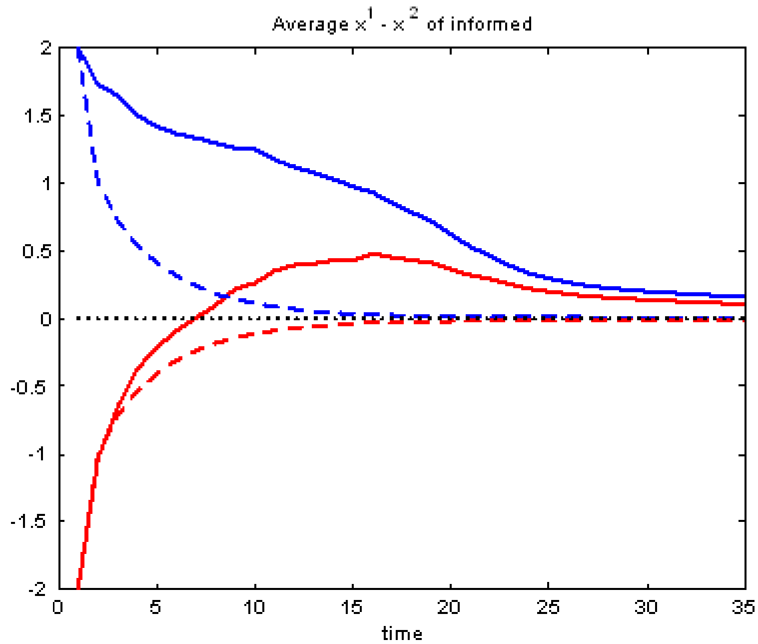

Figure 3: Trades of informed agents under public and private information

the first (more valuable) assets than they will eventually end up with and "overshoot" to have $\left(x^1 - x^2\right) > 0$, and then eventually decrease their holdings to the efficient allocation. The incentives of the rich agents are different (solid blue line). They want to hold on to their endowment of the valuable good until the information will be revealed. These strategies also reveal how informed agents receive a positive lifetime utility from having private information. Consistent with Theorem 1, $x^1 - x^2 \to 0$ for all agents and all lines asymptote to zero.

We also see that it takes longer to achieve efficiency in the economy with private information. For example, in period $t = 15$, in the case of public information the efficiency is essentially achieved, while for our private information economy, the agents are still far from the efficient allocations.

The next graph, Figure 4 illustrates how efficiency is achieved more slowly under the case of private information (solid line) than in the case of public information (dotted line). Here we plot the standard deviation of the difference in endowments in the economy.

Finally, we provide the graph of the volume of trades, which we define as the average size of asset trades, in our environment with private information, Figure 5. One could have

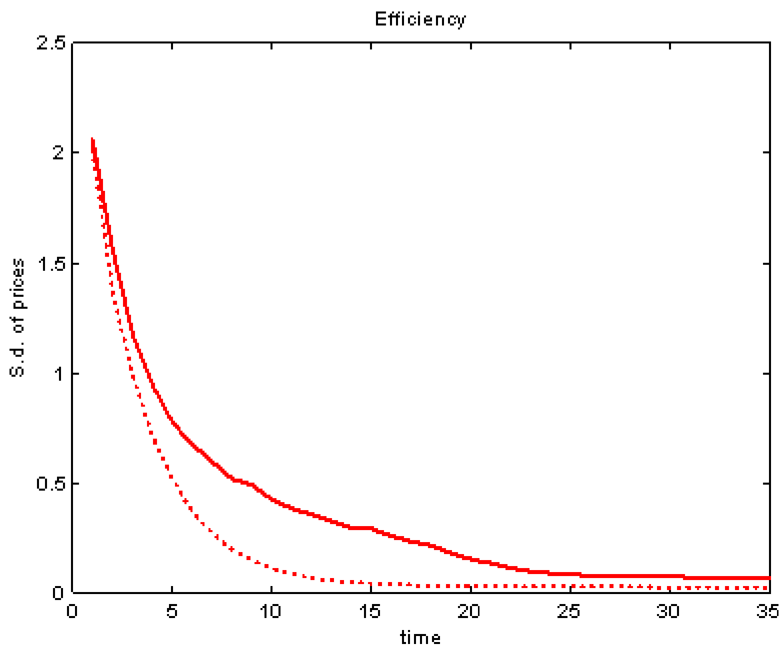

Figure 4: Efficiency

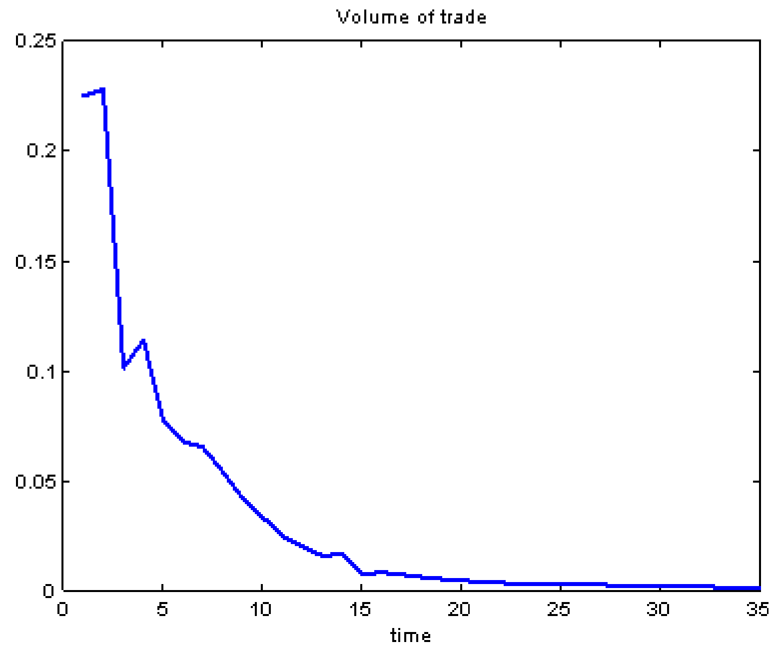

Figure 5: Volume of trades

conjectured that the volume of trade will be higher in this environment as uninformed agents strategically experiment to learn the true state of the world. This turns out not to be the case, and the volume of trade in the environment with private information (in the graph) is virtually identical to the volume in the environment with public information. The intuition for this is as follows. As can be seen in Figure 3, the informed rich agents wait for some time to trade, and if they trade in the short run they trade small amounts. They act as counterparties for the uninformed agents who strategically experiment with relatively small trades.

7 Conclusion

We provide a theory of asset pricing in an environments which are characterized by two frictions: (1) private information; (2) decentralized trade. These frictions often go hand in hand – it is reasonable to think that the decentralized markets (such as, for example, over the counter markets) also have large amounts of private information. While the analysis of asset pricing under asymmetric information in centralized markets is well developed (see, e.g, a comprehensive examination in Brunnermeier, 2001), ours is one of the first papers to develop such theory in the decentralized environments.

Our results on convergence to the efficient allocation, learning by uninformed agents, and dynamics of trade are very general and do not rely on specific functional forms. The reason for this generality is that we employ a novel argument by constructing a trade in which the uninformed agents can experiment and learn the true state. We also provide a numerical simulation that illustrates and extends the theoretical part of the paper.

8 Appendix

8.1 Technical Preliminaries

Here, we prove a number of preliminary results that will be useful throughout the appendix. Lemma 3 shows that the per period and lifetime utilities converge. Lemma 4 shows that endowments converge to a compact set in the interior of R_+^2 with probability arbitrarily close to one. This set is needed mainly for technical reasons, to ensure that maximization problems used in the proofs are well defined. Lemma 5 is a technical result from probability theory that will be useful when we want to combine a statement which holds with positive probability in the long run with a statement which holds with probability arbitrarily close to one.

The lemma that follows is a simple result that shows that the per-period utility converges to the lifetime utility for sufficiently large t both unconditionally and conditional on a given realization of the signal s. The proof uses Lemma 2 and simple algebraic manipulations.

Lemma 3 *For any* $\varepsilon > 0$

$$\lim_{t\to\infty} P\left(|u_t - v_t| \geq \varepsilon\right) = 0$$

and

$$\lim_{t\to\infty} P\left(|u_t - v_t| \geq \varepsilon|s\right) = 0$$

for $s \in \{s_1, s_2\}$.

Proof. First we show that for any $\varepsilon > 0$

$$\lim_{t\to\infty} P(|v_t - E[v_{t+1} \mid \mathcal{F}_t]| \geq \varepsilon) = 0. \tag{18}$$

As argued in Lemma 2, v_t is a bounded supermartingale and converges almost surely to v^∞. Let $y_t \equiv E[v_{t+1} \mid \mathcal{F}_t]$. Since a bounded martingale is uniformly integrable (see Williams 1991), we get $y_t - v_t \to 0$ almost surely. Then note that almost sure convergence implies convergence in probability, so $\lim_{t\to\infty} P\left(|y_t - v_t| \geq \varepsilon\right) = 0$ for all $\varepsilon > 0$. Rewrite equation (7) as

$$(1-\gamma)\, u_t = \gamma\left(v_t - E\left[v_{t+1} \mid \mathcal{F}_t\right]\right) + (1-\gamma)\, v_t.$$

This gives

$$u_t - v_t = \frac{\gamma}{1-\gamma}\left(v_t - E\left[v_{t+1} \mid \mathcal{F}_t\right]\right),$$

which combined with (18) shows that $\lim_{t\to\infty} P\left(|v_t - u_t| \geq \varepsilon\right) = 0$.

To prove the second part of the lemma notice that

$$P\left(|v_t - u_t| \geq \varepsilon\right) = \sum_{k=1,2} P\left(s_k\right) P\left(|v_t - u_t| \geq \varepsilon | s_k\right),$$

where $P\left(s_k\right) = 1/2$ for $k = 1, 2$. Therefore, given any $\eta > 0$, $P\left(|v_t - u_t| \geq \varepsilon\right) < \eta$ implies $P\left(|v_t - u_t| \geq \varepsilon | s_k\right) < 2\eta$ for $k = 1, 2$. ■

The next lemma uses market clearing and the convergence of the lifetime utility v_t to show that we can always find a compact set X in the interior of R_+^2, such that the allocations x_t are in that set with a sufficiently high probability in the long run. In the rest of the appendix, we construct compact sets with this property on various occasions by using the next lemma, to set bounds on the utility gains that agents of different types derive from trades. The proof is a somewhat tedious constructive argument that uses the convergence of utilities and the market clearing conditions.

Lemma 4 *For any $\varepsilon > 0$ and for any $s \in \{s_1, s_2\}$, there exists a time T and a compact set $X \subset R_+^2$ which lies in the interior of R_+^2, such that $P\left(x_t \in X | s\right) \geq 1 - \varepsilon$ for all $t \geq T$.*

Proof. Pick any $\eta > 0$ and choose T so that $P\left(|u_t - v_t| > \eta | s\right) \leq \varepsilon/4$ for all $t \geq T$, i.e., so that u_t is sufficiently close to v_t. This can be done by Lemma 3.

Next, we use goods market clearing to show that $P(x_t^j > 8/\varepsilon | s) \leq \varepsilon/4$ for each good $j = 1, 2$. To prove this, notice that

$$1 = \int x_t^j\left(\omega\right) dP\left(\omega | s\right) \geq \int_{x_t^j(\omega) \geq 4/\varepsilon} x_t^j\left(\omega\right) dP\left(\omega | s\right) \geq \frac{4}{\varepsilon} P(x_t^j \geq 4/\varepsilon | s),$$

which implies that

$$P(x_t^j \geq \frac{4}{\varepsilon} | s) \leq \frac{\varepsilon}{4}.$$

Moreover, let $\bar{v}$ be the upper bound on the agents' lifetime utility coming from the boundedness of the utility function. Choose $a \in R$ such that

$$\frac{\bar{v} - U\left(x, \delta\right)}{\bar{v} - a} \leq \frac{\varepsilon}{8},$$

for all possible initial values of x and δ. Such an a exists because $U(x,\delta) > -\infty$ for all possible initial values of x and δ, as initial endowments are strictly positive by Assumption 3 and the set of initial values is finite. Then notice that $U(x,\delta) \leq E\{v_t|\mathcal{F}_0\}$ for all initial values of x and δ, because an agent always has the option to refuse any trade. Moreover

$$E\{v_t|\mathcal{F}_0\} \leq P(v_t < a|\mathcal{F}_0)\, a + (1 - P(v_t < a|\mathcal{F}_0))\, \bar{v}.$$

Combining these inequalities gives

$$P(v_t < a|\mathcal{F}_0) \leq \frac{\bar{v} - U(x,\delta)}{\bar{v} - a} \leq \frac{\varepsilon}{8}.$$

Taking unconditional expectations shows that $P(v_t < a) \leq \varepsilon/8$. Since $P(s) = 1/2$ it follows that $P(v_t < a|s) \leq \varepsilon/4$. Next, let us use the inequality

$$\begin{aligned}
&P\Big(v_t \geq a \text{ and } |u_t - v_t| \leq \eta \text{ and } x_t^j \leq 8/\varepsilon \text{ for } j = 1,2|s\Big) \geq \\
&1 - P(v_t < a|s) - P(|u_t - v_t| > \eta|s) - \sum_{j=1}^{2} P\left(x_t^j < \frac{4}{\varepsilon}|s\right) \geq \\
&1 - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} - 2\frac{\varepsilon}{4} = 1 - \varepsilon. \qquad (19)
\end{aligned}$$

Define the set $\tilde{X} = \{x : U(x,\delta) \geq a - \eta \text{ for some } \delta \in [0,1]\}$. We want to show that $\tilde{X}$ is closed. To see this, note that $\tilde{X} = \tilde{X}^1 \cup \tilde{X}^2$ where $\tilde{X}^j \equiv \left\{x : U(x,\delta) \geq a - \eta \text{ for some } \delta \in [0,1], x^j \geq x^{-j}\right\}$. Consider $\tilde{X}^1$. Observe that

$$\tilde{X}^1 = \left\{x : U(x,1) \geq a - \eta,\ x^1 \geq x^2\right\}. \qquad (20)$$

To see that this is true, suppose $x \in \tilde{X}^1$. This implies that for some δ $\pi(\delta)u(x^1) + (1 - \pi(\delta))u(x^2) \geq a - \eta$. Since $x^1 \geq x^2$ and $\pi(\delta)$ is increasing, it also must be true that $\pi(1)u(x^1) + (1 - \pi(1))u(x^2) \geq a - \eta$. The set on the right-hand side of (20) is closed by continuity of U, therefore, $\tilde{X}^1$ is closed. Analogously, $\tilde{X}^2$ is closed, and therefore, $\tilde{X}$ is closed. Then we can define

$$X = \tilde{X} \cap (0, 8/\varepsilon]^2.$$

Notice that $x \notin \tilde{X}$ if $x^j = 0$ for some j, and $(0, 4/\varepsilon]^2$ is bounded. Therefore, X is compact and lies in the interior of R^2_+. Moreover

$$\left\{\omega : v_t(\omega) \geq a \text{ and } |u_t(\omega) - v_t(\omega)| \leq \eta \text{ and } x_t^j(\omega) \leq 4/\varepsilon \text{ for } j = 1, 2\right\}$$
$$\subset \{\omega : x_t(\omega) \in X\},$$

given that $v_t(\omega) \geq a$ and $|u_t(\omega) - v_t(\omega)| \leq \eta$ imply $u_t(\omega) \geq a - \eta$. Therefore, by (19), the set X satisfies the desired inequality $P(x_t \in X|s) \geq 1 - \varepsilon$. ■

The following is a basic probability result which will be useful throughout the appendix.

Lemma 5 *Given any $s \in \{s_1, s_2\}$, suppose there are two sets A and B, two scalars $\lambda, \eta > 0$ and a period T, such that $P((x_t, \delta_t) \in A|s) > \lambda$ and $P((x_t, \delta_t) \in B|s) > 1 - \eta$ for all $t \geq T$. Then, $P((x_t, \delta_t) \in A \cap B|s) > \lambda - \eta$ for all $t \geq T$.*

Proof. The probability $P((x_t, \delta_t) \in A|s)$ can be decomposed as

$$P((x_t, \delta_t) \in A|s) = P((x_t, \delta_t) \in A \cap B|s) + P((x_t, \delta_t) \in A \cap B^c|s),$$

where B^c is the complement of B. Moreover,

$$P((x_t, \delta_t) \in A \cap B^c|s) \leq P((x_t, \delta_t) \in B^c|s) < \eta.$$

Therefore,

$$P((x_t, \delta_t) \in A \cap B|s) = P((x_t, \delta_t) \in A|s) - P((x_t, \delta_t) \in A \cap B^c|s) > \lambda - \eta.$$

■

8.2 Proof of Proposition 1

First, we will prove a lemma that will be used for the proofs of several propositions. The first part of the Proposition shows that for any two informed agents whose marginal rates of substitution differ by at least η, there is a trade z that achieves a utility gain of at least Δ for both of them. Moreover, this utility gain Δ can be chosen so that it is independent of the allocations of the agents. The second part of the Proposition shows that for T sufficiently large informed agents would accept any trade that offer them a utility gain of at least Δ.

Let us define the function

$$\mathcal{M}(x,\delta) \equiv \frac{\pi(\delta)\,u'(x^1)}{(1-\pi(\delta))u'(x^2)},$$

which gives the *ex ante* marginal rate of substitution between the two assets for an agent with the endowment-belief pair (x,δ).

Lemma 6 *Let X be a compact set which lies in the interior of R_+^2.*

(a) For any $\eta > 0$, $\theta > 0$ and any $r \in (0,\eta)$, there are a $\Delta > 0$ and an $\varepsilon^* > 0$ such that given any two informed agents with belief $\bar{\delta} \in \{0,1\}$ and endowments $x_A, x_B \in X$ which satisfy $\mathcal{M}\left(x_B,\bar{\delta}\right) - \mathcal{M}\left(x_A,\bar{\delta}\right) \geq \eta$, the trade $z = \varepsilon^*(1,-p)$ with $p = \mathcal{M}\left(x_A,\bar{\delta}\right) + r$, satisfies the following properties: $||z|| < \theta$ and

$$U(x_A - z,\bar{\delta}) - U(x_A,\bar{\delta}) \geq \Delta \tag{21}$$

$$U(x_B + z,\bar{\delta}) - U(x_B,\bar{\delta}) \geq \Delta \tag{22}$$

(b) Let $B = \left\{x : x \in B, \mathcal{M}\left(x,\bar{\delta}\right) \geq \mathcal{M}\left(x_A,\bar{\delta}\right) + \eta\right\}$. If there is a $\lambda > 0$ such that $P\left(x_t \in B, \delta_t = \bar{\delta} \mid s\right) > \lambda$ for all $t \geq T$, then for any $\psi > 0$ there is a $\hat{T}$ such that $\chi_t(z|s) > \lambda - \psi$ for $t \geq \hat{T}$.

Proof. *Part (a).* Since X is compact and lies in the interior of R_+^2, there exists an $\bar{\varepsilon} > 0$ such that for all $\varepsilon \leq \bar{\varepsilon}$ the trade $z = \varepsilon(1,-p)$ satisfies $\|z\| < \theta$ and is feasible for all $x \in X$ (that is, $x - z \geq 0$ and $x + z \geq 0$). In the rest of the proof we focus on $\varepsilon < \bar{\varepsilon}$. For any $x_A \in X$,

$$\begin{aligned} & U(x_A - z,\bar{\delta}) - U(x_A,\bar{\delta}) \\ = \; & \pi(\bar{\delta})u'(x_A^1)\varepsilon - (1-\pi(\bar{\delta}))u'(x_A^2)p\varepsilon + \frac{1}{2}\left(\pi(\bar{\delta})u''(y^1) + (1-\pi(\bar{\delta}))u''(y^2)p^2\right)\varepsilon^2 \end{aligned}$$

for some $\left(y^1,y^2\right) \in \left[x_A^1, x_A^1+\varepsilon\right] \times [x_A^2 - p\varepsilon, x_A^2]$. Since $p > \mathcal{M}\left(x_A,\bar{\delta}\right)$, we can rewrite the above expression as

$$U(x_A - z,\bar{\delta}) - U(x_A,\bar{\delta}) = (1-\pi(\bar{\delta}))u'(x_A^2)\left(p - \mathcal{M}\left(x_A,\bar{\delta}\right)\right)\varepsilon + \frac{1}{2}\left[\pi(\bar{\delta})u''(y^1) + (1-\pi(\bar{\delta}))u''(y^2)p^2\right]\varepsilon^2$$

Let

$$D'_A = \min_{x \in X}(1-\pi(\bar{\delta}))u'(x^2)r$$

and

$$D_A'' = \min_{\left\{y,p:\exists x\in X,(\tilde{y}^1,\tilde{y}^2)\in[x^1,x^1+\bar{\varepsilon}]\times[x^2-p\bar{\varepsilon},x^2],p=\mathcal{M}\left(x,\bar{\delta}\right)+r\right\}} \frac{1}{2}\left[\pi(\bar{\delta})u''(\tilde{y}^1)+(1-\pi(\bar{\delta}))u''(\tilde{y}^2)p^2\right]$$

Note that $D_A' > 0$ from compactness and interiority of X, while D_A'' can be either positive or negative. There must exist some $\varepsilon_A \in (0,\bar{\varepsilon})$ s.t. for all $\varepsilon \leq \varepsilon_A$, $D_A'\varepsilon + D_A''\varepsilon^2 > 0$. Let $\Delta_A = D_A'\varepsilon_A + D_A''\varepsilon_A^2$. By construction, for all $x \in X$,

$$U(x-z,\bar{\delta}) - U(x,\bar{\delta}) \geq \Delta_A > 0.$$

Analogously we construct Δ_B,ε_B, D_B', D_B''.

Let $\varepsilon^* = \min\{\varepsilon_A,\varepsilon_B\}$ and let $\Delta = \min\left\{D_A'\varepsilon^* + D_A''\varepsilon^{*2}, D_B'\varepsilon^* + D_B''\varepsilon^{*2}\right\}$. By construction ε^* and Δ satisfy (21) and (22), completig the proof of part (a).

Part (b). Pick any $x_A \in X$. If there is a $\lambda > 0$ such that $P\left(x_t \in B, \delta_t = \bar{\delta} \mid s\right) > \lambda$ for all $t \geq T$, by Lemmas 3 and 5, for any $\eta > 0$ we can find a $\hat{T}$ such that $P(|v_t - u_t| < \Delta, x_t \in B, \delta_t = \bar{\delta}|s) > \lambda - \eta$ for all $t \geq \hat{T} \geq T$. Any agent with $|v_t - u_t| < \Delta$, $x_t \in B$ and $\delta_t = \bar{\delta}$ is strictly better off accepting trade z, given that

$$U(x_t + z,\bar{\delta}) \geq U(x_t,\bar{\delta}) + \Delta = u_t + \Delta > v_t.$$

This shows that $\chi_t\left(z|s\right) > \lambda - \eta$, completing the proof of part (b). ■

We can now prove the main result of this section.

Proof of Proposition 1. We prove (8) in two steps. First, we show that there exists a sequence $\kappa_t(s)$ such that

$$\lim_{t\to\infty} P\left(|(\mathcal{M}\left(x_t,\delta_t\right)) - \kappa_t(s)| > \varepsilon, \delta_t = \delta^I\left(s\right)|s\right) = 0 \text{ for all } \varepsilon > 0. \tag{23}$$

Second, we show that $\kappa_t(s)$ converges to a constant.

Step 1. Suppose (23) is not true. Then for all $\kappa > 0$ there are $\zeta, \eta > 0$ such that for infinitely many t

$$P\left(|\mathcal{M}\left(x_t,\delta_t\right) - \kappa| > \zeta, \delta_t = \delta^I\left(s\right) \mid s\right) > \eta.$$

The last expression can be rewritten as

$$P\left(|\mathcal{M}\left(x_t,\delta_t\right) - \kappa| > \zeta, \delta_t = \delta^I\left(s\right) \mid s\right) > \eta P\left(\delta_t = \delta^I\left(s\right) \mid s\right). \tag{24}$$

Without loss of generality, we choose $\eta < 2/3$.

Let us show that when (24) holds, we can construct two sets $A, B \subset \Omega$ with the following property: in some period t there is a positive mass of informed agents in both sets, the agents in set A have marginal rates of substitution above some constant κ_t^*, the agents in set B have marginal rates of substitution below $\kappa_t^* - \zeta$, and all these agents are sufficiently close to their long run expected utility. Let X be a compact set in the interior of R_+^2, such that $P(x_t \in X \mid s) \geq 1 - \alpha\eta/2$ for all $t \geq T$, for some T (such a set exists by Lemma 4). Moreover, given that there is a mass α of informed agents in period 0 and all informed agents remain informed, we have $P\left(\delta_t = \delta^I(s) \mid s\right) \geq \alpha$. Then, using Lemma 5 there must be infinitely many periods such that, for all κ, the probability that the marginal rate of substitution is away from κ is sufficiently high:

$$P\left(\left|\mathcal{M}\left(x_t, \delta_t\right) - \kappa\right| > \zeta, \delta_t = \delta^I\left(s\right), x_t \in X \mid s\right) > \eta P\left(\delta_t = \delta^I\left(s\right) \mid s\right) - \alpha\eta/2 \geq \alpha\eta/2,$$

or the probability that the marginal rate of substitution is close to κ is sufficiently low:

$$P\left(\left|\mathcal{M}\left(x_t, \delta_t\right) - \kappa\right| \leq \zeta, \delta_t = \delta^I\left(s\right), x_t \in X \mid s\right) < \alpha\eta/2 \qquad \text{for all } \kappa. \tag{25}$$

For any such period t, let $m = \inf\left\{\kappa : P\left(\mathcal{M}\left(x_t, \delta_t\right) \leq \kappa, \delta_t = \delta^I\left(s\right), x_t \in X \mid s\right) \geq \alpha\eta/2\right\}$ and let κ_t^* be $\kappa_t^* = m + \zeta/2$.

By definition, the probability that the marginal rate of subsitution is below κ_t^* is sufficiently high

$$P\left(\mathcal{M}\left(x_t, \delta_t\right) \leq \kappa_t^*, \delta_t = \delta^I\left(s\right), x_t \in X \mid s\right) \geq \alpha\eta/2 \tag{26}$$

and

$$P\left(\mathcal{M}\left(x_t, \delta_t\right) \leq \kappa_t^* - \zeta, \delta_t = \delta^I\left(s\right), x_t \in X \mid s\right) < \alpha\eta/2.$$

Moreover, (25) implies

$$P\left(\mathcal{M}\left(x_t, \delta_t\right) \in [\kappa_t^* - \zeta, \kappa_t^* + \zeta), \delta_t = \delta^I\left(s\right), x_t \in X \mid s\right) < \alpha\eta/2.$$

Combining the last two inequalities gives

$$P\left(\mathcal{M}\left(x_t, \delta_t\right) < \kappa_t^* + \zeta, \delta_t = \delta^I\left(s\right), x_t \in X \mid s\right) < \alpha\eta.$$

Then, given that $P\left(\delta_t = \delta^I(s), x_t \in X | s\right) \geq \alpha - \alpha\eta/2$, we obtain

$$\begin{aligned}
&P\left(\mathcal{M}(x_t, \delta_t) \geq \kappa_t^* + \zeta, \delta_t = \delta^I(s), x_t \in X \mid s\right) = \\
&P\left(\delta_t = \delta^I(s), x_t \in X \mid s\right) - P\left(\mathcal{M}(x_t, \delta_t) < \kappa_t^* + \zeta, \delta_t = \delta^I(s), x_t \in X \mid s\right) \geq \\
&\alpha - \alpha\eta/2 - \alpha\eta = \alpha(1 - \eta 3/2) > 0. \qquad (27)
\end{aligned}$$

Given the set X and the $\zeta > 0$ defined above, let Δ be the lower bound defined in Lemma 6 and choose a $\hat{T} \geq T$ such that the utilities converged sufficiently:

$$P\left(|v_t - u_t| < \frac{1}{2}\alpha\eta\Delta, x_t \in X \mid s\right) > \frac{1}{2}\min\{\alpha\eta/2, \alpha(1 - \eta 3/2)\} \qquad (28)$$

for all $t \geq \hat{T}$.

Let t be any period $t \geq \hat{T}$ such that (24) holds. We are now ready to define our two sets

$$\begin{aligned}
A &= \left\{\omega : \mathcal{M}(x_t, \delta_t) \leq \kappa_t^*, \delta_t = \delta^I(s), |v_t - u_t| < \frac{1}{2}\alpha\eta\Delta, x_t \in X\right\}, \\
B &= \left\{\omega : \mathcal{M}(x_t, \delta_t) \geq \kappa_t^* + \zeta, \delta_t = \delta^I(s), |v_t - u_t| < \frac{1}{2}\alpha\eta\Delta, x_t \in X\right\}.
\end{aligned}$$

Using (26), (27), (28) and Lemma 5 it follows that $P(A \mid s) > \alpha\eta/2, P(B \mid s) > 0$. Notice that both A and B are $\mathcal{F}_t$-measurable.

This step constructs a profitable deviation that shows that agents of sets A and B can trade with each other and increase their utility.

By Lemma 6, any agent in B can find a trade z^* improving the utility of this agent

$$U(x_t(\omega) - z^*, \delta_t(\omega)) - U(x_t(\omega), \delta_t(\omega)) \geq \Delta$$

and the utility of any agent in set A, for all $\tilde{\omega} \in A$

$$U(x_t(\tilde{\omega}) + z^*, \delta_t(\tilde{\omega})) - U(x_t(\tilde{\omega}), \delta_t(\tilde{\omega})) \geq \Delta.$$

It remains to show that offering trade z^* is a profitable deviation for any agent who reaches set B at time t. By accepting the trade z^* the agents in A get an expected payoff

$$U(x_t(\tilde{\omega}) + z^*, \delta_t) \geq u_t(\tilde{\omega}) + \Delta > v_t(\tilde{\omega}).$$

Since $v_t(\tilde{\omega})$ is their equilibrium expected payoff, this implies that the trade is accepted by all agents in A, which implies that it is accepted with probability $\chi_t(z^*|s) > \alpha\eta/2$. Suppose an agent in B offers z^* at t and stops trading from $t+1$ on (whether or not the trade is accepted at t). The expected payoff of this strategy is

$$U(x_t(\omega), \delta_t(\omega)) + \chi_t(z^*|s)(U(x_t(\omega) - z^*, \delta_t(\omega)) - U(x_t(\omega), \delta_t(\omega))) > u_t(\omega) + \frac{1}{2}\alpha\eta\Delta > v_t(\omega).$$

Since there is a positive mass of agents in B, and $v_t(\omega)$ is their equilibrium expected payoff, this leads to a contradiction.

Step 2. Suppose $\kappa_t(s)$ does not converge to a constant. This implies that there exists $\zeta > 0$ s.t. for any $\tilde{T}$ there exists t', t'' with $\tilde{T} < t' < t''$ s.t. $|\kappa_{t'}(s) - \kappa_{t''}(s)| \geq \zeta$. Choose Δ, z^* by Lemma 6 and let $\tilde{T}$ be such that for all $t > \tilde{T}$

$$P\left(|v_t - u_t| < \frac{1}{2}\gamma^{t''-t'}\alpha\Delta, x_t \in X, \delta = \delta^I(s) \mid s\right) > \frac{1}{2}\alpha. \tag{29}$$

It is possible to do by Lemma 3. Consider any agent in period t' for whom (29) is satisfied. Consider the following deviation for that agent starting from t'. The agent rejects any offer for all $t \geq t'$ and makes $(0,0)$ offer if he is a proposer in all periods except t'', when he makes an offer z^*. The probability of the game continuing up until period t'' is $\gamma^{t''-t'}$, while probability of being matched with an informed agent and being chosen a proposer is $\frac{1}{2}\alpha$, which shows that expected payoff from such strategy is at least $u_t + \frac{1}{2}\gamma^{t''-t'}\alpha\Delta > v_t$. ■

8.3 Proof of Proposition 2

Proposition 2 states that marginal rates of substitution converge for uninformed agents when $\kappa(s_1) \neq \kappa(s_2)$. The proof relies on constructing a deviation that would yield a positive utility to the uninformed agents if marginal rates of substitution failed to converge. This deviation consists of making two offers in subsequent periods. The first offer allows agents to learn the signal if his offer is accepted. If the offer is accepted, the agent makes a second offer in the following round that gives him a positive utility gain by trading with the informed agents.

Before turning to the proof, it is useful to derive several preliminary results. Lemma 7 shows how to construct a small trade which allows the uninformed agent to learn the underlying signal s in the learning phase. Finally, Lemma 8 shows that the beliefs of uninformed agents tend to stay away from zero when the signal is s_1 and away from 1 when the signal is s_2. This lemma will be used to ensure that when the uninformed agent deviates, his learning offer allows him to get sufficiently close to the truth.

8.3.1 Experimentation and learning when $\kappa(s_1) \neq \kappa(s_2)$

The next lemma shows that there exists a trade z that will be accepted with a sufficiently high probability in one state and rejected with a sufficiently high probability in the other state.

Lemma 7 *Suppose $\kappa(s_1) > 1 > 1/\kappa(s_2)$. For any $\theta > 0$ and any $\eta > 0$:*

(i) there is a period T and a trade z, with $\|z\| < \theta$, such that $\chi_t(z|s_1) > \alpha - \eta$, and $\chi_t(z|s_2) < \eta$ for all $t \geq T$;

(ii) there is a period T and a trade z, with $\|z\| < \theta$, such that $\chi_t(z|s_2) > \alpha - \eta$ and $\chi_t(z|s_1) < \eta$ for all $t \geq T$.

Proof. We prove part (i), the proof of part (ii) is symmetric.

Step 1. We start with the usual step — ensure that allocations of informed agents end up with sufficiently high probability in an interior compact set with given properties.

Since the marginal rate of substitution of the informed agents converge, by Lemma 1, given any $\eta > 0$, we can apply Lemmas 4 and 5, and find a compact set X in the interior of R_+^2, a positive scalar ζ, and a time $\hat{T}$ such that

$$P(\mathcal{M}(x_t, 1) \geq 1 + \zeta, x_t \in X|s_1) > \alpha - \eta/2 \text{ for all } t \geq \hat{T}. \tag{30}$$

Define the set

$$A = \{x : \mathcal{M}(x, 1) \geq 1 + \zeta, x \in X\},$$

i.e., the set of allocations in X at which the informed agents in state s_1 have marginal rate of substitution sufficiently above 1. Analogously, define the symmetric set

$$\hat{A} = \{x = (x^2, x^1) : (x^1, x^2) \in A\}.$$

In a symmetric equilibrium, the informed agents behave in a symmetric way, conditional on the signals s_1 and s_2, so that $P(x_t \in A, \delta_t = 1 \mid s_1) = P(x_t \in \hat{A}, \delta_t = 0 \mid s_2)$. Therefore, (30) implies

$$P(x_t \in \hat{A}, \delta_t = 0 \mid s_2) > \alpha - \eta/2 \text{ for all } t \geq \hat{T}, \tag{31}$$

i.e., the allocations of the informed agent if the state of the world is s_2 are in the set $\hat{A}$ with the relevant probability.

Step 2. We now construct a trade z that will be accepted by the informed agents in state s_1 with high enough probability.

Proceeding as in the proof of Lemma 6, we can find a lower bound Δ for the utility gain and choose ε small enough that the trade $z = (\varepsilon, -\varepsilon)$ satisfies $\|z\| < \theta$ and

$$x + z \geq 0 \text{ and } U(x + z, 1) - U(x, 1) \geq \Delta \text{ for all } x \in A,$$

i.e., it is interior and improves utility of the informed agents in state s_1.

The only issue we need to address in this step is to show that if the utilities of a positive mass of informed agents converges in the long run, such agents will accept the relevant trade.

Given that $P\left(x_t \in A, \delta_t = 1 \mid s_1\right) > \alpha - \eta/2$ for all $t \geq \hat{T}$, from (30), we can apply Lemmas 3 and 5 and find a T such that $P\left(|v_t - u_t| < \Delta, x_t \in A, \delta_t = 1|s_1\right) > \alpha - \eta/2 - \eta/2 = \alpha - \eta$ for all $t \geq T$. Any agent with $|v_t - u_t| < \Delta$, $x_t \in A$ and $\delta_t = 1$ is strictly better off accepting trade z, given that

$$U(x_t + z, 1) \geq U(x_t, 1) + \Delta = u_t + \Delta > v_t.$$

This shows that $\chi_t\left(z|s_1\right) > \alpha - \eta$ for all $t \geq T$.

Step 3. This step is the most difficult and important step. We need to show that we can choose T large enough so that the trade is accepted in s_2 with sufficiently low probability by all (both informed and uninformed) agents: $\chi_t\left(z|s_2\right) < \eta$ for all $t \geq T$.

First, as in the argument in Step 2, and using symmetry we show that the opposite of trade z will be accepted by the informed agents. Take any $\hat{x} = \left(x^2, x^1\right) \in \hat{A}$, then given that $x = \left(x^1, x^2\right) \in A$ we have $x + z \geq 0$ and

$$\begin{aligned} U(\hat{x} - z, 0) - U(\hat{x}, 0) &= (1 - \phi)\, u(x^2 - \varepsilon) + \phi u(x^1 + \varepsilon) - (1 - \phi)\, u(x^2) - \phi u(x^1) \\ &= U(x + z, 1) - U(x, 1) \geq \Delta, \end{aligned}$$

i.e., the opposite of trade z is utility improving for the informed agents in state s_2.

Second, suppose, by contradiction, that, for some $\eta > 0$ that the probability of the offer z being accepted in state s_2 is sufficiently high: $\chi_t\left(z|s_2\right) > \eta$ for infinitely many periods. Given that $P(x_t \in \hat{A}, \delta_t = 0|s_2) > \alpha - \eta/2$ for all $t \geq \hat{T}$, we can apply Lemmas 3 and 5 and find a $\bar{T}$ such that the utility of the informed agents in the state s_2 converged sufficiently – $P(|v_t - u_t| < \eta\Delta, x_t \in \hat{A}, \delta_t = 0|s_2) > \alpha - \eta$ for all $t \geq \bar{T}$.

Pick a period $t \geq \bar{T}$ such that $\chi_t(z|s_2) > \eta$. An informed agent with $|v_t - u_t| < \eta\Delta$, $x_t \in \hat{A}$ and $\delta_t = 0$ is strictly better off making the offer z, consuming x_t if the offer is rejected, and consuming $x_t - z$ if the offer is accepted, given that

$$\begin{aligned}(1 - \chi_t(z|s_2))\,U(x_t, 0) + \chi_t(z|s_2)\,U(x_t - z, 0) &\geq \\ U(x_t, 0) + \eta\,(U(x_t - z, 0) - U(x_t, 0))\,\Delta &\geq u_t + \eta\Delta > v_t.\end{aligned}$$

Since this behavior dominates the equilibrium strategy and there is a positive mass of informed agents with $|v_t - u_t| < \Delta$, $x_t \in \hat{A}$ and $\delta_t = 0$, we have a contradiction ■

8.3.2 A bound on incorrect beliefs

The following lemma shows that conditional on the signal s_1, the probability that the belief of the uninformed agent gets close to 0 is small. That is, the uninformed agents can only be very wrong with a small probability. This lower bound will be useful when we construct our profitable deviation in the proof of Proposition 2.

Lemma 8 *For each $\varepsilon \in (0, 1)$ the bound on the incorrect beliefs for all t is given by:*

$$P(\delta_t < \varepsilon \mid s_1) < \frac{\varepsilon}{1 - \varepsilon},$$

$$P(\delta_t > 1 - \varepsilon \mid s_2) < \frac{\varepsilon}{1 - \varepsilon}.$$

Proof. We prove the first inequality, the proof of the second is analogous. Since $\delta_t(\omega)$ are equilibrium beliefs, they must be consistent with Bayesian updating and must satisfy, by definition, $\delta = P(s = s_1 \mid \delta_t(\omega) = \delta)$. This implies $P(s_1 \mid \delta_t < \varepsilon) < \varepsilon$, which implies $P(s_2 \mid \delta_t < \varepsilon) > 1 - \varepsilon$ and

$$\frac{P(s_1 \mid \delta_t < \varepsilon)}{P(s_2 \mid \delta_t < \varepsilon)} < \frac{\varepsilon}{1 - \varepsilon}.$$

Applying Bayes' rule

$$\frac{P(s_1 \mid \delta_t < \varepsilon)}{P(s_2 \mid \delta_t < \varepsilon)} = \frac{P(\delta_t < \varepsilon \mid s_1)\,P(s_1)}{P(\delta_t < \varepsilon \mid s_2)\,P(s_2)}.$$

Combining the last two equations, and using $P(s_1) = P(s_2) = 1/2$, gives

$$P(\delta_t < \varepsilon \mid s_1) < \frac{\varepsilon}{1-\varepsilon} P(\delta_t < \varepsilon \mid s_2) \leq \frac{\varepsilon}{1-\varepsilon},$$

where the last inequality follows from $P(\delta_t < \varepsilon \mid s_2) \leq 1$. ■

We can now turn to the main result of this section.

Proof of Proposition 2.

Suppose, by contradiction, that there exists an $s \in \{s_1, s_2\}$, a $\zeta > 0$ and a $\xi > 0$ such that for infinitely many t

$$P\left(\left|\frac{\phi(s)u'(x_t^1)}{(1-\phi(s))\,u'(x_t^2)} - \kappa(s)\right| > \zeta \mid s\right) > \xi. \tag{32}$$

Without loss of generality, suppose that for infinitely many t

$$P\left(\frac{\phi(s_1)u'(x_t^1)}{(1-\phi(s_1))\,u'(x_t^2)} - \kappa(s_1) > \zeta \mid s_1\right) > \xi. \tag{33}$$

The other cases can be treated analogously.

Lemma 4 shows that there exists a compact set X s.t. for t sufficiently large allocations of most agents (which we will make precise below) lie in X. From Proposition 1, marginal rates of substitution of the most informed agents differ by no more than $\zeta/2$ from $\kappa(s_1)$. Then Lemma 6 shows that there trade z^* s.t. if an uninformed agent's MRS satisfies (32) and lies in X, an offer $z^* = \varepsilon^*(1, -(\kappa(s_1) + \frac{3}{4}\zeta))$, if accepted, gives him utility gain of Δ if $s = s_1$. Consider the following deviation of the uninformed agent in period T for whom $x_t \in X$ and (33) is satisfied: in period T the agent makes an offer $\hat{z}$, which we will describe shortly. If offer $\hat{z}$ is accepted, in period $T+1$, the agent makes offer z^*, if he is selected as a proposer. For all $t \geq T+2$, and in period $T+1$ if offer $\hat{z}$ was rejected in period T, the uninformed agent makes $(0,0)$ offer when a proposer, and rejected all offers if a receiver. Suppose that offer $\hat{z}$ can be chosen so that for all $x \in X$

$$\left|U\left(x + \hat{z}, \bar{\delta}\right) - U\left(x, \bar{\delta}\right)\right| < \lambda \quad \text{for } \bar{\delta} \in \{0,1\} \tag{34}$$

for some $\lambda > 0$.

Let $\chi_t(z|s)$ be the probabilities that offer $z \in \{\hat{z}, z^*\}$ are accepted in period t in state $s \in \{s_1, s_2\}$. Then the expected payoff $\hat{v}$ in period T from such deviation is

$$\begin{aligned}\hat{v} &= \delta_T \left[\begin{array}{c} \chi_T(\hat{z}|s_1)\frac{1}{2}\gamma\chi_{T+1}(z^*|s_1)U(x_T+\hat{z}+z^*,1) \\ +\chi_T(\hat{z}|s_1)\left(1-\frac{1}{2}\gamma\chi_{T+1}(z^*|s_1)\right)U(x_T+\hat{z},1)+(1-\chi_T(\hat{z}|s_1))U(x_T,1) \end{array}\right] \\ &\quad +(1-\delta_T)\left[\begin{array}{c} \chi_T(\hat{z}|s_2)\frac{1}{2}\gamma\chi_{T+1}(z^*|s_2)U(x_T+\hat{z}+z^*,0) \\ +\chi_T(\hat{z}|s_2)\left(1-\frac{1}{2}\gamma\chi_{T+1}(z^*|s_2)\right)U(x_T+\hat{z},0)+(1-\chi_T(\hat{z}|s_2))U(x_T,0) \end{array}\right]\end{aligned} \tag{35}$$

The first term in the square brackets is the probability of offers $\hat{z}$ being accepted, game continuing to period $T+1$, the agent being selected a proposer in period $T+1$ and the offer z^* being accepted. The other two terms, respectively, are the probability of offer $\hat{z}$ being accepted but agent not being able to make a trade z^*; and offer $\hat{z}$ not being accepted.

Let Q be the maximum utility loss if the state is s_2 and agent's offer z^* is accepted:

$$Q \equiv \max_{x\in X} U\left(x,0\right) - U\left(x+z^*,0\right).$$

Using triangular inequality, (35) can be rewritten as

$$\begin{aligned}\hat{v} &\geq \delta_T U(x_T,1)+(1-\delta_T)U(x_T,0) \\ &\quad +\delta_T\chi_T(\hat{z}|s_1)\frac{1}{2}\gamma\chi_{T+1}(z^*|s_1)\Delta+(1-\delta_T)\chi_T(\hat{z}|s_2)\frac{1}{2}\gamma\chi_{T+1}(z^*|s_2)Q-\lambda\end{aligned}$$

Note that per period utility $u_T = \delta_T U(x_T,1)+(1-\delta_T)U(x_T,0)$ and let $\hat{\Delta}_T = v_T - u_T$. As long as we can show that

$$\delta_T\chi_T(\hat{z}|s_1)\frac{1}{2}\gamma\chi_{T+1}(z^*|s_1)\Delta+(1-\delta_T)\chi_T(\hat{z}|s_2)\frac{1}{2}\gamma\chi_{T+1}(z^*|s_2)Q > \hat{\Delta}_T+\lambda, \tag{36}$$

the deviation we consider attains a higher expected payoff than the equilibrium payoff v_T. To prove this, we show next that there exists T large enough so that

1. *The measure of uninformed s.t.* $P\left(x_T \in X, \delta_T \geq \frac{\xi}{4}, \hat{\Delta}_T \leq \lambda, \left|\frac{\phi(s_1)u'(x_t^1)}{(1-\phi(s_1))u'(x_t^2)} - \kappa(s_1)\right| > \zeta \mid s_1\right)$ *is bounded away from zero.* This step ensures that the measure of agents who can undertake profitable deviation is positive. Lemmas 4, 8, and 3 implies that for T sufficiently large probability of each of the terms in the brackets is sufficiently close

to 1, i.e. $P(x_T \in X \mid s_1) \geq 1 - \frac{\xi}{8}$, $P\left(\delta_T \geq \frac{\xi}{4} \mid s_1\right) \geq 1 - \frac{\xi}{3}$, $P(\hat{\Delta}_t \leq \lambda \mid s_1) \geq 1 - \frac{\xi}{8}$, which together with (32) implies

$$P\left(x_T \in X, \delta_T \geq \frac{\xi}{4}, \hat{\Delta}_T \leq \lambda, \left|\frac{\phi(s_1)u'(x_t^1)}{(1-\phi(s_1))\, u'(x_t^2)} - \kappa(s_1)\right| > \zeta \mid s_1\right) > \frac{\xi}{3}.$$

2. *There exists trade z^* s.t. $\chi_{T+1}(z^*|s_1) \geq \alpha/2$.* From Proposition 1 it follows from the large T the measure of informed agents satisfies

$$P\left(x_T \in X, \left|\frac{\phi(s_1)u'(x_T^1)}{(1-\phi(s_1))\, u'(x_T^2)} - \kappa(s_1)\right| < \frac{\zeta}{2} \mid s_1\right) \geq \frac{3}{4}\alpha.$$

   Part (b) of Lemma 6 shows that if T is large enough, at least $\frac{2}{3}$ of such agents accept trade z^*, which implies that $\chi_{T+1}(z^*|s_1) \geq \alpha/2$.

3. *There exists trade $\hat{z}$ for which (34) holds which properties that $\chi_T(\hat{z}|s_2) < \eta$ and $\chi_T(\hat{z}|s_1) > \alpha - \eta$ for $\eta > 0$ satisfying*

$$\eta < \frac{\left(2\lambda - \frac{1}{8}\xi\gamma\alpha^2\Delta\right)}{\left(\frac{1}{2}\gamma Q - \frac{1}{8}\xi\gamma\alpha\Delta\right)} \tag{37}$$

   This follows from Lemma 7. Note that $\eta > 0$ since $\lambda, Q < 0, \Delta > 0$

   When these three conditions are satisfied, deviation increases payoff of the positive measure of agents since

$$\begin{aligned} & \delta_T\chi_T(\hat{z}|s_1)\frac{1}{2}\gamma\chi_{T+1}(z^*|s_1)\Delta + (1-\delta_T)\chi_T(\hat{z}|s_2)\frac{1}{2}\gamma\chi_{T+1}(z^*|s_2)Q - \hat{\Delta}_T - \lambda \\ \geq\ & \frac{\xi}{4}(\alpha-\eta)\frac{1}{2}\gamma\frac{\alpha}{2}\Delta + \eta\frac{1}{2}\gamma Q - 2\lambda \\ >\ & 0 \end{aligned}$$

   where the last inequality follows from (37).

■

8.3.3 Proof of Proposition 3

As in the proof of Proposition 2, we need to find trades that allow the agent to learn the true state. The next lemma shows that the uninformed agent can always find a trade that reveals *some* information about the state s, that is, a trade z such that the probabilities of acceptance are sufficiently different in the two states:

$$\left|\chi_t\left(z|s_2\right)-\chi_t\left(z|s_1\right)\right|>\eta.$$

The difference from Lemma 7 is that there the trade could be chosen to reveal almost perfect information about s_1, as we could make the probability of acceptance arbitrarily close to zero in state s_2. So in that case, one round of experimentation was enough to achieve information on s_1 with any degree of precision. Here instead, the agent may need to experiment for several rounds before being sufficiently well informed. This will affect the proof of convergence below.

Lemma 9 *Suppose $\kappa(s_1)=\kappa(s_2)=1$. Suppose there is a non-vanishing mass of agents with marginal rate of substitution different from 1, that is, for some $s\in\{s_1,s_2\}$ there is a $\zeta>0$ and a $\xi>0$ such that*

$$P\left(\left|\mathcal{M}\left(x_t,\delta_t\right)-1\right|>\zeta|s\right)>\xi$$

for infinitely many t. Then, there exists $\eta>0$ s.t. for any $\theta>0$ such that, for infinitely many t there is a trade z with $\|z\|<\theta$ and

$$\left|\chi_t\left(z|s_2\right)-\chi_t\left(z|s_1\right)\right|>\eta. \tag{38}$$

Proof. The proof is by contradiction. Suppose that without loss of generality

$$P\left(\mathcal{M}\left(x_t,\delta_t\right)<1-\zeta|s\right)>\xi \tag{39}$$

for infinitely many t. By lemma 4 and Proposition 1 it is possible to find a compact set X and T s.t. for $s\in\{s_1,s_2\}$

$$P\left(\mathcal{M}\left(x_t,\delta_t\right)<1-\zeta,x_t\in X|s\right)>\xi/2 \text{ for infinitely many } t\geq T,$$

$$P\left(\mathcal{M}\left(x_t,\delta_t\right)\geq 1-\zeta/2,\delta_t=1,x_t\in X|s\right)>\frac{3}{4}\alpha \text{ for all } t\geq T.$$

Consider a sequence of trades $z_\varepsilon=\varepsilon\left(1,-\left(1-\zeta/3\right)\right)$. By part (b) of Lemma 6, there exists $\hat{T}_\varepsilon\geq T$, s.t. for all $t\geq\hat{T}_\varepsilon$,

$$\chi_t(z_\varepsilon|s)\geq\alpha/2 \tag{40}$$

for all s.

Consider the following deviation. Suppose he makes the offer z_ε and after that round stops trading. His expected utility is:

$$
\begin{aligned}
\hat{U} &= \delta_t \left[(1-\chi_t(z_\varepsilon|s_1)) U(x_t,1) + \chi_t(z_\varepsilon|s_1) U(x_t+z_\varepsilon,1)\right] + \\
&\quad (1-\delta_t)\left[(1-\chi_t(z_\varepsilon|s_2)) U(x_t,0) + \chi_t(z_\varepsilon|s_2) U(x_t+z_\varepsilon,0)\right] \\
&= U(x_t,\delta_t) + \delta_t \chi_t(z_\varepsilon|s_1)\left[U(x_t+z_\varepsilon,1) - U(x_t,1)\right] \\
&\quad + (1-\delta_t)\chi_t(z_\varepsilon|s_2)\left[U(x_t+z_\varepsilon,0) - U(x_t,0)\right] \\
&= U(x_t,\delta_t) + \delta_t \chi_t(z_\varepsilon|s_1)\left[\phi u'(x_t^1) - (1-\phi)(1-\zeta/3)\, u'(x_t^2)\right]\varepsilon \\
&\quad + (1-\delta_t)\chi_t(z_\varepsilon|s_2)\left[(1-\phi)u'(x_t^1) - \phi(1-\zeta/3)\, u'(x_t^2)\right]\varepsilon + o(\varepsilon) \\
&= U(x_t,\delta_t) + \chi_t(z_\varepsilon|s_1)\left[\pi(\delta_t)u'(x_t^1) - (1-\pi(\delta_t))(1-\zeta/3)\, u'(x_t^2)\right]\varepsilon \\
&\quad + (\chi_t(z_\varepsilon|s_2) - \chi_t(z_\varepsilon|s_1))(1-\delta_t)\left[(1-\phi)u'(x_t^1) - \phi(1-\zeta/3)\, u'(x_t^2)\right]\varepsilon + o(\varepsilon) \\
&= U(x_t,\delta_t) + \chi_t(z_\varepsilon|s_1)\left[\pi(\delta_t)u'(x_t^1) - (1-\pi(\delta_t))(1-\zeta/3)\, u'(x_t^2)\right]\varepsilon \\
&\quad \times \left[1 + \frac{(\chi_t(z_\varepsilon|s_2) - \chi_t(z_\varepsilon|s_1))(1-\delta_t)\left[(1-\phi)u'(x_t^1) - \phi(1-\zeta/3)\, u'(x_t^2)\right]}{\chi_t(z_\varepsilon|s_1)\left[\pi(\delta_t)u'(x_t^1) - (1-\pi(\delta_t))(1-\zeta/3)\, u'(x_t^2)\right]}\right] + o(\varepsilon)
\end{aligned}
\tag{41}
$$

In what follows we show that there exists $\eta > 0$,$\bar{\varepsilon} > 0$ s.t. for all $\varepsilon < \bar{\varepsilon}$

$$
|\chi_t(z_\varepsilon|s_2) - \chi_t(z_\varepsilon|s_1)| \geq \eta.
$$

Suppose it was not true. Then there exists a sequence of $\varepsilon \to 0$ s.t. $\lim_{\varepsilon \to 0} \chi_t(z_\varepsilon|s_2) - \chi_t(z_\varepsilon|s_1) = 0$. Then it is possible to pick $\bar{\varepsilon}$ s.t. for all $\varepsilon \leq \bar{\varepsilon}$

$$
\left[1 + \frac{(\chi_t(z_\varepsilon|s_2) - \chi_t(z_\varepsilon|s_1))(1-\delta_t)\left[(1-\phi)u'(x_t^1) - \phi(1-\zeta/3)\, u'(x_t^2)\right]}{\chi_t(z_\varepsilon|s_1)\left[\pi(\delta_t)u'(x_t^1) - (1-\pi(\delta_t))(1-\zeta/3)\, u'(x_t^2)\right]}\right] \geq 0.9
$$

Then (41) becomes

$$
\begin{aligned}
\hat{U} - U(x_t,\delta_t) &\geq 0.9\chi_t(z_{\bar{\varepsilon}}|s_1)\left[\pi(\delta_t)u'(x_t^1) - (1-\pi(\delta_t))(1-\zeta/3)\, u'(x_t^2)\right]\bar{\varepsilon} + o(\bar{\varepsilon}) \\
&\geq 0.9\frac{\alpha}{2}\left[\pi(\delta_t)u'(x_t^1) - (1-\pi(\delta_t))(1-\zeta/3)\, u'(x_t^2)\right]\bar{\varepsilon} + o(\bar{\varepsilon})
\end{aligned}
$$

where the second line follows from (40) as long as $t \geq T_{\bar{\varepsilon}}$.

Choose $\varepsilon \leq \bar{\varepsilon}$ s.t.

$$\begin{aligned} & 0.9\frac{\alpha}{2}\left[\pi(\delta_t)u'(x_t^1) - \left(1-\pi(\delta_t)\right)(1-\zeta/3)\,u'(x_t^2)\right]\bar{\varepsilon} + o(\bar{\varepsilon}) \\ \geq \quad & \frac{\alpha}{4}\left[\pi(\delta_t)u'(x_t^1) - \left(1-\pi(\delta_t)\right)(1-\zeta/3)\,u'(x_t^2)\right]\varepsilon \end{aligned}$$

Let

$$D = \frac{\alpha}{4}\min_{x_t,\delta_t}\left[\pi(\delta_t)u'(x_t^1) - \left(1-\pi(\delta_t)\right)(1-\zeta/3)\,u'(x_t^2)\right]$$

s.t $x_t \in X$ and (39).

From Lemmas 3 and 4 choose $T \geq T_\varepsilon$ large enough so that for $t \geq T$ $\Pr(x_t \in X, \mathcal{M}(x_t,\delta_t) < 1-\zeta, v_t - U(x_t,\delta_t) < \frac{1}{2}D\varepsilon|s_1) > \xi/4$. Then the construction of D and ε shows that all such agents are better off deviating in period t. Therefore we obtain a contradition and (38) must be satisfied. ■

Proof of Proposition 3.

Suppose that (11) is not true, and there is some uppder bound $\bar{J}$ s.t. for any t, there are t', t'' $t \leq t', t'' \leq t+\bar{J}$ s.t. for t' and t'' (10) is satisfied. The proof proceeds following the steps of the proof of Proposition 2. The main difference is that when $\kappa(s_1) = \kappa(s_2)$ there is no single trade $\tilde{z}$ acceptance of which reveals the signal to the uninformed. Instead, we use the insights from Lemma 9 to show that as long as probability of accepting offer $\tilde{z}$ is sufficiently different in the two states, agent can learn the state sufficiently well. Without loss of generality, assume that 38 implies that

$$\chi_{t_k}(\tilde{z}|s_1) > \chi_{t_k}(\tilde{z}|s_2) + \eta \tag{42}$$

for all $t_k \geq T$. Formally, consider the following strategy. An agent after some large T follows the following deviation. Let M be the number of periods after T when t_k occurs J times (i.e. (10) is satisfied J times). From the assumptions of the lemma, $M \leq J\bar{J}$. For the next M periods he makes an offer $\tilde{z}$ whenever he is chosen as a proposer in period t_k, and rejects all offers otherwise. If offer $\tilde{z}$ is accepted J times, he makes an offer z^*, and makes no other trades after that. The probability of him Defining $\lambda, \Delta, \hat{\Delta}, Q$ as in the proof of Proposition 2. Since the probability of being selected as a proposer J times is 2^{-J}, and the probability of acceptace of trade $\hat{z}$ J times in state s is $\prod\limits_{T \leq t_k \leq T+M} \chi_{t_k}(\hat{z}|s)$, equation

(36) becomes

$$\delta_T \prod_{T\leq t_k\leq T+M} \chi_{t_k}(\hat{z}|s_1)2^{-J}\gamma^M\chi_{T+1}(z^*|s_1)\Delta+(1-\delta_T)\prod_{T\leq t_k\leq T+M} \chi_{t_k}(\hat{z}|s_2)2^{-J}\gamma^M\chi_{T+M+1}(z^*|s_2)Q > \hat{\Delta}_T+J\lambda. \tag{43}$$

Use (42) in (43) to obtain

$$\begin{aligned}
&\prod_{T\leq t_k\leq T+M} \chi_{t_k}(\hat{z}|s_2)2^{-J}\gamma^M\left[\delta_T\chi_{T+M+1}(z^*|s_1)\Delta+(1-\delta_T)\chi_{T+M+1}(z^*|s_2)Q\right] \\
&+J\left[\delta_T\eta 2^{-J}\gamma^M\chi_{T+M+1}(z^*|s_1)\Delta-\lambda\right] \\
>\ &\hat{\Delta}_T
\end{aligned} \tag{44}$$

Let assume that $Q<0$ and

$$\delta_T\chi_{T+M+1}(z^*|s_1)\Delta+(1-\delta_T)\chi_{T+M+1}(z^*|s_2)Q\leq 0$$

since otherwise the proof proceeds trivially by setting $J=0$. In this case we can rewrite (44) as

$$\begin{aligned}
&\prod_{T\leq t_k\leq T+M} \chi_{t_k}(\hat{z}|s_2)2^{-J}\gamma^M\left[\delta_T\chi_{T+M+1}(z^*|s_1)\Delta+(1-\delta_T)Q\right] \\
&+J\left[\delta_T\eta 2^{-J}\gamma^M\chi_{T+M+1}(z^*|s_1)\Delta-\lambda\right]-\hat{\Delta}_T \\
\geq\ & 2^{-J}\gamma^M\left[\delta_T\chi_{T+M+1}(z^*|s_1)\Delta+(1-\delta_T)Q+J\delta_T\eta\chi_{T+M+1}(z^*|s_1)\Delta\right]-J\lambda-\hat{\Delta}_T
\end{aligned}$$

Our goal is to show that the expression above is strictly positive for a positive measure of agents.

Step 1. Choose T large enough so that

$$P\left(\mathcal{M}\left(x_t,\delta_t\right)<1-\zeta, x_t\in X|s\right)>3\xi/4 \text{ for infinitely many } t\geq T, \tag{45}$$

$$P\left(\mathcal{M}\left(x_t,\delta_t\right)\geq 1-\zeta/2, \delta_t=1, x_t\in X|s\right)>\frac{3}{4}\alpha \text{ for all } t\geq T.$$

and $\chi_t(z^*|s_1)\geq\alpha/2$ for all $t\geq T$. The way to do that is described in the proof of Lemma 9.

Step 2. Choose J so that for all $\delta \geq \xi/4$,

$$\left[\delta\frac{\alpha}{2}\Delta + (1-\delta)Q + J\delta\eta\frac{\alpha}{2}\Delta\right] \geq \theta$$

for some $\theta > 0$. By Lemma 8 and (45) the measure of uninformed agents in period $t_k \geq T$ satisfies

$$P\left(\mathcal{M}\left(x_{t_k}, \delta_{t_k}\right) < 1-\zeta, x_{t_k} \in X, \delta_{t_k} \geq \xi/4|s\right) \geq 5\xi/12$$

Step 3. Choose trade $\tilde{z}$ sufficiently small so that λ satisfies

$$2^{-J}\gamma^M\left[\delta_T\chi_{T+M+1}(z^*|s_1)\Delta + (1-\delta_T)Q + J\delta_T\eta\chi_{T+M+1}(z^*|s_1)\Delta\right] - J\lambda < \theta/4.$$

This is possible to do by 9.

Step 4. Choose $\tilde{T} \geq T$ large enough so that $\hat{\Delta}_{\tilde{T}} < \theta/4$, which can be done by Lemma 3.

This construction implies that the deviation we consider gives a utility gain of at least $\theta/2$ for $5\xi/12$ of the uninformed agents, which leads to a contradiction. ■

8.4 Proof of Proposition 4

The following lemma shows that at any time t we can start from the equilibrium joint distribution of endowment and beliefs, $\Gamma_t\left(.|s_1\right)$, and eliminate symmetric masses of agents with $\delta < 1/2$ and $\delta \geq 1/2$. By this process, we end up with a distribution of endowment and beliefs $\tilde{\Gamma}_t\left(.|s_1\right)$, where every agent has $\delta \geq 1/2$, and the average endowments of goods 1 and 2 are equal.

Lemma 10 *Let $W = R_+^2 \times [1/2, 1]$ be the set of endowment-belief pairs such that $\delta \geq 1/2$. For any Borel set $F \subseteq W$ let F^C denote the set $\left\{\left(x^1, x^2, \delta\right) : \left(x^2, x^1, 1-\delta\right) \in F\right\}$ and let*

$$\tilde{\Gamma}_t\left(F|s_1\right) \equiv P\left(\left(x_t, \delta_t\right) \in F|s_1\right) - P\left(\left(x_t, \delta_t\right) \in F^C|s_1\right).$$

$\tilde{\Gamma}_t$ is a measure on W with the following property

$$\int_W \left(x^2 - x^1\right) d\tilde{\Gamma}_t = 0 \textit{ for } j = 1, 2.$$

Proof. Notice that by construction $\delta_t \geq 1/2$ if $(x_t, \delta_t) \in F$. Given the definition of δ_t and given that x_t is $\mathcal{F}_t$-measurable, we then have $P\left(s_1|F\right) \geq 1/2$. Moreover, Bayes' rule implies that

$$\frac{P(s_1|F)}{P(s_2|F)} = \frac{P(F|s_1)\,P(s_1)}{P(F|s_2)\,P(s_2)},$$

which, together with $P(s_1|F) \geq 1/2$ and $P(s_1) = P(s_2)$ implies $P(F|s_1) \geq P(F|s_2)$ (this can hold with equality only if all the points in F, except a set of zero measure under both $P(.|s_1)$ and $P(.|s_2)$, have $\delta = 1/2$). By symmetry, we have $P(F|s_2) = P\left(F^C|s_1\right)$, and thus $P(F|s_1) \geq P\left(F^C|s_1\right)$. This shows that $\tilde{\Gamma}_t(F|s_1) \geq 0$ for all $F \subseteq W$.

Recall that $\Gamma_t(F|s_1) = P((x_t, \delta_t) \in F|s_1)$ and let $\Gamma_t^C(.|s_1)$ be defined as

$$\Gamma_t^C(F|s_1) = \Gamma_t\left(F^C|s_1\right).$$

We will use Γ_t, Γ_t^C and $\tilde{\Gamma}_t$ as shorthand for $\Gamma_t(.|s_1), \Gamma_t^C(.|s_1)$ and $\tilde{\Gamma}_t(.|s_1)$. Using market clearing we have

$$\int_{x^2>x^1} \left(x^2 - x^1\right) d\Gamma_t + \int_{x^2=x^1} \left(x^2 - x^1\right) d\Gamma_t - \int_{x^2<x^1} \left(x^1 - x^2\right) d\Gamma_t = \int \left(x^2 - x^1\right) d\Gamma_t = 0,$$

and the middle term in the first expression is zero. By construction $\int_{x^2<x^1} \left(x^1 - x^2\right) d\Gamma_t = \int_{x^1<x^2} \left(x^2 - x^1\right) d\Gamma_t^C$, so, substituting, we have

$$\int_{x^2>x^1} \left(x^2 - x^1\right) \left(d\Gamma_t - d\Gamma_t^C\right) = 0.$$

Decomposing the term on the right-hand side givesd

$$\int_{\substack{x^2>x^1\\ \delta\geq 1/2}} \left(x^2 - x^1\right) \left(d\Gamma_t - d\Gamma_t^C\right) + \int_{\substack{x^2>x^1\\ \delta< 1/2}} \left(x^2 - x^1\right) \left(d\Gamma_t - d\Gamma_t^C\right) = 0.$$

Again, by construction, $\int_{\substack{x^2>x^1\\ \delta<1/2}} \left(x^2 - x^1\right) \left(d\Gamma_t - d\Gamma_t^C\right) = \int_{\substack{x^1\geq x^2\\ \delta\geq 1/2}} \left(x^1 - x^2\right) \left(d\Gamma_t^C - d\Gamma_t\right)$ (where the cases $x^1 = x^2$ and $\delta = 1/2$ are allowed because in the first case $x^1 - x^2 = 0$, in the second case $d\Gamma_t^C - d\Gamma_t = 0$). We conclude that

$$\int_W \left(x^2 - x^1\right) \left(d\Gamma_t - d\Gamma_t^C\right) = \int_W \left(x^2 - x^1\right) d\tilde{\Gamma}_t = 0.$$

■

Lemma 11 *For any $\varepsilon > 0$ there exists $\bar{n}$ s.t. for all $n \geq \bar{n}$*

$$\int_{W, x^2 \leq n, x^1 \leq n} \left| x^2 - x^1 \right| d\tilde{\Gamma}_t \leq \varepsilon$$

Proof. Let $y_n = x^1$ if $x^1 \leq n$ and $y_n = 0$ otherwise. The sequence y_n monotone and converges to x^1 a.e. Then $0 \leq y_n \uparrow x^1$. Monotone convergence theorem (see e.g. Billingsley (1995) Theorem 16.2) which implies $\int_W y_n d\tilde{\Gamma}_t \uparrow \int_W x^1 d\tilde{\Gamma}_t$. Using this result we have there exists some $\bar{n}$ s.t. for all $n \geq \bar{n}$

$$-\int_{W, x^1 \leq n} x^1 d\tilde{\Gamma}_t \leq -\int_W x^1 d\tilde{\Gamma}_t + \varepsilon \tag{46}$$

moreover since $x^2 \geq 0$

$$\int_{W,\ x^1 \leq n} x^2 d\tilde{\Gamma}_t \leq \int_W x^2 d\tilde{\Gamma}_t.$$

So given any ε we can find an $\bar{n}$ such that for all $n \geq \bar{n}$

$$\int_{W,\ x^1 \leq n} \left(x^2 - x^1\right) d\tilde{\Gamma}_t \leq \int_W \left(x^2 - x^1\right) d\tilde{\Gamma}_t + \varepsilon \leq \varepsilon.$$

Analogously we can prove that

$$\int_{W,\ x^1 \leq n} \left(x^2 - x^1\right) d\tilde{\Gamma}_t \leq \int_W \left(x^2 - x^1\right) d\tilde{\Gamma}_t - \varepsilon \geq -\varepsilon$$

whcih complete the proof of the lemma. ■

Proof of Proposition 4. The proof is by contradiction. Suppose equilibrium exists. From Lemma 11 for any $\tilde{\varepsilon}$ there is n s.t.

$$\tilde{\varepsilon} \geq \int_{W,\ x^1 \leq n} \left(x^1 - x^2\right) d\tilde{\Gamma}_t. \tag{47}$$

From Lemma 4, for any $\varepsilon > 0$ there is a set X s.t. for t sufficiently large $\Pr(x_t \in X) \geq 1 - \varepsilon$. Propositions 1 and 3 show that marginal rates of substitution of agents in set X converges in probability to 1. The idea is to use (47) together with these facts to arrive to a contradiction. For this purpose, it is useful to divide all agents from the set $W_n = \left\{\omega : \omega \in W, x^2(\omega) \leq n\right\}$ into four groups and to evaluate integral $\int \left(x^1 - x^2\right) d\tilde{\Gamma}_t$ for each of the groups as $t \to \infty$.

Group 1: Informed agents, whose allocations are in W_n and in X, and for whom MRS converged sufficiently closely to 1. For all such agents $x_1 > x_2$.

Group 2: Uninformed agents, whose allocations are in W_n and in X, and for whom MRS converges sufficiently closely to 1. For all such agents $x_1 \approx x_2$.

Group 3: All agents, whose allocations are in W_n and in X for whom MRS did not converge by period t. The measure of such agents goes to zero.

Group 4: All agents whose allocations are in W_n but not in X. The meaure of such agents is at most ε, and $x^1 - x^2 \geq -n$.

It is clear that as $t \to \infty$, for Group 2 agents $\int \left(x^1 - x^2\right) d\tilde{\Gamma}_t \to 0$, and the measure of agents in Group 3 goes to zero, while for Group 4 agents $\int \left(x^1 - x^2\right) d\tilde{\Gamma}_t \geq -n\varepsilon$. We can obtain a contradition to (47) if we can show that the integral for Group 1 is greater than $\tilde{\varepsilon} - n\varepsilon$. The key to the argument is to show that there is a sufficient mass of informed agents in Group 1 whose endowment is good 1 exceeds endowment of good 2.

In which follows we should how to set $\tilde{\varepsilon}, n, \varepsilon$ so that as $t \to \infty$, (47) is violated.

Group 1. From Lemma 4, there is a compact set X^I, s.t. for all t sufficiently large, $\Pr(x_t \in X^I | s_1) \geq 1 - \alpha/8$. For $\zeta \geq 0$ define

$$d(\zeta) = \min x^1 - x^2$$

s.t.

$$x \in X^I$$

$$\left| \frac{\phi(s_1) u'(x^1)}{(1 - \phi(s_1)) u'(x^2)} - 1 \right| \leq \zeta.$$

By the theorem of maximum, $d(\zeta)$ is a continuous function. Since $d(0) > 0$, there exists $\bar{\zeta} > 0$ s.t. $d(\bar{\zeta}) > 0$.

$d(\bar{\zeta})$ is the minumum utility gain that a significant fraction of informed agents reach if their marginal rates of subsitution converge to 1.

For $\tilde{\varepsilon}$, choose any $\tilde{\varepsilon} \in (0, \alpha d(\zeta)/8)$. Equation (46) implies that for any $\tilde{\varepsilon}$ there is an $\bar{n}$ s.t. for all $n \geq \bar{n}$

$$\tilde{\varepsilon} \geq \int_{W, \left\{\omega : x_t^2(\omega) > n\right\}} x^2 d\tilde{\Gamma}_t \geq n \int_{W \setminus W_n} d\tilde{\Gamma}_t = n \Pr(W \setminus W_n). \tag{48a}$$

Let $n = \max\{\bar{n}, 8\tilde{\varepsilon}/\alpha\}$. For such n, (48a) implies that

$$\Pr(W \backslash W_n) \leq \alpha/8. \tag{49}$$

Choose $\varepsilon \in \left(0, \frac{\alpha d(\bar{\zeta})}{24n}\right)$ and let X_ε be a compact set from Lemma 4.

Let $A_1 = \left\{\omega : \omega \in W_n, x_t(\omega) \in X_\varepsilon, \delta_t(\omega) = \delta^I(s_1), \left|\frac{\phi(s_1)u'(x^1)}{(1-\phi(s_1))u'(x^2)} - 1\right| \leq \bar{\zeta}, s = s_1\right\}$

Note that by construction $\left\{\omega : x_t(\omega) \in X^I, s = s_1\right\} \subset \{\omega : x_t(\omega) \in X_\varepsilon, s = s_1\}$ for any ε. This implies that $P\left(\omega : x_t(\omega) \in X_\varepsilon, \delta_t(\omega) = \delta^I(s_1), s = s_1\right) \geq \frac{7}{8}\alpha$. From Proposition 1, for all t sufficiently large,

$$P\left(\omega : x_t(\omega) \in X_\varepsilon, \delta_t(\omega) = \delta^I(s_1), \left|\frac{\phi(s_1)u'(x^1)}{(1-\phi(s_1))u'(x^2)} - 1\right| \leq \bar{\zeta}\right) \geq \frac{7}{8}\alpha - \varepsilon \geq \frac{6}{8}\alpha$$

Since all the informed agents are in set W, equation (49) implies that

$$P\left(\omega : \omega \in W_n, x_t(\omega) \in X_\varepsilon, \delta_t(\omega) = \delta^I(s_1), \left|\frac{\phi(s_1)u'(x^1)}{(1-\phi(s_1))u'(x^2)} - 1\right| \leq \bar{\zeta}\right) \geq \frac{5}{8}\alpha.$$

By construction

$$\int_{A_1} \left(x^1 - x^2\right) d\tilde{\Gamma}_t \geq \frac{5}{8}\alpha d(\bar{\zeta})$$

Group 2. For $\eta \geq 0$

$$\theta(\eta) = \max_{x \in X_\epsilon} x^2 - x^1$$

s.t.

$$\left|\frac{\pi(\delta)u'(x^1)}{(1-\pi(\delta))u'(x^2)} - 1\right| \leq \eta$$

$$\delta \in [0.5, 1]$$

Since $\theta(\eta)$ is continuous, increasing and $\theta(0) = 0$, there must exist $\bar{\eta}$ s.t.

$$\theta(\bar{\eta}) \leq \frac{\alpha d(\bar{\zeta})/8}{1-\alpha}.$$

For t sufficiently large, by Propostion 3, there exists t s.t. $P\left(x_t(\omega) \in X_\varepsilon, \left|\frac{\pi(\delta_t)u'(x_t^1)}{(1-\pi(\delta_t))u'(x_t^2)} - 1\right| \leq \theta(\bar{\eta}), | \ s_1\right) \geq 1 - \varepsilon$.

Let $A_2 = \left\{\omega : \omega \in W_n, x_t(\omega) \in X_\varepsilon, \delta_t(\omega) \in [0.5, \delta^I(s_1)), \left|\frac{\pi(\delta_t)u'(x_t^1)}{(1-\pi(\delta_t))u'(x_t^2)} - 1\right| \leq \bar{\eta}\right\}$

Then

$$\begin{aligned}\int_{A_2}\left(x^1-x^2\right)d\tilde{\Gamma}_t &\geq -\theta(\bar{\eta})\int_{A_2}d\tilde{\Gamma}_t \\ &\geq -\theta(\bar{\eta})(1-\alpha)\geq -\frac{1}{8}\alpha d(\bar{\zeta})\end{aligned}$$

Group 3. $A_3=\{\omega:\omega\in W_n, x_t(\omega)\in X_\varepsilon, \omega\notin A_1\cup A_2\}$. Then

$$\int_{A_3}\left(x^1-x^2\right)d\tilde{\Gamma}_t\geq -n\int_{A_3}d\tilde{\Gamma}_t\geq -2\varepsilon n$$

Group 4. Let $A_4=\{\omega:\omega\in W_n, x_t(\omega)\notin X_\varepsilon\}$.

$$\int_{A_4}\left(x^1-x^2\right)d\tilde{\Gamma}_t\geq -n\varepsilon$$

Finally we can re-write (47) for t sufficiently large as

$$\begin{aligned}0 &\geq \int_{W,x^2\leq n,x^1\leq n}\left(x^1-x^2\right)d\tilde{\Gamma}_t-\tilde{\varepsilon} \\ &= \sum_{j=1}^{4}\int_{A_i}\left(x^1-x^2\right)d\tilde{\Gamma}_t-\frac{1}{8}\alpha d(\zeta) \\ &\geq \frac{5}{8}\alpha d(\bar{\zeta})-\frac{1}{8}\alpha d(\bar{\zeta})-2\varepsilon n-\varepsilon n-\frac{1}{8}\alpha d(\zeta) \\ &= \frac{3}{8}\alpha d(\bar{\zeta})-3\varepsilon n \\ &\geq \frac{1}{4}\alpha d(\bar{\zeta})>0.\end{aligned}$$

■

9 Computational Appendix

This appendix describes computational algorithms we used to compute numerical examples in Section 6.

We compute an equilibrium in which agent's strategy in period t depends on his allocation of assets inherited from the previous period, x_{t-1}, his beliefs about the probability of signal s_1, δ_{t-1}, and the distribution of beliefs and endowments of other agents $\Gamma_t(\cdot|s)$

in that period for $s = \{s_1, s_2\}$. Notice that an individual agent cannot affect the distribution $\{\Gamma_t(\cdot|s)\}_{t=0}^{\infty}$ since agent's actions are observable only to a measure zero of agents. Therefore, each agent treats the sequence $\{\Gamma_t(\cdot|s)\}_{t=0}^{\infty}$ as given, and the dependence on that sequence can be summarize by the calendar time t, so that the state of each agent is (x, δ, t).

At the beginning of period t, an agent has assets x_{t-1} and beliefs δ_{t-1} and chooses his optimal strategy σ_t to maximize the payoff $W(x_{t-1}, \delta_{t-1}, t)$:

$$\begin{aligned} W(x_{t-1}, \delta_{t-1}, t) &= \max_{\sigma_t} (1-\gamma) E\left\{ U(x_t(\sigma_t), \delta_t(\sigma_t)) \middle| \Pr(s = s_1) = \delta_{t-1} \right\} \\ &\quad + \gamma E\left\{ W(x_t(\sigma_t), \delta_t(\sigma_t), t+1) \middle| \Pr(s = s_1) = \delta_{t-1} \right\}. \end{aligned}$$

An implication of the expression above is that agent's best response strategy $\sigma^* = \{\sigma_t^*\}_{t=1}^{\infty}$ in the infinitely repeated game consists of a sequence of the best responses σ_t^* in a static game where agent's payoff is given by $(1-\gamma)\, U(\cdot, \cdot) + \gamma W(\cdot, \cdot, t+1)$. Therefore, if a sequence of payoffs $\{W(\cdot, \cdot, t)\}_{t=1}^{\infty}$, one can find equilibrium strategies of agents by the following recursive procedure:

1. Start with the initial distribution $\Gamma_0(\cdot|s)$ and compute a static Bayesian Nash equilibrium of this game with payoffs $(1-\gamma)\, U(\cdot, \cdot) + \gamma W(\cdot, \cdot, 1)$;

2. Use equilibrium strategies to compute the distribution in the next period, $\Gamma_1(\cdot|s)$; compute static Bayesian Nash equilibrium for the period $t = 1$;

3. Repeat the above procedure for periods $t = 2, 3, ...$

The two crucial ingredients of this procedure are: (1) finding the sequence of payoffs $\{W(\cdot, \cdot, t)\}_{t=1}^{\infty}$; and (2) finding an equilibrium in a static game with an arbitrary distribution of beliefs and endowments $\Gamma(x, \delta|s)$ and payoffs $((1-\gamma)\, U + \gamma W)$.

Now we describe a general procedure to compute an equilibrium in our games. Then, we discuss some further simplifications we used for computations in Section 6.

For computational purposes, we discretize the state space and the set of offers that agents can make as follows. We fix a grid size (the step of the grid) for the offers to be h_z and for the beliefs to be h_δ. We set the bound for the size of the maximal allowed offer as $\bar{z}$, and the set of allowable offers consists is given by $\mathbf{Z} = Z \times Z$, with $Z \equiv \{\pm n h_z : |n h_z| \leq \bar{z},\ n \in \mathbb{N}\}$, where $\mathbb{N}$ is a set of natural numbers. Similarly, allocations of agents take values on a set $\mathbf{X} = X \times X$, with $X \equiv \{\pm nh_z : |nh_z| \leq \bar{x},\ n \in \mathbb{N}\}$ where $\bar{x}$ is a bound on agent's allocations. Agent's beliefs take values on a set $\Delta \equiv \{0, h_\delta, 2h_\delta, ..., 1\}$.

9.1 Finding an equilibrium in a static game

The first step is to compute an equilibrium in a static, one shot game for some distribution $\Gamma : \mathbf{X} \times \Delta \to [0,1]$ and payoffs $W : \mathbf{X} \times \Delta \to \mathbb{R}$. For this purpose we adopt the algorithm of Fudenberg and Levine (1995) to our Bayesian game. This algorithm computes an approximate equilibrium for a static game, where a degree of approximation depends on a parameter κ. The algorithm has a property as $\kappa \to \infty$ the equilibrium strategies in the approximate equilibrium converge to an equilibrium in the original game[9].

1. Start with the initial guess of a probability that an offer z occurs in equilibrium if the state $s = s_1$: $\psi_0 : \mathbf{Z} \to [0,1]$, $\sum_{z \in \mathbf{Z}} \psi_0(z) = 1$, and $\psi_0(z) > 0$ for all z.

2. For any offer $z = \left(z^1, z^2\right)$ use Bayes' rule to find a posterior belief of any agent with a prior belief δ who receives an offer z:

$$\delta'(\delta, z) = \frac{\delta \psi_0(\left(z^1, z^2\right))}{\delta \psi_0((z^1, z^2)) + (1-\delta)\psi_0((z^2, z^1))}.$$

   If δ' falls outside of the grid point, we round it to the closest point on Δ. Since $\psi_0(z) > 0$ for all z, this rule is well defined.

3. Find the probability χ that an offer z is accepted in state 1. $\chi : \mathbf{Z} \to [0,1]$; $\chi(z) = \sum \Gamma(x, \delta | s_1)$ where the summation is over all $(x, \delta) \in \mathbf{X} \times \Delta$ s.t. $W(x+z, \delta'(\delta, z)) \geq W(x, \delta'(\delta, z))$.

4. Use Bayes' rule to find a posterior of the agent who makes the offer z if such an offer is accepted, δ_a, and a posterior if it is rejected, δ_r:

$$\delta_a(\delta, z) = \begin{cases} \frac{\delta \chi(\left(z^1, z^2\right))}{\delta \chi((z^1, z^2)) + (1-\delta)\chi((z^2, z^1))}, & \text{if } \delta \chi(\left(z^1, z^2\right)) + (1-\delta)\chi(\left(z^2, z^1\right)) > 0 \\ \delta, & \text{otherwise} \end{cases}$$

[9] See Section 3 of Fudenberg-Levine (1995) for a formal statement and a proof.

$$\delta_r(\delta, z) = \begin{cases} \frac{\delta(1-\chi((z^1,z^2)))}{\delta(1-\chi((z^1,z^2)))+(1-\delta)(1-\chi((z^2,z^1)))}, \\ \text{if } \delta(1-\chi((z^1,z^2))) + (1-\delta)(1-\chi((z^2,z^1))) > 0 \\ \delta, \text{ otherwise.} \end{cases}$$

If δ'' falls outside of the grid point, we round it to the closest point on Δ.

5. Find a utility $w(z; x, \delta)$ of the agent (x, δ) if he makes an offer z:

$$\begin{aligned} w(z; x, \delta) &= (\delta\chi((z^1, z^2)) + (1-\delta)\chi((z^2, z^1)))W(x - z, \delta_a(\delta, z)) \\ &+ (1 - (\delta\chi((z^1, z^2)) + (1-\delta)\chi((z^2, z^1))))W(x, \delta_r(\delta, z)) \end{aligned}$$

If the offer $(x - z) \notin \mathbf{X}$, let $w(z; x, \delta)$ be a large negative number, $-\underline{w}$.

6. Define a strategy of an agent with (x, δ) as $\sigma_m(z; x, \delta)$:

$$\sigma_m(z; x, \delta) = \frac{\exp(\kappa w(z; x, \delta))}{\sum_{z' \in \mathbf{Z}} \exp(\kappa w(z'; x, \delta))} \tag{50}$$

Here $\sigma_m(z; x, \delta)$ is the probability that agent (x, δ) makes an offer z.

7. Find a probability of each offer $\sigma_m(z) = \sum_{(x,\delta) \in \mathbf{X} \times \Delta} \sigma_m(z; x, \delta)$. If $||\sigma_m - \psi_0||$ is less than the chosen precision, finish the procedure. Otherwise, let $\psi_1 = \frac{1}{2}\psi_0 + \frac{1}{2}\sigma_m$ and go to Step 1 (for subsequent iterations use $\psi_{n+1} = \frac{n}{n+1}\psi_n + \frac{1}{n+1}\sigma_m$ and repeat the procedure until $||\sigma_m - \psi_n||$ is less than the chosen precision).

In the procedure above, (50) ensures that, for all z, $\sigma_m(z) > 0$ and, since $\psi_0(z) > 0$, $\psi_n(z) > 0$ for all z, n. This ensures that Bayes rule for updating agent's beliefs in Step 2 is well defined.

In computations in Section 6 we further reduce computational complexity by restricting out of the equilibrium beliefs for some offers. We start by considering what is the *lowest* probability that an offer z can be accepted in *any* equilibrium. This probability, $\chi^{\min}(z)$ is defined as $\chi^{\min}(z) = \sum \Gamma(x, \delta | s_1)$ where summation is over all $(x, \delta) \in \mathbf{X} \times \Delta$, s.t. $\min_{\tilde{\delta} \in [0,1]} \left\{ W(x + z, \tilde{\delta}) - W(x, \tilde{\delta}) \right\} > 0$. Next, we follow Steps 3-5 to compute $w(z; x, \delta)$. We define $\sigma_m(z; x, \delta) = 1$ if $z = \arg\max_{z'} w(z'; x, \delta)$ and 0 otherwise and set $\chi_0(z) = \sum_{(x,\delta) \in \mathbf{X} \times \Delta} \sigma_m(z; x, \delta)$. Then we restrict the set of allowed offers to $\tilde{\mathbf{Z}} \equiv \{z \in \mathbf{Z} : \chi_0(z) > 0\}$. With these restrictions we use the iterative procedure described above. This procedure restricts all out of equilirbium beliefs to $\arg\min_{\delta'} \left\{ W(x + z, \delta') - W(x, \tilde{\delta}) \right\}$.

Any offer in a set $\tilde{\mathbf{Z}}$ is accepted at least with a probability $\chi^{\min}$, which means that any offers in a set $\mathbf{Z}\backslash\tilde{\mathbf{Z}}$ are dominated by some offer in a set $\tilde{\mathbf{Z}}$ both on and off the equilibrium path (where of equilibrium beliefs are constracted as the ones which imply the smallest probability of the offer being accepted).

9.2 Finding a sequence of payoffs $\{W(\cdot,\cdot,t)\}_{t=1}^{\infty}$ and an equilibrium of the dynamic game

To compute an equilibrium of a dynamic game, we truncate the game at period T. We assume that if the game has not ended before period T, it ends with probability 1 in period $T+1$.

1. Make a guess on the distribution of beliefs and endowments $\left\{\Gamma_t^0(\cdot,\cdot|s_1)\right\}_{t=1}^{T}$.

2. Let $W_{T+1}^0(\cdot,\cdot) = U(\cdot,\cdot)$. Use the procedure in Section 9.1 to compute equilibrium strategies for a static game with a payoff W_{T+1}^0 and distribution Γ_T^0. Obtains functions ψ, σ_m, $\chi^{\min}$, w.

3. Compute the payoff at the beginning of the period T. For this purpose, let W^m and W^r be, resectively, the payoffs the agents who make and and receive offers. Then

$$W^r(x,\delta) = \sum_{z\in\tilde{\mathbf{Z}}} \psi(z) \max\left\{W_{T+1}^0\left(x+z,\delta'(\delta,z)\right), W_{T+1}^0\left(x,\delta'(\delta,z)\right)\right\}$$

For any $\Gamma(x,\delta|s_1) > 0$ compute utility of the agent who makes an offer as

$$W^m(x,\delta) = \sum_{z\in\tilde{\mathbf{Z}}} \sigma_m(z;x,\delta)w(z;x,\delta)$$

or,

$$\begin{aligned} W^m(x,\delta) \quad = \quad & \max\{\max_{z\in\mathbf{Z}\backslash\tilde{\mathbf{Z}}}(\delta\chi^{\min}(\left(z^1,z^2\right)) + (1-\delta)\chi^{\min}(\left(z^2,z^1\right)))W(x-z,\delta_a(\delta,z)) \\ & +(1-(\delta\chi^{\min}(\left(z^1,z^2\right)) + \\ & +(1-\delta)\chi^{\min}(\left(z^2,z^1\right))))W(x,\delta_r(\delta,z)), \max_{z\in\tilde{Z}} w(z;x,\delta)\} \end{aligned}$$

The beginning of period T payoff is then $\frac{1}{2}W^m + \frac{1}{2}W^r$.

4. Set $W_T^0 = \gamma\left(\frac{1}{2}W^m + \frac{1}{2}W^r\right) + (1-\gamma)\,U$, and returnt to Step 2 until the whole sequence $\left\{W_t^0\right\}_{t=1}^T$ is computed.

5. Start with the initial distribution $\Gamma_1(\cdot,\cdot|s_1)$ and W_1^0 from Step 2 and compute the equilibrium in a one shot game using the algorithm in Section 9.1. Compute

$$\begin{aligned}
\Gamma_2^1(\tilde{x},\tilde{\delta}|s_1) \quad = \quad & \frac{1}{2}\sum_{\substack{\{x,\delta,z:x-z=\tilde{x}\\ \delta_a(\delta,z)=\tilde{\delta}\}}} \sigma^m(z;x,\delta)(\delta\chi(\left(z^1,z^2\right)) + (1-\delta)\chi(\left(z^2,z^1\right)))\Gamma_1(x,\delta|s_1) \\
& + \frac{1}{2}\sum_{\substack{\{x,\delta,z:x=\tilde{x}\\ \delta_r(\delta,z)=\tilde{\delta}\}}} \sigma^m(z;x,\delta)(1-(\delta\chi(\left(z^1,z^2\right)) + (1-\delta)\chi(\left(z^2,z^1\right))))\Gamma_1(x,\delta|s_1) \\
& + \frac{1}{2}\sum_{\substack{\{x,\delta,z:\delta'(\delta,x)=\tilde{\delta},x+z=\tilde{x}\\ W(x+z,\delta'(\delta,z))\geq W(x,\delta'(\delta,z))\}}} \chi(z)\Gamma_1(x,\delta|s_1) \\
& + \frac{1}{2}\sum_{\substack{\{x,\delta,z:\delta'(\delta,x)=\tilde{\delta},x=\tilde{x}\\ W(x+z,\delta'(\delta,z))<W(x,\delta'(\delta,z))\}}} \chi(z)\Gamma_1(x,\delta|s_1)
\end{aligned}$$

   Here, the first term is the transition probabilities of all makers whose offers are accepted, the second term is the transition probabilities of all makers whose offers are rejected, the third term is transition probabilities of all receivers who accept offers and the fourth term is the transition probabilities of all receivers who reject offers. $\Gamma(x,\delta|s_2)$ can be obtained from $\Gamma(x,\delta|s_1)$ using symmetry of equilibrium.

6. Go to Step 5 until the whole sequence $\left\{\Gamma_t^1\right\}_{t=1}^\infty$ is computed.

7. If $||\Gamma^1 - \Gamma^0||$ ($||\Gamma^{n+1} - \Gamma^n||$ in subsequent iterations) is less than chosen precision, finish the procedure. Otherwise, proceed to Step 1.

9.3 Further simplifications with exponential utility function

The procedure described above can be further simplified by assuming exponential utility function $u(x) = -\exp(-x)$ and allowing agents to have any (both positive or negative) x in all periods. In this case the strategies of any agent depend on $\left(x^1 - x^2, \delta, t\right)$, which reduces the number of state variables. To see that this is the case, consider a payoff for any agent (x,δ) in period t by following some strategy σ:

$$
\begin{aligned}
W(x,\delta,t)(\sigma) &= E\left\{\sum_{k=0}^{\infty}(1-\gamma)^k \left[\begin{array}{c} \pi(\delta_{t+k}(\sigma_{t+k}))u(x^1_{t+k}(\sigma_{t+k})) \\ +(1-\pi(\delta_{t+k}(\sigma_{t+k})))u(x^2_{t+k}(\sigma_{t+k})) \end{array}\right] \mid \Pr(s=s_1)=\delta\right\} \\
&= E\left\{\sum_{k=0}^{\infty}(1-\gamma)^k \left[\begin{array}{c} \pi(\delta_{t+k}(\sigma_{t+k}))u(x^1+\sum_{m=0}^{k} z^1_{t+m}(\sigma_{t+m})) \\ +(1-\pi(\delta_{t+k}(\sigma_{t+k})))u(x^2+\sum_{m=0}^{k} z^2_{t+m}(\sigma_{t+m})) \end{array}\right] \mid \Pr(s=s_1)=\delta\right\} \\
&= \exp(-x^2)E\left\{\sum_{k=0}^{\infty}(1-\gamma)^k \left[\begin{array}{c} \pi(\delta_{t+k}(\sigma_{t+k}))u(\left(x^1-x^2\right)+\sum_{m=0}^{k} z^1_{t+m}(\sigma_{t+m})) \\ +(1-\pi(\delta_{t+k}(\sigma_{t+k})))u(\sum_{m=0}^{k} z^2_{t+m}(\sigma_{t+m})) \end{array}\right] \mid \Pr(s=s_1)=\right.
\end{aligned}
$$

Consider any two strategies, σ' and σ'', s.t. $W(x,\delta,t)(\sigma') \geq W(x,\delta,t)(\sigma'')$ for some (x,δ). Since we do not impose bounds on asset holdings x_t, the same strategies σ' and σ'' are feasible for all agents. But then the last expression implies that $W(\tilde{x},\delta,t)(\sigma') \geq W(\tilde{x},\delta,t)(\sigma'')$ for all $\tilde{x}$ s.t. $\tilde{x}^1-\tilde{x}^2 = x^1-x^2$.

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