Optimal Fragile Financial Networks*

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Abstract

We study a financial network characterized by depositors, banks and their shareholders. Belonging to a financial network is beneficial for both the depositors and banks' shareholders since the return to investment increases with the number of banks connected. However, the network is fragile since banks, which invest on behalf of the depositors, have an incentive to gamble with depositors' money when not sufficiently capitalized. The bankruptcy of a bank negatively affects the banks connected to it in the network. First, we compute the social planner solution. This efficient financial network is characterized by a core-periphery structure. Second, we analyze the decentralized solution showing that participating in a fragile financial network is ex-ante optimal when the probability of bankruptcy is sufficiently low, giving rationale of financial fragility as a rare phenomenon. Finally, we analyze the efficiency of the decentralized financial network. On the one hand, if the probability of bankruptcy is sufficiently low the structure of the decentralized financial network coincides with the efficient one. However, in the decentralized network some banks will gamble as compared to the socially preferred outcome. On the other hand, when the probability of bankruptcy is sufficiently high, the decentralized network does not necessarily coincide with the optimal one, and inefficiencies may arise.

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1 Introduction

It is sometimes observed that financial systems turn out to be fragile. With this economists mean that an adverse shock is able to cause the collapse of the entire financial system, implying large consumption losses for investors. As a consequence, it is usually claimed that fragility should be avoided altogether. This paper challenges this view showing under what conditions such fragility could be indeed an optimal feature of financial networks.

We characterize a financial network with the presence of banks, consumers, and banks' shareholders. The economy is made of several regions (countries, sectors) each with its own representative bank. Consumers need to deposit their endowment into banks to take advantage of the investment opportunities in the economy. Shareholders, which are different agents from depositors, provide bank capital and decide the type of investment the bank will choose. Banks have two types of asset in which they can invest. One asset is safe and the other asset is risky. The second asset is risky since it delivers the same return as the safe asset if it succeeds and nothing if it fails, however it gives private benefits to banks' shareholders. The second asset is clearly dominated for depositors and it is a pure gambling asset. Once the type of investment is chosen, both the deposits and the bank capital will be invested in that type of asset. It turns out that poorly capitalized banks find it convenient to gamble. This mimics the usual risk-shifting problem due to limited liability.

Banks also choose whether to join or not the financial network, which structure is fully anticipated. Participating in the network is beneficial for both the depositors and banks' shareholders since the return to both types of investment increases with the number of banks connected. This captures the idea that more connections give access to more investment opportunities. However, the decision of belonging to the network entails a trade-off. The possible bankruptcy of a bank that gambles affects the connected banks in the network. We make the assumption that when a bank fails all the directly linked banks will get the same zero return, that is they will be bankrupt as well. Even if this is an unrealistic assumption, it allows us to focus on an extremely strong form of financial fragility.

We also allow for the possibility of bank capital transfers among banks. These transfers can be interpreted as direct investment in the bank capital of other banks, or exchange of (asymmetric) cross holding of bank capital. The only aim of these bank capital transfers is to solve the moral hazard problem. That is, a safe bank could find it convenient to transfer its bank capital to a gambling bank in 'exchange' of financial stability.

The first-best (efficient) solution is achieved by avoiding the moral hazard problem. A social planner that allocates bank capital in each bank, can guarantee the first-best allocation if a sufficient amount of aggregate bank capital is available in the economy. Otherwise, a constrained first-best allocation will be achieved (i.e., allowing some banks to gamble). In this case, the lower the aggregate bank capital, the lower is the total expected payoff that the (now fragile) financial network can achieve.

Since the transmission of a crisis depends on the links established by the failing bank, contagion is greater the larger is the number of links. Then the optimal network structure is not necessarily the fully connected one. In particular, the efficient structure is characterized by a core-periphery shape. The core includes all the banks that invest in the safe asset and form a complete network structure among them. The periphery includes all the banks that gamble that can be connected among themselves and/or with the core according to the parameters values. It turns out that, for given aggregate bank capital, when the risk of bankruptcy is sufficiently low the efficient structure becomes the fully connected one. Otherwise, the higher the probability of default the less connected the periphery is to the core.

In the decentralized environment the bank capital is allocated randomly across the different banks. Same banks now can have too few capital and will gamble. In the decentralized network the (unconstrained) efficient allocation cannot be reached unless all the individual bank capital endowments are high. However, for a probability of bankruptcy arbitrarily low we show that (i) joining a fragile financial network is ex-ante optimal, (ii) the structure of the decentralized financial network is the same as the constrained efficient one, and (iii) the fragile network delivers a total expected payoff arbitrarily close to the efficient one. The last two results are obtained also considering bank capital transfers. The intuition is the following. When the probability of failure is sufficiently low, the cost of financial fragility becomes lower than the cost of bank capital transfer, and this will make depositors and shareholders willing to take the risk of financial instability.

Accordingly, financial fragility is a rare event since banks and depositors will not enter such network, unless it is ex-ante convenient (that is, unless the probability of default is low). Consequently, the fragile financial networks can be 'optimal' since they are ex-ante Pareto-improving with respect to the autarky situation. However, even when the decentralized network has the same structure of the efficient one, it could be that some banks invest in the gambling asset in the decentralized network while they would choose the safe asset in the efficient one. That is, the core of the efficient network can be larger than the core of the decentralized network. The reason is that, when the probability of default is low, the structure of the network matters more than the type of investments in determining the expected payoffs.

Finally, when the probability of default is sufficiently high, the decentralized network does not necessarily coincide with the constrained efficient one. In particular, it exists a range of values of the probability of default, under which the social planner finds it optimal to link a safe bank with a gambling bank. This occurs when the expected losses of the former are lower than the expected gains of the latter. However, this effect is not internalized by the safe banks that, in the decentralized network, will severe the link with the gambling banks. In this case, the decentralized network is characterized by an inefficiently low degree of connectivity.

The theory of networks has been successfully applied in several economics fields. However, few attempts have been made to use such theory to understand the working of financial systems (see Allen and Babus [2] for a recent survey). One exception is represented by Leitner's [13] model, which gives a rationale of financial networks that are able to spread contagion. We share with Leitner the same goal, extending however his approach in two directions. On the one hand, we model a decentralized formation of the financial network specifying the incentives of the agents involved. On the other hand, we provide a new rationale for fragile financial network formation.

Financial networks in Leitner's [13] model induce private bailouts because of the threat of contagion. Given that formal commitments are impossible, financial networks may be ex-ante optimal since banks can obtain mutual insurance. The bailouts are implemented by means of money transfers between banks. The idea behind Leitner's model is that banks can be surprised by an unexpected liquidity shock which is able to make bankrupt at least one bank in the system (see Allen and Gale, [3]). The possibility that this original failure can spread to the entire system give the rationale of belonging to a financial system. We show that we do not need necessarily the presence of lack of commitment or money transfers among banks, to give rationale of an ex-ante optimal financial network.

Nier et al. [14] consider the link between network models and financial stability. Their approach is to take the network structure as given and study how an exogenous shock is transmitted through the network. Gai and Kapadia [9] study an analytical model of contagion in financial networks with arbitrary structure. However, they assume that the network forms randomly and exogenously, leaving aside issues related to the endogenous network formation, the optimal network structure and its efficiency.

Gale and Kariv [10] show that trade that is restricted to happen on a network generates an efficient outcome as far as the agents are sufficiently connected in the network. Babus [5] considers a model of endogenous network formation, where banks form links with each other in order to reduce the risk of contagion. Similarly to Leitner [13], in her model the financial network serves as an insurance mechanism.

Outside the network literature, various contributions have analyzed financial fragility and contagion. Our approach shares the same scope with the strand of literature that model contagion as the outcome due to the presence of financial links among banks. In particular, banks are connected through interbank deposit markets that are desirable ex-ante, but during a crisis the failure of one institution can have negative payoff effects on the institutions to which it is linked (see Rochet and Tirole, [15]; Allen and Gale, [3]; Aghion, Bolton and Dewatripoint, [1]; Freixas, Parigi and Rochet, [8]). A common feature of these models is the reliance on some exogenous unexpected shock that causes a financial crisis to spill over into other financial institutions. In Allen and Gale [3] and Aghion, Bolton and Dewatripont [1] financial contagion is due to an unexpected aggregate liquidity shortage. Allen and Gale [3] also find that the more connected the interbank deposit market is the more resilient is the system to contagion. Freixas, Parigi and Rochet [8] model financial contagion as a solvency shock to a particular bank. They find that, similarly to Allen and Gale [3], the degree of interbank connections enhance the resiliency of the banking system to withstand the insolvency problem.

More recently, Brusco and Castiglionesi [6] have attempted to model contagion in the banking system without relying on unexpected shocks. All events are anticipated and contractible by the agents. The present model captures some features of the Brusco and Castiglionesi's approach. In particular, the possibility of banks bankruptcy comes from the banks' gambling behavior, which occurs when banks are not sufficiently capitalized. The liquidity coinsurance mechanism (built as in Allen and Gale [3] to face idiosyncratic liquidity shocks) implies that a bank's default will cause the linked banks to fail. Thus, the more connected is the banking system the larger the extent of contagion. If this is the correct reason that more links mean more contagion, one has to wonder how robust this result is when endogenous links are considered. The present paper directly addresses this issue.

On general ground, our model is in line with the results of Allen and Gale [4]. They show that financial default is not always best avoided. Their result holds with incomplete contracts, but it crucially depends on the presence of complete markets for aggregate uncertainty. We do not consider aggregate uncertainty, but we extend the analysis to financial fragility. In our model there are only idiosyncratic shocks (i.e., the gambling behavior of the banks) and contracts are incomplete since it is not possible to contract out bank's moral hazard behavior. When a bank defaults all the linked banks will be bankrupt as well. However, this does not necessarily represent a market failure given that the incidence of financial fragility can be constrained efficient.

The paper is organized as follows. Section 2 sets up the model. Section 3 presents an example of the network formation. Section 4 analyzes the planner problem, characterizing the constrained first-best solution. Section 5 shows the financial network formation and studies its structure. Section 6 characterizes the efficiency of the decentralized financial network. Section 7 presents a discussion of the modeling choices. Section 8 contains the conclusions, and the Appendix contains the proofs.

2 The Model

There are three dates $t = 0, 1, 2$ and one divisible good called 'dollars' (\$). The economy is divided into n regions, each with its own representative bank. Let $N = \{1, 2, ..., n\}$ be the set of the regions and banks. Each region is populated by a continuum of consumers endowed with 1 \$ at $t = 0$. However, they consume at $t = 2$. In order to access the investment opportunities of the economy, each consumer has to deposit her endowment in the representative bank of the region she belongs to.

Each representative bank i randomly receives an endowment $e_i \in [\underline{e}, \overline{e}]$ of dollars, which represents the bank capital and it is owned by banks' shareholders (or investors). Consumers and investors are different type of agents in the economy. The pair (N, e), with $N = \{1, 2, ..., n\}$ and $e = (e_1, e_2, ..., e_n)$ is called an **economy**.

We also allow for transfer of bank capital across banks. Let $x_i = e_i + t_i$ be the bank capital for a bank $i \in N$ after transfers have been made (i.e., t_i is the transfer and can be positive or negative). Then each bank expects to have $1 + x_i$ dollars to invest in $t = 1$. A vector of bank capitals $x = (x_1, x_2, ..., x_n)$ is called feasible for a given economy (N, e) if (i) $x_i \geq 0$ for all i, and (ii) $\sum_{i \in N} x_i = \sum_{i \in N} e_i$. Let $\mathcal{X}$ denote the set of all feasible vectors of bank capital for a given economy (N, e). The sequence of events is reported in Table 1.

Table 1. Sequence of events

Time	Events
$t = 0$	1. Bank's capital is realized;
	2. Financial network is chosen.
$t = 1$	1. Bank's capital transfers are made;
	2. Projects are chosen and investments are undertaken.
$t = 2$	1. Projects cash flows are realized;
	2. Depositors are paid.

Banks have access to two types of project (the investment can be made either in one or the other type of project):

1. The safe project b yields $R > 1$ dollars in $t = 2$ per dollar invested in $t = 1$.

2. The 'gambling' project g yields $R > 1$ dollars with probability η, and 0 dollars with probability $(1 - \eta)$, in $t = 2$ per dollar invested in $t = 1$. We assume $\eta R > 1$. This project yields also a private benefit $B > 0$ to bank's shareholders.

Private benefits are realized by banks' shareholders at the moment of the investment (so they do not have dollar value, consider them as perks or investment in family business). Then the gambling asset becomes a simple device to mimic the risk shifting problem that characterizes financial institutions protected by limited liability. Conditional on the bank choosing the gambling asset, notice that as $\eta \to 1$ the moral hazard problem vanishes. This is true since as $\eta \to 1$ the probability of the bank going bankrupt becomes negligible.

Let $K_i \subseteq N$ be the set of banks to whom bank i is directly linked, then the number of banks connected to bank i is $k_i \in \{0, 1, ..., n - 1\}$. The vector $K = (K_1, K_2, ...K_n)$ captures the interdependence among the banks, and it represents the financial network. We restrict ourselves to undirected networks, i.e. bank i is related to bank j if and only if bank j is related to bank i. Formally, $i \in K_j$ if and only if $j \in K_i$. Let $\mathcal{K}$ denote the set of all possible financial networks for a given economy (N, e).

We assume that the per unit return of the investment (both safe and gambling) is increasing in the number of banks linked with the investing bank i. We indicate the increase of return for each unit of investment with the function $f(k_i)$ with $f'(k_i) > 0$, and $f''(k_i) \leq 0$ for all $k_i \in [0, n - 1]$. We assume that $f(0) = 1$ and $f(n - 1) = \rho$, with $\rho > 1$, that is $f(k_i) \in [1, \rho]$.

The return of bank i for each unit invested is then equal to $f(k_i)R$. Consequently a bank that makes the investment in autarky ($k_i = 0$) will obtain the lowest return R, while a bank that is connected with all the other banks ($k_i = n-1$) will achieve the highest return ρR. This assumption has the natural interpretation that more connections give access to more investment opportunities. However, the increase of return is assumed to occur at decreasing rate. That is, the higher is the number of connections in place and the less valuable is the marginal node since, in this case, it represents an investment opportunity similar to those already existing and then less useful for the diversification purpose.[1]

Let $s_i \in \{b, g\}$ be the choice of project of a bank i. The vector s denotes the investment strategy profile, that is $s = \{s_i\}_{i \in N}$. Let $\mathcal{S}$ denote the set of all possible investment profiles for a given economy (N, e). For given network $K \in \mathcal{K}$ and strategy profile $s \in \mathcal{S}$, let $p_i(K, s)$ be the probability that the project chosen by bank i succeeds. This probability needs to take into account the possible use of the gambling asset in the financial network K. We assume

$$p_i(K, s) = \prod_{j \in K_i \cup \{i\}} \pi_j\left(s_j\right), \tag{1}$$

where $\pi_j\left(s_j\right)$ is defined as

$$\pi_j(s_j) = \begin{cases} 1 & \text{if } s_j = b, \\ \eta & \text{otherwise.} \end{cases}$$

Note that, when bank i and all its neighbors are investing in the safe project b, then the probability of success is 1. However, if bank i or one of its neighbors is investing in the gambling asset, while the rest are investing in the safe asset, the system will not collapse with probability η. This happens since the safe projects work and the gambling asset realizes on positive payoffs. If there are exactly two neighbors who are investing in the gambling asset, then the probability of success is η^2, and so on and so forth.

We are implicitly assuming that once the project of a bank fails, its neighbors will also be bankrupt with probability equal to 1. This implies that we are going to analyze the properties of a network with a very strong form of fragility. Consequently, if we find conditions under which it can be optimal or even efficient to joint such network, this implies that the same conditions still apply for networks characterized by less strong fragility. In other words, we put ourselves in the

[1] In a risk averse environment, the function $f(k)$ would represent the gain in utility coming from diversification. An alternative, but formally equivalent, motivation of the benefit of belonging to a financial network is discussed in Section 7.1.

worst possible scenario to show our results.[2]

Finally, notice that among the strategies of the banks there is no possibility of avoiding the investment. Since the banks can choose to be connected or not to the network, it is always possible for them to be disconnected and invest in autarky. Given that $R > \eta R > 1$ banks and depositors will always choose to invest.

We assume all the agents are risk-neutral. Accordingly, investors in bank i choosing a strategy s_i expect the following payoff (i.e., amount of dollars)

$$m_i(K, x, s) = \begin{cases} p_i(K, s) f(k_i) R x_i & \text{if } s_i = b, \\ p_i(K, s) f(k_i) R x_i + B & \text{otherwise.} \end{cases} \tag{2}$$

The expected payoff is clearly determined by the bank capital after transfers (that is, $x_i = e_i + t_i$), the financial network K chosen at date 0 with the corresponding return to investment $f(k_i)R$, and the strategies of all the other banks s_{-i}, with $s = (s_i, s_{-i})$. Note that equation (2) means that investors in bank i get the profit from their corresponding investment x_i times the probability that the project succeeds. Furthermore when investors in bank i gamble, they obtain private benefits B independently of the gambling asset being successful or not. The expected amount of dollars for depositors in bank i is given by

$$M_i(K, x, s) = p_i(K, s) f(k_i) R. \tag{3}$$

For the given network K and strategy profile $s = (s_i, s_{-i})$, let $g_i\left(K, s_{-i}\right)$ denote the number of gambling neighbors of bank i. To avoid abuse of notation, we will simply make use of g_i instead of $g_i\left(K, s_{-i}\right)$, unless this simplification would lead to confusion. By definition, $g_i \in [0, k_i]$. Then the probability of success $p_i(K, s)$ can be written as

$$p_i(K, s) = \begin{cases} \eta^{g_i} & \text{if } s_i = b, \\ \eta^{g_i + 1} & \text{otherwise.} \end{cases}$$

Let us analyze the incentives that investors in bank i have to choose the safe asset or to gamble, for a given financial network K chosen at $t = 0$. Investors will place the bank's resources in the safe asset whenever the possible bank capital loss incurred in gambling is higher than the private benefits. Then for given $f(k_i)$ and s_{-i} investors in bank i will invest in the safe asset if

[2] Clearly, a more attractive and realistic assumption is that the performance of each bank depends on its investment decisions. This would generate a more complicated probability distribution over returns. However, with a less fragile financial system, the main results of the paper would carry over.

and only if

$$\eta^{g_i} f(k_i) R x_i \geq \eta^{g_i+1} f(k_i) R x_i + B,$$

which implies

$$x_i \geq \frac{B}{(1-\eta)\eta^{g_i} f(k_i) R} \equiv I^*[k_i, \eta, g_i].$$

Clearly, banks with relatively low level of bank capital have incentive to invest in the gambling asset while relatively high capitalized banks do not have this incentive. Then if $x_i \in [I^*(k, \eta, g_i), \overline{e}]$ depositors in bank i expect to get $\eta^{g_i} f(k_i)R$. Otherwise, if $x_i \in [\underline{e}, I^*(k_i, \eta, g_i))$, their expected payoff is $\eta^{g_i+1} f(k_i)R$.

Note that the risk of capital loss vanishes when the probability of default becomes negligible and in this case more an more banks find it convenient to gamble. This is because the cut-off value $I^*[k_i, \eta, g_i]$ becomes bigger and bigger as η converges to 1.

Finally, notice that the cut-off value $I^*(k_i, \eta, g_i)$ is decreasing in k_i, increasing in g_i, and increasing in η if and only if $\eta > \frac{g_i}{1+g_i}$. The level of capital $I^*[k_i, \eta, g_i]$ makes investors in bank i indifferent between investing safe or gamble. As far as g_i is different from zero, the expected payoffs in the two investments will increase due to an increase in η. Then we have to look at the marginal effect of an increase in η on investing safe or gambling. For example, if the increase in the expected payoff from investing safe is larger than the increase in the expected payoff of gambling, then $I^*[k_i, \eta, g_i]$ decreases. The intuition is that the bank needs less bank capital in order to be indifferent between investing safe and gambling. Formally, when η increases, the increase in the investors expected payoff for investing safe is $g_i \eta^{g_i - 1} f(k_i) R x_i$ and for gambling is $(g_i + 1)\eta^{g_i} f(k_i) R x_i$. Thus, when the ratio $\frac{g_i}{(g_i+1)\eta}$ is smaller than 1 the cut off value $I^*[k_i, \eta, g_i]$ is increasing in η. Otherwise, it is decreasing in η. This implies that if $g_i = 0$ the cut-off value is always increasing for any η, while if $g_i > 0$ the cut-off will also be bigger and bigger as η approaches 0.

We next turn our attention on an example to show the intuition behind the financial network formation. The aim is to highlight the different forces that are behind the network formation in the Leitner's [13] model and ours.

3 Network Formation: An Example

In Leitner's model there is one (safe) asset with return $R > 1$ that has a cost of one dollar. In order to realize the return R, all the linked agents in the network need to invest the one dollar. Assume there are two risk-neutral agents: Agent 1 (randomly chosen) has an endowment $e_1 = 2$ and agent 2 has an endowment equal to $e_2 = 0$.

First-best (efficient) allocation. The social planner reaches efficiency by transferring one unit of endowment from agent 1 to agent 2 getting a return R for both agents. In this case the total return is higher than not doing the transfer, that is $R+R > 1+R+0$. In other words, the loss of return of the first agent $(1+R-R)$ is lower than the gain of return of the second agent $(R-0)$.

No linked agents. In this case agent 1 decides whether to make a transfer or not. Without transfer she gets $1+R$ and agent 2 gets 0. If she makes the transfer she will get R and also agent 2 will get R. So no transfer will be made, and agent 2 gets 0.

Linked agents. In this case agent 1 needs agent 2 to invest. Without the transfer agent 1 gets 2 and agent 2 gets 0, as agents can always choose not to invest, and in a Nash equilibrium agent 1 will see that agent 2 cannot invest. If the transfer is made we have that both agents get R, as the social planner coordinates both agents to invest (note that, no one investing is also a Nash equilibrium and both agents get 1). Then, when $R \geq 2$, agent 1 bails out agent 2 through the transfer and being linked achieves efficiency. When $R < 2$ there is no transfer in case of the network, or, in other words, agent 1 does not bail out agent 2.

Network formation. Let φ_i^{nl} and φ_i^l be the return achieved by agent i when she is not linked and when she is linked, respectively. When the agents are not linked, agent 1 will get $\varphi_1^{nl} = 1+R$. When the agents are linked, the payoff of agent 1 depends on making or not the transfer. We have $\varphi_1^l = R$ when the transfer is made, and $\varphi_1^l = 2$ when the transfer is not made. Accordingly, in both cases we have $\varphi_1^{nl} > \varphi_1^l$. Consequently, agent 1 would not enter in the network if she knew her endowment. The assumption in Leitner's model is that agents may agree to join the network *before* they know the realization of their endowments. This uncertainty makes the agents willing to be connected.

Assume there are only two realizations (equally likely) of (e_1, e_2): $(2, 0)$ and $(0, 2)$. Then if the agents are connected they will get a total expected return of $2R$ if $R \geq 2$ (since the transfer is made) and a total expected return of 2 if $R < 2$ (since no transfer is made). If they are not connected they would get a total expected return of $1+R$.

Since $2R > 1+R > 2$, it is always first-best (or Pareto) efficient to join the network, make the transfer, and have both agents invest one unit. The implementability of the first-best depends on the value of R. If $R < 2$ the first-best is not implementable. If $R \geq 2$ the network is implementable (or, according to Leitner, it is second-best). Although participating in the network and bailing out is Pareto efficient, the network is implementable (or second-best) only for a range of values of R. However, in a completely decentralized framework, agents will *never* build a network if they know their endowment in advance, even if the network is implementable (that is, even if $R \geq 2$).

The financial network in Leitner [13] plays the role of a coordinating device for agents affected by lack of commitment. This can be achieved by means of transfers, which in a decentralized economy could be implemented by private bailouts. The network allows ex-post bailout in case some of the agents (randomly chosen) do not have enough resources. Events that cause such agents being short of resources in financial systems can be rationalized by unexpected liquidity shocks (see Allen and Gale, [3]). Therefore, agents have to agree on bailing out each other before their endowments are realized.

We show that the uncertainty on endowments, and consequently the ex-post transfers, is not a necessary condition to give rationale for a fragile financial network. Depositors and investors can find it optimal to belong to a fragile financial network even knowing their endowment and without making ex-post bail out.

Consider an economy with two banks. Since $n = 2$, we have $k_i \in \{0, 1\}$ for $i = 1, 2$. The investment return is then $f(0)R = R$ in autarky, and $f(1)R = \rho R > R$ when the two banks are connected. The first bank is endowed with a level of bank capital equal to $e_1 = I^*(0, \eta, 0) + \varepsilon$ and the second bank has a bank capital equal to $e_2 = I^*(1, \eta, 0) - \varepsilon$. Here, $\varepsilon > 0$ and recall that $I^*(0, \eta, 0) = \frac{B}{(1-\eta)R}$ and $I^*(1, \eta, 0) = \frac{B}{(1-\eta)\rho R}$. In other words, the first bank has enough bank capital to invest in the safe asset even when in autarky, while the second bank has a level of bank capital not large enough to invest safe even when the banks are connected and the first bank invests in the safe asset. Since $I^*(1, \eta, 1) = \frac{B}{(1-\eta)\eta\rho R}$, and assuming $\eta\rho \geq 1$, we have that

$$I^*(1, \eta, 0) < I^*(1, \eta, 1) < I^*(0, \eta, 0). \tag{4}$$

We assume that the values of the parameters are such that the aggregate bank capital is

$$e_1 + e_2 = I^*(1, \eta, 0) + I^*(0, \eta, 0) = 2I^*(1, \eta, 1).$$

The two banks have an amount of dollars to invest equal to $1 + e_1$ and $1 + e_2$, respectively. That is, the amount of deposits (equal to 1 dollar) plus the bank capital.

Social Planner. The first-best is given by both banks choosing the safe asset, since the social planner does not value private benefits. Given this, the project will yield the highest profit if the banks are connected as $\rho > 1$. In order to achieve this outcome, the social planner is assumed to choose the bank capital, the network and, if necessary, to coordinate the banks in the safe investment. We have to clarify that the social planner acts as a substitute of the random draw of bank capital. The planner pools the overall bank capital endowment, equal to $2I^*(1, \eta, 1)$, and he assigns $I^*(1, \eta, 1)$ to each bank connecting them in the financial network. In this way banks' investors find it optimal to invest (without profitable individual deviations) in the safe asset getting a return $\rho R I^*(1, \eta, 1)$ and depositors in both banks get ρR. The total amount of money that the social planner achieves is equal to

$$2\rho R I^*(1, \eta, 1) + 2\rho R = \rho R(2 + e_1 + e_2).$$ [3]

Not Linked. In this case the return of investment is equal to R. First note that investors in bank 1 will never make a transfer to bank 2 since the success of their project is independent of the investment decision of bank 2. Investors in bank 1 will invest in the safe project with a payoff equal to $[I^*(0, \eta, 0) + \varepsilon]R$, and depositors in bank 1 will get R. Investors in bank 2 gamble getting an expected payoff equal to $[I^*(1, \eta, 0) - \varepsilon]\eta R + B$, consequently depositors in bank 2 expect to get $\eta R > 1$.

Linked. In this case the return to investment is equal to $\rho R > R$. However, when the two banks belong to the financial network the investment decision of one bank affects the probability of success of the other bank. Since bank 2 always gambles, this behavior reduces the possibility

[3] In this example the planner does not need to coordinate the banks investment decision since the aggregate bank capital $2I^*(1, \eta, 1)$ is sufficient to make the two bank willing to invest safe even if the other is gambling. So that investing safe is a dominant strategy. However, for lower level of aggregate bank capital, this outcome is not guaranteed. For example, if the aggregate bank capital is $2I^*(1, \eta, 0)$, and the planner equally splits it among the two banks, then the two banks will invest in the safe asset if they expect the other bank to do the same. Otherwise, if they expect the other bank to gamble they will gamble as well since the individual bank capital is less than $I^*(1, \eta, 1)$. Under the assumption that the planner can coordinate the investment decision, he would obtain the same allocation as in the text. For lower levels of aggregate bank capital the planner cannot reach the first-best even with coordination. In this case we have a constrained first-best. Assume for example that aggregate bank capital is $I^*(0, \eta, 0)$. In this case the planner achieves the highest expected payoff putting all the bank capital in one bank and nothing in the other. The former invests safe while the latter gambles. If the planner does not link together the two banks, the total expected payoff is $R[1 + \eta + I^*(0, \eta, 0)]$. If the planner chooses to link the two banks then the total expected payoff is $\eta\rho R[2 + I^*(0, \eta, 0)]$. It is efficient to choose the fragile financial system as long as $\eta\rho > \frac{1+\eta+I^*(0,\eta,0)}{2+I^*(0,\eta,0)}$, which is always satisfied under the assumption that $\eta\rho > 1$.

of success for the depositors and investors in bank 1. A way to avoid this event is to make a bank capital transfer to bank 2. We have four cases that bank 1 faces: a) investing safe without transfer; b) gambling without transfer; c) investing safe with transfer; d) gambling with transfer. Let us examine the four cases in turn.

Case a). Investors in bank 1 expect to get $[I^*(0,\eta,0)+\varepsilon]\eta\rho R$, while the expected payoff for investors in bank 2 is $[I^*(1,\eta,0)-\varepsilon]\eta\rho R+B$. Depositors in bank 1 and bank 2 expect to get $\eta\rho R$.

Case b). Investors in bank 1 always prefer to invest in the safe asset since, by definition, we have $x_1=e_1>I^*(1,\eta,1)$. Then, case b) cannot be an equilibrium.

When a transfer is made from bank 1 to bank 2 (in order to solve its moral hazard problem) we have to check if investors in bank 2 still have incentive to gamble (no commitment to use the transfer properly). We assume that when a bank is indifferent between gambling or safe, it will choose safe.

Case c). Since investors in bank 1 are planning to transfer and play safe, they will choose the minimal transfer that will make bank 2 change from choosing the gambling asset to choosing the safe asset. This will result in $x_2=I^*(1,\eta,0)$, and, consequently, the transfer has to be equal to $t_1=x_2-e_2=I^*(1,\eta,0)-[I^*(1,\eta,0)-\varepsilon]=\varepsilon$. Capital in bank 1 is then $x_1=e_1-t_1=I^*(0,\eta,0)$. Investors in bank 1 get $\rho RI^*(0,\eta,0)$, while investors in bank 2 get $\rho RI^*(1,\eta,0)$. Depositors in both banks get ρR.

Case d). Since bank 1 is planning to gamble, a transfer equal to ε will not induce investors in bank 2 switching from gambling to invest in the safe asset, as by definition, x_2 would be equal to $I^*(1,\eta,0)$, which is smaller than $I^*(1,\eta,1)$. In this case investors in bank 1 will have to make a transfer resulting in a level of capital for bank 2 of at least $I^*(1,\eta,1)$. Since the total amount of capital in the economy is $2I^*(1,\eta,1)$, this will result in bank 1 keeping at most $I^*(1,\eta,1)$ as its own capital. Then the transfer has to be at least

$$I^*(0,\eta,0)-I^*(1,\eta,1)+\varepsilon=I^*(1,\eta,1)-I^*(1,\eta,0)+\varepsilon$$

given that $I^*(1,\eta,0)+I^*(0,\eta,0)=2I^*(1,\eta,1)$. Take $t_1=I^*(1,\eta,1)-I^*(1,\eta,0)+\varepsilon+\delta$, with δ very close to zero (since the bigger the δ the smaller the payoff for investors in bank 1). Depositors in bank 1 expect to get $\eta\rho R$ and the investors expected payoff in bank 1 is $\eta\rho R[I^*(1,\eta,1)-\delta]+B$. Thus, depositors prefer investing in the safe asset than gambling when the transfer is made. However, it may not be necessary preferred for investors in bank 1.

Let us compare equilibria a), c) and d). The one yielding the highest expected payoff for investors will be the continuation equilibrium after banks 1 and 2 have joined the network. Investors in bank 1 prefer a) to c) if and only if

$$\eta[I^*(0,\eta,0)+\varepsilon] \geq I^*(0,\eta,0). \tag{5}$$

As η is arbitrarily high, this expression converges to $\varepsilon \geq 0$. On the other hand, they will prefer a) to d) if and only if

$$\eta\rho R[I^*(0,\eta,0)+\varepsilon] > \eta\rho R[I^*(1,\eta,1)-\delta]+B,$$

which implies

$$B(\eta\rho-1)+(\varepsilon+\delta)\eta\rho R(1-\eta) > B(1-\eta).$$

Again, as η is arbitrarily high, this expression always holds true. So if investors in bank 1 bear the decision of making the bank transfer, they will never make it (ex-ante) for sufficiently high η.[4]

Network formation. In order to see what the incentives of being in a network are, we have to confront the two possible equilibrium situations: the autarky and equilibrium a).

Given that bank 1 depositors get R in autarky and expect to get $\eta\rho R$ when the banks are linked, they will prefer to join the network if and only if $\eta\rho > 1$, which is true. Investors in bank 1 will prefer to join the network with respect being in autarky if and only if

$$\eta\rho R[I^*(0,\eta,0)+\varepsilon] > R[I^*(0,\eta,0)+\varepsilon],$$

which always holds for $\eta\rho > 1$. Therefore, both depositors and investors in bank 1 find it optimal to enter the network. Depositors in region 2 always prefer the network given that they obtain $\eta\rho R$ greater than ηR obtained in autarky. Investors in region 2 also prefer to join the network since

$$\eta\rho R[I^*(1,\eta,0)-\varepsilon]+B > \eta R[I^*(1,\eta,0)-\varepsilon]+B.$$

[4] The reader could note that depositors in bank 1 would prefer instead equilibrium c) to a). This is true under the assumption that the investors pay the tranfer ε. If the transfer would be paid by depositors they would prefer equilibrium a) to c) if and only if

$$(1-\eta)\rho R \leq \varepsilon.$$

Accordingly, also depositors would oppose the tranfer for sufficiently high η. What is crucial for equilibrium a) to be preferred to equilibrium c) is that the transfer has to be relatively more costly than the risk of financial instability.

To sum up, equilibrium a) is optimal for all the agents in the economy and it will be preferred to autarky or equilibrium c) if η is sufficiently high.

Finally, let us consider the efficiency of the equilibria. The total amount of dollars that the social planner achieves is $\rho R(2+e_1+e_2) = 2\rho R[1+I^*(1,\eta,1)]$. Note that this is the same amount of total dollars that equilibrium c) delivers. However, according to (5), we know that when η is high enough the bank capital transfer becomes relatively more costly than the risk of financial instability, and equilibrium c) would not be played in a decentralized environment.

In autarky the financial system would deliver a total expected amount of dollars equal to $R+\eta R+[I^*(0,\eta,0)+\varepsilon]R+[I^*(1,\eta,0)-\varepsilon]\eta R$. Notice that the last expression is necessarily strictly less than the first-best outcome. Indeed, the maximum amount of dollars that the financial system can achieve when both banks are in autarky is when both of them invest in the safe asset. In this case, the financial system gets $R(2 + e_1 + e_2) = 2R[1 + I^*(1, \eta, 1)]$ dollars, which is strictly less than the efficient amount.

Under equilibrium a) instead the total expected amount of dollars that the financial network achieves is

$$2\eta\rho R + [I^*(0, \eta, 0) + \varepsilon]\eta\rho R + [I^*(1, \eta, 0) - \varepsilon]\eta\rho R = 2\eta\rho R[1 + I^*(1, \eta, 1)].$$

If $\eta = 1$, the total dollars available would be exactly the efficient amount of dollars that a social planner would deliver. For $\eta \to 1$, the financial network characterized by equilibrium a) is able to deliver an expected amount of dollars close to the efficient one. That is, as $\eta \to 1$ equilibrium a) gets arbitrarily close to the first-best. Furthermore, the equilibrium network structure is equal to the efficient one (both banks being linked).

The difference between equilibrium a) and the efficient one lies in the investment strategies chosen by the two banks. While in the efficient network both banks invest in the safe asset, in the decentralized network one of the banks is gambling. However, for small probability of bankruptcy, the structure of the network becomes more relevant in determining the total expected payoff of the system.

This example shows that a fragile financial network can deliver the first-best outcome when moral hazard problem is nearly negligible. In other words, joining a fragile financial network not only can be an ex-ante optimal decision but it could allow to achieve an expected payoff arbitrarily close to the first-best solution (i.e., unconstrained efficient). We are going to characterize this result in a more general setting.

4 Constrained First-best Solution

Before running the analysis, we need to introduce some definitions. Let (N, e) be a given economy, as defined in Section 2. An allocation is a vector (K, x, s), where: (i) $x \in \mathcal{X}$, (ii) $K \in \mathcal{K}$, and (iii) $s \in \mathcal{S}$. An allocation thus specifies a reallocation of initial endowments of capital x, a network K and an investment decision s for each bank.

Let the function $m(K, x, s)$ be as defined in equation (2). An allocation (K, x, s) is an *Investment Nash Equilibrium* (INE) for a given economy (N, e) if

$$m_i(K, x, s) \geq m_i(K, x, (s_{-i}, \tilde{s}_i)) \qquad \text{for all } i \text{ in } N,$$

with $\tilde{s}_i \in \{b, g\}$. In other words, an allocation is an INE for a given economy if taking the financial network and capital as given there are no unilateral profitable deviations in the choice of the investment asset, with no possibility for further transfers of bank capital. Note that, if an allocation is an INE for a given economy it has to hold that

$$x_i \geq I^*(k_i, \eta, g_i) \text{ if } s_i = b \qquad \text{for all } i \in N.$$

The constrained first-best solution is characterized by the social planner problem, which is defined as follows.

Definition 1 *Given an economy (N, e), an allocation (K^*, x^*, s^*) is a constrained first-best (CFB) if it maximizes*

$$\sum_{i \in N} p_i(K, s) f(k_i) R\left(x_i + 1\right) \tag{6}$$

subject to

$$x_i \geq 0 \text{ for all } i \in N, \tag{7}$$

$$\sum_{i \in N} x_i = \sum_{i \in N} e_i, \tag{8}$$

$$x_i \geq I^*(k_i, \eta, g_i) \text{ if } s_i = b. \tag{9}$$

We will refer to x^* as the optimal capital allocation, K^* as the optimal financial network and s^* as the optimal investment decision.

Note that we are assuming that the planner is able to perfectly transfer the initial endowments of capital across banks, fix a financial network and suggest investment plans to the banks. We allow banks to unilaterally deviate from investment decisions (see constraint (9)). This restricts the social planner problem in a way that moral hazard has to be taken into account. Finally, note that the social planner does not take into account private benefits B, and only maximizes the total expected amount of money that is generated in the economy. Indeed, the expression

$$p_i(K,s)f(k_i)R\,(x_i+1)$$

does not include B even when $s_i = g$. On the contrary, the payoff $m_i(K,x,s) + M_i(x,K,x)$ as defined in equations (2) and (3) includes B when $s_i = g$.

We first characterize the optimal distribution of bank capital made by the social planner in the CFB allocation.

Proposition 1 *Let (K^*, x^*, s^*) be a CFB for a given economy (N, e). Then,*

1. *For every bank i such that $s_i^* = g$: if there exists another bank j with $k_j^* \geq k_i^*$ such that either $s_j^* = b$ and $g_j \leq g_i$, or $s_j^* = g$ and $g_j < g_i$, then $x_i^* = 0$.*
2. *For every bank i such that $s_i^* = b$ and $g_i > 0$: if there exist another bank j with $k_j^* \geq k_i^*$ such that either $s_j^* = b$ and $g_j < g_i$ or $s_j^* = g$ with $g_j < g_i - 1$, then $x_i^* = I^*(k_i^*, \eta, g_i)$.*

Proposition 1 states that once the minimal bank capital that induces to choose the safe asset is met, the planner will distribute the extra amount of bank capital into the nodes that yield a higher return, either because they are better connected or because they face a smaller risk of bankruptcy.

We then establish the shape of the efficient financial network. The following proposition states that the CFB allocation consists of a core-periphery network structure. In this structure, the *core* banks choose the safe asset and are all connected to each other. The *peripheral* banks choose the gambling asset and can eventually be connected to some core banks and some peripheral banks depending on the value of the parameters. Note that the structure where all banks play safe, is a special core-periphery structure where the periphery consists of no banks, that is, where all banks are in the core.

Proposition 2 *Let (K^*, x^*, s^*) be a CFB for a given economy (N, e). Then, for every pair of banks i and j such that $s_i^* = s_j^* = b$ we have that $i \in K_j^*$ and $j \in K_i^*$.*

The intuition is as follows. When two banks are choosing the safe asset, it is always better to have them connected than unconnected. This is true since one more neighbor always increases investment opportunities. Given that both banks are choosing the safe asset, linking them together does not impose any additional risk. Indeed, if a bank has enough bank capital to choose the safe asset in a given financial network, then the same capital will be sufficient to avoid the gambling behavior if the bank has one more neighbor that invests in the safe asset.

Note that the aggregate amount of bank capital available in the economy restricts the number of banks that will invest in the safe asset in the CFB allocation. It is easy to see that, as far as

$$\sum_{i \in N} e_i \geq n I^*(n-1, \eta, 0)$$

the CFB consists of $x_i \geq I^*(n-1, \eta, 0)$, for all i, $K_i^* = N \backslash \{i\}$, and $s_i^* = b$. In other words, as far as the planner has enough aggregate bank capital, the CFB allocation is equal to the (unconstrained) first-best where moral hazard is completely avoided and all banks are connected investing in the safe asset. When the planner does not have enough aggregate bank capital, i.e. when

$$\sum_{i \in N} e_i < n I^*(n-1, \eta, 0),$$

the structure of the CFB network depends on the relative magnitude between the risk of gambling and the benefits of the financial network.

On the one hand, the higher the η the lower the risk of contagion, so the CFB yields structures where the periphery is more and more connected, until in the limit the CFB financial network is the complete network. On the other hand, the lower the η the higher the risk of contagion, and then the CFB allocations yield structures where the periphery is less and less connected, until in the limit the CFB financial network only connects banks that invest in the safe asset. That is, the CFB is characterized by the complete network on the core and the empty network on the periphery. This is formally stated in the following two propositions.

Proposition 3 *Let (K^*, x^*, s^*) be a CFB for a given economy (N, e) and assume that $\eta \geq \dfrac{f(n-2)}{f(n-1)}$. Then*

1. *$K_i^* = N \backslash \{i\}$ for all $i \in N$.*

2. *Let c^* be the biggest number in $\{1, 2, ..., n\}$ such that*

$$\sum_{i \in N} e_i \geq c^* I^*(n-1, \eta, n-c^*),$$

then the number of banks investing in the safe asset in the CFB is equal to c^*.

The intuition is straightforward. When η is high enough, the risk of bankruptcy is sufficiently low such that it cannot outweigh the advantages of portfolio diversification represented by $f(k)$. Therefore, connecting two banks always yields more money than leaving them unconnected. Note that it is convenient to add a gambling bank into the network as long as

$$\eta f(k_i + 1) > f(k_i).$$

Consequently, if the condition

$$\eta \geq \frac{f(n-2)}{f(n-1)}$$

is satisfied, then it is optimal to add the gambling banks into the network until the $(n-1)th$ bank (i.e., the last bank). Thus, the structure that maximizes the total expected amount of money generated in the economy is the complete network structure.

Proposition 4 *Let (K^*, x^*, s^*) be a CFB for a given economy (N, e) and assume that $\eta < \dfrac{1}{f(1)}$. Then*

1. *$g_i^* = 0$ for all i such that $s_i^* = g$.*
2. *Assume further that $0 < \eta < \dfrac{1}{\rho + (n-1)(\rho - 1)(\bar{e} + 1)}$. Then*
   - *(a) $K_i^* = \emptyset$ for all i such that $s_i^* = g$.*
   - *(b) Let c^* be the biggest number in $\{1, 2, ..., n\}$ such that*

$$\sum_{i \in N} e_i \geq c^* I^*(c^* - 1, \eta, 0),$$

then the number of banks investing in the safe asset in the CFB is equal to c^*.

As before, the same intuition applies. When η is very low, the risk of bankruptcy can outweigh the advantages of financial diversification and it becomes optimal not to connect a bank investing in the gambling asset to another bank. However, we have to differentiate between linking a gambling bank to another gambling bank or to a safe one.

Note that it is not optimal to connect two gambling banks whenever

$$\eta^{g_i} f(k_i) > \eta^{g_i + 1} f(k_i + 1) \Longrightarrow f(k_i) > \eta f(k_i + 1).$$

Since $f(k_i)$ is quasiconcave, if the condition

$$\eta < \frac{f\left(0\right)}{f\left(1\right)} = \frac{1}{f(1)}$$

holds, then it is never optimal to connect two gambling banks.

However, the planner could find it optimal to link a gambling bank to a safe bank if the expected gain of the former are higher than the expected loss of the latter. Proposition 4 establishes a sufficient condition under which this possibility does not occur. Note that the condition

$$\eta < \frac{1}{\rho + \left(n-1\right)\left(\rho - 1\right)\left(\overline{e}+1\right)}$$

can be written as $\eta\rho - 1 + (n-1)\eta(\rho-1)(\overline{e}+1) < 0$. Consequently, the condition implies that $\eta\rho < 1$.

On the one hand, note that when the planner links a gambling bank to a safe bank the expected gain is strictly less than $(n-1)\eta(\rho-1)(\overline{e}+1)R$. To see why, notice that a gambling bank i can expect to gain at most

$$\eta\left[f(k_i) - 1\right]\left(e_i + 1\right)R$$

when it is connected to k_i safe banks. Since $f(.)$ is increasing and $e_i < \overline{e}$, the expected gain is strictly smaller than $\eta(\rho-1)(\overline{e}+1)R$. The number of banks that are in the periphery can be at most $(n-1)$ since one bank has to invest in the safe asset. Then, considering any possible way of joining any number of gambling banks to safe banks, the total expected gain cannot be larger than $(n-1)\eta(\rho-1)(\overline{e}+1)R$.

On the other hand, the expected loss for a safe bank to be connected to g_i gambling banks is

$$\left[\eta^{g_i} f(k_i + g_i) - f(k_i)\right]\left(e_i + 1\right)R.$$

Given that $f(.)$ is increasing and $\eta < 1$, the minimum expected loss is given by $[\eta\rho - 1]\left(e_i + 1\right)R$, which is strictly less than $[\eta\rho - 1]R$. If the minimum expected loss plus the maximum expected gain is still negative, that is

$$(\eta\rho - 1)R + (n-1)\eta(\rho-1)(\overline{e}+1)R < 0,$$

then it cannot be optimal to connect a gambling bank with a safe bank.

Consider again the two-banks case discussed in Section 3. Since $n = 2$, we have

$$\frac{f\left(n-2\right)}{f\left(n-1\right)} = \frac{f\left(0\right)}{f\left(1\right)} = \frac{1}{\rho}.$$

Assume first that η is high, such that $\eta\rho \geq 1$. The CFB allocation will depend on the aggregate amount of bank capital. When $e_1+e_2 \geq 2I^*(1,\eta,0)$ the two banks are linked investing in the safe asset. In this case the CFB allocation is an unconstrained first-best. When $I^*(1,\eta,1) < e_1+e_2 \leq 2I^*(1,\eta,0)$ the two banks being linked is still an optimal financial network, but there is not enough capital for both banks to invest safe. One of the two banks invests optimally in the gambling asset, imposing the risk of contagion on the other bank. Finally, when $0 < e_1 + e_2 \leq I^*(1,\eta,1)$, the CFB allocation still recommends both banks to be connected investing in the gambling asset.

Now assume that η is low, such that $\eta\rho < 1$. Also in this case the CFB allocation will depend on the amount of aggregate bank capital. When $e_1 + e_2 \geq 2I^*(1,\eta,0)$ the two banks will still be linked investing in the safe asset. However, when $I^*(1,\eta,1) \leq e_1 + e_2 < 2I^*(1,\eta,0)$ the two banks being linked might not be an optimal financial network anymore since one bank invests in the gambling asset. It would be optimal to connect them if and only if

$$\eta \geq \frac{e_1 + e_2 + 1}{\rho(e_1 + e_2 + 2) - 1} \equiv \overline{\eta}.$$

Otherwise, if $\eta < \overline{\eta}$, it is optimal to be disconnected. Note that

$$\overline{\eta} > \frac{1}{\rho + (\rho - 1)(\overline{e} + 1)},$$

where $\eta < \dfrac{1}{\rho + (\rho - 1)(\overline{e} + 1)}$ is the sufficient (but not necessary) condition for an empty periphery in Proposition 4. Finally, when $0 \leq e_1 + e_2 < I^*(1,\eta,1)$, there is not enough aggregate bank capital to guarantee at least one bank investing safe, and then it is optimal to disconnect the two banks.

4.1 Examples of Efficient Network Structures

In this Section we provide examples of efficient network structures. In particular, we consider the four banks case since it has been widely used in previous banking literature (see Allen and Gale [3]; Brusco and Castiglionesi, [6]).

Let

$$\sum_i e_i = E$$

be the aggregate amount of bank capital in the economy. Fix the number of banks to be equal to four. If the aggregate bank capital satisfies the condition $E \geq 4I^*(3,\eta,0)$, then, the efficient network structure is the complete one with everybody linked with everybody else and investing in the safe asset. That is, the core of the financial network is made of four banks.

When the aggregate bank capital is such that $E < 4I^*(3, \eta, 0)$, then avoiding moral hazard in the complete network structure is no longer possible. Consequently, the core of the financial network will be made of three (or less) banks.

Figure 1 shows four core-periphery network structures under the assumption that the core is made of three banks. In particular, the two banks represented at the top and the one at the bottom right are investing in the safe asset while the one at the bottom left is investing in the gambling asset.

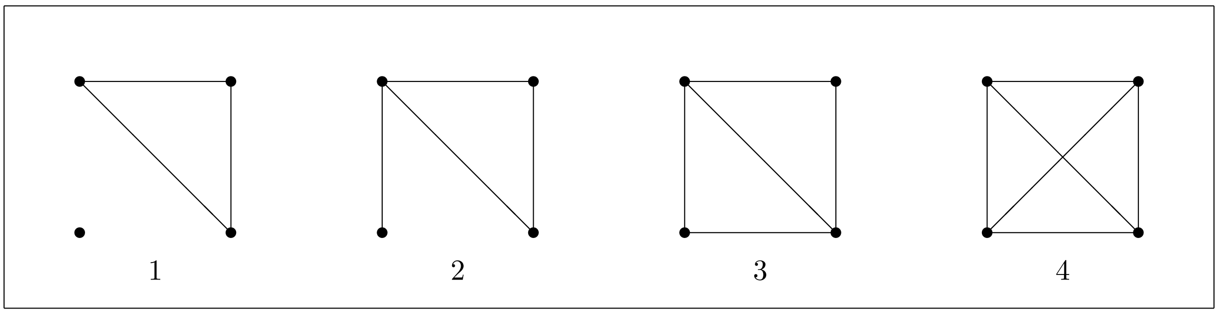

FIGURE 1

Clearly, the efficient structure depends both on the total capital available and on how η relates to the different values of the function $f(k)$. Let us now be more specific about the aggregate bank capital available and the values of η and $f(k)$. Table 2 reports the minimal bank capital requirements for network structures 1, 2, 3 and 4 represented in Figure 1.

Table 2. Minimum aggregate capital for network structures 1, 2, 3 and 4

Structure	Minimum Capital
1	$3I^*(2, \eta, 0) = E_1$
2	$2I^*(2, \eta, 0) + I^*(3, \eta, 1) = E_2$
3	$I^*(2, \eta, 0) + 2I^*(3, \eta, 1) = E_3$
4	$3I^*(3, \eta, 1) = E_4$.

In order for structure 1 to be feasible (i.e., to be an INE), the aggregate bank capital has to be at least E_1. Otherwise, it would not be feasible. The same reasoning applies to the other structures. For each feasible network structure with three banks in the core, there are different ways of allocating the aggregate bank capital. The optimal choice depends on how the values of $\eta f(3)$ and $f(2)$ are related.

Recall that the social planner has to meet the minimal individual capital requirement since the CFB has to be an INE. Then he has to allocate the remaining capital, if any, to the banks with the highest return (given the network). Assume that $\eta f(3) > f(2)$. This implies that a bank with three neighbors, with one of them investing in the gambling asset, obtains higher returns of a bank with two neighbors investing in the safe asset.

Under the assumption that $\eta f(3) > f(2)$ we have $E_1 > E_2 > E_3 > E_4$. The social planner will choose then the structure that maximizes the total expected amount of money generated by the network. Table 3 specifies the total expected amount of money generated by each structure in Figure 1 under the assumption that $\eta f(3) > f(2)$.

Table 3. Total expected amount of money generated by structures 1, 2, 3 and 4

Structure	**Total Expected Payoff**
1	$f(2)[E+3]+\eta$
2	$\eta f(3)[E+1-2I^*(2,\eta,0)]+$ $+2f(2)[I^*(2,\eta,0)+1]+\eta f(1)$
3	$\eta f(3)[E+2-I^*(2,\eta,0)]+$ $+f(2)[I^*(2,\eta,0)+1]+\eta f(2)$
4	$\eta f(3)[E+4]$

From Table 3 it is clear that the total expected amount of money generated by structure 4 is larger than the one generated by structure 3, which is larger than the one generated by structure 2, which is larger than the one generated by structure 1. Consequently, whenever feasible, the planner would choose structure 4. This is the case when $E_4 < E < 4I^*(3,\eta,0)$.

Under the conditions $\eta f(3) > f(2)$ and $E_4 < E < 4I^*(3,\eta,0)$, the planner chooses network structure 4 with the corresponding optimal capital allocation. Intuitively, when η is high enough the optimal network structure is the complete one. Note the difference with the case in which the core of the complete network is made of four banks. The complete network in structure 4 is fragile since one bank is gambling, while before the complete network was safe. The *same* complete structure can be characterized by different degrees of financial fragility. If $E_4 > E$ then the core of the networks has to be made of two banks (or less).

The same reasoning would apply if we have $f(2) > \eta f(3)$. In this case the inequality between the minimum bank capitals in Table 2 becomes $E_1 < E_2 < E_3 < E_4$. If $f(2) > \eta f(3)$ and $E_1 < E < 4I^*(3,\eta,0)$ then the planner chooses network structure 1 for $\eta < \frac{1}{f(3)+3(f(3)-1)(\overline{e}+1)}$, by Proposition 4. In such a case, the optimal network structure recommends to link only banks that are not investing in the gambling asset. Also in this case, if $E_1 > E$ then the core of the networks has to be made of two banks (or less).

Figure 2 represents core-periphery structures for four banks where the core is made of two banks. In particular, the two banks represented at the top are the core ones, while the two banks at the bottom are the periphery ones. The latter banks could be disconnected (as in structures 5 until 11) or connected (as in structures 12 until 18). Moreover, the periphery banks could be not connected to the core (as in structures 5 and 12) or connected to one or both core banks.

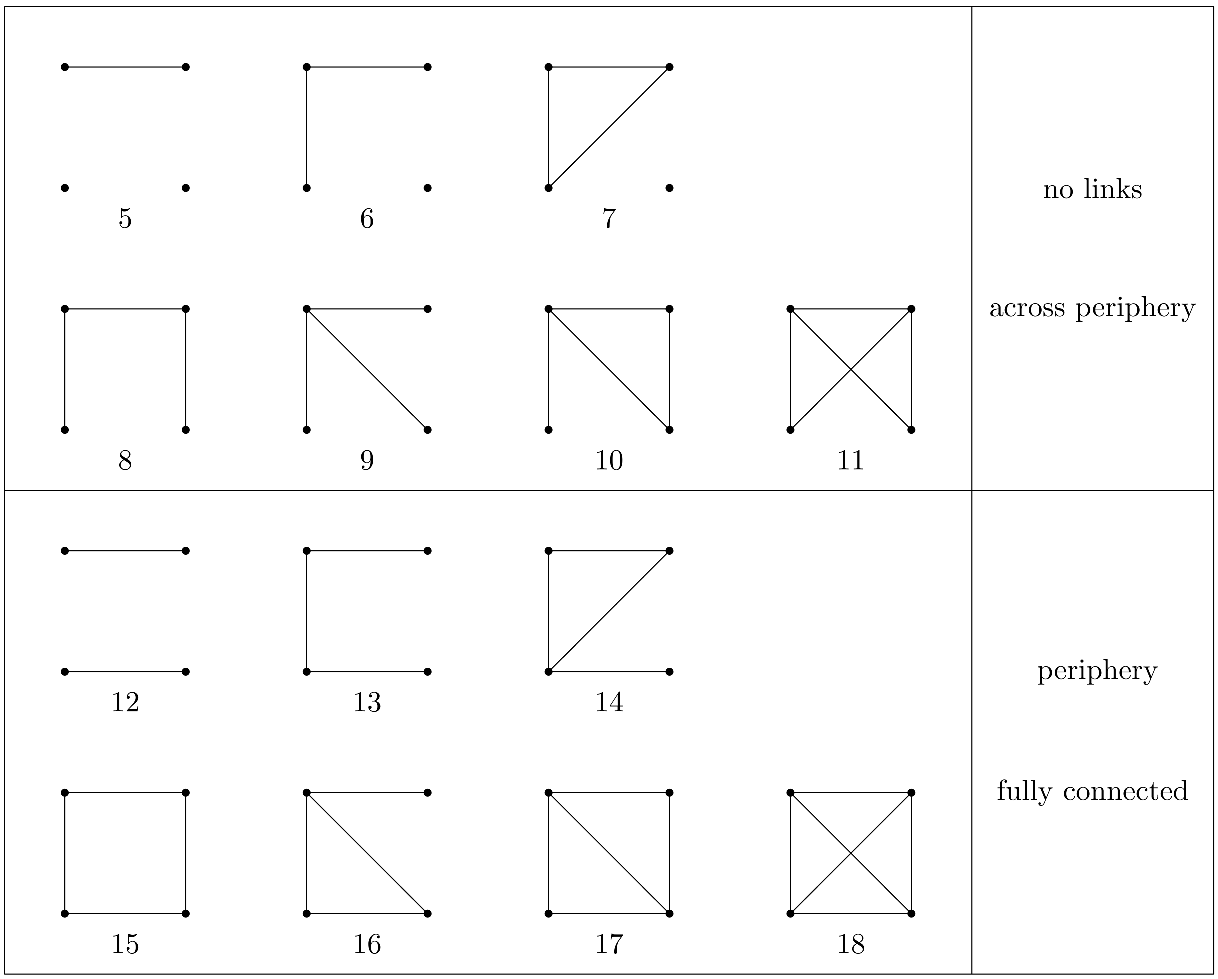

FIGURE 2

Note that structure number 2 is the same as structures 10, 14 and 16. Also structure 3 coincides with structure 17. Structure 4 is equal to structure 18. The difference lies in the investment decisions taken by the banks. The core in structures 2, 3 and 4 is larger than the core in other structures.

Again, under the conditions $\eta f(3) > f(2)$ and $E < E_4$, the optimal network structure would be 18 (for the same reason as before). This means that the planner would choose the fully connected structure also in this case. However, structure 18 implies a financial fragility higher than the one in structure 4 since in the former structure there are two banks gambling while in the latter there was only one bank doing so. Again, the same (fully connected) structure can imply different degree of financial fragility.

However, for different parameter values, other structures can arise as the optimal one. For example, let us consider network structure 15, which is the incomplete structure analyzed in Allen and Gale [3]. As already seen, assuming $E < E_1$ and $\eta f(3) < f(2)$, no structure with a core made of three banks can be an INE (there is not enough capital available). Recall that the function $f(k)$ is quasiconcave. This implies that the ratio $\frac{f(k)}{f(k+1)}$ is increasing in k and $f(k) > \frac{f(k+1)+f(k-1)}{2}$. Furthermore, assume that η satisfies the following condition

$$\frac{f(1)}{f(2)} < \eta < \frac{f(2)}{f(3)}.$$

Note that the condition $\eta f(2) > f(1)$ guarantees that whenever there is a bank connected to at most one other bank, it is better to connect it to another bank even if the new bank is gambling. This implies that structure 8 yields an expected amount of money higher than structure 6. For the same reason, structure 6 is preferred to structure 5; structure 15 is preferred to structures 8, 12 and 13; and structure 16 is preferred to structure 9.

Moreover, since $f(1) > \frac{f(2)+f(0)}{2}$, structure 8 is preferred to structure 7, and, since $f(2) > \frac{f(1)+f(3)}{2}$, structure 15 is preferred to structure 14. It remains to compare then structures 10, 11, 15, 16, 17 and 18. Note that $\eta f(3) < f(2)$ implies that $I^*(3,\eta,2) > I^*(2,\eta,1)$, and that $f(1) < \eta f(2)$ implies that $I^*(1,\eta,0) > I^*(2,\eta,1)$. Let

$$\tilde{E} \equiv \min\{I^*(2,\eta,1) + I^*(3,\eta,2), I^*(2,\eta,1) + I^*(1,\eta,0)\}.$$

If the aggregate of bank capital E lies in the interval $\left[2I^*(2,\eta,1), \tilde{E}\right)$ then structures 10, 11, 16, 17 and 18 are not feasible anymore. So the CFB allocation is given by structure 15 when

1. $\frac{f(1)}{f(2)} < \eta < \frac{f(2)}{f(3)}$, and

2. $2I^*(2,\eta,1) \leq E < \tilde{E}$ and $E < 3I^*(3,\eta,1)$.

Note that this is not the unique case in which structure 15 could be the CFB allocation. Finally, two observations. First, contingent on the failing of *one* bank, the incomplete structure

15 is more resilient than the complete structure 18. Indeed, when one of the gambling banks fails, a safe bank will survive in structure 15 while all the system will collapse in structure 18.

Second, assessing financial fragility in networks with different structures is not so easy since they are characterized by different probability of default. For example, the probability of failure of the entire system in structure 18 is $1-\overline{\eta}^2$, while the same probability in structure 15 is $(1-\underline{\eta})^2$, with $\underline{\eta} < \frac{f(2)}{f(3)} < \overline{\eta}$. Accordingly, financial fragility is higher in structure 18 if

$$\overline{\eta} < \sqrt{\underline{\eta}(2-\underline{\eta})}.$$

Otherwise, structure 15 is characterized by higher financial fragility. The implication is that an higher connectivity of the financial network does not necessarily decrease financial fragility. The relationship between network connectivity and fragility can go either way.

5 Financial Network Formation

In this section we analyze the decentralized financial network formation. To fulfill such task, we start with the concept of INE. Given an INE, we define a Transfer Nash Equilibrium (TNE) where the network structure is taken as given and agents know that, once transfers are realized, banks play an INE in choosing the investment. Finally, using the notion of pairwise stability (Jackson and Wolinsky, [11]), the agents choose the network taking into account the TNE and the INE. In other words, given the sequence of events described in Table 1, we solve the model backwards.

Let the investors payoff function $m(K, x, s)$ be as defined in (2). Recall that an allocation (K, x, s) is an INE for a given economy if taking the financial network and capital as given there are no unilateral profitable deviations in the investment choice. Recall also that an INE allocation has to satisfy for all $i \in N$ that

$$s_i = \begin{cases} b, & \text{if } x_i \geq I^*(k_i, \eta, g_i), \\ g, & \text{otherwise.} \end{cases}$$

In order to analyze the decentralized network formation, we have to specify the transfers that lead to a particular allocation of bank capital x. So far, we did not specify any rule about transfers. We have just assumed that a bank i can only give a transfer to another bank j that is connected to it in the financial network.

Our equilibrium concept will select the allocations where banks *do not* have any further incentive to transfer money through the financial network.

Definition 2 *Given an INE* (K, x, s) *and a bank* i*, another allocation* $(K, \tilde{x}_i, s(\tilde{x}_i))$ *is called a* short-sighted profitable deviation *for a bank* i *from* (K, x) *if there exists a transfer* $0 < t_i \leq x_i$ *and a set of neighbors* $J \subseteq G_i$ *such that:*

1. $\tilde{x}_i(K, x) = x_i - t_i$, $\tilde{x}_j(K, x) = x_j + t_{ij} \geq I^*(k_j, \eta, g_j - |J \cap G_j|)$, with $\sum_{j \in J} t_{ij} = t_i$, for all $j \in J$, and $\tilde{x}_r(K, x) = x_r$ for all $r \notin J$;

2. $$s_i(\tilde{x}_i) = \begin{cases} b, & \text{if } x_i - t_i \geq I^*(k_i, \eta, g_i - |J|), \\ g, & \text{otherwise,} \end{cases}$$
   $s_j(\tilde{x}_i) = b$ for all $j \in J$ and $s_r(\tilde{x}_i) = s_r$ for all $r \notin J$;

3. $m_i(K, \tilde{x}_i, s(\tilde{x}_i)) > m_i(K, x, s)$ and $m_j(K, \tilde{x}_i, s(\tilde{x}_i)) \geq m_j(K, x, s)$ for all $j \in J$.

Note that a profitable deviation for a bank i occurs if it can make transfers to a set of neighbors in the network to avoid those neighbors' moral hazard, expecting these neighbors to accept the transfer. If the transfer does not avoid the moral hazard problem bank i cannot gain anything from making such transfers. Therefore that case is not considered in a profitable deviation.

To keep things tractable, we will assume that after making a transfer to a set of neighbors J, bank i can anticipate changes to the investment decisions corresponding only to that set of neighbors J receiving the transfer and itself. Hence, $s(\tilde{x}_i)$ is restricted to $s_r(\tilde{x}_i) = s_r$ for all $r \notin J$. Note that a short-sighted profitable deviation for a bank i given an INE is not necessarily an INE itself, as some neighbors of banks in J could decide to move to the safe asset when the banks in J do so.

An allocation (K, x, s) is a *Transfer Nash Equilibrium* (TNE) if it is an INE and for all $i \in N$ there is no short-sighted profitable deviation from (K, x). Let the set $\mathcal{T}(K)$ denote the set of all possible TNE capital reallocations and investment strategies, once the financial network has been fixed to be equal to K. Formally,

$$\mathcal{T}(K) = \{(x, s) \in \mathcal{X} \times \mathcal{S} \text{ such that } (K, x, s) \text{ is a TNE}\}.$$

Note that in the definition of the TNE there is no specification about the order or dynamics of transfers. If we allow banks to make transfers to one of their neighbors that are choosing the gambling asset at any point, a TNE is a stable point in a short-sighted way. However, there are no guarantees that a free process of making transfers might end up in a cycle of allocations instead of a single allocation. Nevertheless, we can find conditions under which the transfers are never made. In other words, we can identify the conditions under which for every K there exists a strategy $s(K) \in \mathcal{S}$ such that the allocation $(K, e, s(K))$ is a TNE.

Recall that $\underline{e}$ and $\overline{e}$ are the lower and upper bounds for individual bank capital endowment, respectively. We have the following

Proposition 5 *Given the economy (N, e) and η, assume that one of the following conditions is satisfied:*

1. $\underline{e} \geq \frac{B}{(1-\eta)\eta^{n-1}R}$, *or*

2. $\eta \geq 1 - \frac{B}{2\overline{e}\rho R}$.

Then, for any $s \in \mathcal{S}$ such that (K, e, s) is an INE it is true that (K, e, s) is a TNE.

The intuition of the proposition is as follows. Condition 1 means that all the banks on every possible network are sufficiently capitalized, since $\frac{B}{(1-\eta)\eta^{n-1}R} \geq \frac{B}{(1-\eta)f(k_i)R}$ for any $f(k_i)$. This implies that all the banks invest in the safe asset, consequently there is no need of any bank capital transfer.

Condition 2 implies that everybody plays gamble in every possible financial network, and that two banks never have enough bank capital to make at least one of them choosing the safe asset. This can be achieved either with low bank capital levels (note that if $\overline{e} \leq \frac{B}{2\rho R}$, Condition 2 is satisfied for every $\eta \geq 0$) or with η high enough for any level of bank capital. Either way, whenever Condition 2 holds, there are no incentives for any bank capital transfer.

Recall that η sufficiently high means that the risk of bankruptcy of the gambling banks is low. The only reason it might be profitable for a bank to initiate transferring bank capital to its neighbors is to avoid their gambling behavior. However, if the gambling behavior is not too risky, i.e. there is a low probability of bankruptcy, banks will not find it worthwhile to give away bank capital in exchange of financial stability.

We define an **economy without transfers** an economy (N, e), and the given η, in which either Condition 1 or Condition 2 in Proposition 5 is satisfied.

As already anticipated, our notion for network formation is an adapted version of *pairwise stability* introduced by Jackson and Wolinsky [11]. Before giving the definition we need to introduce some notation. Given a network K, we can define a new network $K \cup ij$ resulting from adding a link joining banks i and j to the existing network K. Formally, $K \cup ij = \left(\tilde{K}_1, ..., \tilde{K}_n\right)$ such that $\tilde{K}_i = K_i \cup \{j\}$, $\tilde{K}_j = K_j \cup \{i\}$ and $\tilde{K}_r = K_r$ for all $r \neq i, j$. On the contrary, for any two banks i and j connected in K, let $K \backslash ij$ denotes the resulting network from severing the link joining banks i and j from K. Formally, $K \backslash ij = \left(\tilde{K}_1, ..., \tilde{K}_n\right)$ such that $\tilde{K}_i = K_i \backslash \{j\}$, $\tilde{K}_j = K_j \backslash \{i\}$ and $\tilde{K}_r = K_r$ for all $r \neq i, j$.

Definition 3 *An allocation (K, e, s) is pairwise stable without transfers (PSWT) if the following holds:*

1. For all i and j directly connected in K: $m_i(K, e, s) \geq m_i(K \backslash ij, e, \tilde{s})$ and $m_j(K, e, s) \geq m_j(K \backslash ij, e, \tilde{s})$ for all allocations $(K \backslash ij, e, \tilde{s})$ that are INE.

2. For all i and j not directly connected in K: if there is an INE $(K \cup ij, e, \tilde{s})$ such that $m_i(K, e, s) < m_i(K \cup ij, e, \tilde{s})$, then $m_j(K, e, s) \geq m_j(K \cup ij, e, \tilde{s})$.

The definition of PSWT captures two ideas that directly derive from the notion of pairwise stability. The first idea refers to the network internal stability: No pair of banks directly connected in the current financial network individually gain from severing their financial link. This idea implicitly states that any of the two banks could sever the link unilaterally. The second idea establishes the network external stability: If one bank could gain from creating a link with another bank, it has to be that the other bank cannot gain from that link. This idea implicitly assumes that both banks have to agree in order to create a new link. The willingness of one bank in creating a new link is not enough to change the network structure.

Note that we define the equilibrium as pairwise stable without transfers since (i) the banks are assuming that no transfers are going to be made once the network is formed and (ii) the resulting outcome is going to be an INE fixing all bank capitals to be equal to the initial endowments. From Proposition 5, pairwise stability without transfers makes sense in the context of an economy without transfers. If none of the conditions that guarantee that no transfers are taking place, some banks in the financial system could be willing to transfer bank capital to their neighbors. In such a case, it is not guaranteed that the system will rest in a TNE once the dynamic of transfers has started, or it will rest in a cycle of transferring behavior.[5]

Definition 4 *An allocation (K, e, s) is a decentralized equilibrium without transfers (DEWT) if it is INE, TNE and PSWT.*

We proceed to describe the set of decentralized equilibria for a given economy (N, e) by means of the following proposition.

[5] One could impose additional conditions on the problem, but the spirit of the paper would remain the same.

Proposition 6 *Assume that η and (N, e) define an economy without transfers. Then, a DEWT is a core-periphery structure, i.e., if (K^e, e, s^e) is a DEWT, then, for every pair of banks i and j such that $s_i^e = s_j^e = b$, we have that $i \in K_j^e$ and $j \in K_i^e$.*

On the one hand, a bank agrees to be connected to any neighbor that is choosing the safe asset since this decision entails no extra risk of bankruptcy. On the other hand, if a bank plays safe any other bank would like to be connected to it for the same reason. Since links are expected to be beneficial for both participating banks, two banks choosing safe will normally be connected. Therefore a core-periphery structure appears where all banks choosing the safe asset are connected among themselves. The connectivity of banks choosing gambling (the low capitalized banks) depends on how the parameters of the model relate.

Note that Proposition 6 does not imply that the network structure in a DEWT is the same as in the optimal one. The core in the CFB might have a different size than the core in any DEWT.

6 Financial Network Efficiency

With the presence of moral hazard the (unconstrained) first-best cannot be reached when the aggregate bank capital available is not sufficiently large. However, when the probability of bankruptcy is sufficiently low, we show that the structure of the decentralized network is equal to the efficient one. Furthermore, the total payoff delivered by the decentralized network is arbitrarily close to the efficient one. We have the following

Proposition 7 *If η and (N, e) define an economy without transfers, and $\eta \geq \dfrac{f(n-2)}{f(n-1)}$ then the only network structure for any DEWT is the complete network structure.*

This result states formally the idea that when the risk of financial contagion is sufficiently low, it is always worthwhile to take the risk of being connected to a low capitalized bank in order to obtain the advantages resulting from investment diversification.

Note that Proposition 7, even if it establishes that the decentralized network is equal to the optimal one, it does not imply that investment decisions are the same in both networks. In the DEWT the investment decisions might be suboptimal, as the example in Section 3 showed, since some banks can gamble while in the efficient network they would invest in the safe asset. However, when η tends to 1, investment profiles yield the same total expected amount of money provided they have the same network structure. In other words, as the moral hazard problem vanishes, the only factor that determines the expected payoff *is* the network structure.

Proposition 8 *If η and (N, e) define an economy without transfers, and $\eta < \frac{1}{f(1)}$ the only network structure for any DEWT is a core-periphery structure where the periphery has no links.*

Proposition 8 states that when the probability of default is sufficiently high, the decentralized financial network has zero probability of contagion. Only banks that are choosing the safe asset have connections in the decentralized network. While Proposition 8 establishes that $\eta < \frac{1}{f(1)}$ is sufficient to observe an empty periphery in the decentralized network, Proposition 4 stated that the same condition is not sufficient for the empty periphery to be optimal.

The condition $\eta < \frac{1}{f(1)}$ in Proposition 8 guarantees that no safe bank wants to be connected to a gambling bank. However, we have seen that the expected gains for the gambling bank could outweight the expected loss for the safe bank if they get connected, in which case the planner finds optimal to link them together. As a consequence, gambling banks are (inefficiently) under-connected in the decentralized network when

$$\eta \in [\frac{1}{\rho + (n-1)(\rho - 1)(\bar{e} + 1)}, \frac{1}{f(1)}].$$

Finally, in order to fully establish the efficiency of the decentralized network structures, we have to combine the conditions in Propositions 7 and 8 with the conditions stated in Proposition 5.

Condition 1 in Proposition 5 implies that everybody plays safe in every possible network. This means that the fully connected network will be formed and everybody chooses the safe asset. This is indeed the (unconstrained) first-best outcome.

Condition 2 in Proposition 5 implies that everybody plays gamble in every possible network. On the one hand, if the condition in Proposition 7 holds, the complete network will be formed, which is the same structure as in the CFB. Again, this does not mean that the optimal investment profile is to choose the gambling asset for all banks, since the social planner could pool the bank capitals inducing one (or some) bank to choose the safe investment. On the other hand, if the condition in Proposition 8 holds, the resulting structure is the empty network since all banks are periphery banks. This structure could not coincide with the CFB structure if by pooling all the bank capital endowments the planner can get at least one bank choosing the safe asset.

Overall, when η is sufficiently high (according to the condition in Proposition 7), we have established that the decentralized financial network is characterized by a structure that coincides either with the (unconstrained) first-best structure or the CFB structure. On the contrary, when η

is sufficiently low (according to the condition in Proposition 8), the structure of the decentralized network is not the same as the CFB structure.

7 Discussion

7.1 Alternative Motivation for the Network Benefits

We motivated the benefit of the financial network with the assumption that the return to investment increases with the number of banks connected. Alternatively, another rationale for the benefits of establishing financial links among banks can be found also in the banking literature. If banks face idiosyncratic liquidity shocks due to consumers' consumption preferences (Diamond and Dybvig, [7]), and as long as there is no aggregate liquidity shortage, the uncertainty arising from these shocks can be eliminated by establishing financial links (Allen and Gale, [3]). Moreover, banks in autarky would need to invest more resources in short term liquidity to prevent high idiosyncratic liquidity shocks, and consequently less resources can be invested in more profitable long term projects. The same result holds when banks are affected by moral hazard problem (Brusco and Castiglionesi, [6]).

To capture this feature we could assume that the per unit cost of the investment (both safe and gambling) is decreasing in the number of banks linked with the investing bank $i \in N$, where $K_i \subseteq N$ is still the set of banks to whom bank i is directly linked. Again, the number of banks connected to bank i is $k_i \in \{0, 1, ..., n-1\}$. We indicate the cost of investment with the function $C(k_i)$ with $C'(k_i) < 0$, and $C''(k_i) \leq 0$ for all $k_i \in [0, n-1]$. We assume that $C(0) = 1$ and $C(n-1) = c$, with $0 < c < 1$, that is $C(k) \in [c, 1]$. Consequently a bank that makes the investment in autarky ($k = 0$) faces the highest investment cost, while a bank that is connected with all the other banks ($k = n-1$) faces the lowest investment cost.

The assumption has the natural interpretation that more connections give access to more opportunities for risk-sharing the idiosyncratic liquidity shocks. Also here the reduction in the cost of the investment occurs at decreasing rate. That is, the higher is the number of established connections and the less valuable is the marginal node since, in this case, it represents a bank with idiosyncratic liquidity shock similar to those banks already in the network and then less useful for the risk-sharing purpose.

With a little change in notation, in particular considering $\frac{R}{C(k_i)}$ instead of $f(k_i)R$, this assumption preserves the same mathematical structure of the one used in the previous analysis and consequently all our results go through.

7.2 Linkages Externalities vs. Network Externalities and Policy Implication

In this paper we study the optimality of financial networks, analyzing the contagious effect of the links *directly* established among banks. Accordingly, we focus on the externalities that arise from the linkages that banks voluntary establish. We ruled out from the analysis possible systemic effects or network externalities. For example, the bankruptcy of a certain number of banks can cause the collapse of the entire financial network because it could affect the payment system (see Freixas, Parigi and Rochet [8] and Kahn and Santos [12]). Since we consider this an interesting topic to analyze in the framework of network formation, we leave it for future research.

However, this paper points out that these systemic effects are not necessary in order to have inefficiencies in a financial network. When the probability of default is sufficiently high, we found that banks could be inefficiently under-connected in the decentralized network. This imply that banks sever the links even when they should not, and the network structure can become inefficiently empty. A policy intervention is then granted. Nevertheless, in these circumstances injecting liquidity through open market operation or discount window interventions may be ineffective because they do not address the fundamental cause.

This ineffectiveness has been evident during the recent subprime mortgage crisis, where banks feared hidden losses in their counterparts. Banks became reluctant to lend one another. Interbank lending rates started to rise and soon the market for short term lending dried up. This paper may help explain this kind of phenomena in interbank markets.

When the risk associated with the lending of funds is too high, connections become too costly relative to the benefits they bring and banks sever the financial links. This could be inefficient, and the kind of intervention that is needed is to restore confidence. This would be achieved most directly by means of bank capital injection (for example, in the form provided by sovereign wealth funds).

8 Conclusion

We present a model where fragile financial networks can be optimal when aggregate bank capital is not large enough to avoid any problem of moral hazard. From a decentralized point of view, banks can rationally join a fragile financial network as far as the risky asset fails with a sufficiently low probability. We characterize the set of optimal financial networks as core-periphery structures that get more and more connected as the probability of failure becomes smaller. The financial networks resulting from a decentralized process of network formation are also core-periphery structures provided there are no bank capital transfers, although they may not be exactly equal to the optimal ones. Furthermore, the investment profiles in the decentralized system are in general different from the efficient. However, our main conclusion is that the decentralized system works close enough to the social planner's solution when the probability that the network collapses is negligible. On the contrary, when the same probability is sufficiently high, inefficient structures of the decentralized financial network arise.

Appendix

Proof of Proposition 1. The first statement is proved by contradiction. Assume that (K^*, x^*, s^*) is a CFB for (N, e) such that there is a bank i with $s_i^* = g$ and $x_i^* > 0$, and there is another bank j with $k_j^* \geq k_i^*$, $s_j^* = b$ and $g_j \geq g_i$ (the proof of the case for $s_j^* = g$ and $g_j < g_i$ is equivalent and therefore omitted). Take $(\hat{x}, K^*, s^*)$ a new allocation for the same economy (N, e), where the network and investment strategies are the same. The allocation of capital differs in bank j receiving all the capital endowment of bank i. Formally, $\hat{x}_i = 0$, $\hat{x}_j = x_j^* + x_i^*$, and $\hat{x}_r = x_r^*$ for all $r \neq i, j$. We show that $(\hat{x}, K^*, s^*)$ yields a higher expected total money generated in the economy. Therefore, the initial allocation (K^*, x^*, s^*) cannot be a solution to the planners problem and statement 1 follows. Note that if (K^*, x^*, s^*) is a CFB then it has to be an INE. Therefore, as j is choosing the safe asset, we have that $x_j^* \geq I^*(k_j^*, \eta, g_j)$. By definition of $\hat{x}$, we have that $\hat{x}_j > x_j^* \geq I^*(k_j^*, \eta, g_j)$ and therefore the allocation $(\hat{x}, K^*, s^*)$ is an INE. Furthermore,

$$\begin{aligned} & \sum_{r \in N} p_r(K^*, s^*) f(k_r^*) R\left(\hat{x}_r + 1\right) - \sum_{r \in N} p_r(K^*, s^*) f(k_r^*) R\left(x_r^* + 1\right) \\ = \quad & R x_i^* \left[\eta^{g_j} f(k_j^*) - \eta^{g_i + 1} f(k_i^*)\right] \geq 0, \end{aligned} \tag{10}$$

as $k_j^* \geq k_i^*$, the function $f(k)$ is increasing in k and $g_j \leq g_i$. By equation (10), the allocation $(\hat{x}, K^*, s^*)$ yields a higher expected total money in the economy and therefore (K^*, x^*, s^*) was not a CFB.

The second statement is also proved by contradiction. Assume that (K^*, x^*, s^*) is a CFB for (N, e) but there is a bank i such that $s_i^* = b$ and $x_i^* > I^*(k_i^*, \eta, g_i)$, but there is another bank j with $k_j^* \geq k_i^*$ such that $s_j^* = b$ and $g_j < g_i$ (the proof of the case for $s_j^* = g$ and $g_j < g_i - 1$ is equivalent and therefore omitted). Take $(\hat{x}, K^*, s^*)$ a new allocation for the same economy (N, e), where the network and investment strategies are the same. The allocation of capital differs in bank j receiving all *extra* capital endowment of bank i. Formally, $\hat{x}_i = I^*(k_i^*, \eta, g_i)$, $\hat{x}_j = x_j^* + x_i^* - I^*(k_i^*, \eta, g_i)$, and $\hat{x}_r = x_r^*$ for all $r \neq i, j$. We show now that $(\hat{x}, K^*, s^*)$ yields a higher expected total money generated in the economy. Therefore, the initial allocation (K^*, x^*, s^*) cannot be a solution to the planner's problem and statement 2 follows.

As before, since (K^*, x^*, s^*) is a CFB then it has to be an INE. This means that, as $\hat{x}_j > x_j^* \geq I^*(k_j^*, \eta, g_j)$, the allocation $(\hat{x}, K^*, s^*)$ is also an INE. Furthermore,

$$\begin{aligned} & \sum_{r \in N} p_r(K^*, s^*) f(k_r^*) R\left(\hat{x}_r + 1\right) - \sum_{r \in N} p_r(K^*, s^*) f(k_r^*) R\left(x_r^* + 1\right) \\ = \quad & R[x_i^* - I^*(k_i^*, \eta, g_i)] \left[\eta^{g_j} f(k_j^*) - \eta^{g_i} f(k_i^*)\right] \geq 0, \end{aligned} \tag{11}$$

given that $k_j^* \geq k_i^*$, and the function $f(k)$ is increasing in k and $g_j \leq g_i$. By equation (11), the allocation $(\hat{x}, K^*, s^*)$ yields a higher expected total money in the economy and therefore (K^*, x^*, s^*) was not a CFB. ■

Proof of Proposition 2. The proof is by contradiction. Assume that (K^*, x^*, s^*) is a CFB for (N, e) but there are two unconnected banks i and j such that $s_i^* = s_j^* = b$. Take a new allocation $(x^*, \hat{K}, s^*)$ for the same economy (N, e), where the allocation of bank capital and strategies are the same but the network structure only adds the link of banks i and j. Formally, $\hat{K}_i = K_i^* \cup \{j\}$, $\hat{K}_i = K_j^* \cup \{i\}$, and $\hat{K}_r = K_r^*$ for all $r \neq i, j$. We show now that the allocation $(x^*, \hat{K}, s^*)$ yields a higher expected total money generated in the economy. Therefore, the initial allocation (K^*, x^*, s^*) cannot be a solution to the planners problem and the statement of Proposition 2 follows.

Note first that, by definition of $\hat{K}$, $p_r(\hat{K}, s^*) = p_r(K^*, s^*)$, for all $r \in N$, and

$$\hat{k}_r = \begin{cases} k_r^* + 1, & \text{if } r = i \text{ or } r = j \\ k_r^*, & \text{otherwise.} \end{cases}$$

On the other hand, if (K^*, x^*, s^*) is a CFB then it has to be an INE. Therefore, as both i and j are choosing the safe asset, we have that $x_i^* \geq I^*(k_i^*, \eta, g_i)$ and $x_j^* \geq I^*(k_j^*, \eta, g_j)$. By definition of the function $I^*(., \eta, .)$, it is true then that $x_i^* \geq I^*(k_i^* + 1, \eta, g_i)$ and $x_j^* \geq I^*(k_j^* + 1, \eta, g_j)$. Therefore, the allocation $(x^*, \hat{K}, s^*)$ is an INE. Furthermore,

$$\begin{aligned} & \sum_{r \in N} p_r(\hat{K}, s^*) f(\hat{k}_r) R\left(x_r^* + 1\right) - \sum_{r \in N} p_r(K^*, s^*) f(k_r^*) R\left(x_r^* + 1\right) \\ = \quad & \eta^{g_i} R\left(x_i^* + 1\right)\left[f(k_i^* + 1) - f(k_i^*)\right] + \eta^{g_j} R\left(x_j^* + 1\right)\left[f(k_j^* + 1) - f(k_j^*)\right] \geq 0, \end{aligned} \tag{12}$$

since the function $f(k)$ is increasing in k. By equation (12), the allocation $(x^*, \hat{K}, s^*)$ yields a higher expected total money in the economy and therefore (K^*, x^*, s^*) was not a CFB. ■

Proof of Proposition 3. We prove statement 1 by contradiction. Assume that (K^*, x^*, s^*) is a CFB allocation where there are at least two banks i and j not directly connected in K^*. Take a new allocation $(x^*, \hat{K}, s^*)$ for the same economy (N, e), where the allocation of capital and strategies are the same but the network structure only adds the link of banks i and j. Formally, $\hat{K}_i = K_i^* \cup \{j\}$, $\hat{K}_j = K_j^* \cup \{i\}$, and $\hat{K}_r = K_r^*$ for all $r \neq i, j$. We show that there is an INE that yields at least the payment of this allocation $(x^*, \hat{K}, s^*)$, which is a higher expected total money generated in the economy for η high enough. Therefore, the initial allocation (K^*, x^*, s^*) cannot be a solution to the planner's problem and statement 1 follows.

First note that if (K^*, x^*, s^*) is a CFB then it has to be an INE. If at least one of them, for example i, is choosing safe (the equivalent applies for j), it means that $x_i^* \geq I^*(k_i^*, \eta, g_i)$ and the other one is playing gamble (by Proposition 2 we know that they cannot be both investing safe, otherwise (K^*, x^*, s^*) would not be a CFB allocation). Note that $I^*(k_i^*, \eta, g_i) \geq I^*(k_i^*+1, \eta, g_i+1)$ if and only if $\eta f(k_i^*+1) \geq f(k_i^*)$, which is true for $\eta \geq \dfrac{f(n-2)}{f(n-1)}$.

Recall that $f(k_i)$ is quasiconcave, then the ratio $\dfrac{f(k_i)}{f(k_i+1)}$ is increasing in k_i and

$$\frac{f(n-2)}{f(n-1)} \geq \frac{f(k_i)}{f(k_i+1)} \text{ for } k_i \leq n-2.$$

Therefore, the allocation $(x^*, \hat{K}, s^*)$ could be an INE, in which case,

$$\begin{aligned} & \sum_{r \in N} p_r(\hat{K}, s^*) f(\hat{k}_r) R(x_r^*+1) - \sum_{r \in N} p_r(K^*, s^*) f(k_r^*) R(x_r^*+1) \\ = \; & \eta^{g_i} R(x_i^*+1)\left[\eta f(k_i^*+1) - f(k_i^*)\right] + \eta^{g_j+1} R\left(x_j^*+1\right)\left[f(k_j^*+1) - f(k_j^*)\right], \end{aligned} \tag{13}$$

if i chooses the safe asset, or

$$\begin{aligned} & \sum_{r \in N} p_r(\hat{K}, s^*) f(\hat{k}_r) R(x_r^*+1) - \sum_{r \in N} p_r(K^*, s^*) f(k_r^*) R(x_r^*+1) \\ = \; & \eta^{g_i+1} R(x_i^*+1)\left[\eta f(k_i^*+1) - f(k_i^*)\right] + \eta^{g_j+1} R\left(x_j^*+1\right)\left[\eta f(k_j^*+1) - f(k_j^*)\right], \end{aligned} \tag{14}$$

if i chooses the gambling project. Note that both (13) and (14) are greater than zero since $\eta \geq \dfrac{f(n-2)}{f(n-1)} \geq \dfrac{f(k_i)}{f(k_i+1)}$ by assumption and the function $f(k)$ is increasing in k. Therefore, the allocation $(x^*, \hat{K}, s^*)$ yields a higher expected total money in the economy and therefore (K^*, x^*, s^*) was not a CFB. If $(x^*, \hat{K}, s^*)$ is not an INE it is because j would choose the safe asset as well, once $\hat{K}$ is given, or, in the case when both i and j choose the gambling asset in (K^*, x^*, s^*) an INE would select at least one of them choosing the safe asset. In all these cases, the new INE will yield a higher total amount of expected money higher than the one in $(x^*, \hat{K}, s^*)$. This proves statement 1.

Once established that the only optimal financial network for η big enough is the complete network structure, it is easy to see that c^* as defined in statement 2 is the highest number of banks that can choose safe given the financial network being equal to the complete network structure. Any allocation with a lower number of banks choosing the safe project yields a lower total expected money in the economy. ■

Proof of Proposition 4. We prove statement 1 by contradiction. Assume that (K^*, x^*, s^*) is a CFB allocation where there is at least one bank j with $s_j^* = g$ and $g_i^* \neq 0$. Let $i \in K_j^*$

with $s_i^* = g$. Take $(x^*, \hat{K}, s^*)$ a new allocation for the same economy (N, e), where the allocation of capital and strategies are the same but the network structure only severs the link of banks i and j. Formally, $\hat{K}_i = K_i^* \backslash \{j\}$, $\hat{K}_j = K_j^* \backslash \{i\}$, and $\hat{K}_r = K_r^*$ for all $r \neq i, j$. Again, we show that this allocation $(x^*, \hat{K}, s^*)$ yields a higher expected total money generated in the economy for η low enough. As before, it is easily seen that if $(x^*, \hat{K}, s^*)$ is not an INE, i or j or both prefer choosing the safe asset in $\left(x^*, \hat{K}\right)$. Such a case yields a higher expected total amount of money than $(x^*, \hat{K}, s^*)$. Therefore, the initial allocation (K^*, x^*, s^*) cannot be a solution to the planner's problem and statement 1 follows.

By definition,

$$\begin{aligned}
& \sum_{r \in N} p_r(\hat{K}, s^*) f(\hat{k}_r) R\left(x_r^* + 1\right) - \sum_{r \in N} p_r(K^*, s^*) f(k_r^*) R\left(x_r^* + 1\right) \\
= \;& \eta^{g_i} R\left(x_i^* + 1\right) [f(k_i^* - 1) - \eta f(k_i^*)] + \eta^{g_j} R\left(x_j^* + 1\right) [f(k_j^* - 1) - \eta f(k_j^*)], \qquad (15)
\end{aligned}$$

which is greater than zero for $0 < \eta < \dfrac{1}{f(1)}$ since $f(k)$ is quasiconcave. This means that, if (K^*, x^*, s^*) is a CFB allocation, two banks choosing the gambling asset cannot be connected. Therefore, for any allocation (K^*, x^*, s^*) to be CFB: $x_j^* = g$ implies $g_j = 0$ for $0 < \eta < \dfrac{1}{f(1)}$. This proves statement 1.

We prove now that no bank choosing the safe asset can be connected to a gambling bank in an optimal allocation, for

$$0 < \eta < \frac{1}{\rho + (n-1)[\rho - 1](\bar{e} + 1)} < \frac{1}{\rho} < \frac{1}{f(1)}.$$

Assume that a bank i is choosing the safe project in the CFB allocation (K^*, x^*, s^*), with $g_i^* > 0$. As we have just shown, $g_j = 0$ for any $j \in G_i$ if

$$\eta < \frac{1}{\rho + (n-1)[\rho - 1](\bar{e} + 1)}$$

since

$$\frac{1}{\rho + (n-1)[\rho - 1](\bar{e} + 1)} < \frac{1}{f(1)}.$$

Take a new allocation $(x^*, \hat{K}, s^*)$ for the same economy (N, e), where the allocation of capital and strategies are the same but the network structure only severs all the risky links of bank i. Formally, $\hat{K}_i = K_i^* \backslash \{G_i\}$, $\hat{K}_j = K_j^* \backslash \{i\}$, for all $j \in G_i$ and $\hat{K}_r = K_r^*$ for all $r \notin G_i$. Again, we show that this allocation $(x^*, \hat{K}, s^*)$ is an INE and yields a higher expected total money generated in the economy for η low enough. Therefore, the initial allocation (K^*, x^*, s^*) cannot be a solution to the planner's problem and statement 2 follows.

First note that if (K^*, x^*, s^*) is a CFB then it has to be an INE. If i and j are choosing the gambling project, the new allocation is also INE as x_i^* and x_j^* are greater or equal to zero. If i is choosing the safe asset, $x_i^* \geq I^*(k_i^*, \eta, g_i)$. Note that $I^*(k_i^* \eta, g_i) \geq I^*(k_i^* - g_i, \eta, 0)$ if and only if $f(k_i^* - g_i) \geq \eta f(k_i^*)$, which is true for $\eta\rho < 1$ since, given that $f(k_i)$ is increasing, $\frac{1}{\rho} \leq \frac{f(k_i - g_i)}{f(k_i)}$ for $k_i \geq g_i$. On the other hand, $x_j^* < I^*\left(k_j^*, \eta, g_j\right) < I^*\left(k_j^* - 1, \eta, g_j\right)$ for any $j \in G_i$. Hence, the allocation $(x^*, \hat{K}, s^*)$ is an INE. Then,

$$\begin{aligned} & \sum_{r \in N} p_r(\hat{K}, s^*) f(\hat{k}_r) R\left(x_r^* + 1\right) - \sum_{r \in N} p_r(K^*, s^*) f(k_r^*) R\left(x_r^* + 1\right) \\ = \;& R\left(x_i^* + 1\right)\left[f(k_i^* - g_i) - \eta^{g_i} f(k_i^*)\right] + \eta \sum_{j \in G_i} R\left(x_j^* + 1\right)\left[f(k_j^* - 1) - f(k_j^*)\right] \geq \\ \geq \;& R[1 - \eta\rho + \eta(\bar{e} + 1)(n-1)(1-\rho)]. \end{aligned} \tag{16}$$

Recall that $x_i^* \geq 0$. The inequality is true, since, as $1 \leq g_i \leq k_i^* \leq n-1$, and by increasiness of $f(.)$, we have (i) $f(k_i^* - g_i) - \eta^{g_i} f(k_i^*) \geq 1 - \eta\rho$, (ii) $0 \geq f(k_j^* - 1) - f(k_j^*) \geq 1 - \rho$, and (iii) $x_j^* \leq \bar{e}$ for each j. Note that $1 - \eta\rho + \eta(n-1)(1-\rho)(\bar{e}+1) \geq 0$ since $\eta < \frac{1}{\rho + (n-1)(\rho-1)(\bar{e}+1)}$. Therefore, if (K^*, x^*, s^*) is a CFB allocation, it has to be that peripheral agents have no links.

If the optimal financial network for $0 < \eta < \frac{1}{\rho + (n-1)(\rho-1)(\bar{e}+1)}$ is the complete network structure linking only banks that choose the safe project, it is easy to see that c^* as defined in statement 2 is the highest number of banks that can choose safe under such a network structure. Any allocation with a lower number of banks choosing the safe project yields a lower total expected money in the economy. ■

Proof of Proposition 5. Given an economy (N, e) fix any network K. We prove that for any (K, e, s) that is an INE there are no short-sighted profitable deviations for any bank i as far as either conditions 1 or 2 are satisfied. Fix a bank i such that K_i contains at least one neighbor j choosing the gambling asset. If we cannot find such an i it means that no bank can have a short-sighted profitable deviation. This case is true whenever condition 1 is satisfied. Now assume that condition 1 is not satisfied and such a bank i has at least one neighbor j choosing the gambling asset in the INE (K, e, s).

If there were a short-sighted profitable deviation for bank i there has to be a transfer $0 < t_i \leq e_i$ that is bounded depending on the choices of bank i before and after t_i. Notice that the bank transfer is directed in principle towards all the neighbor gambling banks $j \in J$, however feasibility implies that $t_i = \sum_{j \in J} t_{ij}$.

Note that $\frac{B}{(1-\eta)\rho R} = I^*(n-1,\eta,0) \leq \frac{B}{(1-\eta)\eta^{g_i} f(k_i) R}$ for any g_i and k_i. Condition 2 therefore means that $2\overline{e} \leq I^*(k_i,\eta,g_i)$ for every g_i and k_i, or, in other words, that $\overline{e} < I^*(k_i,\eta,g_i)$. This means that when condition 2 holds no bank i is choosing the safe asset before the transfer takes place. Then, we have to consider 2 different cases:

1. Bank i chooses the gambling asset before the transfer, but chooses the safe asset after the transfer. This means that $e_i < I^*(k_i,\eta,g_i)$, $e_i - t_i \geq I^*(k_i,\eta,g_i-|J|)$ and $e_j + t_{ij} \geq I^*(k_j,\eta,g_j-1-|J\cap G_j|)$, where $e_j < I^*(k_j,\eta,g_j)$ since (K,e,s) is an INE and t_i generates a short-sighted profitable deviation. If Condition 2 is satisfied, the inequalities $e_i - t_i \geq I^*(k_i,\eta,g_i-|J|)$ and $e_j + t_{ij} \geq I^*(k_j,\eta,g_j-|J\cap G_j|)$ for all $j \in J$ cannot simultaneously hold since they imply that

$$e_i + \sum_{j\in J} e_j \geq I^*(k_i,\eta,g_i-|J|) + \sum_{j\in J} I^*(k_j,\eta,g_j-1-|J\cap G_j|),$$

   a contradiction with the fact that

$$e_i + \sum_{j\in J} e_j \leq \overline{e} + |J|\overline{e} < |J| I^*(n-1,\eta,0) < I^*(k_i,\eta,g_i-|J|) + \sum_{j\in J} I^*(k_j,\eta,g_j-1-|J\cap G_j|).$$

2. Assume now that bank i chooses the gambling asset before and after the transfer. This implies that $e_i < I^*(k_i,\eta,g_i)$, $e_i - t_i < I^*(k_i,\eta,g_i-|J|)$ and $e_j + t_{ij} \geq I^*(k_j,\eta,g_j-|J\cap G_j|)$, where $e_j < I^*(k_j,\eta,g_j)$ since (K,e,s) is an INE and t_i generates a short-sighted profitable deviation. Again, if Condition 2 is satisfied, the inequality $e_j + t_{ij} \geq I^*(k_j,\eta,g_j-|J\cap G_j|)$ cannot be satisfied. Recall that $\sum_{j\in J} t_{ij} = t_i \leq e_i$. Therefore, $e_j + t_{ij} \geq I^*(k_j,\eta,g_j-|J\cap G_j|)$ implies that

$$e_i + \sum_{j\in J} e_j \geq \sum_{j\in J} I^*(k_j,\eta,g_j-|J\cap G_j|),$$

   a contradiction with the fact that

$$e_i + \sum_{j\in J} e_j \leq \overline{e} + |J|\overline{e} \leq |J| I^*(n-1,\eta,0) < \sum_{j\in J} I^*(k_j,\eta,g_j-|J\cap G_j|).$$

   Hence, there is no short-sighted profitable deviation for any possible case. ■

Proof of Proposition 6. The assumption on (N,e) and η guarantees that an INE without transfers is also a TNE. Assume by contradiction that (K^e,e,s^e) is a DEWT, but there exist two banks i and j such that $s_i^e = s_j^e = b$ but $i \notin K_j^e$, and therefore $j \notin K_i^e$. We prove that $(K^e \cup ij, e, s^e)$

is an INE and that both $m_i(K^e \cup ij, e, s^e) > m_i(K^e, e, s^e)$ and $m_j(K^e \cup ij, e, s^e) > m_j(K^e, e, s^e)$, contradicting the fact that (K^e, e, s^e) is a PSWT, and therefore it cannot be a DEWT.

Note first that, since (K^e, e, s^e) is a DEWT, it has to be an INE. This means that both $e_i \geq I^*(k_i, \eta, g_i)$ and $e_j \geq I^*(k_j, \eta, g_j)$. Furthermore, it has to be $I^*(k_i, \eta, g_i) \geq I^*(k_i + 1, \eta, g_i)$ and $I^*(k_j, \eta, g_j) \geq I^*(k_j + 1, \eta, g_j)$ given that $f(k)$ is increasing in k. This implies that $e_i \geq I^*(k_i + 1, \eta, g_i)$ and $e_j \geq I^*(k_j + 1, \eta, g_j)$. Therefore, $(K^e \cup ij, e, s^e)$ is also an INE. Finally, note that

$$m_i(K^e \cup ij, e, s^e) = \eta^{g_i} f(k_i + 1) Re_i > \eta^{g_i} f(k_i) Re_i = m_i(K^e, e, s^e),$$

and

$$m_j(K^e \cup ij, e, s^e) = \eta^{g_j} f(k_j + 1) Re_j > \eta^{g_j} f(k_j) Re_j = m_j(K^e, e, s^e),$$

since the function $f(k)$ is increasing in k. Therefore, (K^e, e, s^e) as defined above cannot be a DEWT. ■

Proof of Proposition 7. The assumption on (N, e) and η guarantees that an INE without transfers is also a TNE. Assume by contradiction that (K^e, e, s^e) is a DEWT, but there exist two banks i and j such that $i \notin K_j^e$, and therefore $j \notin K_i^e$. We prove that there exists an $(K^e \cup ij, e, \tilde{s})$ that is an INE such that $m_i(K^e \cup ij, e, \tilde{s}) > m_i(K^e, e, s^e)$ and $m_j(K^e \cup ij, e, \tilde{s}) > m_j(K^e, e, s^e)$, contradicting the fact that (K^e, e, s^e) is a PSWT, and therefore it cannot be a DEWT. We distinguish two cases: In the first one, a bank chooses the safe asset while the other chooses the gambling asset. In the second one, both banks choose the gambling asset. Note that since we know that a DEWT is a core-periphery structure we do not need to check the case of both banks choosing the safe asset.

Consider the first case, where $s_i^e = b$ and $s_j^e = g$. Note first that, since (K^e, e, s^e) is a DEWT, it has to be an INE. This means that $e_i \geq I^*(k_i, \eta, g_i)$ and $e_j < I^*(k_j, \eta, g_j)$. Given that $f(k)$ is increasing in k, $e_i \geq I^*(k_i + 1, \eta, g_i)$. Nevertheless, $e_j \geq I^*(k_j + 1, \eta, g_j)$ could also be true or not. Therefore, $(K^e \cup ij, e, s^e)$ could also be an INE, in which case

$$m_i(K^e \cup ij, e, s^e) = \eta^{g_i + 1} f(k_i + 1) Re_i > \eta^{g_i} f(k_i) Re_i = m_i(K^e, e, s^e),$$

for $\eta \geq \dfrac{f\,(n-2)}{f\,(n-1)} \geq \dfrac{f\,(k_i)}{f\,(k_i+1)}$, and

$$m_j(K^e \cup ij, e, s^e) = \eta^{g_j + 1} f(k_j + 1) Re_j + B > \eta^{g_j + 1} f(k_j) Re_j + B = m_j(K^e, e, s^e),$$

since the function $f(k)$ is increasing in k. In the case when $e_j \geq I^*(k_j + 1, \eta, g_j)$ there is an INE where at least j switches from choosing the gambling asset in s^e to choosing the safe asset in

$s = \tilde{s}$. This case is even more profitable for banks i and j than in $(K^e \cup ij, e, s^e)$, so we have $m_i(K^e \cup ij, e, \tilde{s}) \geq m_i(K^e \cup ij, e, s^e)$ and $m_j(K^e \cup ij, e, \tilde{s}) \geq m_j(K^e \cup ij, e, s^e)$.

Consider now the second case, when both $s_i^e = s_j^e = g$. The least profitable case would be when $(K^e \cup ij, e, s^e)$ is also an INE. Otherwise we can find an INE where at least one of them switches from investing in the gambling asset to investing in the safe asset. Note that

$$m_i(K^e \cup ij, e, s^e) = \eta^{g_i+2} f(k_i+1) Re_i + B > \eta^{g_i+1} f(k_i) Re_i + B = m_i(K^e, e, s^e),$$

and

$$m_j(K^e \cup ij, e, s^e) = \eta^{g_j+2} f(k_j+1) Re_j + B > \eta^{g_j+1} f(k_j) Re_j + B = m_j(K^e, x^e, s^e),$$

again for $\eta \geq \dfrac{f(n-2)}{f(n-1)}$ and given that $f(k)$ is increasing and quasiconcave in k. ■

Proof of Proposition 8. The assumption on (N, e) and η guarantees that an INE without transfers is also a TNE. Assume by contradiction that (K^e, e, s^e) is a DEWT, but there exist two banks i and j such that $i \in K_j^e$, and therefore $j \in K_i^e$, with at least one of them choosing the gambling asset. We prove that there exists an $(K^e \backslash\, ij, e, \tilde{s})$ that is INE and either $m_i(K^e \backslash\, ij, e, \tilde{s}) > m_i(K^e, e, s^e)$ or $m_j(K^e \backslash\, ij, e, \tilde{s}) > m_j(K^e, e, s^e)$, contradicting the fact that (K^e, e, s^e) is a PSWT, and therefore it cannot be a DEWT. We distinguish two cases: In the first one, a bank chooses the safe asset while the other chooses the gambling asset. In the second one, both banks choose the gambling asset. Since a DEWT has a core-periphery structure, we do not need to check the case of both banks choosing the safe asset.

Consider the first case, where $s_i^e = b$ and $s_j^e = g$. Note first that, since (K^e, e, s^e) is a DEWT, it has to be an INE. This means that $e_i \geq I^*(k_i, \eta, g_i)$ and $e_j < I^*(k_j, \eta, g_j)$. On the one hand, note that $I^*(k_i, \eta, g_i) \geq I^*(k_i, \eta, g_i)$ if and only if $\eta \leq \dfrac{f(k_i - 1)}{f(k_i)}$. By assumption

$$\eta < \frac{1}{f(1)} \leq \frac{f(k_i - 1)}{f(k_i)} \text{ for } k_i \geq 1$$

and therefore $e_i \geq I^*(k_i - 1, \eta, g_i - 1)$. On the other hand, given that $f(k)$ is increasing in k, we have $I^*(k_j, \eta, g_j) \leq I^*(k_j - 1, \eta, g_j)$ and therefore $e_j < I^*(k_j - 1, \eta, g_j)$. Therefore, $(K^e \backslash ij, e, s^e)$ is also an INE. Furthermore,

$$m_i(K^e \backslash ij, e, s^e) = \eta^{g_i - 1} f(k_i - 1) Re_i > \eta^{g_i} f(k_i) Re_i = m_i(K^e, e, s^e),$$

for $\eta < \dfrac{1}{f(1)} \leq \dfrac{f(k_i - 1)}{f(k_i)}$.

Consider now the second case, when both $s_i^e = s_j^e = g$. Since (K^e, e, s^e) is an INE we know that both $e_i < I^*(k_i, \eta, g_i)$ and $e_j < I^*(k_j, \eta, g_j)$. Like before, the assumption $\eta < \dfrac{1}{f(1)}$ that, since $f(k)$ is quasiconcave, implies both $\eta \leq \dfrac{f(k_i - 1)}{f(k_i)}$ and $\eta \leq \dfrac{f(k_j - 1)}{f(k_j)}$, means that $I^*(k_j, \eta, g_i) \geq I^*(k_j - 1, \eta, g_i - 1)$ and $I^*(k_j, \eta, g_j) \geq I^*(k_j - 1, \eta, g_j)$. Therefore, $(K^e \backslash ij, e, s^e)$ could also be an INE, in which case

$$m_i(K^e \backslash ij, e, s^e) = \eta^{g_i} f(k_i - 1) Re_i + B > \eta^{g_i+1} f(k_i) Re_i + B = m_i(K^e, e, s^e),$$

and

$$m_j(K^e \backslash ij, e, s^e) = \eta^{g_j} f(k_j - 1) Re_j + B > \eta^{g_j+1} f(k_j) Re_j + B = m_j(K^e, e, s^e),$$

again given that $\eta < \dfrac{1}{f(1)}$ and $f(k)$ is quasiconcave in k. Otherwise, we can find an INE where at least one of them switches from investing in the gambling asset to investing in the safe asset, earning more than $m_i(K^e \backslash ij, e, s^e)$ and $m_j(K^e \backslash ij, e, s^e)$ respectively, and therefore earning more than in the proposed DEWT (K^e, x^e, s^e). ■

References

[1] Aghion, P., P. Bolton and M. Dewatripont (2000), 'Contagion Bank Failures in a Free Banking System', *European Economic Review,* **44**: 713-718.

[2] Allen, F. and A. Babus (2008), 'Networks in Finance' in P. Kleindorfer and J. Wind (eds), '*Network-based Strategies and Competencies'*, Wharton School Publishing, forthcoming.

[3] Allen, F. and D. Gale (2000), 'Financial Contagion', *Journal of Political Economy*, **108**: 1-33.

[4] Allen, F. and D. Gale (2004), 'Financial Intermediaries and Markets', *Econometrica*, **72**: 1023-1061.

[5] Babus, A. (2007), 'The Formation of Financial Networks', Manuscript, Erasmus University of Rotterdam.

[6] Brusco, S. and F. Castiglionesi (2007), 'Liquidity Coinsurance, Moral Hazard, and Financial Contagion', *Journal of Finance*, **62**: 2275 - 2302.

[7] Diamond, D. and P. Dybvig (1983), 'Bunk Runs, Deposit Insurance and Liquidity', *Journal of Political Economy*, **91**: 401-419.

[8] Freixas, X., B. Parigi and J.C. Rochet (2000), 'Systemic Risk, Interbank Relations and Liquidity Provision by the Central Bank', *Journal of Money, Credit and Banking*, **32**: 611-638.

[9] Gai, P. and S. Kapadia (2008), 'Contagion in Financial Networks', Manuscript, Bank of England.

[10] Gale, D. and S. Kariv (2007), 'Financial Networks', *American Economic Review, Papers and Proceedings*, **97**(2): 99-103.

[11] Jackson, M. O. and A. Wolinsky (1996), 'A Strategic Model of Social and Economic Networks', *Journal of Economic Theory*, **71**: 44-74.

[12] Kahn, C. and J. Santos (2007), 'Endogenous Financial Fragility and Prudential Regulation', Manuscript, University of Illinois and Fed New York.

[13] Leitner, Y. (2005), 'Financial Networks: Contagion, Commitment, and Private Sector Bailouts', *Journal of Finance*, **6**: 2925-2953.

[14] Nier, E., J. Yang, T. Yorulmazer, and A. Alerton (2007), 'Network Models and Financial Stability', *Journal of Economic Dynamics and Control*, **31**: 2033-2060.

[15] Rochet, J.C. and J. Tirole (1996), 'Interbank Lending and Systemic Risk', *Journal of Money, Credit and Banking*, **28**: 733-762.