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ABSTRACT

We explore the consequences for asset pricing of admitting a bequest motive into an otherwise standard overlapping generations model where agents trade equity and perpetual debt securities. Prices of securities are seen to be approximately 50% higher in an economy with bequests as compared to an otherwise identical one where bequests are absent. Robust estimates of the equity premium are obtained in several cases where the desire to leave bequests is modest relative to the desire for old age consumption.

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1. Introduction

This paper explores the implications of bequests for the statistical pattern of equilibrium stock and bond returns. It does so in the context of a "behavioral style" model in which households make their consumption and savings decisions not only to smooth consumption over their saving and dis-saving years, but also to provide for "indirect consumption" in their old age in the form of gifts and bequests. We say the elderly are motivated by a well defined "joy of giving".

But what motivates the bequeathing of economic property? While a casual consideration of bequests naturally assumes that they exist because of parents' altruistic concern for the economic status of their offspring, results in Hurd (1989) and Kopczuk and Lupton (2004), among others (see also Wilhelm (1996), Laitner and Juster (1996), Altonji et al. (1997), and Laitner and Ohlsson (2001)), suggest otherwise: households with children do not in general exhibit behavior more in accord with a bequest motive than childless households. As a result, the literature is presently largely agnostic as to bequest motivation, attributing bequests to general idiosyncratic, egoistic reasons.[1] The model we propose to explore, however, is sufficiently general to be consistent both with purely egoistic and purely altruistic, concern-for-offspring based motivations. Qualitatively, it also nests a model of unintended bequests arising from a period of abbreviated retirement (positive probability of premature death).

[1] These empirical results will lead us to eschew the perspective of Barro (1974), who postulates that each generation receives utility from the consumption of the generations to follow, in favor of a more general formulation.

While the motivation for bequests is not yet precisely understood, there is little dispute as to their pervasiveness and significance for household capital accumulation. Kotlikoff and Summers (1981) present evidence that roughly 46% of household wealth arises from intergenerational transfers, although Modigliani's (1988) analysis points to a more modest 20% estimate.[2] Other studies place inherited wealth as a proportion of household wealth in the range of 15% – 31%.[3] Using a more general statistical methodology, Kopczuk and Lupton (2004) estimate that 70% of the elderly population has a bequest motive, which directly motivates 53% of the wealth accumulation in single person, elderly U. S. households. None of these estimates is so small as to imply that bequests should be ignored in a discussion of asset pricing regularities. Yet to our knowledge, the implications of bequests for such regularities have not yet been explored in the applied literature.

A consideration of bequests mandates that our study be undertaken in an OLG context. Agents live for three periods. In the first period, while young, they consume their income and neither borrow nor lend. We adopt this convention as a parsimonious device for acknowledging that, with a steep expected future income profile, the young do not wish to lend, and cannot borrow because they have no assets to offer as collateral. In the second, high wage, middle-aged period of their lives they consume, save for old age and receive bequests of securities from the then old who were born one period

[2] We discuss the basis of this wide discrepancy in estimates in the calibration section of the paper. The estimates themselves come from converting flows of bequests into stocks of capital. Alternatively one may estimate life cycle savings and compare this with accumulated wealth. Under this latter method, the estimates of Kotlikoff and Summers (1981) and Modigliani (1988) become, respectively, 81% and 20%.

earlier. In the third and final period of their lives, as elderly, they consume out of savings and themselves leave the residual as a bequest of securities, the value of which is modeled as directly providing them utility.

We further refine the behavior of the elderly in a number of alternative ways. In the simplest version of the model, the consumption of the old is fixed, with the entire residual value of savings going to bequests. For the old aged the only source of risk is therefore bequest risk. In making this assumption we appeal to the fact that a substantial component of old aged spending is medically determined. It is thus related to the state of a person's health and uncorrelated with the business cycle. Other components of old aged consumption, such as vacations, entertainment and housing, are also largely a function of the state of an elderly person's health. Particularly for the well-to-do, fluctuations in the value of their wealth invested in the stock market play but a secondary role in determining overall spending, a fact that is confirmed by the low empirical correlation between the direct consumption of the old and the return on the stock market. As a first approximation it is thus reasonable to exclude the direct consumption of the old consumers from consideration in examining the relevant Euler equations. Fixing old age direct consumption has this effect. Subsequent versions of the model jointly endogenize the choice between old-age direct consumption versus indirect consumption in the form of bequests. In summary, we explore the asset pricing

[3] This range of estimates is drawn from Menchick and Martin, (1983), Modigliani (1988), Hurd and Mundaca (1989), Gale and Scholz (1994), and Laitner and Juster (1996).

and financial structure cum welfare implications of the aforementioned family arrangements.

Intuition suggests that bequests may provide a possible route to the resolution of some of the most celebrated anomalies in financial economics; viz., the risk free and equity premium puzzles. At first appearances, the logic with respect to the equity premium and risk free rate puzzles is particularly straightforward. Within the context of the representative consumer, time separable preferences paradigm of, e.g., Grossman and Shiller (1981), Hansen and Singleton (1983), and Mehra and Prescott (1985), it is the very low covariance of aggregate consumption growth with equity returns that constitutes a major stumbling block to explaining the mean equity premium; vis-à-vis consumption risk, stocks are simply too good a hedging instrument to command a return much in excess of that on risk free securities.

In the OLG model considered here, however, the magnitude of a household's bequests – and the indirect utility thereby created – are, by an accounting identity, perfectly positively correlated with the prices of and returns to securities.[4] With regard to "bequest risk", equity securities, in particular, constitute an especially poor hedge, a fact that suggests high equilibrium equity and low risk free returns. This reasoning further implies that as investors value their bequests more highly, the premium should rise.

Quite surprisingly, the results of the analysis profoundly refute this logic. While it is not difficult to generate a high equity premium in the model, the premium actually declines as investors value their bequest more and more highly. We detail the intuition behind this phenomenon in the discussion below.

1.1 Related Literature

The theoretical antecedents of this work are many. Since not all agents in our model hold securities, it is directly related to the literature emphasizing the limited participation of some households in the financial markets. Mankiw and Zeldes (1991) emphasize that it should be the risk preferences and consumption risk of the stockholding class that matter for equilibrium security returns. Although 52 percent of the U.S. adult population held stock directly or indirectly in 1998, as compared to 36 percent in 1989, substantial stock holdings remain largely concentrated in the portfolios of the wealthiest few. Brav, Constantinides, and Geczy (2002) and Vissing-Jorgenson (2002) find evidence that per capita consumption growth can explain the equity premium with a relatively high coefficient of relative risk aversion (CRRA) once we account for limited stock market participation.[5] In addition, wealthy investors may be infra marginal in the equity markets if their wealth is tied up in private equity. See, for example, Blume and Zeldes (1993) and Haliassos and Bertaut (1995).

Other studies have proposed models which increase the covariance of equity returns with the growth rate of aggregate consumption, effectively by using the growth rate of aggregate consumption as a proxy for the aggregate stock market return in the Euler equations of consumption. Epstein and Zin (1991), in particular, introduce a

[4] And, of course, real estate. Our model does not attempt to explicitly model real estate as a differential asset.

recursive preference structure that emphasizes the timing of the resolution of uncertainty. Although their preference ordering is defined over consumption alone, the stock market return enters directly into the Euler equations defining optimal consumption if labor income is ignored. Lastly, Bakshi and Chen (1996) introduce a set of preferences defined over consumption and wealth (they argue that this captures the 'spirit of capitalism') that also have the effect of increasing the covariance of equity returns with consumption growth, broadly defined.

The presence of financial market incompleteness connects us to another well developed branch of the literature. Bewley (1982), Mankiw (1986) and Mehra and Prescott (1985) suggest the potential of enriching the asset pricing implications of the representative agent paradigm by relaxing the implicit complete markets paradigm. More recently, Constantinides and Duffie (1996) confirm that incomplete markets can substantially enrich the implications of the representative household. Their main result is a proposition demonstrating, by construction, the existence of a household income process, consistent with calibrated aggregate dividend and income processes such that equilibrium equity and bond price processes match the analogous observed price processes for the U.S. economy. Unlike the household specific heterogeneity introduced in Constantinides and Duffie (1996), the OLG model considered here emphasizes only the heterogeneity across age cohorts. Whereas introducing household specific heterogeneity may enhance the explanatory power of the model, we eschew this option

[5] Brav, Constantinides, and Geczy (2002) point out, however, that the statistical evidence is weak and the results highly sensitive to experimental design.

in order to highlight the role of the indirect consumption of the old aged in the form of gifts and bequests. See Kocherlakota (1996) for an excellent review of the drawbacks to relying purely on incomplete-markets phenomena.

The outline of the paper is as follows: Section 2 details the simplest model formulation and presents the calibration. That agents receive utility directly from the magnitude of their bequests represents a departure from the standard Arrow-Debreu equilibrium construct: the level of consumption provided by the bequests simultaneously provides utility to two distinct agents, the old who bequeath the bequests and the middle-aged who receive them. The resultant existence issues are addressed. In Section 3 we present the results of numerically computing equilibrium security prices and returns for a wide class of reasonable parameterizations. Robustness issues are explored in Section 4 where we also generalize the model by allowing the old to undertake the consumption-bequest choice. Section 5 concludes the paper.

2. The Model, Equilibrium and Calibration

2.1. Model Description

As in Constantinides et al. (2002), we consider an overlapping generations, pure exchange economy in which each generation lives for three periods, as young, middle aged and old. Each generation is modeled as a representative consumer, a choice that implicitly ignores consumer heterogeneity within a generation in favor of exploring the implications of heterogeneity across generations in as parsimonious a construct as possible.

Income (output) in this model is denominated in terms of a single consumption good, and may be received either as wages, dividends or interest payments. Accordingly, there are two securities traded which are in positive net supply (no claims to labor income are traded), a single equity claim and $b > 0$ consol bonds. Each bond pays one unit of the consumption good every period in perpetuity (aggregate interest payments are thus b) and q_t^b denotes its period t, ex-coupon price. We view the bond as a proxy for long-term government debt.

The single equity security represents a claim to the stochastic aggregate dividend stream $\{d_t\}$. We interpret the dividend as the sum total of all private capital income including corporate dividends, corporate bond interest and net rents. We denote by q_t^e the ex-dividend share price in period t. In equilibrium, the stock and bond are instruments by which the economic participants can alter their income profiles across dates and states.

Lastly, we postulate the existence of a one period, risk free discount security, with period t price $q_t^{r_f}$ in zero net supply. The payoff profile associated with such a security issued in some arbitrary period t is

$$\begin{array}{cc} t & t+1 \\ -q_t^{r_f} & 1 \end{array}$$

While the formal presence or absence of this security does not alter the equilibrium allocations in any way, we include it in order to assess the economy's implied risk free rate. In what follows, we detail only the most basic version of the model; elaborations are detailed in subsequent sections.

A representative consumer born in period t receives deterministic wage income W_0 when young. He enters and concludes the young period of his life with zero holdings of securities; in effect, $c_{t,0} \equiv W_0$, where $c_{t,0}$ denotes the consumption of a young agent born in period t. As noted earlier, this requirement is a simple way of capturing the fact that wage income alone does not collateralize loans in modern economies, and that under our calibration, the wage profile of a representative consumer is sufficiently steep that it is non-optimal for him to save.

In the second period of his life, as middle aged, the period-t-born agent receives a stochastic wage income, $\tilde{W}^1_{t+1}$, and a stochastic bequest of securities from the then old generation born in period t-1; we denote the latter by $\tilde{B}_{t-1,2}$. Out of this aggregate wealth, the middle aged agent chooses the number of equity securities, $z^e_{t,1}$, consol bonds, $z^b_{t,1}$, and risk free securities, $z^{r_f}_{t,1}$ he wishes to acquire in order to finance his old-age consumption and bequests, and his (residual) level of middle aged consumption. Accordingly, his budget constraint assumes the form

$$(1) \qquad c_{t,1} + q^e_{t+1} z^e_{t,1} + q^b_{t+1} z^b_{t,1} + q^{r_f}_{t+1} z^{r_f}_{t,1} \leq \tilde{W}^1_{t+1} + \tilde{B}_{t-1,2}$$

where $c_{t,1}$ denotes the consumption of a middle aged agent born in period t.

In the final period of his life the period-t born-agent receives no wage income, consumes a fixed level of consumption $\bar{c}_2$, and bequeaths the remainder; that is,

$$\text{(2)} \qquad \tilde{B}_{t,2} = z^{e}_{t,1}\ (\tilde{q}_{t+2} + \tilde{d}_{t+2}) + z^{b}_{t,1}\ (\tilde{q}^{b}_{t+2} + 1) + z^{r_f}_{t,1} - \bar{c}_2 .$$

In effect, the elderly in this model sell a portion of their security holdings to the middle aged to finance their old-age consumption $\bar{c}_2$, and pass down the residual value as a gift.

Taking prices as given, the decision problem faced by a representative agent born in period t is

$$\text{(3)} \qquad \underset{\left\{z^{e}_{t,1},\ z^{b}_{t,1},\ z^{r_f}_{t,1}\right\}}{\text{Max}} \ E\left\{\sum_{i=0}^{2}\beta^{i}u(c_{t,i}) + \beta^{2}Mv(\tilde{B}_{t,2})\right\}$$

$$\text{s.t.} \quad c_{t,0} \leq W_0$$

$$c_{t,1} + q^{e}_{t+1}\ z^{e}_{t,1} + q^{b}_{t+1}\ z^{b}_{t,1} + q^{r_f}_{t+1}\ z^{r_f}_{t,1} \leq \tilde{W}^{1}_{t} + \tilde{B}_{t-1,2}$$

$$\tilde{c}_{t,2} + \tilde{B}_{t,2} \leq (\tilde{q}^{e}_{t+2} + \tilde{d}_{t+2})\ z^{e}_{t,1} + (\tilde{q}^{b}_{t+2} + 1)\ z^{b}_{t,1} + z^{r_f}_{t,1}$$

$$\tilde{c}_{t,2} \equiv \bar{c}_2$$

$$0 \leq z^{e}_{t,1} \leq 1,$$

$$0 \leq z^{b}_{t,1} \leq b,$$

$$0 \leq z^{r_f}_{t,1}.$$

In the above formulation, $u(\cdot)$ denotes the agent's utility-of-consumption function and $v(\cdot)$ his utility-of-bequests function. The constant M is the relative weight assigned to the utility of bequests. Both $u(\cdot)$ and $v(\cdot)$ are assumed to display all the basic properties sufficient for problem (3) to be well defined: they are continuously differentiable, strictly concave, increasing, and satisfy the Inada conditions. The postulated bequest function $v(\cdot)$ is sufficiently general to encompass both altruistic and egoistic bequest motivations.

2.2. Optimality Conditions and Equilibrium

Let $\tilde{Y}_t$ denote the period t aggregate income. By construction, the economy's overall budget constraint satisfies:

$$(4) \qquad \tilde{Y}_t = \tilde{W}^1_t + W_0 + b + \tilde{d}_t.$$

In equilibrium, only the middle aged hold securities and their optimal holdings are determined by the tradeoff between their marginal utility of consumption as middle aged and the expected marginal benefit to granting one additional unit of indirect consumption in the form of a bequest. Taking prices as given, the middle aged agent's optimal holdings of equity, bonds, and risk free assets satisfy, respectively, the following three equations:

$$(5) \qquad z^e_{t,1} : u_1(c_{t,1})q^e_t = \beta E_t \left\{ M v_1(\tilde{B}_{t,2})[q^e_{t+1} + d_{t+1}] \right\}$$

$$(6) \qquad z^b_{t,1} : u_1(c_{t,1})q^b_t = \beta E_t \left\{ M v_1(\tilde{B}_{t,2})[q^b_{t+1} + 1] \right\}$$

$$(7) \qquad z^{r_f}_{t,1} : u_1(c_{t,1})q^{r_f}_t = \beta E_t \left\{ M v_1(\tilde{B}_{t,2}) \right\}$$

where the (conditional) expectations are taken over all realizations of the economy's aggregate state variables, $\tilde{Y}_{t+1}$ and $\tilde{W}^1_{t+1}$.

While superficially similar to the asset pricing equations in any standard dynamic model, these equations are fundamentally different in one important respect. In any period, the two agents whose utility is determined by the financial markets are the old generation (via the granting of a “declaimed” consumption bequest packaged as a portfolio of marketable securities) and the middle aged generation (who receive the bequests). Both of those agents receive utility from the same portfolio of securities or, more precisely, its consumption equivalent. This feature represents a departure from the standard Arrow-Debreu model and we might anticipate this fact to have asset pricing implications.

Market clearing conditions for the three securities are as follows:

(8) $\quad z^{e}_{t,1} = 1,\ z^{b}_{t,1} = b,$ and $z^{r_f}_{t,1} = 0.$

Equilibrium in this economy is defined as follows:

Definition: Equilibrium for the economy described by problem (3) and market clearing conditions (8) is a triple of time stationary security pricing functions $q^{e}(Y_t, W^1_t)$, $q^{b}(Y_t, W^1_t)$ and $q^{r_f}(Y_t, W^1_t)$ which satisfy equations (9) – (11):

(9) $\quad u_1(W^1_t + d_t + b - \overline{c}_2)\, q^{e}(Y_t, W^1_t)$

$$= \beta \int Mv_1\ (q^{e}(Y_{t+1}, W^1_{t+1}) + d(Y_{t+1}, W^1_{t+1}) + bq^{b}(Y_{t+1}, W^1_{t+1}) + b - \overline{c}_2)$$

$$[q^{e}(Y_{t+1}, W^1_{t+1}) + d(Y_{t+1}, W_{t+1})]\ dF(Y_{t+1}, W^1_{t+1}; Y_t, W^1_t)$$

(10) $\quad u_1(W^1_t + d_t + b - \overline{c}_2)\, q^{b}(Y_t, W^1_t)$

$$= \beta \int Mv_1\ (q^{e}(Y_{t+1}, W^1_{t+1}) + d(Y_{t+1}, W^1_{t+1}) + bq^{b}(Y_{t+1}, W^1_{t+1}) + b - \overline{c}_2)$$

$$[q^{b}(Y_{t+1}, W^1_{t+1}) + 1]\ dF(Y_{t+1}, W^1_{t+1};\ Y_t, W^1_t),\ \text{and}$$

(11) $\quad u_1(W^1_t + d_t + b - \overline{c}_2)\ q^{r_f}(Y_t, W^1_t)$

$$= \beta \int Mv_1 \left(q^e(Y_{t+1}, W^1_{t+1}) + d(Y_{t+1}, W^1_{t+1}) + bq^b(Y_{t+1}, W^1_{t+1}) + b - \bar{c}_2)\right]$$

$$dF(Y_{t+1}, W^1_{t+1}; Y_t, W^1_t),$$

where F(;) denotes the conditional density function on the economy's aggregate state variables.

Specializing the economy even further, we assume that the joint stochastic evolution of $(\tilde{Y}_t, \tilde{W}^1_t)$ is governed by a discrete Markov process, with no absorbing states. Our benchmark calibration recognizes that output and the total wage bill are highly positively correlated in the U.S. economy. A number of variations are considered which differ only with respect to the assumed correlation structure between $\tilde{Y}_t$ and $\tilde{W}^1_t$.

It was argued in the introduction that asset prices are higher in the presence of bequests than in a standard consumption-savings setting and the basis for this assertion is directly apparent in equations (9) – (11): there is no utility cost today of paying more for a security as higher prices only mean greater bequests in a stationary equilibrium (see also Geanakoplos et al. (2003)). Unlike in a standard OLG setting, q^e_t, q^b_t, and $q^{r_f}_t$ do not appear in the marginal utility expressions on the left hand side of, respectively, equations (9), (10), and (11). As the "auctioneer" calls out an increasing set of prices, the marginal utility of period t consumption does not increase to reduce demand. The effect of price increases on the suppression of demand is thus greatly reduced, a fact that suggests the possibility of explosive price behavior. That prices are likely to be higher under a bequest equilibrium relative to a pure consumption savings context says nothing about relative return behavior, however. An explicit solution of (9) - (11) is therefore required.

Following Constantinides et al. (2002), we specify four admissible states representing two possible values of output mixed with two possible values of the wage endowment of the middle aged. The two preference functions are assumed to be of the standard form, $u(c_{t,i}) = \frac{(c_{t,1})^{1-\gamma_C}}{1-\gamma_C}$, i = 0, 1, 2, and $v\left(B_{t-1,2}\right) = \frac{(B_{t-1,2})^{1-\gamma_B}}{1-\gamma_B}$ with $\gamma_c = \gamma_B$ in the benchmark cases. With these specifications, the equations defining the equilibrium functions may be simplified as follows:

$$(9)' \quad \frac{q^e(j)}{(W^1(j)+d(j)+b-\overline{c}_2)^{\gamma_C}} = \beta\sum_{k=1}^{4} \frac{M(q^e(k)+d(k))\ \ \pi_{jk}}{(q^e(k)+d(k)+bq^b(k)+b-\overline{c}_2)^{\gamma_B}}$$

$$(10)' \quad \frac{q^b(j)}{(W^1(j)+d(j)+b-\overline{c}_2)^{\gamma_C}} = \beta\sum_{k=1}^{4} \frac{M(q^b(k)+1)\pi_{jk}}{(q^e(k)+d(k)+bq^b(k)+b-\overline{c}_2)^{\gamma_B}}$$

$$(11)' \quad \frac{q^{r_f}(j)}{(W^1(j)+d(j)+b-\overline{c}_2)^{\gamma_C}} = \beta\sum_{k=1}^{4} \frac{M\ \ \pi_{jk}}{(q^e(k)+d(k)+bq^b(k)+b-\overline{c}_2)^{\gamma_B}}$$

where the states are indexed j = 1,2,3,4 and $d(j) = Y(j) - W^1(j) - W_0 - b$, and π_{jk} represents the probability of passing from state j to k. With the above specifications, equilibrium in this bequest-driven economy can be shown to exist:

Theorem 2.1 Suppose that $u(\cdot) = v(\cdot)$ is of the CRRA family of utility functions with parameter γ and that $(Y(j), W^1(j))$ follows a level stationary N state Markov chain. Suppose also that $\theta(j) \equiv d(j)+b-\overline{c}_2 > 0\ \forall j$, $d(j) > 1\ \forall j$ and that $2\beta ML < 1$.

Define

$$\Psi = \max_{1 \le j \le N} d(j)$$

$$L = \max_{1 \le j \le N} \sum_{k=1}^{N} \pi_{jk} \left(\frac{w^1(j) + \theta(j)}{\theta(k)} \right)^{\gamma}$$

Then there exists a solution to $(9)' - (11)'$ in $A \subseteq R_{+}^{2N}$

where $A = \left\{ x_1, \ldots, x_N, y_1, \ldots, y_N) : 0 \le x_i \le \Psi, 0 \le y_i \le \Psi \right\}$, provided $2\ \beta\ M\ L < 1$.[6, 7]

Proof: See Appendix 1.

By the following homogeneity property, the numerical search for the equilibrium price functions can be substantially simplified: if $\{(q^e(j),\ q^b(j), q^{r_f}(j))\ j = 1,2,3,4\}$ constitutes an equilibrium for an economy defined by $\{(Y(j),\ W^1(j),\ W_0,\ b, \bar{c}_2):\ j = 1,2,3,4\}$, then for any $\lambda > 0$, $\{(\lambda q^e(j),\ q^b(j), q^{r_f}(j)):\ j=1,2,3,4\}$ is an equilibrium for the economy defined by $\{(\lambda Y(j),\ \lambda W^1(j),\ \lambda W_0,\ \lambda b,\ \lambda \bar{c}_2):\ j=1,2,3,4\}$.[8] Returns are thus unaffected if the economy is scaled up or down.

---

[6] Note that once the existence of $q^e(j)$ and $q^b(j)\ j = 1, 2, 3, 4,$ is guaranteed, $q^{r_f}(j)$ follows from $(11)'$ directly.

[7] The key assumption is that $\theta(j) > 0$, for all j, which is required for the continuity of the equilibrium mapping. It may be relaxed to the requirement that $q^e(\underline{j}) + q^b(\underline{j}) + \theta(\underline{j}) > 0$ where $\underline{j}$ is that state defined as $d(\underline{j}) = \min_j(d(j))$.

[8] If $\gamma_C \neq \gamma_B$, then the economy with scaled output, wages, interest payments and old aged consumption will have the same prices as the unscaled economy but with M altered to $M\lambda^{\gamma_C - \lambda_B}$, where λ is the scaling factor.

2.3 Calibration

In this section we select parameter values for the period utility and bequest function while also specifying the joint stochastic process on Y_t and W^1_t.

There are eleven parameter values to be selected: $\{(Y(j), W^1(j))$: j=1,2,3,4$\}$, W_0, b, $\overline{c}_2$, β, M, γ_C and $\gamma_B\}$. In light of the homogeneity property, for an arbitrary E(Y), $\{Y(j), W^1(j)$: j=1,2,3,4$\}$, W_0, b, and $\overline{c}_2$ can be chosen to replicate the fundamental ratios: $\frac{\sigma_Y}{E(Y)}, \frac{\sigma_{W^1}}{E(W^1)}, E\left\{\frac{W_0}{Y}\right\}, E\left\{\frac{W^1+W_0}{Y}\right\}, E\left\{\frac{b}{Y}\right\}$ and $E\left\{\frac{\overline{c}_2}{Y}\right\}$. With a period corresponding to 20 years, and a maximum of five or six reliable non-overlapping 20 year periods in U.S. real GDP and aggregate wage data, it is difficult to conclusively fix the output and middle aged wage coefficients of variation. Somewhat arbitrarily, both are chosen at .25 (see Constantinides et al. (2002) for an elaboration).

The remaining ratios, however, can be established with more confidence. Consistent with U.S. historical experience, we set the share of income to interest on U.S. government debt as $E\left\{\frac{b}{Y}\right\} = 0.03$. Depending on the historical period and the manner by which single proprietorship income is imputed, the average share of income to wages, $E\left\{\frac{W^1+W_0}{Y}\right\}$ is generally estimated (U.S. data) to lie in the range (0.60, 0.75). For most of our examples, we match the ratio $E\left\{\frac{W^1+W^0}{Y}\right\} = 0.67$.

This leaves W_0 and $\bar{c}_2$; they are chosen in order to replicate the U.S. age-consumption expenditure profile in Fernandez-Villaverde and Krueger (2002; figure 4.1.1), where we interpret our three period lifetimes as corresponding roughly to the 0-20, 20-60 and 60-80 age cohorts detailed there. For the benchmark calibration, in particular, their data suggests $E\left\{\frac{\bar{c}_2}{Y}\right\} \approx 0.20$ and $E\left\{\frac{W_0}{Y}\right\} \approx 0.20$.[9]

Following Constantinides et al. (2002) we postulate a transition matrix Π of the form:

$$
\begin{array}{cc}
 & \begin{array}{cccc} (Y(1),W^1(1)) & (Y(2),W^1(2)) & (Y(3),W^1(3)) & (Y(4),W^1(4)) \end{array} \\
\begin{array}{c} (Y(1),W^1(1)) \\ (Y(2),W^1(2)) \\ (Y(3),W^1(3)) \\ (Y(4),W^1(4)) \end{array} & \begin{bmatrix} \phi & \Pi & \sigma & H \\ \Pi+\Delta & \phi-\Delta & H & \sigma \\ \sigma & H & \phi-\Delta & \Pi+\Delta \\ H & \sigma & \Pi & \phi \end{bmatrix}
\end{array}
$$

Choices of ϕ, Π, σ, H and Δ determine the following important correlations $\rho(Y_t, Y_{t-1})$, $\rho(Y_t, W_t^1)$ and $\rho(W_t^1, W_{t-1}^1)$. The table below contains the precise values:

[9] Fernanadez-Villaverde and Krueger (2002) present data on per capita consumption on a quarterly basis from year 20 to year 80. Aggregating these quantities into the 20-60 and 60-80 age ranges plus adopting the convention that quarterly consumption in years 1-20 coincides with year 20 first quarter consumption yields the indicated proportions.

Table 1

Correlation Structures and Associated Parameter Values

$corr(Y_t, Y_{t-1})$ and $corr(W_t^1, W_{t-1}^1)$	$corr(Y_t, W_t^1)$	ϕ	Π	σ	H	Δ
0.1	0.1	0.5298	0.0202	0.0247	0.4253	0.01
0.1	0.8	0.8393	0.0607	0.0742	0.0258	0.03
0.8	0.1	0.5496	0.0004	0.0034	0.4466	0.03
0.8	0.8	0.8996	0.0004	0.0034	0.0966	0.03

Taking all these requirements into account yields the following benchmark calibration: $Y_t \in \{118,200, 78,598\}$, $W_t^1 \in \{55,850, 33,450\}$, $\bar{c}_2 = 19,000$, $W_0 = 20,000$ with these quantities employed in conjunction with any of the four probability structures detailed in Table 1. All the major ratios detailed earlier are thereby replicated.

Next, we fix β =0.44 (corresponding to a β^{annual} = 0.96 which is consistent with a 4% real return on capital) for all cases and, in the benchmark calibrations, $\gamma_C = \gamma_B$=2, which is within the acceptable range of estimates provided by micro studies.

It remains to calibrate the parameter M.

2.4 Choosing a Value for the Bequest Parameter M

The parameter M, by governing the extent to which the middle-aged desire to bequeath, substantially influences both the relative and absolute level of equilibrium security prices. Note that M does not itself determine or even influence the actual consumption path of a representative cohort; that is fixed —c_0, $\bar{c}_2$, $\tilde{Y}_t$, $\tilde{w}_t^1$— by the prior calibration. Given this setting, we select a value for M in order that the share of existing wealth that is being inherited, $\left(\dfrac{B_{t-1,2}}{q_t^e + d_t + b(q_t^b + 1)}\right)$ respects the data.

Estimates on the value of this share differ widely. As noted in the introduction, Kotlikoff and Summers (1981) place the share of existing wealth that has been inherited in the range of [.5, .8], while Modigliani (1986) arrives at a more modest estimate of 20%. A very simple third estimate can be garnered from estate tax data under a number of simplifying assumptions: (i) following McGrattan and Prescott (2000) net corporate indebtedness is approximately zero, (ii) more than 90% of business capital, at market prices, is traded in the public equity market (also McGrattan and Prescott (2000)), (iii) all gifted equity capital is received as bequests rather than as in-vivos transfers, and (iv) all bequeathed equity is associated with estates of size resulting in the filing of an estate tax return. Under these assumptions, the ratio of the value of bequests as a proportion of CRSP aggregate market value is roughly analogous to our quantity $\left(\frac{B_{t-1,2}}{q_t^e + d_t + b(q_t^b + 1)}\right)$. Table 2 below supplies the relevant data for a selection of the years for which data is available.

Table 2
U.S. Equity Bequests as a Proportion of U.S. Equity Market Value
Select Years(ii)

(A) Year	(B)(i) Value of Equity Component of Bequest	(C) CRSP Aggregate Equity Market Value	(D) (B)/(C)
1931	1.909	21.577	0.08857
1938	1.273	40.680	0.0313
1950	1.773	85.701	0.0207
1961	6.766	383.720	0.01805
1970	10.495	643.326	0.0163
1977	12.483	1002.450	0.0124
1991	27.087	4072.320	0.00665
1996	44.151	8497.241	0.005196
2001	77.343	14,419.260	0.005466

(i) all values measured in billions of dollars
(ii) Source: IRS Estate Tax Returns, Publication 764; indicated years.

The value of annually bequeathed stock generally declined as a percentage of aggregate stock market value until the 1990s, when it stabilized at roughly 0.006%. On the basis of a 20 year time horizon, and assuming stationary-in-levels asset values, this represents a total equity bequest equal to 12% of aggregate stock market valuations; if 1977 is used as the base, the ratio rises to 25%, while in 1950 the fraction is roughly 45%. The substantially lower figures for more recent years are puzzling and may reflect either an increased use of tax avoidance schemes (e.g., generation-skipping trusts) by the very wealthy who own the lion's share of equity in the U.S. or the broader ownership of stocks in small estates exempt from taxation. In any event, we choose M so that $\left(\frac{B_{t-1,2}}{q_t^e + d_t + b(q_t^b + 1)}\right) \in [0.25, 0.5]$. While this is difficult to attain when $E\left\{\frac{\bar{c}_2}{Y_t}\right\} = 0.20$, it is easily achieved when the joint bequest-old age consumption level is determined endogenously in the model.

In what follows we solve equations (9)′ – (11)′ numerically for the indicated parameterizations. In order to get a feel for how the model behaves, we allow $\bar{c}_2$, M, and $\gamma_C=\gamma_B$ to vary. Since the results depend very little, either qualitatively or quantitatively, on the choice of transition matrix, we typically only report results for cases corresponding to $\phi = 0.5298$.

3. First Results: The Case of Constant Old Aged Consumption

Much of the intuition provided by this model is evident in the $c_{t,2} = \overline{c}_2$ case. This perspective was justified earlier by arguing that the consumption of the old aged is governed by their health status, a circumstance that is likely to be unrelated to the business cycle, especially for those with large equity holdings. Fixing old-age consumption as a constant reflects this viewpoint in a parsimonious way.

Table 3 provides a basic set of results for an uncontroversial set of parameters. The risk aversion parameter γ_C is fixed at $\gamma_C = 2$, and M is chosen to be $M = \frac{1}{10}$. It seems intuitively reasonable that agents would value their bequests less highly than their own consumption.

Table 3

Basic Financial Statistics: First Benchmark Parameterization

	U.S. Data			Benchmark Model $\bar{c}_2=20{,}000$, $M=\frac{1}{10}$, $\phi= 0.5298^{(j)}$ $\gamma_C=\gamma_B=2$		
	(a)[ii]	(b)		(a)	(b)	
Return on Equity	6.98	16.54		5.84	12.19	
Return on Long Bond	n.a	n.a		3.24	15.15	
Risk Free Return	0.80	5.67		2.00	17.53	
Equity Premium	6.18	16.67		3.84	9.93	
		Max	Min		Max	Min
Equity/Output[iii]		1.33	0.48	0.11	0.19	0.04
Bequests/Assets[iv]		0.50	0.25	0.35	0.79	-0.61[v]

(i) For this set of parameters, the corresponding middle aged consumption and bequests in states j=1,2,3,4 are: $c_1(1) = 89{,}200$; $c_1(2) = 49{,}598$; $c_1(3) = 89{,}200$; $c_1(4) = 49{,}598$; $B(1) = 49{,}672$; $B(2) = 8{,}624$; $B(3) = 96{,}044$; $B(4) = 21{,}794$.

(ii) (a) is the unconditional mean while (b) is the unconditional standard deviation annualized in the manner described in Footnote (10). All returns are real. U.S. data from Mehra and Prescott (1985), and Mehra (1998).

(iii) This ratio is defined as $\frac{q^e(j)}{Y(j)}$; U.S. data for the period 1945-1993 as provided in Mehra (1998).

(iv) This ratio is defined as $\frac{q^e(j)+d(j)+b(q^b(j)+1)-\bar{c}_2}{q^e(j)+d(j)+b(q^b(j)+1)}$.

(v) The negative minimum bequest/asset ratio must be interpreted as an instance in which the middle aged subsidize the consumption of their parents, something that is familiar to many families. This situation arises when asset prices are so low in some states that old age wealth falls short of $\bar{c}_2$. For the calibrations considered in this paper, such an event occurs exclusively in state 2.

The benchmark case displays considerable success in replicating some aspects of the data. In particular, the mean return on equity and its standard deviation match the data quite well.[10] The equity premium is a robust 3.84%, attributable in large measure to a relatively low risk free rate. While smaller than the estimates of Mehra and Prescott (1985), it is nevertheless consistent with more recent experience. Lastly, the bequests/assets ratio falls comfortably within the range of empirical estimates. On the negative side, the equity/output ratio is too low, but this simply reflects the comparatively low valuation placed on bequests (M= 1/10): there is insufficient demand for the equity security, which, in all cases, constitutes most of the bequests in value terms. More significantly, the relative standard deviations display a perverse ordering relative to the data: as a security's payment schedule becomes less variable, the standard deviation of its return (and thus necessarily its relative equilibrium price volatility) increases.

This latter observation is difficult to reconcile with standard intuition, although it is likely to be related to the identical pattern in observed price variation. For equities, in particular, $\max_{j=1,2,3,4} q^e(j) \Big/ \min_{j=1,2,3,4} q^e(j) = 2.95$. For the risk free security, the corresponding ratio is 11.4. Indeed, these results seem to suggest that if a particular security (under

[10] The reader is cautioned to keep in mind how these returns are computed and the consequent qualifications to any of the interpretations. For the equity security the annualized mean return was computed as $\frac{1}{20}\left\{\sum_{j=1}^{4}\varphi_j\sum_{k=1}^{4}\pi_{jk}\ \log\left(\frac{q^e(k)+d(k)}{q^e(j)}\right)\right\}$ with the mean returns of the other securities computed analogously. In the above expression φ_j denotes the stationary probability of state j. The 20 year standard deviation of the equity return was computed as $\left\{\sum_{j=1}^{4}\varphi_j\left(\sum_{k=1}^{4}\pi_{jk}\ \log\left(\frac{q^e(k)+d(k)}{q^e(j)}\right)-\sum_{j=1}^{4}\varphi_j\ \sum_{k=1}^{4}\pi_{jk}\ \log\left(\frac{q^e(k)+d(k)}{q^e(j)}\right)\right)^2\right\}^{1/2}$ while the corresponding annualized standard deviation satisfied $SD^{equity}_{annuity} = \frac{1}{\sqrt{20}}\ SD^{equity}_{20\ year}$. Again, the return standard deviations for the other securities were computed in an identical fashion.

our parameterization, the equity security) provides the overwhelming majority of bequest utility, that security will display the greater relative price stability irrespective of the volatility of its associated "dividend payment." In a world where agents derive utility directly from bequests, the notion of risk is blurred. Cass and Pavlova (2000) argue for an analogous ambiguity in a related model.[11]

While 3.84% is a large premium relative to that obtained in a similarly calibrated representative agent model, our initial intuition was that it should be enormously larger because of the near perfect correlation of bequests and equity returns. That this fails to be so follows directly from the fact that the argument of a bequest function includes asset prices which are themselves large relative to their associated payments: ceteris paribus, the marginal utility of bequests is thus computed with respect to a less concave portion of the utility surface, causing the agent to act in a less risk averse manner, a phenomenon that to some extent offsets the influence of the high correlation.

Security prices are, in fact, extremely high relative to what they would be in an identically parameterized model where the middle aged agent is saving exclusively for old aged consumption. This follows from the properties of steady state equilibrium in a pure consumption savings context: an increase in the price of the equity security reduces middle aged consumption, a 'brake' that is not present in the bequest driven formulation considered here.

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[11] In a standard Lucas (1978) asset pricing model with log utility where the representative agent trades a risk free bond and a stock, Cass and Pavlova (2000) introduce a simple linear transformation by which the stock becomes the risk free asset and the bond the risky one in the sense that its payment is now the uncertain one. While their model context is very different from the one considered here, they present a similar instance of the more variable return security having the lesser associated payment variation.

3.1 Comparative Dynamics: Equilibrium Consequences of Parameter Changes

We first attempt to improve model performance – especially along the $q^e(j)/Y(j)$ dimension - by increasing the bequest weight M. As bequests become more important, we would expect equilibrium security prices to be bid up. As noted above, however, as security prices rise, middle-aged investors do not suffer a concomitant loss in consumption: the value of the bequests they themselves receive grows in lockstep with their desire to bestow them. As a result, equilibrium asset prices would be expected to rise much more dramatically than in a standard consumption-savings context where savers increase their preference for second-period consumption (as a result, e.g., of an increase in β).

The results in Table 4 confirm these conjectures where, for parsimony, we report the value of aggregate bequests (except for $\bar{c}_2$, the ex ante value of all securities) state by state. Notice that as M increases from M = 0.001 to M = 1, the values of the securities (and bequests) increase enormously.[12] Since security payments are unaltered, attendant to these price increases is a simultaneous across-the-board reduction in the rates of return; in the case of the risk free security, expected returns in fact may become negative when M sufficiently exceeds one (not reported). State (2) is exceptional in that low asset prices lead to negative bequests (for small M), something that we interpret as a transfer from the middle aged to the elderly.

[12] With prices rising yet $\bar{c}_2$ fixed, the E(B/A) ratio will naturally approach one, as observed.

Table 4

Effects of Changes in M on Equilibrium Security Prices and Returns and Bequests

$\bar{c}_2 = 20{,}000$, $\gamma_C = \gamma_B = 2$, $\phi = 0.5298$

$(c_1(1), c_2(1)) = (c_1(3), c_2(3)) = (79{,}200,\ 20{,}000)$

$(c_1(2), c_2(2)) = (c_1(4), c_2(4)) = (39{,}598,\ 20{,}000)$ [(i)]

	M=0.001	M=0.01	M=0.1	M=1
$q^e(1)$	773	5400	22,422	70,287
$q^e(2)$	21	176	3,172	16,797
$q^e(3)$	99	822	11,426	58,188
$q^e(4)$	229	1594	6,557	20,317
$q^b(1)$	0.03	0.22	1.27	4.71
$q^b(2)$	0.001	0.02	1.85	6.70
$q^b(3)$	0.01	0.07	6.12	22.37
$q^b(4)$	0.01	0.07	0.38	1.38
$B(1)$ [(ii)]	24,218	29,415	49,573	107,779
$B(2)$	-16,225	-16,019	7,539	20,643
$B(3)$	45,871	46,778	75,548	171,054
$B(4)$	6,406	7,942	13,826	30,598
$E(r^e)$	23.14%	13.47%	5.84%	2.54%
$\sigma(r_e)$	16.03%	15.01%	12.19%	12.5%
$E(r^b)$	23.75%	13.73%	3.24%	1.22%
$\sigma(r_b)$	21.87%	18.72%	15.15%	13.33%
$E(r^f)$	23.72%	13.55%	2.00%	-0.48%
$\sigma(r_f)$	21.94%	19.07%	17.53%	15.24%
$E(r_p)$	-0.58%	-0.05%	3.84%	3.02%
$\sigma(r_p)$	7.26%	5.32%	9.93%	5.36%
$E(q^e/Y)$	0.01	0.02	0.11	0.59
$E(B/A)$ [(iii), (iv)]	-0.59	-0.51	0.35	0.71

[(i)] Since old age consumption is fixed, there are only two consumption states.

[(ii)] Bequests in states j=1,2,3,4.

[(iii)] Bequest/value of assets ratio averaged across all states.

[(iv)] The negative values of E(B/A) reported are due to the negative bequests observed in state 2. In that particular state the (B/A) ratio is negative and large in absolute value.

Notice that the equity premium is first increasing with M and then declines. Two conflicting phenomena are responsible for this effect. The first was mentioned in the introduction as our initial intuition: as an agent's desire to bequeath increases (larger M), stocks become progressively less desirable as a vehicle for this goal due to the high positive correlation of their returns with the bequest itself. Offsetting this mechanism is the fact that as security prices rise, the argument of the bequest function simultaneously increases, thereby pushing the agent on to a less concave portion of his bequest utility curve where he becomes more tolerant of risk. Eventually the latter effect dominates the former and the premium declines, as observed.

We would therefore anticipate that the decline in the premium would begin to occur at lower asset price levels (lower M values) if the agent's $\overline{c}_2$ were less (it is only the value surplus above $\overline{c}_2$ that matters for bequest risk tolerance). This is confirmed in Table 5 where we repeat the prior exercise with $\overline{c}_2 = 10{,}000$: the premium begins to decline with M = 0.01.

Table 5
Effects of Changes in M on Equilibrium Return Patterns

All Parameters as in Table 4 except $\overline{c}_2 = 10{,}000$

	M = 0.001	M = 0.01	M = 0.1	M = 1
$E(r^e)$	17.23%	12.23%	6.51%	2.70%
$\sigma(r^e)$	36.83%	22.02%	12.08%	10.65%
$E(r_B)$	6.28%	5.26%	3.19%	1.45%
$\sigma(r_B)$	23.23%	20.57%	15.07%	11.50%
$E(r_f)$	7.84%	6.25%	2.88%	0.24%
$\sigma(r_f)$	34.77%	30.09%	19.7%	13.21%
$E(r_p)$	9.39%	5.98%	3.63%	2.45%
$\sigma(r_p)$	3.53%	8.51%	9.71%	5.04%

While an increase in M is very similar in its effects on return statistics to an increase in β in the standard Mehra and Prescott (1985) model (mean security returns decline in either case), these latter observations suggest that the equivalence is not perfect. In particular, an increase in β will also uniformly reduce the premium in the Mehra and Prescott (1985) model while an increase in the bequest preference parameter M will only reduce it uniformly when $\overline{c}_2$ is small relative to the bequest valuation itself (as in Table 5).

We note also that the standard deviation of security returns declines with an increase in M. To see why this is to be expected, first recall that the marginal rate of substitution is $M\left(\frac{c_1(j)}{B(k)}\right)^{\gamma}$. With $c_1(j)$ unaffected by M and B(k) increasing with M for all states, it is clear that the coefficient of variation of the MRS diminishes. It follows that price and return volatility will diminish as well. Lastly we observe that for low values of M (M = 0.001 and M = 0.01) the inverted volatility ranking is absent: $\sigma_{r_e} > \sigma_{r_f}$. That this should be so is attributable to the low security valuations attendant to low M values: $\overline{c}_2$ is no longer relatively insignificant vis-à-vis bequest values, a precondition for the inverted volatility results to hold.

Although we formally consider only a steady state analysis, it is clear from these remarks that an increased propensity to bequeath increases security prices, although the consequences for the premium are ambiguous. But what might precipitate such an increased propensity? It is natural to suggest that a preference shift in favor of bequests could be motivated by a more pessimistic outlook on the part of parents for the economic prospects of their children – with the resulting increase in the parental "M". The 1990s in the U.S., where real wages generally stagnated, and wealth accumulation thus became more difficult for those initially lacking it, suggests itself as such a period. In that light, the simple model considered here offers an explanation for the run up in securities prices observed at that time. All of these results depend on the fact that, in equilibrium, the desire to give bequests does not, ceteris paribus, diminish the consumption of those giving them.

It remains to consider the consequences of increases in $\overline{c}_2$, the old-aged consumption level, and γ, the bequest risk-time preference parameter.[13] The same basic phenomenon noted immediately above will be seen to drive the observed results. We consider first changes in $\gamma_c = \gamma_B = \gamma$, and Table 6 presents the results for a representative set of cases. Note that we maintain the lower old-age consumption level of Table 5 because the benchmark level of $\overline{c}_2 = 20{,}000$ leads to non existence of equilibrium for $\gamma = 3, 5, 7$, etc.

[13] Under standard formulations with CRRA utility, an increase in γ will induce the agent to smooth his consumption more thoroughly across both states and time periods. For the extreme formulations considered in this paper, consumption smoothing across time periods is totally unaffected by γ. For this reason we will henceforth refer to γ exclusively as the bequest risk parameter.

Table 6

Effect on Equilibrium Security Prices and Returns of Changes in γ

$\overline{c}_2 = 10{,}000$, $M = 0.1$, $\phi = 0.5298$, $Y(1)$, $Y(2)$, $W^1(1)$, $W^1(2)$ as in Table 1, $\gamma_c = \gamma_B = \gamma$

Panel A: Equilibrium Prices and Price Ratios

	$\gamma=1$	$\gamma=2$	$\gamma=3$	$\gamma=5$
$(c_1(1), c_2(1))$	(89,200, 10,000)	(89,200, 10,000)	(89,200, 10,000)	(89,200, 10,000)
$(c_1(2), c_2(2))$	(49,598, 10,000)	(49,598, 10,000)	(49,598, 10,000)	(49,598, 10,000)
$(c_1(3), c_2(3))$	(89,200, 10,000)	(89,200, 10,000)	(89,200, 10,000)	(89,200, 10,000)
$(c_1(4), c_2(4))$	(49,598, 10,000)	(49,598, 10,000)	(49,598, 10,000)	(49,598, 10,000)
$q^e(1)$	4,910	13,322	31,483	162,366
$q^e(2)$	3,637	4,818	5,813	8,934
$q^e(3)$	6,081	13,451	28,432	141,202
$q^e(4)$	2,782	4,565	6,504	10,742
$q^b(1)$	0.28	1.00	2.76	14.02
$q^b(2)$	1.52	3.35	5.14	7.09
$q^b(3)$	2.27	8.95	24.51	109.29
$q^b(4)$	0.17	0.36	0.57	0.92
$E(q^e/Y)$	0.04	0.09	0.17	0.71
$E(B/A)^{(i)}$	0.63	0.73	0.78	0.85

Panel B: Equilibrium Return Statistics

	$\gamma=1$	$\gamma=2$	$\gamma=3$	$\gamma=5$
$E(r^e)$	9.06%	6.51%	4.67%	2.39%
$\sigma(r^e)$	10.52%	12.08%	16.65%	28.39%
$E(r_B)$	5.73%	3.19%	2.06%	1.26%
$\sigma(r_B)$	16.03%	15.07%	18.10%	28.48%
$E(r_f)$	6.06%	2.88%	0.39%	-3.64%
$\sigma(r_f)$	18.94%	19.70%	22.33%	32.29%
$E(r_p)$	3.00%	3.63%	4.28%	6.03%
$\sigma(r_p)$	8.85%	9.71%	8.95%	6.67%

(i) $B/A \equiv$ Bequests (j)/value of all assets in state j.

From Panel A we see that equity and bond prices increase robustly in all states as γ rises, sometimes by a factor of more than thirty in the case of the equity security and more than fifty in the case of the consol (both comments apply to state 1). By arguments similar to those presented for the case of an increasing M, the average equity/output ratio naturally increases and the average B/A ratio asymptotically approaches one (prices are rising and security payments are fixed).[14] As the payoffs to all these securities are unaffected by changes in γ, these price increases cause all returns to decline, even to less than zero in the case of the discount bond (equity and consol bond average returns will remain positive). Consistent with our intuition from the standard consumption-savings model, equity returns fall less rapidly than risk free returns, giving rise to an increasing premium as γ increases. The other prominent stylized fact is the concomitant increase in the standard deviations of all security returns (although not reported, for all securities the range of the state contingent expected returns increases as well).

How do we interpret these results? That all security prices must be bid up across the board is best seen from a slight rewriting of the equity asset pricing equation (9) as follows:

$$(12) \qquad q^e(j) = \beta M(W^1(j) + d(j) + b - \bar{c}_2)^\gamma \sum_{k=1}^{4} \frac{(q^e(k) + d(k))\, \pi_{jk}}{(q^e(k) + d(k) + b(q^b(k) + 1) - \bar{c}_2)^\gamma}$$

As is readily apparent, as γ increases the multiplier $\beta M(W^1(j) + d(j) + b - \bar{c}_2)^\gamma$ increases robustly, since in all of our cases $(W^1(j) + d(j) + b - \bar{c}_2) > 1$. It is as though the agent "valued the future" more fully, and clearly the only way that equality can be maintained is for $q^e(j)$ to increase for all j; similar remarks apply to the other securities. Relative to the standard consumption-savings case where the multiplier assumes the

[14] With $\bar{c}_2$ = 10,000, however, the B/A ratio is counterfactually high in all cases.

form $\beta M(W^1(j) + d(j) + b - \bar{c}_2 - q^e(j) - bq^b(j))^\gamma$, the offsetting presence of $q^e(j)$ and $q^b(j)$ is absent: prices must rise even more to restore equilibrium and they do, but at a differential rate in order to restrain bequest volatility. Stated most simply, the only way an investor can respond to a heightened distaste for bequest volatility is to increase his demand for all securities, i.e., to have more wealth in every state. This objective is in fact accomplished: in the case of γ=1, the ratio $\max_j B(j) / \min_j B(j)$ is approximately 30, while for γ=5 it falls to 22. In this sense an increase in γ is qualitatively very similar to an increase in M and thus much, though not all, of our intuition carries over from that earlier analysis.

An exception to this general theme is the pattern of volatilities: as M increases the return volatilities decline while as γ increases return volatilities increase for all securities. To understand the distinction we return again to the pricing kernel, $M\left(\frac{c(j)}{B(k)}\right)^\gamma$: ceteris paribus, an increase in $\gamma > 1$ will increase the volatility of the kernel, while an increase in M, by increasing B(k) alone will diminish it. In the former case, the consequent increased price volatility leads to an increase in equilibrium return volatility, with the opposite effects being observed in the latter. Informally, if an agent's bequest parameter M increases, he will want to bequeath large bequests in every state irrespective of whether such increases are accompanied by higher bequest volatility. In the event of an increased γ, however, the middle aged agent acquires the added desire to stabilize the level of bequests across states. As a result he will be even more reluctant to hold risky assets in the states in which they have low price prospects next period (and thus their returns will be relatively lower), while accepting higher returns in states where price prospects next period are already bright on an expected basis. The range and standard deviation of returns thus increases. We note that the same phenomenon is observed in Lucas (1978) or Mehra and Prescott (1985) style asset pricing models: a higher CRRA of the representative agent leads to higher equilibrium return volatility across the board.

We conclude this section by considering the consequences of an increase in $\overline{c}_2$, the level of planned old-aged consumption (Table 7) for a representative parameterization.

Table 7

Effect on Equilibrium Security Prices and Returns of Changes in $\bar{c}_2$

M=0.1, ϕ=0.5298, $\gamma_c=\gamma_B$=3.1, Y(1), Y(2), $W^1(1)$, $W^1(2)$ as in Table 1

Panel A: Equilibrium Prices and Price Ratios

	$\bar{c}_2=1$	$\bar{c}_2=1{,}000$	$\bar{c}_2=11{,}700$	$\bar{c}_2=21{,}000$	$\bar{c}_2=31{,}000$
$q^e(1)$	20,225	20,998	34,526	62,884	142,742
$q^e(2)$	4,266	4,399	6,133	8,236	11,122
$q^e(3)$	16,773	17,610	31,413	57,467	126,646
$q^e(4)$	5,039	5,150	6,850	9,355	13,095
$q^b(1)$	1.71	1.81	3.06	5.61	12.35
$q^b(2)$	3.49	3.65	5.41	6.83	8.19
$q^b(3)$	13.23	14.13	27.06	46.58	91.10
$q^b(4)$	0.40	0.42	0.61	0.83	1.13
$E(q^e/Y)$	0.11	0.11	0.18	0.31	0.65
$E(B/A)$	1.00	0.97	0.76		

Panel B: Equilibrium Return Statistics

	$\bar{c}_2=1$	$\bar{c}_2=1{,}000$	$\bar{c}_2=11{,}700$	$\bar{c}_2=21{,}000$	$\bar{c}_2=31{,}000$
$E(r^e)$	5.81%	5.7%	4.46%	3.32%	2.27%
$\sigma(r^e)$	14.61%	14.79%	17.07%	19.98%	24.84%
$E(r_B)$	2.77%	2.69%	1.96%	1.49%	1.12%
$\sigma(r_B)$	17.03%	17.1%	18.39%	20.7%	25.09%
$E(r_f)$	1.72%	1.59%	0.14%	-1.33%	-3.18%
$\sigma(r_f)$	21.46%	21.54%	22.53%	24.31%	28.48%
$E(r_p)$	4.09%	4.11%	4.32%	4.65%	5.45%
$\sigma(r_p)$	10.18%	16.08%	8.7%	7.36%	6.12%

In this model, $\bar{c}_2$ serves the role of a "subsistence level": bequest utility is obtained only if the value of residual assets exceeds $\bar{c}_2$. As a result, its increase will induce the agent to behave in a more bequest-risk averse manner. In this sense, the results concerning the return and price statistics of an increase in $\bar{c}_2$ are qualitatively identical to those of an increase in γ: in both cases average returns on all securities fall and their volatilities rise, and the state contingent prices rise. The interpretive stories also remain the same: the middle agent's sensitivity to bequest risk increases by pushing his consumption pattern to lie on a more concave portion of his utility function. The observed price increases are not as dramatic as in some of our earlier results, reflecting in part the fact that an increased $\overline{c}_2$ actually diminishes the middle aged agent's utility. Of greater significance is the failure of equilibrium to exist for certain ranges of $\overline{c}_2$. In particular, this occurs when $\overline{c}_2$ experiences a further increase of 520 to $\overline{c}_2$ = 31,520: equilibrium does not exist.

4. Endogenous Consumption of the Old

Endogenizing the joint consumption-bequest decision of the old is accomplished by appending to the equilibrium characterization (10)-(12), the state by state condition (13) $u_1(c_2(j)) = MV_1(B(j))$, a feature that at once provides four additional equations (j = 1,2,3,4) while demanding four additional equilibrium quantities ($c_2(j)$: j = 1,2,3,4). With this generalization we revisit the benchmark case.

Table 8

Endogenous vs. Fixed Old Age Consumption Security Prices and Returns

ϕ=0.5298, γ=2, M=1 (as applicable)

Panel A: Prices, Bequests, Consumption

	$c_2(j) \equiv \overline{c}_2 = 20{,}000$	$c_2(j)$, $B(j)$ Endogenous	$c_2(j)$ Endogenous; No Bequests[i]
$(c_1(1), c_2(1))$	79,200, 20,000	(48,698, 50,503)	(39,859, 59,341)
$(c_1(2), c_2(2))$	39,598, 20,000	(37,534, 22,064)	(37,293, 22,305)
$(c_1(3), c_2(3))$	79,200 20,000	(42,473, 56,737)	(25,561, 73,639)
$(c_1(4), c_2(4))$	39,598, 20,000	(32,655, 29,943)	(26,128, 33,470)
B(1)	107,779	50,502	n.a
B(2)	20,643	22,064	n.a
B(3)	171,054	56,727	n.a
B(4)	30,598	26,943	n.a
$q^e(1)$	70.287	51,111	14,178
$q^e(2)$	16,797	30,522	12,298
$q^e(3)$	58,188	36,715	5,424
$q^e(4)$	20,317	24,512	6,454
$q^{r_f}(1)$	1.50	0.88	0.40
$q^{r_f}(2)$	0.86	0.77	0.71
$q^{r_f}(3)$	2.87	0.86	0.28
$q^{r_f}(4)$	0.45	0.45	0.19
E(B/A)	0.71	0.50	n.a
$E(q^e/Y)$	1.13	0.36	n.a

Panel B: Return Statistics

$E(r^e)$	2.54%	2.86%	7.17%
$\sigma(r^e)$	12.50%	5.62%	3.92%
$E(r_B)$	1.22%	1.99%	4.88%
$\sigma(r_B)$	13.33%	6.59%	2.98%
$E(r_f)$	-0.48%	1.74%	5.48%
$\sigma(r_f)$	15.24%	6.38%	1.38%
$E(r_p)$	3.02%	1.12%	1.69%
$\sigma(r_p)$	5.36%	1.88%	3.35%

(i) The model underlying these statistics is formally presented in footnote (9).

The results are presented in Table 8. For the leftmost case, old age consumption is fixed and bequests are paramount for asset pricing. In the middle case (equation (13) applies), bequests and old age-middle age consumption smoothing jointly influence equilibrium. Relative to the prior case, old aged investors sell a larger fraction of their securities to the middle aged while bequeathing the rest. In the right most column, equilibrium security prices and returns are exclusively determined by inter and intra temporal consumption smoothing (no bequests).[15]

As we move across the table from left to right, therefore, bequests progressively recede in importance. Note that for each security type, the associated payments are invariant across the three cases.[16] Focusing first on Panel A, bequests and asset prices uniformly decline as bequests become less significant. The decline is particularly dramatic (on a proportional scale) when bequests are eliminated entirely, a fact directly attributable to

[15] This corresponds to the constrained problem detailed in Constantinides et al. (2002): middle aged agents accumulate securities purely to finance their retirement consumption (no bequests). The latter is accomplished by selling their security accumulation ex dividend to the then middle aged agents. More formally, the maximization problem of the period-t-born agent is:

$$\text{Max E}\left(\sum_{j=0}^{2}\beta^{j}u(c_{t,j})\right)$$

$$c_{t,0} \leq W_0$$

$$c_{t,1} + q^{e}_{t+1}\ z_{t,1} + \ q^{b}_{t+1}\ z^{b}_{t,1} + q^{r_f}_{t+1}\ z^{r_f}_{t,1} \leq \tilde{W}^{1}_{t}$$

$$c_{t,2} \leq q^{e}_{t+2}\, z_{t,1} + q^{b}_{t+2}\, z^{b}_{t,1} + q^{r_f}_{t+2}\, z^{r_f}_{t,1}$$

$$0 \leq z^{e}_{t,1} \leq 1$$

$$0 \leq z^{b}_{t,1} \leq b$$

$$0 \leq z^{r_f}_{t,1}$$

the large influence bequests have on the equilibrium steady state security prices: unlike savings for old age consumption which entails an actual (steady state) cost for the middle aged, bequests do not impinge upon middle aged consumption. As a further consequence of declining bequests, old age consumption increases, but not by the full magnitude of the bequest reduction because prices are lower. That bequests and old age consumption coincide in real terms for the middle case is a direct implication of constraint (13), since M = 1 and $\gamma_c = \gamma_B$

A number of other idiosyncratic features of Table 8 are worth exploring. For one, the equity price is consistently highest in state one. It is this state that corresponds simultaneously to the highest output level and the highest possible middle aged wage level. While not the highest attained value, dividends in this state are much higher than in a majority of the other states. With a relatively persistent dividend steam and a high level of income (wages) with which to purchase securities, it is not surprising that these two effects conspire to bid equity prices up to uniquely high levels. Although state three experiences the highest dividend per se, resources for purchasing securities are much lower.

Comparing the endogenous bequest and no bequest cases, it is also interesting to observe that middle aged consumption is higher in the former and old age consumption higher in the latter. This is not surprising as bequests provide more resources to the middle aged. Furthermore, consumption appears to be less smooth intertemporally under the no bequest regime: comparing the endogenous and no bequest cases, in every

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[16] In all cases the state contingent dividend is d(1) = 40,350, d(2) = 748, d(3) = 62,750, d(4) = 23,148.

state middle aged consumption is lower and old age consumption higher in the latter case. This phenomenon follows again from the observation that the effect of bequests is to shift consumption to the middle aged; they do not have to save fully for old age consumption, and thus can more easily enjoy more consumption as middle aged. In effect, bequests are equivalent to costless borrowing.[17] As a result, middle aged investors have much higher wealth in the bequest case and bid up securities prices to much higher levels as observed.

Turning now to Panel B, two striking regularities are evident. First, as bequests diminish in importance to the agent (again moving from left to right across the columns) expected returns rise dramatically (by a factor of five in the case of the risk free asset) while volatilities decline by a similarly large extent. That expected returns should increase is formally attributable to lower security prices (Panel A) in tandem with unchanging security payments. With the prices of future income streams declining, expected returns rise. That volatilities decline is more difficult to explain intuitively. Variation in security returns essentially reflects variation in the expected utility value of some quantity tomorrow relative to today at the margin. In the left most column, this quantity is bequests; in the right most it is old age consumption. Since bequests are "self financing" they are much less constrained and thus asset prices respond much more dramatically to changes in income and expectations under the former scenario.

[17] We have to be careful of this interpretation in that there is no agency or individual in the model from whom the middle aged might borrow. It is intended to be construed in the sense that a gift is equivalent to a loan that never needs repayment.

Qualitatively, the observed statistical pattern of securities moving from left to right in Table 8 is identical to that observed with decreasing M in Table 4.

Another important truth to emerge from a study of the benchmark cases is that the addition of a bequest motive, in the manner we have chosen to model it, is no panacea for the equity premium puzzle. Relative to the pure consumption savings (rightmost) case, the premium actually declines when bequests are admitted and endogenized (1.69% vs. 1.12%). Of the three benchmark cases, it is the fixed old age consumption (leftmost) context that best explains the data, although the pattern of relative return volatilities across the three securities is at variance with reality (for reasons detailed in Section 3). What is attractive about this case is that its non trivial premium results principally from the low risk free rate.

Notice finally that bequest endogeneity allows the model to better approximate the bequest/asset ratio: in the M=1 case (Table 8, center column), $E\left\{\frac{B}{A}\right\} = 0.50$. While at the upper range of reasonable values, it is a substantial improvement over the fixed consumption ($\overline{c}_2 = 20{,}000$) benchmark. As would be expected, lowering M brings this ratio to a more acceptable level, with the added benefit of a somewhat higher equity premium.

4.1 Comparative Dynamics Revisited

The data for various M and various γ in an environment of endogenous bequests is presented in Tables 9 and 10, respectively. Basically, most all of the qualitative relationships detailed for the fixed old age consumption case, and their underlying justifications, carry over to this more general setting.

Table 9

Effects of Changes in M on Equilibrium Prices, Returns, and Bequests

Endogenous Old Age Consumption

ϕ=0.5298, γ=2[(i)]

Panel A: Prices, Bequests, Consumption

	M=0.001	M=0.01	M=0.1	M=1
$(c_1(1), c_2(1))$	(40,484, 58,716)	(41,678, 57,522)	(44,451, 54,750)	(48,698, 59,503)
$(c_1(2), c_2(2))$	(37,310, 22,288)	(37,344, 22,254)	(37,421, 22,177)	(37,534, 22,064)
$(c_1(3), c_2(3))$	(26,767, 72,433)	(29,062, 70,137)	(34,362, 64,387)	(42,473, 56,727)
$(c_1(4), c_2(4))$	(26,580, 33,018)	(27,449, 32,149)	(29,485, 30,113)	(32,655, 26,943)
B(1)	1,857	5,752	17,313	50,503
B(2)	705	2,225	7,012	22,064
B(3)	2,291	7,014	20,503	56,727
B (4)	1,044	3,218	9,522	26,943
$q^e(1)$	15,273	17,672	25,473	51,111
$q^e(2)$	12,867	14,101	18,021	30,523
$q^e(3)$	6,218	8,019	14,269	36,715
$q^e(4)$	6,977	8,130	11,910	24,513
$q^{r_f}(1)$	0.42	0.47	0.60	0.88
$q^{r_f}(2)$	0.71	0.72	0.74	0.77
$q^{r_f}(3)$	0.31	0.37	0.54	0.86
$q^{r_f}(4)$	0.21	0.23	0.30	0.45
E(B/A)	0.03	0.09	0.24	0.50
$E(q^e/Y)$	0.11	0.13	0.18	0.36

Panel B: Return Statistics

	M=0.001	M=0.01	M=0.1	M=1
$E(r^e)$	6.56%	6.00%	4.73%	2.86%
$\sigma(r^e)$	8.94%	8.02%	6.48%	5.62%
$E(r_B)$	4.52%	4.12%	3.25%	1.99%
$\sigma(r_B)$	11.24%	10.32%	8.47%	6.59%
$E(r_f)$	5.04%	4.54%	3.41%	1.74%
$\sigma(r_f)$	10.03%	9.19%	7.65%	6.38%
$E(r_p)$	1.51%	1.46%	1.33%	1.12%
$\sigma(r_p)$	3.88%	3.61%	2.96%	1.88%

(i) All other parameters as in Table 4.

Table 10

Effects of Changes in γ on Equilibrium Prices, Returns, and Bequests

Endogenous Old Age Consumption

ϕ=0.5298, M=0.1[i]

Panel A: Prices, Bequests, Consumption

	γ=1	γ=2	γ=3	γ=5
$(c_1(1), c_2(1))$	(41,521, 57,679)	(44,451, 54,750)	(43,386, 55,814)	(39,729, 59,471)
$(c_1(2), c_2(2))$	(39,021, 20,577)	(37,421, 22,177)	(35,748, 23,850)	(33,730, 25,868)
$(c_1(3), c_2(3))$	(27,380, 71,820)	(34,362, 64,387)	(35,757, 63,443)	(35,020, 64,180)
$(c_1(4), c_2(4))$	(24,880, 34,718)	(29,485, 30,113)	(30,933, 28,665)	(31,333, 28,265)
B(1)	5,768	17,313	25,907	37,524
B(2)	2,058	7,012	11,070	16,322
B(3)	7,182	20,503	29,448	40,495
B(4)	3,472	9,522	13,305	17,834
$q^e(1)$	18,187	25,473	33,257	45,046
$q^e(2)$	15,504	18,021	18,552	17,136
$q^e(3)$	11,148	14,269	16,579	17,722
$q^e(4)$	10,844	11,910	13,670	16,746
$q^{r_f}(1)$	0.42	0.60	0.82	1.13
$q^{r_f}(2)$	0.56	0.74	0.83	0.90
$q^{r_f}(3)$	0.35	0.54	0.70	0.90
$q^{r_f}(4)$	0.26	0.30	0.35	0.43
E(B/A)	0.09	0.24	0.32	0.39
$E(q^e/Y)$	0.15	0.18	0.21	0.25

Panel B: Return Statistics

	γ=1	γ=2	γ=3	γ=5
$E(r^e)$	5.36%	4.73%	4.26%	3.82%
$\sigma(r^e)$	5.46%	6.48%	6.99%	7.57%
$E(r_B)$	4.73%	3.25%	2.36%	1.62%
$\sigma(r_B)$	6.46%	8.47%	8.98%	9.15%
$E(r_f)$	4.87%	3.41%	2.34%	1.24%
$\sigma(r_f)$	6.03%	7.65%	8.16%	8.6%
$E(r_p)$	0.49%	1.33%	1.92%	2.59%
$\sigma(r_p)$	1.43%	2.96%	3.33%	3.75%

(i) All other parameters as in Table 4.

As M increases, in particular, asset pries and the value of bequests rise, while expected returns decline. Return volatilities decline as in the fixed old age consumption case as well. All quantities are comparatively smaller in the endogenous consumption case, reflecting the reduced influence of bequests. However, the volatility of the risk free return exceeds that of the equity security as in Table 3 and for the same fundamental reasons.

The comparative results (Tables 6 vs. 10) for an increase in risk aversion are in the same spirit. As in the fixed old age consumption case, greater risk aversion coincides with lower expected returns and higher return volatilities. The equity premium also increases with gamma. For $\theta \geq 2$, E(B/A) uniformly lies within the acceptable range. As in the previous case, all the return statistics are muted relative to their fixed old age consumption counterparts. There are no issues of the non-existence of equilibrium for any of these cases.

Substantial differences can be found in the level and variation in the price and bequest series. Comparing Table (10) with (6), there is seen to be much less variation in bequest levels or asset prices across the four states, a fact that is also manifest in the means and standard deviations of returns across all the securities. This is to be expected: in the former case quantities can adjust more freely. There is thus less need for prices themselves to adjust.

5. Summary and Conclusions

This paper has focused on the influence of bequests on equilibrium security prices and returns. Generally speaking, the effect of bequests is to increase, dramatically, security prices. In a standard consumption-savings context, the purchase of securities to finance future consumption reduces consumption today thereby raising the marginal utility of consumption, which acts as a discouragement to further savings. This latter effect is not present in a bequest-driven model of the type considered here, at least in the steady state, leading to much more powerful income effects. Both asset prices and price volatility are substantially higher.

In general, the addition of a bequest utility function does not a priori allow the model to resolve, in any complete way, the equity premium or return volatility puzzles. With regard to the former, the high covariance of equity returns with bequests – a fact suggestive of a high premium – is neutralized because the resulting high bequest levels make the agent nearly risk neutral vis-à-vis bequest utility risk. A separation of time and risk preferences may allow for a better understanding of this particular phenomenon.

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Appendix 1

Proof of Theorem 2.1: Define the operator T by

$$T(x_1,\ldots, x_N, y_1,\ldots, y_N) =$$

$$\left(\beta M \sum_{k=1}^{N} \pi_{1k} \left(\frac{W^1(1)+\theta(1)}{x_k + b\,y_k + \theta(k)} \right)^{\gamma} \left(x_k + d(k)\right),\ldots, \right.$$

$$\beta M \sum_{k=1}^{N} \pi_{Nk} \left(\frac{W^1(N)+\theta(N)}{x_k + b\,y_k + \theta(k)} \right)^{\gamma} \left(x_k + d(k)\right),$$

$$\beta M \sum_{k=1}^{N} \pi_{1k} \left(\frac{W^1(1)+\theta(1)}{x_k + b\,y_k + \theta(k)} \right)^{\gamma} \left(x_k + 1\right),\ldots,$$

$$\left. \beta M \sum_{k=1}^{N} \pi_{Nk} \left(\frac{W^1(N)+\theta(N)}{x_k + b\,y_k + \theta(k)} \right)^{\gamma} \left(x_k + 1\right) \right).$$

Defined as per above, A is compact in R^{2N}. Furthermore, since $\theta(j) > 0\ \forall j$, T is continuous on A. Clearly, for every $(x_1, \ldots, x_N, y_1, \ldots, y_N) \geq 0$, $T(x_1, \ldots, x_N, y_1, \ldots y_N) \geq 0$. In order to apply Brower's fixed point theorem we need only to show that each entry in the image of T falls short of Ψ. For any x_j,

$$\beta M \sum_{k=1}^{N} \pi_{jk} \left(\frac{W^1(j)+\theta(j)}{\theta(k)} \right)^{\gamma} \left(x_k + d(k)\right)$$

$$\leq \beta M\ (\Psi)\ (2) \sum_{k=1}^{N} \pi_{jk} \left(\frac{W^1(j)+\theta(j)}{\theta(k)} \right)^{\gamma}$$

$$< 2\beta M L\ \Psi < \Psi$$

For any y_j,

$$\beta M\sum_{k=1}^{N}\pi_{jk}\left(\frac{W^1(j)+\theta(j)}{\theta(k)}\right)^{\gamma}\left(y_k+1\right)$$

$$<\beta M\ 2\ \Psi\ L<\Psi$$

Thus $\exists$ a fixed point $(\hat{x}_1,...,\hat{x}_N,\hat{y}_1,...,\hat{y}_N)$ of T on A. Identify

$$\hat{x}_j \equiv q^e(j)$$

$$\hat{y}_j \equiv q^b(j)$$

Then $\left(q^e(j), q^b(j)\right)$ solves (9) and (10.)

Note that since $\theta(j)>0$ and $d(j)>0$ $\forall j$, $\left(q^e(j), q^b(j)\right)>0$ $\forall j$. Lastly, $q^{r_f}(j)>0$ is defined as per (11) once $q^e(j)$, $q^b(j)$ are determined.