Reallocation of Intangible Capital and Secular Stagnation*

Andrea Caggese
Universitat Pompeu Fabra, CREI & Barcelona GSE

Ander Pérez-Orive
Federal Reserve Board

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Abstract

Low interest rates can hurt capital reallocation and reduce aggregate productivity and output in economies that rely strongly on intangible capital. This insight is obtained in a model in which productive credit-constrained firms can only borrow against the collateral value of their tangible assets and there is substantial dispersion in productivity. In a tangibles-intense economy with highly leveraged firms, low rates enable more borrowing and faster debt repayment, reduce misallocation, and increase aggregate output. Conversely, an increase in the share of intangible capital in production reduces the borrowing capacity and increases the cash holdings of the corporate sector, which switches from being a net borrower to a net saver. In this intangibles-intense economy, the ability of firms to purchase intangible capital using retained earnings is impaired by low interest rates, because they increase the price of capital and slow down the accumulation of corporate savings. As a result, the emergence of intangible technologies, even when they replace significantly less productive tangible technologies, may be contractionary.

Keywords: Intangible Capital, Borrowing Constraints, Capital Reallocation, Secular Stagnation
JEL Classification:

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1 Introduction

Real interest rates have decreased in the last decades, while economic growth has fallen short of previous trends, developments that have been linked to a process of 'secular stagnation' (Summers (2015), Eichengreen (2015)). At the same time, the developed world has experienced a technological change towards a stronger importance of information technology and knowledge, human and organizational capital, which has gradually reduced the reliance on physical capital (Corrado and Hulten (2010a)), and which has been linked to the significant decrease in corporate net borrowing (Falato, Kadyrzhanova, and Sim (2014), Döttling and Perotti (2015)).

This paper argues that the increased reliance on intangible capital and the low real interest rates interact to hurt capital reallocation and reduce productivity and output growth. Aggregate productivity depends on an efficient reallocation of resources from declining or exiting firms to new entrants or expanding firms. The rise of intangible capital implies a growing importance of the reallocation of intangible assets such as patents, brand equity, and human and organizational capital. These assets cannot be collateralized, and their acquisition has to be financed mostly using retained earnings. As a result, the corporate sector borrows less, holds an increasing amount of cash, and switches from being a net borrower to a net saver. We show that this shift not only adds additional downward pressure on interest rates, but also alters the dynamic relationship between interest rates and efficiency in the allocation of capital. The decrease in interest rates increases the price of these intangible assets, and reduces the ability of credit constrained expanding firms to purchase them. Lower interest rates also decrease the rate at which non-investing firms can accumulate savings to finance future expansions.

This alternative explanation of secular stagnation is consistent with crucial stylized facts about recent trends in industrialized economies, such as declining interest rates, below-potential growth, and large increases in net corporate savings and asset prices over GDP, and has potentially important policy implications.

We formalize this intuition by developing a stylized model of an economy in which a productive sector uses a technology with tangible capital, intangible capital and labor as complementary factors in the production of consumption goods. We follow Kiyotaki and Moore (2012) in assuming that this sector is populated by a continuum of firms that can only invest occasionally. Firms suffer from financing constraints that prevent them from issuing equity, or from borrowing any amount in excess of the collateral value of their holdings of tangible and intangible capital. They have finite lives, and this prevents them from accumulating enough savings to overcome their financial constraints. In equilibrium, they save as much as possible in non-investing periods, and invest all of their accumulated net savings plus their maximum available borrowing in investing periods. Any residual capital not absorbed by the productive firm sector is used by an unproductive alternative sector, which as a result of being the marginal buyer of capital also prices it. Therefore, aggregate productivity in this economy depends on the ability of growing high-productivity firms to absorb the assets liquidated by the exiting firms. The consumer sector is modelled as overlapping generations of households displaying a realistic life cycle, modelled in a way that enables us to obtain an equilibrium interest rate in the steady state which is not necessarily equal to the household rate of time preference.

We first inspect the analytical solution of a simplified version of the model to describe four channels through which lower interest rates interact with the intensity of intangible capital in firms' production function to affect the steady state equilibrium of our economy. First, a *debt overhang channel* allows net borrowing firms to pay down their debt more easily when interest rates are low, and helps capital reallocation. Conversely, a *savings channel* operates when the firm sector is a net saver, and reductions in the interest rate decrease the speed of accumulation of savings and hurt capital reallocation. Third, lower interest rates that increase the price of tangible and intangible assets reduce the amount of capital firms can purchase for a given amount of net worth and borrowing capacity, a *capital purchase price channel*. Fourth, a lower interest rate increases the present value of the collateral pledged next period, and reduces the size of the downpayment necessary to purchase capital, improving capital reallocation through a *borrowing/collateral value channel*. The analytical solution of the simplified model provides a clear illustration of the main theoretical finding of the paper: in an economy with relatively low collateral value of capital, the negative channels dominate and a drop in the interest rate worsens the allocation of resources and reduces aggregate investment, productivity and output.

In the remaining sections of the paper we calibrate and simulate our full general equilibrium model to study how the parallel developments in the household and the corporate sector have interacted to generate aggregate patterns consistent with the secular stagnation hypothesis. In the household sector, we model a progressive decrease in individuals' rate of time preference, which puts downward pressure on the equilibrium interest rate. We interpret our exercise as a shortcut for a collection of different factors, such as population aging, wealth and income inequality, financial deepening and foreign sector developments, which have contributed to increase households' demand for savings in the last 40 years. In the corporate sector, we introduce a gradual shift in the reliance on intangible capital of firms, from the pre-1980 economy, in which intangible capital accounted only for 20% of aggregate capital, to the post-2010 value of 60% of total capital ($\mu = 0.6$) (Corrado and Hulten (2010a), Falato, Kadyrzhanova, and Sim (2014), Döttling and Perotti (2015)). Since we assume that intangible capital is more productive than tangible capital, this gradual shift is consistent with the notion of the transition to intangible capital as a privately optimal choice of firms adopting more productive technologies.

We find that while the household sector developments in isolation and the corporate sector developments in isolation are both expansionary, the combination of both developments is contractionary. The increase in household net savings puts downward pressure in interest rates and, even though it encourages capital creation and increases high-productivity firms' ability to borrow and pay down their debt, affects capital allocation negatively by increasing capital prices. Firms' increased reliance in a type of capital which attracts less external finance decreases corporate leverage and tightens firms' borrowing constraints significantly. Firms switch from being net borrowers to being net lenders, and the share of output produced by the high-productivity firms drops significantly. The lower corporate borrowing puts downward pressure on interest rates, which amplifies the misallocation of capital through a capital purchase price channel and a savings channel. Despite the fact that capital creation increases strongly, and that the economy is shifting to a higher reliance on a type of capital which is significantly more productive, the output drop caused by the combination of both developments is in excess of 1%.

We interpret this comparative static exercise as capturing the developments in the US economy following the rise in the share of intangible capital and the rise in net household and foreign sector savings in the last 40 years. In this respect, this simple model is remarkably consistent with a series of well documented trends during this period: i) net corporate savings increased as a fraction of GDP; ii) household leverage increased as a fraction of GDP; iii) the real interest rate fell; iv) intra industry dispersion in productivity has increased; and v) output and productivity progressively declined relative to their previous trends. While the importance of the rise in intangible capital for stylized facts i)-iii) has been already shown by Falato, Kadyrzhanova, and Sim (2014) and Döttling and Perotti (2015), this paper is the first to show that it is potentially very important in explaining the low growth.

Related Literature

The secular stagnation hypothesis as an explanation of recent economic trends has been proposed, amongst others, by Summers (2015) and Eichengreen (2015). One prominent example of a formalization of these ideas is Eggertsson and Mehrotra (2014), who show how a persistent tightening of the debt limit facing households can reduce the equilibrium real interest rate and, in the presence of a zero lower bound and sticky prices, generate permanent reductions in output.[1]

Common to most of these accounts of secular stagnation is that the excess savings arise from the household or the foreign sector, but not from a decrease in the demand for those savings from the corporate sector. For example, Summers (2015) and Eichengreen (2015) mention factors such as population aging and a rise in savings of developing economies. An exception is Thwaites (2015), who explains the decrease in interest rates as a result of the decrease in the relative price of investment goods. In our model, a realistically calibrated increase in the use of intangible capital can achieve a substantial decrease in interest rates. More importantly, our paper identifies a novel misallocation effect of endogenously low real interest rates which has important policy implications, different from those of other existing secular stagnation theories.

The rising use of intangible capital has been documented by Corrado and Hulten (2010a), and its relation to the decrease in corporate borrowing and the rise in corporate cash holdings has been shown empirically by Bates et al. (2009). Falato et al. (2014) and Döttling and Perotti (2015) introduce models that describe how the rise in intangibles can lower the equilibrium interest rate by decreasing firms' net borrowing. Giglio and Severo (2012) link the decrease in interest rates caused by the rise of intangibles to the appearance of asset price bubbles. Our contribution to this literature is to describe a mechanism through which the rise in intangibles can have a negative impact on aggregate capital reallocation and growth.

[1] Other recent theoretical papers with alternative explanations of secular stagnation are Bachetta et al (2015) and Benigno and Fornaro (2015).

Finally, our paper is related to the literature on the determinants of aggregate investment. A broad class of investment models predict that lower interest rates reduce the user cost of capital and stimulate investment. However, a large body of empirical research finds very little evidence of this negative relation (e.g. see, Caballero (1999), and Schaller (2007)). More recently, Kothari, Lewellen and Warner (2015), using a multivariate regression framework that includes as additional determinants of investment corporate profits, stock market returns, credit spreads and GDP growth, find a positive relation between lagged risk free interest rates and aggregate investment up to 2 quarters into the future. In our general equilibrium model, aggregate capital and interest rates are both endogenous and they may correlate positively or negatively with each other depending on the relative importance of tangible and intangible factors of production, with a positive relation which prevails, because of the rise in intangibles, during the post-1980 period.

The rest of the paper is organized as follows. Section 2 introduced the empirical evidence that motivates this paper. We describe a very simple model in Section 3 that conveys the basic intuition of the mechanisms we introduced in this paper, and develop a full-fledged general equilibrium extension in Section 4. The steady state and calibration of the general equilibrium model are described in Section 5 and the simulation results in Section 7. Section 9 concludes.

2 Empirical Motivation

In this section we summarize the key stylized facts that motivate our model.

1 - Developed economies are significantly more reliant on intangible capital now than in the 1980s, and this technological shift has been linked to the simultaneous transition of the corporate sector from net debtor to net saver

The developed world has experienced a technological change towards a stronger importance of information technology and knowledge, human and organizational capital, which has gradually reduced the reliance on physical capital (Brown and Petersen (2009), Corrado and Hulten (2010a), Falato, Kadyrzhanova, and Sim (2014)). In the U.S., intangible as a share of total capital went from around 0.2 in the 1970s to 0.5 in the 2000s (Falato, Kadyrzhanova, and Sim (2014)). In parallel, there has been a shift in the net financial position of the nonfinancial corporate sector from a net borrowing position roughly before the year 2000 into a net saving position (Armenter and Hnatkovska (2016), Quadrini (2016), Chen, Karabarbounis and Neiman

(2016), Zetlin-Jones and Shourideh (2016)).

The empirical evidence suggests that these two trends are related. The process of technological change has been linked to a lower availability of collateral for the corporate sector, which has lowered its debt capacity. Brown, Fazzari, and Petersen (2009) document that U.S. firms finance most of their R&D expenditures out of retained earnings and equity issues, an observation in line with the conclusion in Hall (2002) that R&D-intensive firms feature much lower leverage on average than less R&D intensive firms. Gatchev, Spindt and Tarhan (2009) document that, in addition to R&D, also marketing expenses and product development are mostly financed out of retained earnings and equity. This is in contrast to tangible assets, which are mostly financed with debt.[2] The process of technological change has been linked to an increase in the precautionary motives for cash accumulation to avoid future financial shortages (Bates, Kahle, and Stulz (2009), Falato, Kadyrzhanova, and Sim (2014), Falato and Sim (2014), Döttling and Perotti (2015), Begenau and Palazzo (2016)).[3] As a result of lower debt and higher cash holdings, net debt in the nonfinancial corporate sector has decreased significantly, and, under some measures, turned negative.[4]

Furthermore, firm level empirical evidence suggests that the observed link between intangible intensity and high cash holdings is driven by financial frictions. Begenau and Palazzo (2016) introduce evidence showing that an important determinant of the increase in cash holdings of public firms is the increase in frequency of new firms that are very R&D intensive, and suggest that these trends are consistent with a model where cash holdings are driven by financial frictions of the R&D intensive firms and costly equity financing. Similarly, Falato et al. (2014) show

[2] Inventory investment and investment in machinery and equipment are mostly financed with debt, which has been shown empirically to be the case. This is perhaps most clear in the case of leases, which can be interpreted as collateralized debt financing in which the debtor can very easily repossess the leased asset in case of default. The structure of lease contracts, designed to facilitate repossession and redeployment of the leased asset, suggests that they are most useful in the case of assets that are not highly firm specific and can easily find alternative uses. Eisfeldt and Rampini (2009) report that a big share of machinery, equipment, buildings and other structures are financed with leases. Inventory investment and other assets with short maturities under one year attract substantial debt finance in the form of trade credit and bank credit lines (Petersen and Rajan, 1997; Sufi, 2009). Finally, investment in commercial real estate is primarily financed with mortgage loans (Benmelech, Garmaise, and Moskowitz, 2005). Furthermore, these authors find, consistent with the results in this paper, that higher asset redeployability leads to larger loans with longer maturities.

[3] Lack of access to debt financing of firms that rely on intangible capital could be compensated by easy access to equity financing. While easy access to equity financing would be consistent with the observed lower leverage of these firms, it would be harder to reconcile with the remarkable accumulation of cash holdings. A large body of evidence shows that external equity financing is significantly costly (Altinkilic and Hansen (2000), Gomes (2001), and Belo, Lin and Yang (2016)).

[4] Different authors use slightly different measures of liquid financial assets in corporations' balance sheets, leading to different measures of net debt positions. The trends however are robust to alternative definitions of net debt.

empirically that the relation between reliance on intangible capital and cash holdings is stronger among firms for which financing frictions are more severe.

2 - Productivity dispersion has increased in intangibles sectors during recent decades, while it has remained roughly constant in tangibles sectors

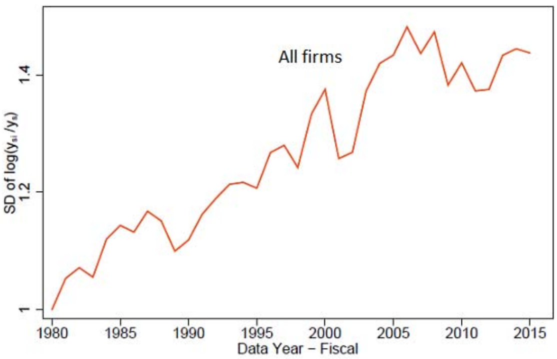
Kehrig (2015) analyzes establishment level manufacturing data from the US Census, and documents a significant increasing trend in the dispersion of productivity across firms within sectors over the last 40 years. The research mentioned above shows that the rising intangible capital share is related to an increase in firm level cash holding to overcame external finance constraints. If the misallocation of resources caused by financial constraints is a factor contributing to the increase in productivity dispersion, we should expected the latter to be more pronounced in sectors with higher intensity of intangible capital.

In order to investigate on the relation between intangible capital and productivity dispersion, we use accounting data of 34,900 US corporations obtained from COMPUSTAT covering the period 1980 to 2015, containing 379,318 firm-year observations. We define intangible capital as the sum of knowledge capital and organizational capital. We follow Falato et al. (2014), and measure the former by capitalizing research and development (R&D) expenses, and the latter by capitalizing Selling, General and Administrative (SG&A) expenses weighted by 0.2.[5][6] The expenditures are capitalized applying the perpetual inventory method with a depreciation rate of 15% for R&D and 20% for SG&A. In order to get a measure for tangible capital, we also use the perpetual inventory method to capitalize tangible capital expenses with a depreciation rate of 15%. We drop firms that are only observed once, firms that are not observed in a continuous time period and exclude regulated, financial and public service firms. We consider sectors at the 2-digit SIC level, and drop those with less than 500 firm-year observations. We measure output by sales, labor input by the number of employees, and total capital by the sum of capitalized tangible and intangible capital.

We consider two alternative productivity measures: labor productivity (y) and TFP (A),

[5] A portion of SG&A expenses capture expenditures that increase the value of intangible capital items such as brand names and knowledge capital. Part of SG&A expenditures however do not affect the value of intangible capital, so Falato et al (2014) follow Corrado, Hulten, and Sichel (2009) and assume the portion relevant to intangible capital is around 0.2.

[6] Falato et al (2014) also consider informational capital. However, they state that their results do not depend on its inclusion. As informational capital can only be measured at the industry level but not at the firm level using Compustat data, we choose not to include this type of capital.



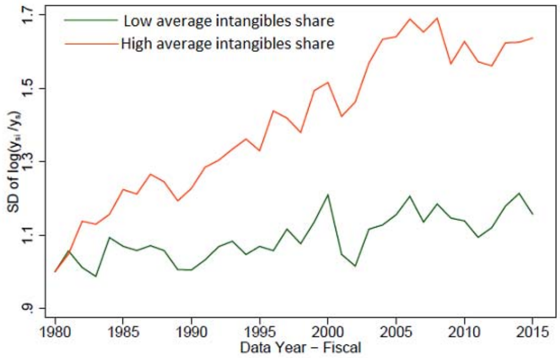


Figure 1: Within industry dispersion in firm-level labour productivity, Compustat Data

which is defined as the residual of a Cobb-Douglas production function with a capital share of income equal to 0.35 (Kurtzman and Zeke (2016)). To control for outliers we drop firms in the first and 99th percentile of the distribution of labor productivity. Our measure of misallocation, the productivity dispersion, is computed by the standard deviation of the difference between the logs of the productivity of firm i, and the aggregate productivity of the industry s firm i operates in.

Figures 1 and 2 plot the dispersion of labor productivity and TFP, respectively, in 2-digit SIC industries over time (normalized by the value in 1980). In both figures the left graph shows average dispersion for all sectors, and it replicates the upward sloping trend already documented by Kehrig (2015) using establishment level data. In the right graph the orange line shows the mean of the dispersion measure across industries (weighted by sales) in the top 50%, and the green line in the bottom 50%, of the distribution of the industry-wide ratio of intangible capital to total capital (averaged across years).[7] Both figures show that the constant rise in the within-industry dispersion of productivity is driven by the sectors with higher average share of intangible capital. This evidence is consistent with the hypothesis that intangible capital

[7]The high intangible share sectors are: Chemicals and Allied Products; Industrial and Commercial Machinery and Computer Equipment; Electronic & Other Electrical Equipment & Components; Transportation Equipment; Measuring, Photographic, Medical, & Optical Goods, & Clocks; Miscellaneous Manufacturing Industries; Wholesale Trade - Durable Goods; Home Furniture, Furnishings and Equipment Stores; Miscellaneous Retail Business Services; Engineering, Accounting, Research, and Management Services.

The low intangible share sectors are: Oil and Gas Extraction; Food and Kindred Products; Paper and Allied Products; Rubber and Miscellaneous Plastic Products; Stone, Clay, Glass, and Concrete Products; Primary Metal Industries; Fabricated Metal Products; Wholesale Trade - Nondurable Goods; General Merchandise Stores; Food Stores; Apparel and Accessory Stores; Eating and Drinking Places.

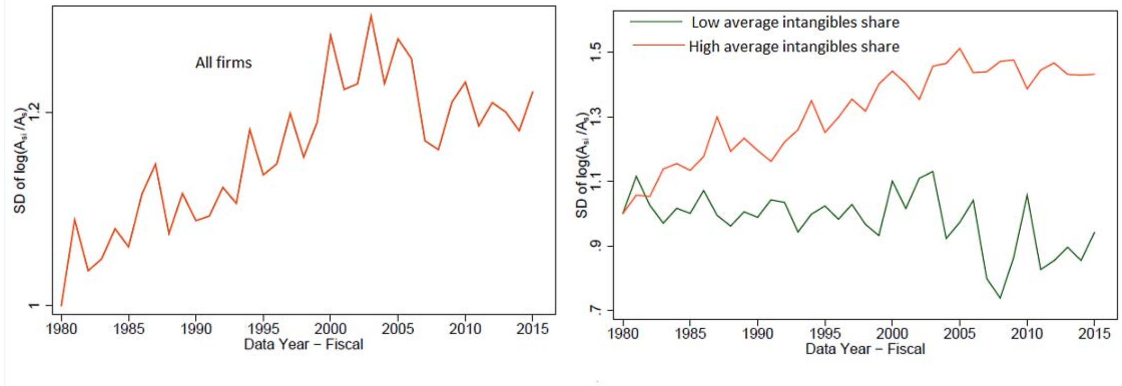


Figure 2: Within industry dispersion in firm-level total factor productivity, Compustat Data

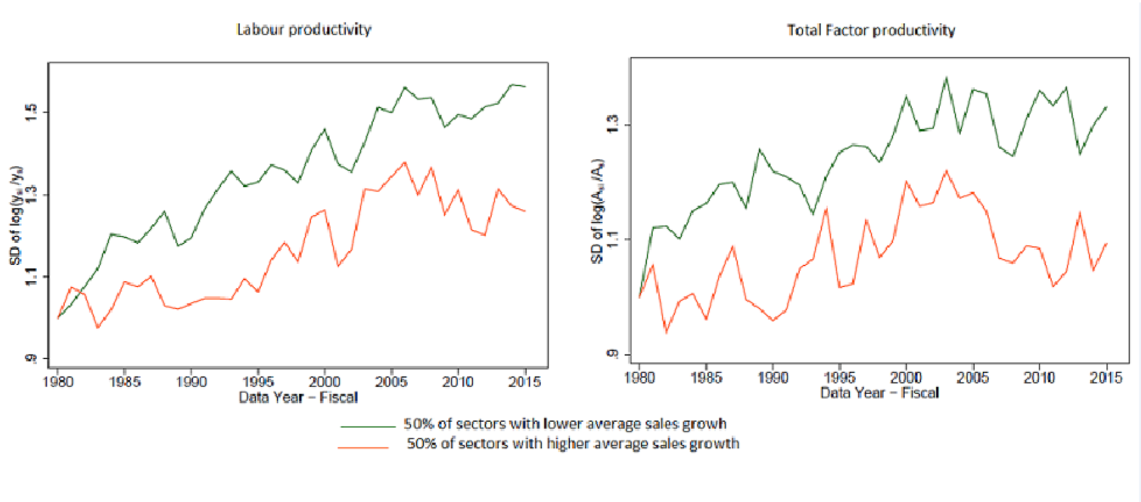


Figure 3: Within industry dispersion in productivity for high growth and low growth sectors, Compustat Data.

exacerbates misallocation problems caused by financial frictions.

One alternative explanation could be that the "high intangible share" sectors do not have a worse allocation of resources, but rather are more dynamic and fast growing, and the increase in dispersion of productivity reflects this higher dynamism. However in Figure 3 we show that sectors with high average sales growth have lower productivity dispersion in the whole sample period.

Furthermore, Table 1 shows regression results where the dependent variable is a measure of productivity dispersion for each 2-digit sector-year observation. Among the regressors we consider: the dummy "High share", which is equal to one if the sector belongs to the 50% 2-digit industries with highest average intangible share, and equal to zero otherwise; a time

Table 1: Intangible share and dispersion in productivity. Regression analysis.

VARIABLES	(1) TFP	(2) TFP	(3) y	(4) y
Time trend		-0.000892		0.00316***
		(0.000837)		(0.000737)
Time trend*High share		0.00558***		0.00370***
		(0.000622)		(0.000548)
High share	0.0897***		0.110***	
	(0.00990)		(0.0144)	
Observations	828	828	828	828
R-squared	0.112	0.632	0.134	0.869
Industry FE	no	yes	no	yes
Year FE	yes	yes	yes	yes

trend; year and sector fixed effects. In columns 1 and 2 the dependent variable is the dispersion in total factor productivity. Column 1 includes year fixed effects, and shows that the dispersion is significantly larger for sectors with higher intangible share. Column 2 includes a time trend, interacted with the High share variable, and both sector and time fixed effects. It shows that the trend in dispersion over time is significantly more positive in the 50% most intangible sectors than in the other sectors, confirming the significance of the result shown in Figures 1 and 2. Similar results, across the two groups of high and low intangibles sectors, are obtained using labor productivity, as shown in columns 3-4.

3 Simple and Intuitive Explanation of the Mechanisms

We introduce in this section the simplest possible model that can describe our proposed mechanisms and deliver analytical results. Our main interest is studying how exogenous interest rate variations affect the allocation of capital and aggregate output depending on the degree of tangibility of capital. This framework is extended in Section 4 in a full-fledged general equilibrium setup that can be used for realistic quantitative analysis.

Consider an infinite-horizon, discrete-time model of an economy. Firms use capital, which is in constant aggregate supply $\overline{K}$, to produce a homogeneous consumption good using a constant returns to scale technology. There are two types of firms, *high-productivity* and *low-productivity*. Efficiency is determined by the share of $\overline{K}$ allocated to high-productivity firms. Here we present the aggregate steady state equilibrium conditions, and introduce the details of the derivation of this simple model in Appendix A.

Aggregate output in the steady state is

$$Y = Y^p + Y^u + Y^e = zK + z^u \left(\overline{K} - K\right), \tag{1}$$

where z captures the productivity of high-productivity firms, and $z^u < z$ captures the productivity of low-productivity firms.

Aggregate capital holdings K of the high-productivity firms, which are assumed to be financially constrained, are

$$K = \frac{A^e(1+r) + Y^e}{q\left(1 - \frac{\theta}{1+r}\right)} \tag{2}$$
$$0 \leq \theta \leq 1$$

where

$$q = \frac{z^u}{r+\xi}, \tag{3}$$

is the price of capital. Low-productivity firms, which are financially unconstrained, have aggregate capital holdings of $\overline{K} - K$, are the marginal buyers of capital, and price it according to their marginal productivity. The parameter ξ captures a pricing wedge (such as a risk premium).[8]

The numerator of (2) captures the total funds available to high-productivity firms to invest, and is assumed to be positive in equilibrium. It is equal to the aggregate net savings or liabilities of the high-productivity firms $A^e(1+r)$, including their return r this period, plus output generated when young Y^e.[9] The denominator of (2) captures the downpayment necessary to purchase one unit of capital. High-productivity firms can borrow using one-period debt up to a fraction θ of the value of capital next period, and have to pay q per unit.

We now describe the four main mechanisms through which interest rates interact with the degree of reliance on intangible capital to affect the allocation of capital and aggregate output. We capture reliance on intangible capital by two features: positive A^e and low θ. Intangible capital is poor collateral (low θ), so firms that rely on intangible capital instead accumulate

[8] In the full general equilibrium model of Section 4, a positive wedge ξ arises endogenously because of capital depreciation and because of decreasing returns to scale in the low-productivity firms' production function.

[9] Equation (2) is derived in Appendix A from the equilibrium of a model in which overlapping generations of firms live for two periods, and receive an endowment of $A^e(1+r) + Y^e$ when they are born.

retained earnings and are more likely to be net savers ($A^e > 0$). Tangible capital has a high collateral value (high θ), so firms that rely on tangible capital are able to borrow more and are more likely to be net borrowers ($A^e < 0$). Importantly, this negative relationship between tangibility of capital θ and financial wealth of productive firms is consistent with the empirical evidence, which we discussed at length in the previous section, and it arises endogenously in the full model derived in Section 4.

If $A^e > 0$, an exogenous increase in r benefits capital allocation by increasing available savings to high-productivity firms to invest. That is the *savings channel.* If $A^e < 0$, an increase in r hurts capital allocation by decreasing available savings to high-productivity firms. That is the *debt overhang channel.* The *capital purchase price channel* is the mechanism through which increases in r hurt capital reallocation by increasing q and making capital more expensive. Finally, the *collateral value channel* is the channel through which increases in r hurt capital reallocation by decreasing the value of firm's collateral (the term $\theta/(1+r)$) and tightening the borrowing constraint.

How do these four channels depend on the intensity of intangible capital? Inspecting dK/dr,

$$\frac{dK}{dr} = \frac{A^e}{q\left(1 - \frac{\theta}{1+r}\right)} + K\left[\frac{1}{r+\xi} - \frac{\theta}{(1+r-\theta)(1+r)}\right]$$

we can identify the four channels described. The first term is positive if $A^e > 0$, capturing the savings channel, and is negative if $A^e < 0$, capturing the debt overhang channel. The first term inside the square brackets captures the capital price channel and is always positive. The second term inside the brackets represents the collateral value channel and is always negative.

How does the tangibility of capital matter for the effect of variations in r on the efficiency of this economy? For clarity of exposition, assume that a tangibles intensive economy is one in which $A^e < 0$ and $\theta > 0$, and an intangibles intensive economy is one in which $A^e > 0$ and $\theta = 0$. Then,

$$sign\left[\frac{dK}{dr}(\text{tangible})\right] = \frac{A^e}{q\left(1 - \frac{\theta}{1+r}\right)} + K\left[\underbrace{\frac{1}{r+\xi}}_{>0} - \underbrace{\frac{\theta}{(1+r-\theta)(1+r)}}_{<0}\right] < 0 \text{ if (6) met,} \tag{4}$$

and

$$sign\left[\frac{dK}{dr}(\text{intangible})\right] = \underbrace{\frac{A^e}{q}}_{>0} + \frac{A^e(1+r)+Y^e}{q}\left[\underbrace{\frac{1}{r+\xi}}_{>0}\right] > 0 \quad \text{always.} \tag{5}$$

$\frac{dK}{dr} > 0$ in an intangibles economy, meaning that a reduction in r is unambiguously contractionary. It is instead most likely expansionary in a tangibles economy, particularly if the responsiveness of q to r is limited (ξ is high) and the borrowing capacity is large (θ is high). More specifically, a decrease in r is unambiguously expansionary if the following condition is satisfied:

$$\theta > \frac{1+2r+r^2}{1+2r+\xi}. \tag{6}$$

Taken together, this analysis suggests that the degree of tangibility of capital in an economy matters importantly for how exogenous variations in the interest rate affect capital allocation and output, and describes four important channels through which these effects occur. It shows that these effects can change sign, and falling interest rates can become contractionary in an economy that relies on intangible capital.

Section 4 analyses these four channels in a full-fledged model in which we endogenize firm financing constraints, firm saving and borrowing, investment, the interest rate, wages, the price of capital, and household consumption and savings.

The Investment Demand Curve

To provide a deeper understanding of how the features of the equilibrium of this economy change as a result of a transition from an economy reliant on tangible capital to one in which intangible capital acquires a larger importance, we represent the equilibrium in the credit market in Figure 4. The main objective is to provide an empirically relevant assessment of the slope of the investment demand curve for different values of θ. To do so, we calibrate the parameters at the annual frequency to be broadly consistent with observed moments of US data. We postpone a more thorough calibration to the full model developed in Section 4. We study a range of the real interest rate between $r = 6\%$ and $r = 0\%$, consistent with the observed evolution of real rates between the early 1980s and the present. We normalize the productivity of low-productivity firms to $z^u = 1$ and the output endowment to $Y^e = 1$. We consider a tangibles economy to feature a pledgeability parameter of capital θ equal to 0.9 and a net borrowing position equivalent to 20% of output ($A^e = -0.2$). We consider an intangibles economy to

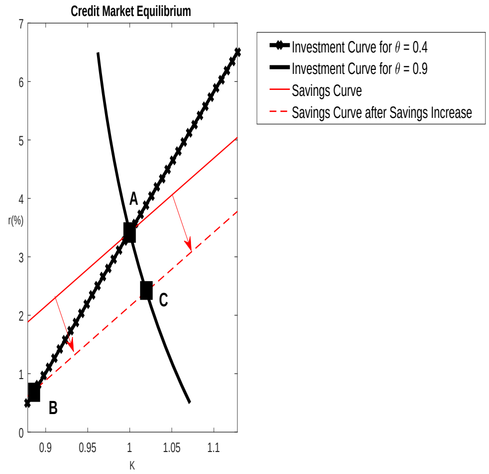


Figure 4: Credit Market Equilibrium.

feature a pledgeability parameter of capital θ equal to 0.4 and a net saving position equivalent to 20% of output ($A^e = 0.2$). The interest rate wedge ξ is set at 20%, and is meant to capture a combination of factors such as risk premia, default premia, and capital depreciation.

In the graph, the upward sloping savings curve captures the combination of the (unmodelled) net savings of the household sector. Higher interest rates induce households to save more, under the empirically realistic assumption that the substitution effect dominates the income effect for them. The demand for capital by the investing firms is equal to the amount borrowed by them plus (minus) the savings (debt) they carry over from the previous period. This curve can be upward or downward sloping depending on the relevance of intangible capital in the production function. In an economy where capital is interpreted to be of a tangible nature ($\theta = 0.9$ and $A^e = -0.2$), an increase in aggregate savings has the effect of lowering interest rates and increasing capital purchases from expanding firms. When there is a shift outwards in the savings curve, the economy moves from point A to point C. The collateral value channel and the debt overhang channel dominate. As a result, a larger share of the capital stock is in the hands of the high-productivity firms, which improves the allocation of resources and increases aggregate productivity and output. Instead, in an economy where capital is interpreted to be of an intangible nature ($\theta = 0.4$ and $A^e = 0.2$), the demand for capital curve is upward sloping due to the strength of the capital price and savings channel. As interest rates rise, firms demand more capital because they have larger savings and because the price of capital is lower. In this case, an outwards shift in the savings schedule generates a decrease in equilibrium capital purchases, because the decrease in interest rates it generates hurts reallocation of capital towards high-productivity firms. The economy moves from point A to point B, worsening the allocation of resources and reducing aggregate productivity and output.

4 General Equilibrium Model

We introduce an infinite-horizon, discrete-time economy populated by an intermediate sector which produces capital, by a final good sector in which firms use labor and capital to produce consumption goods, and by households, who provide labor and own both sectors. There are several important extensions to the simple model analyzed in Section 3, and we describe here the main ones. We introduce an intermediate capital producing sector that allows us to endogenize in equilibrium the aggregate stock of capital. In the final good sector, we model explicitly tangible and intangible capital, and we derive endogenously the accumulation of financial and physical assets of firms that live multiple periods. The household sector is a modelled as a life-cycle framework, which allows us to endogenize the interest rate and study how it is affected by demographic changes and other demand side factors.

4.1 The Capital-producing Sector

A representative firm in this sector chooses investment in tangible and intangible capital, respectively I_t^T and I_t^I, in order to maximize profits:

$$\max_{I^J} q_{J,t} I_t^J - b_t^J \left(\frac{I_t^J}{\varphi}\right)^{\varphi}$$

where $\varphi > 1, b_t^J > 0$, and $q_{J,t}$ is the price of type of capital $J \in \{T, I\}$. We allow for b_t^T and b_t^I to be time varying in order to capture trends in the evolution of the relative price of capital. The first order condition yields $I_t^J = \varphi \left(\frac{q_{J,t}}{b_t^J}\right)^{\frac{1}{\varphi-1}}$, and profits are:

$$\pi_t^J = \frac{q_{J,t}^{\frac{\varphi}{\varphi-1}}}{b_t^{J\frac{1}{\varphi-1}}} (\varphi - 1)$$

At the beginning of period t total capital available is $\overline{K}_t^T$ and $\overline{K}_t^I$. New capital I_t^T and I_t^I is produced and sold in period t, so that the aggregate dividends generated by the capital-producing sectors are:

$$D_t^k = \pi_t^T + \pi_t^I$$

During period t tangible capital and intangible capital depreciate at the rates $0 \leq \delta^T < 1$ and $0 \leq \delta^I < 1$, respectively. And the law of motion of aggregate capital is:

$$\overline{K}_{t+1}^T = I_t^T + (1 - \delta^T)\overline{K}_t^T$$

$$\overline{K}_{t+1}^I = I_t^I + (1 - \delta^I)\overline{K}_t^I$$

4.2 Final Good Sector

There are two types of final good producing firms: high-productivity and low-productivity.

4.2.1 The high productivity firms

There is a continuum of mass 1 of high productivity firms.

Technology and financing opportunities

High-productivity firms produce a final good using a constant returns to scale production function which is Cobb-Douglas in labor and capital. The firms use two different types of complementary capital, tangible and intangible. For simplicity, we assume that they are perfect complements. The production function takes the form:

$$y_t^p = z_t n_t^{(1-\alpha)} \left[\min \left(\frac{k_{T,t}}{1-\mu}, \frac{k_{I,t}}{\mu} \right) \right]^{\alpha} \tag{7}$$

where $0 < \alpha \leq 1$, $0 < \mu < 1$. The terms $k_{T,t}$ and $k_{I,t}$ represent tangible and intangible capital installed in period $t-1$ that produce output in period t. Finally, z_t is a productivity parameter, and n_t is labor. The Leontief production structure implies that in equilibrium intangible capital as a share of total capital in the high-productivity firms is equal to μ. The only other difference between the two types of capital is that we assume tangible capital to have an higher collateral value than intangible capital.

The budget constraint for high-productivity firms is given by the following dividend equation:

$$d_t = y_t^p + (1+r_t)a_{f,t} - a_{f,t+1} - q_{T,t}\left(k_{T,t+1} - (1-\delta^T)k_{T,t}\right) - q_{I,t}(k_{I,t+1} - (1-\delta^I)k_{I,t}) - w_t n_t. \quad (8)$$

where r_t is the interest rate paid or received in date t, $q_{T,t}$, and $q_{I,t}$ are the prices of tangible and intangible capital, respectively, and w_t is the wage. The term $a_{f,t} > 0$ indicates that the firm is a net saver, and $a_{f,t} < 0$ indicates that the firm is a net borrower.

High-productivity firms are subject to frictions in their access to external finance. They are unable to issue equity, which means that dividends are subject to a non-negativity constraint:

$$d_t \geq 0. \quad (9)$$

They can issue one-period riskless debt, subject to the constraint that they can pledge, as collateral, the fractions θ^T and θ^I of tangible capital and intangible capital, respectively. This translates into the following borrowing constraint:

$$a_{f,t+1} \geq -\frac{\theta^T q_{T,t+1} k_{T,t+1} + \theta^I q_{I,t+1} k_{I,t+1}}{1 + r_{t+1}} \quad (10)$$

where $0 < \theta^T \leq 1$ and $0 < \theta^I < \theta^T$. In reality, firms finance part of their investment with equity issues, which could be captured in the model by assuming that dividends can be negative up to a fraction of the firm's value. However, rather than complicating the model further, in the calibration section we consider equity financing by assuming larger values of θ^T and θ^I than are normally assumed in the literature. This assumption is without loss of generality, because assuming instead negative dividends proportional to the firm's value and lower collateral values of capital would not change our qualitative and quantitative results.

From the Leontief structure of the production function it follows that $k_{T,t} = \frac{1-\mu}{\mu} k_{I,t}$. Therefore, from now onwards, we use this result to express all equations as a function of intangible capital only. At the beginning of each period, both types of capital are predetermined and in their optimal ratio $k_{T,t} = \frac{1-\mu}{\mu} k_{I,t}$, and therefore the production function can be written as:

$$y_t^p = z_t n_t^{(1-\alpha)} \left(\frac{k_{I,t}}{\mu}\right)^{\alpha}. \quad (11)$$

After producing, the firm's technology becomes obsolete with probability ψ. In this case, the firm liquidates all its capital, and pays out as dividends all of its savings, including the liquidation value of capital, and exits.

Firms cannot invest every period. More specifically, they can only invest in a given period with probability η. This assumption, in addition to capturing the realistic feature that firms' investment is lumpy (Caballero (1999)), is meant to allow firms to have the opportunity to accumulate significant amounts of liquid savings, in line with the empirical evidence.

Optimization

Firms choose their investment and savings in order to maximize the net present value of their dividends. Let λ_t and ϑ_t be the Lagrange multipliers of constraints (9) and (10), respectively. We define the value functions conditional on investing and not investing, respectively $V^+(k_{I,t}, a_{f,t})$ and $V^-(k_{I,t}, a_{f,t})$, as follows:

$$\begin{aligned} V_t^+(k_{I,t}, a_{f,t}) &= \max_{n_t, dt, a_{f,t+1}, k_{I,t+1}} (1+\lambda_t)d_t + \vartheta_t \left(a_{f,t+1} + \frac{\theta^T q_{T,t+1} k_{T,t+1} + \theta^I q_{I,t+1} k_{I,t+1}}{1+r_{t+1}} \right) \\ &\quad + \frac{1}{1+r_{t+1}} \left[(1-\psi) V_{t+1}(k_{I,t+1}, a_{f,t+1}) + \psi d_{t+1}^{exit} \right], \end{aligned} \tag{12}$$

and

$$\begin{aligned} V_t^-(k_{I,t}, a_{f,t}) &= \max_{n_t, a_{f,t+1}} (1+\lambda_t)d_t + \vartheta_t \left(a_{f,t+1} + \frac{\theta^T q_{T,t+1} k_{T,t+1} + \theta^I q_{I,t+1} k_{I,t+1}}{1+r_{t+1}} \right) \\ &\quad + \frac{1}{1+r_{t+1}} \left[(1-\psi) V_{t+1}(k_{I,t}, a_{f,t+1}) + \psi d_{t+1}^{exit} \right], \end{aligned} \tag{13}$$

where d_{t+1}^{exit} is the dividend in case of liquidation and exit from activity:

$$d_t^{exit} = y_t^p + (1+r_t)a_{f,t} + (1-\delta)q_{T,t}\frac{1-\mu}{\mu}k_{I,t} + (1-\delta)q_{I,t}k_{I,t} - w_t, \tag{14}$$

and $V_{t+1}(k_{I,t+1}, a_{f,t+1})$ is the value function conditional on continuation but before the investment shock is realized:

$$V_{t+1}(k_{I,t+1}, a_{f,t+1}) = \eta V^+(k_{I,t+1}, a_{f,t+1}) + (1-\eta)V^-(k_{I,t+1}, a_{f,t+1}) \tag{15}$$

The firm solves (12) or (13), subject to **(8)**, (9) and (10). We next provide a characterization of high-productivity firms' optimal choices under the assumption that they are permanently financially constrained. We claim $-$ and check later in our calibrated simulations $-$ that in equilibrium the marginal return on capital for high-productivity firms is always higher than its user cost:

$$\frac{\partial y^p_{t+1}}{\partial k_{I,t+1}} = \frac{\alpha z_{t+1} n_{t+1}^{(1-\alpha)}}{\mu}\left(\frac{k_{I,t+1}}{\mu}\right)^{\alpha-1} > \left(q_{T,t}\frac{1-\mu}{\mu} + q_{I,t}\right) - \frac{(1-\delta)\left(q_{T,t+1}\frac{1-\mu}{\mu} + q_{I,t+1}\right)}{1+r_{t+1}}. \tag{16}$$

The implication of assumption (16) for investing firms is that the borrowing constraint (10) is binding, and that firms choose not to pay dividends, so that the equity constraint (9) is also binding. Making $d_t = 0$ in budget constraint (8), using (8) to substitute for $a_{f,t+1}$ in (10), assuming (10) is binding, and solving for $k_{I,t+1}$, we obtain their level of investment:

$$(k_{I,t+1} \mid \text{invest}) = \frac{y^p_t - w_t n_t + (1+r_t)a_{f,t} + (1-\delta)\left(q_{T,t}\frac{1-\mu}{\mu} + q_{I,t}\right)k_{I,t}}{q_{T,t}\frac{1-\mu}{\mu} + q_{I,t} - \left(\theta^T \frac{q_{T,t+1}}{1+r_{t+1}}\frac{1-\mu}{\mu} + \theta^I \frac{q_{I,t+1}}{1+r_{t+1}}\right)}. \tag{17}$$

The right hand side of equation (17) is the maximum feasible investment in intangible capital for a firm. The numerator is the total wealth available to invest. The denominator captures the downpayment necessary to purchase one unit of $k_{I,t+1}$ and $\frac{1-\mu}{\mu}$ units of $k_{T,t+1}$. The term $q_{T,t}\frac{1-\mu}{\mu} + q_{I,t}$ represents the total cost necessary to purchase these amounts of both types of capital, and the term $\theta^T \frac{q_{T,t+1}}{1+r_{t+1}}\frac{1-\mu}{\mu} + \theta^I \frac{q_{I,t+1}}{1+r_{t+1}}$ is the amount that can be financed by borrowing.

Investing firms in equilibrium borrow as much as possible, and:

$$(a_{f,t+1} \mid \text{invest}) = -\left(\theta^T \frac{q_{T,t+1}}{1+r_{t+1}}\frac{1-\mu}{\mu} + \theta^I \frac{q_{I,t+1}}{1+r_{t+1}}\right) k_{I,t+1} < 0. \tag{18}$$

The implication of assumption (16) for non-investing firms is that they will not sell any of their capital, and for these firms the law of motion of capital is:

$$(k_{I,t+1} \mid \text{not invest}) = (1-\delta)k_{I,t}. \tag{19}$$

Regarding the dividend and cash accumulation policy of non-investing firms, the first order condition for cash holdings $a_{f,t+1}$ is:

$$(1+\lambda_t) = (1-\psi)\left[\eta(1+\lambda^+_{t+1}+\vartheta_t) + (1-\eta)\left(1+\lambda^-_{t+1}+\vartheta_t\right)\right] + \psi, \tag{20}$$

Substituting (20) recursively forward, it is clear that if the firm expects ϑ_t to be positive now or in the future, then $\lambda_t > 0$, and a non-investing firm will always retain all earnings and

$d_t = 0$. It is important to note that this is so because there is no cost of holding cash. Cash holdings for non-investing are obtained by substituting $d_t = 0$ and (19) in (8):

$$(a_{f,t+1} \mid \text{not invest}) = y_t^p + (1 + r_t)a_{f,t} - w_t n_t. \tag{21}$$

Equations (18) and (21) determine the wealth dynamics of firms. A firm that invested in period $t-1$ but is not investing in period t has debt equal to $-a_{f,t} = \left(\theta^T \frac{q_{T,t+1}}{1+r_{t+1}} \frac{1-\mu}{\mu} + \theta^I \frac{q_{I,t+1}}{1+r_{t+1}}\right) k_{I,t+1}$. It uses current profits $y_t^p - w_t n_t$ to pay the interest rate on debt $-r_t a_{f,t}$ and to reduce the debt itself. As long as the firm is not investing, the debt $-a_{f,t}$ decreases until the firm becomes a net saver and has $a_{f,t} > 0$. At this point, wealth accumulation is driven both by profits $y_t^p - w_t n_t$ and by interest on savings $r_t a_{f,t}$, until the firm has an investment opportunity and its accumulated wealth $(1 + r_t)a_{f,t}$ is used to purchase capital (see equation (17)). This discussion clarifies that a lower interest rate r_t helps the non-investing firm to repay existing debt (the debt hangover channel), but it slows down the accumulation of savings after the firm has repaid the debt (the savings channel).

Finally, the first order condition for n_t, for both investing and non investing firms, implies that given the wage w_t and its predetermined capital $k_{I,t}$, a firm will choose the profit maximizing level of labor, which determines the optimal capital labor ratio:

$$\frac{k_{I,t}}{n_t} = \mu \left[\frac{w_t}{(1-\alpha)\, z_t}\right]^{\frac{1}{\alpha}} \tag{22}$$

4.2.2 The low productivity firms

There is a mass one of identical low productivity firms who have access to two production functions. Each production function combines capital $k_{uJ,t}$ with specialized labor $n_{uJ,t}^{1-\chi_J}$ using a constant returns to scale technology, where $J = \{I, T\}$ captures the tangibility of the capital used. The total amount $y_t^{\mathbf{u}}$ of the homogeneous final good produced is then:

$$y_t^u = z_t^{u,I} n_{uI,t}^{1-\chi_I} k_{uI,t}^{\chi_I} + z_t^{u,T} n_{uT,t}^{1-\chi_T} k_{uT,t}^{\chi_T},$$

where χ_I and χ_T determine the capital shares. We do not introduce the assumption of perfect complementarity between tangible and intangible capital (which we do introduce for the high productivity firms) to gain tractability in the pricing of capital, as will become clear in the next section. This is without loss of generality.

This sector is assumed to be able to finance capital with equity from the household sector and to pay out all profits as dividends d_t^u to households every period:

$$d_t^u = y_t^u - w_t^{uI} n_{uI,t} - w_t^{uT} n_{uT,t} - q_{I,t}\left(k_{I,t+1}^u - (1-\delta)k_{I,t}^u\right) - q_{T,t}\left(k_{T,t+1}^u - (1-\delta)k_{T,t}^u\right), \quad (23)$$

in addition to remunerating households for their labor services $(w_t^{uI} n_{uI,t} + w_t^{uT} n_{uT,t})$.

The first order conditions for the two types of labor implies that given wages w_t^{uI} and $w_t^{uT}$and a firm's predetermined capital stocks $k_{I,t}^u$ and $k_{T,t}^u$, a low productivity firm will choose the profit maximizing level of each type of labor, which determines the optimal capital labor ratio:

$$\frac{k_{uJ,t}}{n_{uJ,t}} = \left[\frac{w_t^{uJ}}{(1-\chi_J)\, z_t^{u,J}}\right]^{\frac{1}{\chi_J}}. \quad (24)$$

Given that low productivity firms are financially unconstrained, and provided that their marginal return on each of the two types of capital is lower than for the high productivity firms, the low productivity firms are willing to absorb all the capital not demanded by the high productivity firms, at a price equal to their marginal return on capital.

4.2.3 Aggregation of the Firm Sector, and Pricing of Assets

We assume (see section 4.3) that the aggregate supply of all types of labor is normalized to $N = N_{uI} = N_{uT} = 1$. Since all high-productivity firms produce at the optimal capital labor ratio determined by equation (22), and the production function is constant returns to scale, we can aggregate production across firms to obtain:

$$Y_t^p = z_t \left(\frac{K_{I,t}}{\mu}\right)^{\alpha}. \quad (25)$$

The wage is determined in competitive markets by the marginal return of labor:

$$w_t = (1-\alpha)\, z_t \left(\frac{K_{I,t}}{\mu}\right)^{\alpha}, \quad (26)$$

and aggregate wealth W_t of the high-productivity firms at the beginning of period t is:

$$W_t \equiv Y_t^p - w_t + (1+r_t)A_{f,t} + (1-\delta)\left(q_{T,t}\frac{1-\mu}{\mu} + q_{I,t}\right) K_{I,t}. \quad (27)$$

Aggregate capital is determined as follows. A fraction $(1-\psi)$ of high-productivity firms continues activity and a fraction η of those has an investment opportunity. They have a fraction $(1-\psi)\,\eta$ of total wealth W_t, which they use to buy the amount of capital given by equation (17). A fraction ψ of high-productivity firms exits, and is replaced by an equal number of firms with an initial endowment of W_0 and no capital. A fraction η of new entrants invest. Therefore, we define total intangible capital in the hands of investing agents at the end of period t, expressed in aggregate terms, as $\eta K_{I,t+1}^{INV}$, where $K_{I,t+1}^{INV}$ is:

$$K_{I,t+1}^{INV} = \frac{(1-\psi)W_t + \psi W_0}{\left(q_{T,t} - \theta^T \frac{q_{T,t+1}}{1+r_{t+1}}\right)\frac{1-\mu}{\mu} + q_{I,t} - \theta^I \frac{q_{I,t+1}^I}{1+r_{t+1}}} \tag{28}$$

The $(1-\eta)$ fraction of surviving firms that do not have an investment opportunity continue to hold their depreciated capital. Therefore aggregate capital for the next period is equal to:

$$K_{I,t+1} = \eta K_{I,t+1}^{INV} + (1-\delta)\,(1-\psi)\,(1-\eta)\,K_{I,t}, \tag{29}$$

It follows that aggregate tangible capital of the high-productivity firms is equal to:

$$K_{T,t+1}^{*} = \frac{1-\mu}{\mu} K_{I,t+1} \tag{30}$$

Furthermore, we can aggregate the output of low-productivity firms, substituting labor supply $N_{uI} = N_{uT} = 1$, and obtain:

$$Y_t^u = z_t^{u,I}\left(\overline{K}^I - K_{I,t}\right)^{\boldsymbol{\chi}_I} + z_t^{u,T}\left(\overline{K}^T - K_{T,t}\right)^{\boldsymbol{\chi}_T}. \tag{31}$$

The marginal return of capital in the high productivity firms is as follows. In order obtain a marginal increase $\frac{\partial Y_t^p}{\partial K_{I,t}} = \frac{\alpha}{\mu} z_t \left(\frac{K_{I,t}}{\mu}\right)^{\alpha-1}$, these firms purchase one unit of intangible capital and $\frac{1-\mu}{\mu}$ units of tangible capital. The equilibrium described above requires that the high-productivity firms have the highest return on capital, or:

$$\frac{\alpha}{\mu} z_t \left(\frac{K_{I,t+1}^{*}}{\mu}\right)^{\alpha-1} > z_t^{u,I}\chi_I\left(\overline{K}^I - K_{I,t}\right)^{\boldsymbol{\chi}_I - 1} + \frac{1-\mu}{\mu} z_t^{u,T}\chi_T\left(\overline{K}^T - K_{T,t}\right)^{\boldsymbol{\chi}_T - 1}, \tag{32}$$

where the right hand side of this inequality captures the marginal return of one unit of tangible capital and $\frac{1-\mu}{\mu}$ units of intangible capital in the low-productivity firms.

If condition 32 is satisfied, then it follows immediately that the prices of capital are:

$$q_{I,t} = z_I^{u,I} \chi_I \left(\overline{K}^I - \mathbf{K}_{I,t} \right)^{\boldsymbol{\chi_I - 1}} + \frac{1-\delta}{1+r_{t+1}} q_{I,t+1}, \tag{33}$$

and

$$q_{T,t} = z_t^{u,T} \chi_T \left(\overline{K}^T - \mathbf{K}_{T,t} \right)^{\boldsymbol{\chi_T - 1}} + \frac{1-\delta}{1+r_{t+1}} q_{T,t+1}, \tag{34}$$

By substituting (33) and (34) into (32), it follows that:

$$\frac{\alpha}{\mu} z_t \left(\frac{K_{I,t+1}^*}{\mu} \right)^{\alpha-1} > q_{I,t} - \frac{1-\delta}{1+r_{t+1}} q_{I,t+1} + \frac{1-\mu}{\mu} \left(q_{T,t} - \frac{1-\delta}{1+r_{t+1}} q_{T,t+1} \right), \tag{35}$$

which implies that the claim (16) is correct.

To compute aggregate financial assets of the high-productivity firms $A_{f,t+1}$, we take into account that, among the fraction $1-\psi$ of continuing firms, a fraction $1-\eta$ simply accumulates savings, while a fraction η borrows up to the maximum to invest. Among the fraction ψ of new firms, a fraction η borrows up to the maximum, while the rest save their initial endowment W_0 :

$$\begin{aligned} A_{f,t+1} \quad = \quad & (1-\psi) \left[(1-\eta) \left(Y_t^p + (1+r_t) A_{f,t} - w_t \right) \right] + \psi (1-\eta) W_0 \\ & -\eta \left(\theta^T \frac{q_{T,t+1}}{1+r_{t+1}} \frac{1-\mu}{\mu} + \theta^I \frac{q_{I,t+1}}{1+r_{t+1}} \right) K_{I,t+1}^{INV}. \end{aligned} \tag{36}$$

At the aggregate level total investment $\left(q_{T,t} \frac{1-\mu}{\mu} + q_{I,t} \right) \left(K_{I,t+1} - (1-\psi) K_{I,t} \right)$ is also equal to total resources available to invest:

$$\begin{aligned} \left(q_{T,t} \frac{1-\mu}{\mu} + q_{I,t} \right) \left(K_{I,t+1} - (1-\delta)(1-\psi) K_{I,t} \right) = & \left[(1-\psi) \eta \left(Y_t^p - w_t + (1+r_t) A_{f,t} \right) + \psi \eta W_0 \right] \\ & + \eta \left(\theta^T \frac{q_{T,t+1}}{1+r_{t+1}} \frac{1-\mu}{\mu} + \theta^I \frac{q_{I,t+1}}{1+r_{t+1}} \right) K_{I,t+1}^{INV}. \end{aligned} \tag{37}$$

Substituting (37) into (36) we obtain:

$$A_{f,t+1} = (1-\psi) \left(Y_t^p - w_t + (1+r_t) A_{f,t} \right) + \psi W_0 - \left(q_{T,t} \frac{1-\mu}{\mu} + q_{I,t} \right) \left(K_{I,t+1} - (1-\delta)(1-\psi) K_{I,t} \right), \tag{38}$$

Finally, total dividends paid out by exiting high-productivity firms to households are equal to:

$$D_t^p = \psi\left(Y_t^p - w_t + (1+r_t)A_{f,t} + \left(q_{T,t}\frac{1-\mu}{\mu} + q_{I,t}\right)K_{I,t}\right) - \psi W_0, \tag{39}$$

and the dividends paid by the low-productivity firms are:

$$D_t^u = Y_t^u - w_t^{uI} - w_t^{uT} - q_{I,t}\left[\left(\overline{K}^I - K_{I,t+1}\right) - \left(\overline{K}^I - K_{I,t}\right)\right] - q_{T,t}\left[\left(\overline{K}^T - K_{T,t+1}\right) - \left(\overline{K}^T - K_{T,t}\right)\right], \tag{40}$$

4.3 Households

We consider a life-cycle model with two types of households, young and old, with measures H^y and H^o, respectively, whose sum is normalized to 1. Young households supply 3 types of differentiated labor: high-productivity firm labor (in exchange for wage w_t), low-productivity intangible technology labor (in exchange for wage w_t^{uI}), and low-productivity tangible technology labor (in exchange for wage w_t^{uT}). There is an inelastic aggregate supply of one unit of each type of labor. Young households receive a fraction γ of the aggregate dividends. Households remain young for N periods, and become old after $N+1$ periods, so that there is a constant fraction $\phi = \frac{1}{N}$ of young households for every age between 1 and N, and every period a measure ϕH^y of households becomes old. Old households cannot work, receive a fraction $(1-\gamma)$ of aggregate dividends, and die with probability ϱ. The measure of old households H^o is determined as follows:

$$H^o = (1-\varrho)H^o + \phi H^y, \tag{41}$$

while the measure of young households is:

$$H^y = (1-\phi)H^y + N^y, \tag{42}$$

where N^y is the constant measure of newborn households. From the assumption that $H_t^o + H_t^y = 1$ follows that $N^y = \frac{\phi\varrho}{\phi+\varrho}$, $H_t^o = \frac{\phi}{\phi+\varrho}$, and $H_t^y = \frac{\varrho}{\phi+\varrho}$.

We follow Blanchard (1985) and Yaari (1965) in assuming that households participate in a life insurance scheme when old. The insurance scheme works within a cohort, so that the survivors within a cohort pay the debt of the dying (if they are in debt), or alternatively receive the savings of the dying. An old household begins a period with net debt $(1+r_t)b_t^o$. The insurance contract specifies that the ϱ fraction of old households that die transfer their assets (or debt) $(1+r_t)b_t^o$ to the life insurer. Among the fraction $(1-\varrho)$ of households that survive, if they are net savers ($b_t^o < 0$) then they receive a return $\frac{1}{1-\varrho}(1+r_t)b_t^o$ on their assets, while if they are net debtors ($b_t^o > 0$), they make a payment of $\frac{1}{1-\varrho}(1+r_t)b_t^o$ to the life insurer. For the detailed solution of the households maximization problem, see Appendix B.

5 Steady State

5.1 Equilibrium

We consider a steady state equilibrium and drop reference to the time subscript t. We can compute aggregate household borrowing as:

$$B = B^o + B^y, \tag{43}$$

where savings of the old B^o is:

$$B^o = \frac{\varrho}{\phi+\varrho}\frac{1}{N}\left[\left(\frac{A}{\beta+r\beta-1}+b^{retirement}\right)\frac{(1-\varrho)}{1-(1-\varrho)\beta(1+r)} - \frac{A}{\beta+r\beta-1}\frac{1-\varrho}{\varrho}\right], \tag{44a}$$

and savings of the young B^y is:

$$B^y = \frac{\varrho}{\phi+\varrho}\frac{1}{N}\left[A_1\frac{\gamma d + w + w^{uI} + w^{uT}}{r} - A_2c_N + A_3b^{retirement}\right] \tag{45}$$

A, A_1, A_2, A_3,c_N and $b^{retirement}$ are nonlinear functions of exogenous parameters, where the latter is the borrowing (if positive) and savings (if negative) of a retiring young household. For a detailed derivation of these terms, see Appendix B.

Total output of the high-productivity and low-productivity firms is, respectively,

$$Y^p = z\left(\frac{K_I}{\mu}\right)^{\alpha}, \tag{46}$$

and

$$\mathbf{Y}^u = \mathbf{z}^{u,I}\left(\overline{K}^I - K_I\right)^{\boldsymbol{\chi}_I} + \mathbf{z}^{u,T}\left(\overline{K}^T - K_T\right)^{\boldsymbol{\chi}_T}. \tag{47}$$

Dividends d are given by:

$$d = D^p + D^u + D^k, \tag{48}$$

where

$$D^u = Y^u - w_t^{uI} - w^{uT} - q_I \delta \left(\overline{K}^I - K_I \right) - q_T \delta \left(\overline{K}^T - K_T \right)$$

$$D^p = \psi \left(\alpha z \left(\frac{K_I}{\mu} \right)^\alpha + (1+r) A_f + \left(q_T \frac{1-\mu}{\mu} + q_I \right) K_I \right) - \psi W_0,$$

$$D^k = \frac{q_{T,t}^{\frac{\varphi}{\varphi-1}}}{b^{T \frac{1}{\varphi-1}}} (\varphi - 1) + \frac{q_{I,t}^{\frac{\varphi}{\varphi-1}}}{b^{I \frac{1}{\varphi-1}}} (\varphi - 1)$$

Aggregate cash holdings of the high-productivity firms in the steady state can be obtained by combining (38), (25) and (26) to obtain:

$$A_f = \frac{(1-\psi)\, \alpha z_t \left(\frac{K_I}{\mu} \right)^\alpha + \psi W_0 - \left(q_T \frac{1-\mu}{\mu} + q_I \right) [\psi + \boldsymbol{\delta}(1-\psi)]\, K_I}{[1 - (1-\psi)\,(1+r)]} \tag{49}$$

Aggregate borrowing is equal to aggregate savings, or

$$A_f = B, \tag{50}$$

and by Walras' Law, the aggregate resource constraint is satisfied. In order to determine the aggregate capital of the high-productivity firms, equation (29) in the steady state is equal to:

$$K_I = \eta \frac{(1-\psi) W + \psi W_0}{\left[q_T \left(1 - \frac{\theta^T}{1+r} \right) \frac{1-\mu}{\mu} + q_I \left(1 - \frac{\theta^I}{1+r} \right) \right] [1 - (1-\delta)(1-\psi)\,(1-\eta)]} \tag{51}$$

where W is defined using equation (27) in steady state:

$$W \equiv \alpha z_t \left(\frac{K_I}{\mu} \right)^\alpha + (1+r) A_f + (1-\delta) \left(q_T \frac{1-\mu}{\mu} + q_I \right) K_I \tag{52}$$

We can also express (51) as

$$K_I = \frac{\eta(1-\psi) \left(\alpha z_t \left(\frac{K_I}{\mu} \right)^\alpha + (1+r) A_f \right) + \eta \psi W_0}{\left[q_T \left(1 - \frac{\theta^T}{1+r} \right) \frac{1-\mu}{\mu} + q_I \left(1 - \frac{\theta^I}{1+r} \right) \right] [\delta + \psi\,(1-\delta)] - \left(q_T \frac{\theta^T}{1+r} \frac{1-\mu}{\mu} + q_I \frac{\theta^I}{1+r} \right) \eta (1-\delta)(1-\psi)}, \tag{53}$$

which has an intuitive explanation. The numerator is the aggregate amount of liquid resources of investing firms. The denominator is the downpayment necessary to support one unit of capital in the steady state. It requires the replacement of the depreciated capital and the lost capital of exiting firms (a fraction $\delta + \psi\,(1-\boldsymbol{\delta})$), and can benefit from using existing capital held by the investing firms as collateral (fraction $\eta(1-\delta)(1-\psi)$).

Finally, the prices of capital are determined by recursively iterating forward equations (33) and (34):

$$q_I = \frac{1}{r+\boldsymbol{\delta}} z^{u,I} \chi_I \left(\overline{K}^I - K_I\right)^{\boldsymbol{\chi}_I - \mathbf{1}}, \tag{54}$$

and

$$q_T = \frac{1}{r+\boldsymbol{\delta}} z^{u,T} \chi_T \left(\overline{K}^T - K_T\right)^{\boldsymbol{\chi}_T - \mathbf{1}}, \tag{55}$$

where aggregate capital and investment are given by

$$\overline{K}^J = \frac{I^J}{\boldsymbol{\delta}}, \tag{56}$$

and

$$I^J = \varphi \left(\frac{q_J}{b^J}\right)^{\frac{1}{\varphi-1}} \tag{57}$$

for $J \in \{I, T\}$.

The steady state values of W, A_f, B, K_I, q_I, q_T, and r are jointly determined by equations (43), (49), (50), (51), (52), (54), and (55).

5.2 Discussion

Assuming for simplicity that $q_T = q_I = q$, the collateral value of one unit of capital is $\frac{q}{1+r}\frac{1}{\mu}\left[(1-\mu)\,\theta^T + \mu\theta^I\right]$. Since $\theta^T > \theta^I$, a technology that relies more on tangible capital (lower μ) places a higher weight on the collateral value of tangible capital θ^T, thus increasing the overall collateral value of the firms' capital. Such an economy has a lower downpayment in the denominator of (53) and more capital K_I for a given total wealth at the numerator.

Equation (49) determines financial wealth A_f, which is equal to the net earnings of the productive firms, in the numerator, multiplied by a multiplicative factor $\frac{1}{1-(1-\psi)(1+r)}$, which measures the future value of one unit of wealth saved today by these firms. The net earnings are the endowment of the new firms ψW_0, plus the net earnings of continuing firms. The term $(1-\psi)\,\alpha z_t \left(\frac{K_I}{\mu}\right)^{\alpha}$ is retained earnings, net of wage payments, and is concave in K_I. The term $\left(q_T \frac{1-\mu}{\mu} + q_I\right)\left[\psi + \boldsymbol{\delta}(1-\psi)\right] K_I$ is total expenditures to replace the depreciated capital of

Table 2: Benchmark Calibration - Parameter Choices

Parameter	**Symbol**	**Value**
Discount factor	β	0.95
Capital share, productive firms	α	0.4
Capital share, unproductive firms, tangible capital	χ_I	0.4
Capital share, unproductive firms, intangible capital	χ_T	0.4
Intangible share of total capital	μ	0.20
Unproductive firms, TFP tangible technology	$z_t^{u,T}$	10
Unproductive firms, TFP tangible technology	$z_t^{u,I}$	10
Years households remain young	N	40
Probability of death of old households	ϱ	0.25
Productivity parameter	z	25
Collateral value of tangible capital	θ^T	1
Collateral value of intangible capital	θ^I	0.6
Probability of an investment opportunity	η	0.07
Additional productivity of intangible capital	κ	0.25
Adjustment cost convexity	φ	4
Adjustment cost parameter (intangible)	b_I	0.00018
Adjustment cost parameter (tangible)	b_T	0.00004
Exit probability of high-productivity firms	ψ	0.19
Endowment of new firms	W_0	5
Depreciation of capital	δ	0.15
Share of dividends to young households	γ	50.2%

continuing firms $\delta(1-\psi)K_I$, and the capital liquidated by exiting firms ψK_I, and is linear in K_I. A high average collateral value of capital in a tangible economy increases K_I and makes it likely that the sum of the two last terms is negative, and since ψW_0 is very small, it makes also A_f negative: the productive firms are on aggregate net borrowers. Conversely in an intangible (high μ) economy, A_f is likely to be positive.

The above discussion clarifies that the exogenous assumptions made in the simple model in section 3 are endogenously derived in the full general equilibrium model. Moreover, even though a change in the interest rate affects aggregate capital K_I in (53) through the same four channels identified in the simple model in section 3, it is important to emphasize that the endogeneity of financial assets amplifies the strength of the savings channel. When A_f is positive, a reduction in the interest rate reduces investment both through a reduction in the return on savings rA_f, and through a reduction in the multiplicative factor $\frac{1}{1-(1-\psi)(1+r)}$.

6 Calibration

For the purpose of evaluating the qualitative and quantitative importance of the channels explained above for the real economy, we calibrate the model on US data. Our benchmark calibration, illustrated in Table 2, is meant to capture the US economy during the period immediately preceding 1980, with a small share of intangible capital and high real interest rates. In this respect we follow Falato et al.(2014) in setting $\mu = 0.2$, so that the share of intangible capital over total capital is 20%. We set the share of dividends that are paid to the working age population, γ, so that we obtain a real interest rate $r = 5\%$. The elasticity of output with respect to capital for productive firms α, and for unproductive firms χ_T and χ_I, are set equal to 0.4. The pledgeability parameters of tangible capital θ^T and intangible capital θ^I are equal to 1 and 0.6, respectively. Thus we assume tangible capital to be fully collateralizable, in line with Falato, Kadyrzhanova, and Sim (2014). Moreover we set θ^I at a relatively high value compared to the literature. We do so to capture the fact that in reality firms finance their acquisitions in part with equity issues and other forms of external financing beyond collateralized debt. An alternative approach would have been to assume a value of θ_I much closer to zero, in line with Falato et al (2014), and allow dividends d_t to be negative, with an associated equity issuance cost proportional to the amount financed. This approach would have slightly complicated the model and yielded very similar quantitative results.

The TFP of unproductive firms, $z^{u,T}$ and $z^{u,I}$, is normalized to 10. The TFP of productive firms z_t is modeled as follows:

$$z_t = [1 + (\mu - 0.2)\kappa]\, z, \tag{58}$$

For the benchmark value of $\mu = 0.2$, it follows that $z_t = z$. The parameter κ measures the increase in TFP associated with adopting a more intangible intensive technology. We choose a value of $\kappa = 0.3$, so that an increase in μ from 0.2 to 0.6 raises z_t by 12%. A positive value of κ is consistent with the notion of the rise of intangible capital as a privately optimal choice of firms, and allows us to be able to make conservative and robust statements about the potential for negative effects of the shift to intangibles.

The depreciation factor δ is set equal to 15%. This value is appropriate for both intangible and tangible capital, where the latter is assumed to include also inventories, and is consistent with the depreciation rates used for the perpetual inventory method in section 2. The probability of having an investment opportunity η is set equal to 7%. This value is consistent with the empirical evidence from the lumpy investment literature. The initial endowment of newborn firms W_0 is equal to 5, which corresponds to 2% of average firm output.

For the capital production sector we assume $\varphi = 4$, which implies that in equilibrium, a marginal decline of r by 1 percentage point increases aggregate investment by around 1.5%.

There are four remaining parameters to calibrate in the firms sector: the other two parameters of the capital production function b^T and b^I, the efficiency of productive firms z, and the probability of exit ψ. b^T and b^I determine the aggregate supply of tangible and intangible capital, and their equilibrium prices. Conversely, z and ψ determine the ability of the productive firms to expand, absorb more capital, and increase their leverage, because exiting firms are forced to pay back all their debt, and newborn firms start with positive financial assets. Therefore these four coefficients are jointly chosen to satisfy the four following criteria: first, the relative price of tangible to intangible capital is normalized to one; second, output of all productive firms is roughly 50% of total output; third, the interquartile productivity differential is 2.5, which is consistent with the cross sectional dispersion in productivity for US firms in the 1970s.[10] Fourth, net leverage in the productive firms is around 35%, consistently with Compustat data for the 1975-1980 period. It is worth noticing that the probability of firm exit ψ is calibrated to the relatively high value of 0.19. Even interpreting this as plant rather firm exit the value is still substantially higher than the average plant turnover in the US. One way to justify this high value is to assume that the productive firms are indeed productive because are new and dynamic businesses specializing in very innovative project with high return if successful but also high risk of failure.

Among the parameters in the household sector, the discount factor β is set equal to 0.95. The number of years households are young N is set equal to 40, which corresponds to the working age period between 25 and 65 years old. The death of probability of old households ϱ is set equal to 0.25, in order to match life expectancy pre-1980, and it implies that the old households are the 33% of the total population.

[10] Syverson (2004) examines plant level data from 1977 and finds an average interquartile difference in labour productivity around 2 for 4 digit US manufacturing sectors. Since dispersion of productivity is larger for less narrowly defined sector, a value of 2.5 is probably a very conservative estimate of the dispersion of productivity across all firms.

7 Simulation Results

In this section, we introduce comparative static exercises that capture parallel developments in the household and the corporate sector. In the household sector, we model a progressive decrease in the rate of time preference (an increase in β), which puts downward pressure on the equilibrium interest rate. We interpret this exercise as a shortcut for a collection of different factors, such as financial deepening, wealth and income inequality, and foreign sector developments. In the corporate sector, we introduce a gradual shift in the reliance on intangible capital of firms, from the pre-1980 United States, in which intangible capital accounted only for 20% of aggregate capital ($\mu = 0.2$), to the post-2010 value of 60% of total capital ($\mu = 0.6$) (Corrado and Hulten (2010a), Falato, Kadyrzhanova, and Sim (2014), Döttling and Perotti (2015)).

In order to explain in detail all the different effects at play, we first consider two counterfactual exercises: one in which households' propensity to save gradually increases but the share of intangible capital is constant at $\mu = 0.2$, and one in which the household sector has constant preferences but in the corporate sector μ increases from 0.2 to 0.6. Our ultimate goal is then to analyze these two developments simultaneously and show how they have interacted to generate aggregate patterns consistent with the secular stagnation hypothesis.

In each of these three simulation exercises we report selected equilibrium values of the economy over time. Since we abstract from long run growth considerations, the graphs that show relative changes in total output should be interpreted as deviations from long run trends.

In Figure 5 we introduce the household sector developments in isolation. We increase β gradually from the initial value of 0.95 in 1970 to a final value of 0.9875 in 2010, in an economy in which intangible capital intensity is as in the 1970s ($\mu = 0.2$). The increase in households propensity to save puts downward pressure in the interest rate, which follows a pattern consistent with the empirical evidence and fall from 5% to 1%. This increases the price of capital (middle graph in the 2nd row) and encourages capital creation, so that aggregate tangible and intangible capital stocks increase by more than 5% (middle row, right graph).

The last row of Figure 5 analyzes the implications for reallocation of capital and efficiency. The bottom left graph shows that the high productivity firms, which are financially constrained, are able to expand their capital holdings by 2.5%. High productivity firms are leveraged, and are net borrowers (top row, right graph). Therefore, the decline in the interest rate benefits them

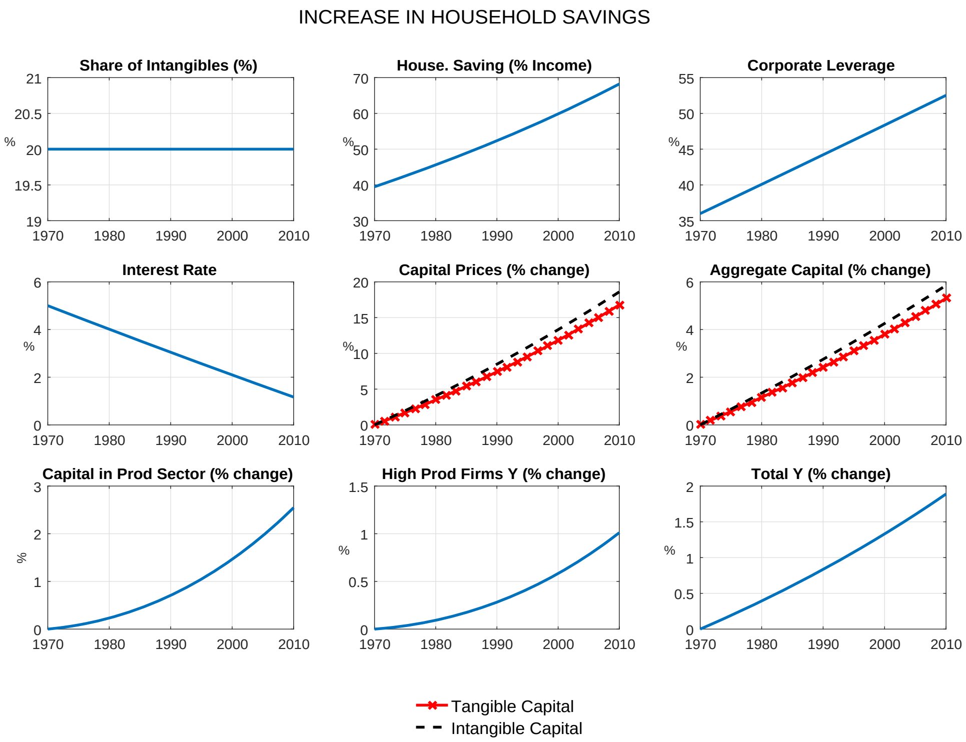


Figure 5: Simulation exercise: households' propensity to save gradually increases but the share of intangible capital is constant at $\mu = 0.2$.

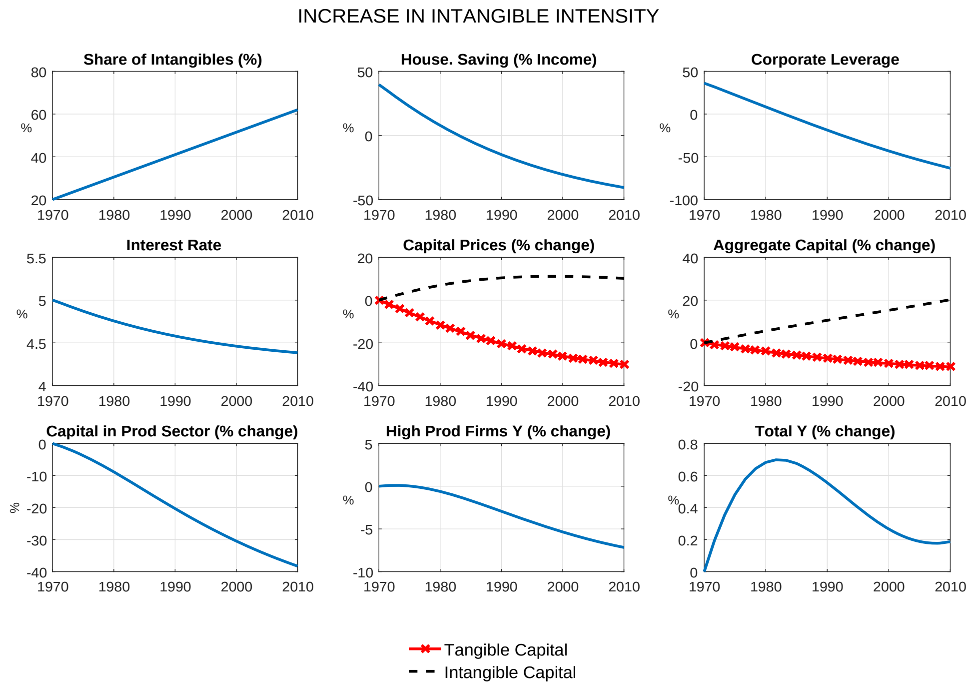


Figure 6: Simulation exercise: the household sector has constant preferences but in the corporate sector μ increases from 0.2 to 0.6.

both because it is easier to pay back the debt (the debt hangover channel) and because they can borrow more when they invest (the collateral value channel). These two channels prevail over the capital price channel, which operates in the opposite direction, and imply that the drop in r benefits high-productivity firms. They can absorb a higher share of existing capital, thus improving the allocation of resources. Overall, output increases by close to 2%, both because of the positive reallocation effect and because of the increase in the overall aggregate capital stock.

In Figure 6 we implement the gradual shift to intangible capital, but do not consider any developments in the household sector. High productivity firms demand progressively more intangible capital, and less tangible capital, thus increasing the price of the former and decreasing the price of the latter.[11] These firms increasingly rely on a type of capital which attracts less ex-

[11] In order to limit the rise of intangible capital prices, we also progressively increase the efficiency in the intangible capital production sector, so to generate a rise of intangibles both in the productive firms as well as in the economy as a whole. The central panel of the second row shows that intangible capital prices increase by around 10%. About half of this increase is due to a higher demand from productive firms, and the remaining is

ternal finance, and this decreases corporate leverage, and tightens firms' borrowing constraints significantly. In fact, firms switch from being net borrowers to being net lenders, consistent with evidence in the U.S. for corporations (Armenter and Hnatkovska (2016), Quadrini (2016), Chen, Karabarbounis and Neiman (2016)). This increase in corporate savings reduces interest rates to ensure that households borrow more and absorb the excess savings. Aggregate capital in the productive firms shrinks by as much as 40%, thus substantially worsening the allocation of resources. This contraction in the aggregate amount of capital allocated to high productivity firms is not only the result of the reduction in their borrowing capacity and the increase in price of intangible capital, but also of the additional negative effect of the endogenous decline in r (described in detail below). Despite the fact that the rise of the intangible share in high productivity firms increases their TFP by up to 12%, their output falls by up to 7% (bottom row, middle graph). Nonetheless, aggregate output grows moderately, by around 0.2%, due to the large increase in aggregate capital, which is mostly allocated to the low productivity firms.

Figure 7 analyses both developments simultaneously. The decrease in rates caused by household developments has increasingly negative effects on capital allocation as the economy shifts into a higher intangible intensity regime. High-productivity firms shift from being net borrowers to net savers, so lower rates benefit them initially by decreasing their debt overhang, but hurt them as they become savers and low rates hurt their ability to accumulate savings. In addition, higher capital prices decrease their ability to purchase capital from exiting firms, worsening capital reallocation. Initially, the increase in capital creation and the fact that the economy is shifting to a higher reliance on a type of capital that is significantly more productive help aggregate output expand. However, the negative reallocation effects eventually prevail, and from its peak in the mid 80s until 2010 output falls gradually by up to 2%.

Figure 8 compares the output dynamics in the 3 exercises analyzed above. The left graph compares the % change of total output of the high productivity firms. In the case of the increase in intangibles intensity (dotted blue line), output falls by 7%. While household sector developments in isolation raise high productivity firms' output by 1%, the combination of household sector developments and an increase in intangibles intensity result in a significantly stronger decline in output, which is as large as 13% (red line). The negative interaction between these

determined by lower interest rates. Morover, the same graph shows that tangible capital prices fall by around 25%, which is consistent with the empirical evidence from the 1970-2010 period.

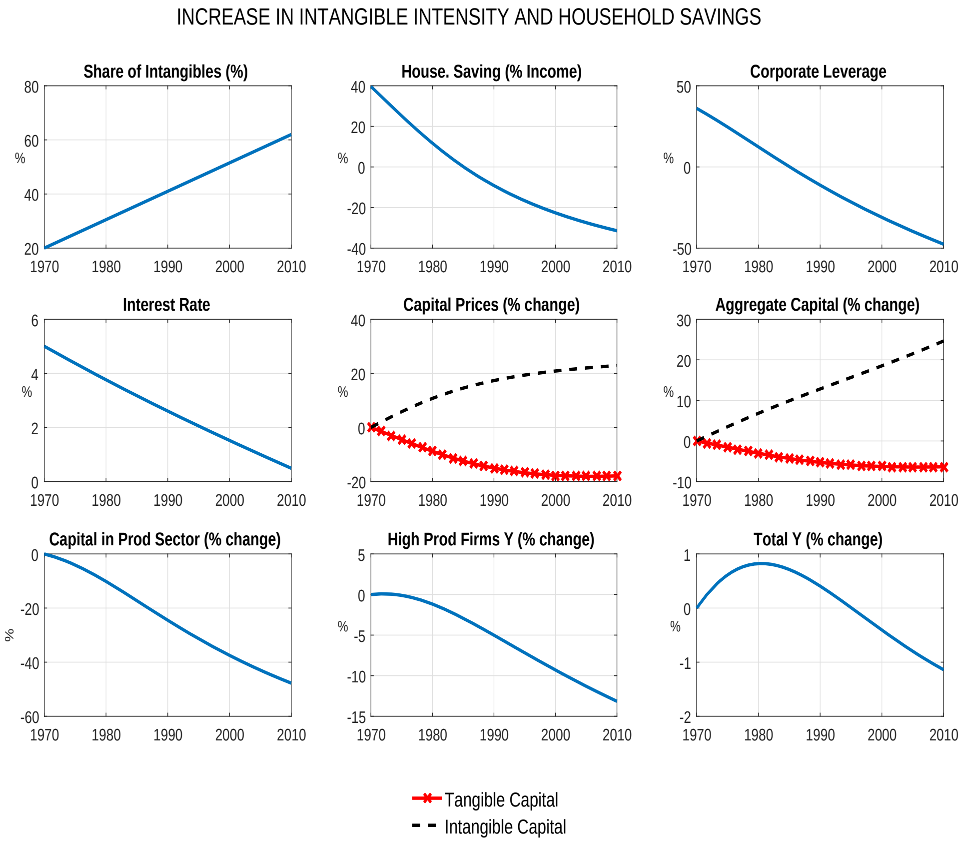


Figure 7: Simulation exercise: households' propensity to save gradually increases and in the corporate sector μ increases from 0.2 to 0.6.

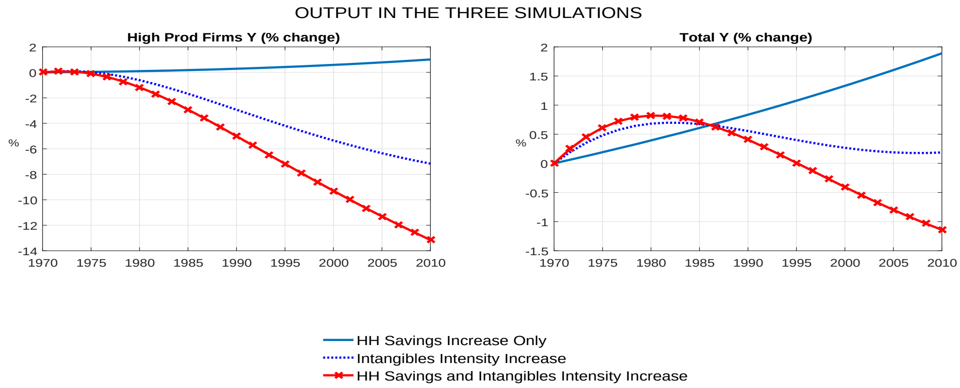


Figure 8: Summary of the three simulation exercises.

developments in the firm and household sectors is also very clear in the right graph, which shows the % changes in aggregate output. While the two effects in isolation are expansionary, they are jointly contractionary, reflecting the dynamics of the four channels illustrated before: a reduction in interest rates, which is expansionary for highly leveraged productive firms, hampers reallocation and growth once the economy relies more on intangible and less collateralizable capital.

The contraction in output happens despite our assumption that intangible capital is more productive, because of a strong misallocation effect in which too many resources are absorbed by low productivity firms. An empirically verifiable consequence of this misallocation is a rise in the dispersion of productivity. Figure 9 shows the dispersion in marginal productivity of capital (MPK, left panel) and total factor productivity (TFP, right panel). This dispersion is virtually constant when household sector developments happen in isolation (blue line). Misallocation does not change because both high-productivity and low-productivity firms expand in roughly equal measure. However when the drop in r is accompanied by the rise in intangibles (red line), there is a strong increase in misallocation. This is especially noticeable in the left panel. High-productivity firms shrink, and their MPK rises, while low-productivity ones expand to absorb the excess capital, and their MPK falls. Figure 9 is consistent with the empirical evidence shown in Figures 1 and 2, which documents that the positive trend in the productivity dispersion within

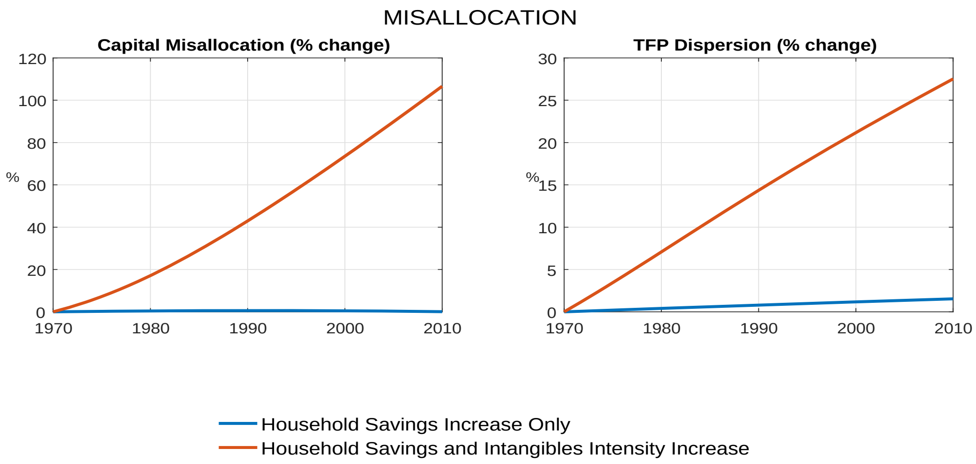


Figure 9: Misallocation the different simulation exercises.

sectors is entirely driven by the sectors with above-median usage of intangible capital.

Finally, figure 10 displays output growth in the simulated economies. The blue line is a counterfactual simulation where there is no increase in misallocation of resources as the share of intangibles increases. The line shows that the higher productivity of intangible capital determines a positive growth rate of output around 0.32%. We interpret this value as potential output growth in excess of the unmodelled long run trend. The green line is aggregate output growth in case of the actual simulation with increase in intangibles intensity. It is initially positive, but the misallocation problems illustrated above quickly reduce it to around zero in the 1980-2010 period. The red line includes also the increase in households demands for savings, and shows a more pronounced decline. Output growth is initially positive and larger than 0.2%, but becomes negative and around -0.13% in the 1990-2010 period.

8 Sensitivity and Robustness Analysis

[to be completed]

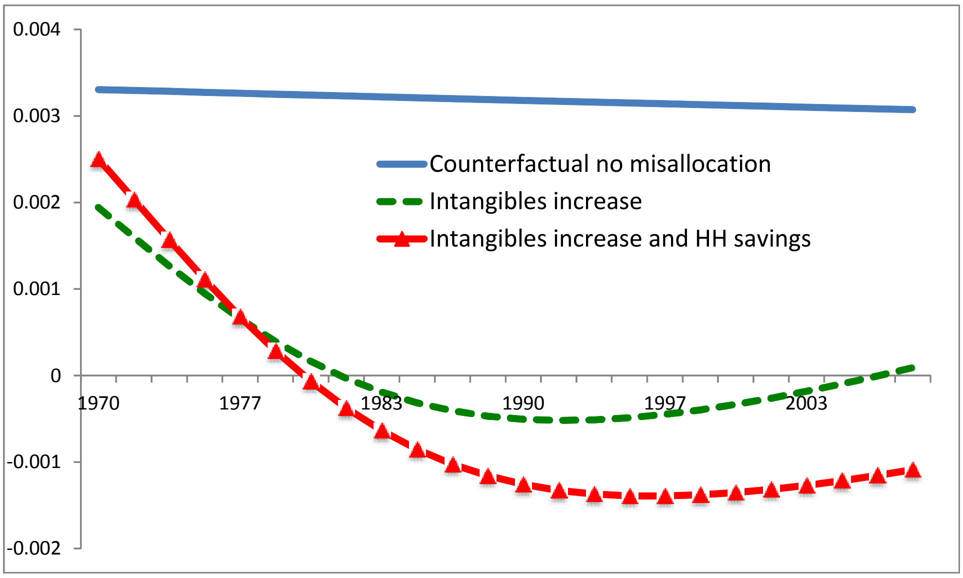


Figure 10: Aggregate output growth in the different simulation exercises.

9 Conclusion

This paper highlights a novel misallocation effect of endogenously low interest rates which has important policy implications. From a quantitative standpoint, our results are consistent with several developments that have taken place in the last 40 years: i) net corporate savings increased as a fraction of GDP; ii) household leverage increased as a fraction of GDP; iii) the real interest rate fell; iv) intra industry dispersion in productivity has increased; and v) output and productivity progressively declined relative to their previous trends. Interestingly, the model shows that even though the shift to intangible technologies was already taking place in the 1970s, its net negative effects on output growth only started to gather pace from mid 1980s onwards. This is consistent with studies that show a decline in dynamism of U.S. businesses starting in the mid 1980s and gathering speed especially from 2000 onwards (Haltiwanger (2015)).

More broadly, our results suggest that the changes in firms' financing behavior brought about by technological evolution might help explain the subpar growth experienced in recent years, because they have occurred during a period of low interest rates. Our insights could be extended to develop interesting policy implications. On one hand, the mechanisms described in this paper, operating mostly through the endogenous reaction of interest rates, suggest that the rise in intangibles might have important implications for monetary policy. On the other hand, the negative externality in households' and firms' excessive saving decisions might introduce a role for a fiscal policy that discourages such saving.

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10 Appendix A - Derivation of the Simple Model

This appendix solves a simple partial equilibrium model that delivers the equations introduced in Section 3. Consider an infinite-horizon, discrete-time model of the final good-producing sector of an economy. Firms use capital, which is in constant aggregate supply $\overline{K}$, to produce a homogeneous consumption good. There are two types of firms, *high-productivity* and *low-productivity*, each composed of a continuum of mass 1. High-productivity firms live for 2 periods, and there are overlapping generations of these types of firms. Efficiency in this economy is determined by the share of $\overline{K}$ allocated to high-productivity firms. Our main interest is studying how exogenous interest rate variations affect the allocation of capital and aggregate output depending on the degree of tangibility of capital. This framework is extended in Section 4 in a full-fledged general equilibrium setup that can be used for realistic quantitative analysis.

10.1 High-productivity Firms

Technology and Financing

High-productivity firms live for two periods, which we denote with y (young) and o (old). An old firm which dies in period $t-1$ leaves a a financial endowment or liability a^e $(-\infty < a^e < \infty)$, which translates into net worth $a^e(1+r_t)$ for the newborn young firm in period t. The young firm is able to produce y^e units of the final good in period t, and has access to a technology to produce the final good in period $t+1$ using the following linear production function:

$$y^p_{t+1} = z_{t+1}k_{t+1}, \tag{59}$$

where k_{t+1} represents capital purchased in t that produces output in $t+1$, and z_{t+1} is a productivity parameter. The firm can borrow b_{t+1} to purchase capital, subject to a constraint:

$$(1 + r_{t+1})\, b_{t+1} \leq \theta q_{t+1} k_{t+1}, \tag{60}$$

where $0 < \theta \leq 1$. The collateral value of capital θ is the parameter in this stylized model that captures capital tangibility. A shift towards a stronger reliance on intangible capital will be captured as a decrease in θ. [12] Firms cannot issue equity.

The budget constraint for a high-productivity young firm is:

$$q_t k_{t+1} = a^e(1 + r_t) + b_{t+1} + y^e. \tag{61}$$

A mature firm realizes output, pays back any debts, sells its holdings of capital, and pays the residual, net of the endowment for the next generation a^e, as a dividend d_{t+1} to its shareholders:

$$d_{t+1} + a^e = y^p_{t+1} + q_{t+1}k_{t+1} - b_{t+1}\,(1 + r_{t+1})\,.$$

Optimal Solution

Productive young firms in $t = 0$ maximize the present value of the dividend d_{t+1}. We claim,

[12] Note that a standard collateral constraint of the form

$$b_{t+1} \leq \frac{\theta q_{t+1} k_{t+1}}{1 + r_{t+1}},$$

would not work when $W_0 < 0$. This is because in that case the firm would have to be assumed to borrow more than q_t per unit of capital:

$$q_t < \frac{\theta q_{t+1}}{1 + r_{t+1}},$$

to be able to purchase capital and pay down the debt, and by making the collateral constraint a function of k_{t+1}, it would make the firm financially unconstrained, and the problem would not have a solution.

and later verify, that their marginal product of capital is greater than its user cost:

$$z_t > q_t - \frac{q_{t+1}}{1 + r_{t+1}} \tag{62}$$

firms are credit constrained ((60) is binding), so that

$$k_{t+1} = \frac{a^e(1 + r_t) + y^e}{q_t - \theta \frac{q_{t+1}}{1+r_{t+1}}}. \tag{63}$$

in which investment is equal to the total wealth available to invest divided by the downpayment necessary to purchase one unit of capital.

10.2 Unproductive Firms

There is a mass 1 of identical firms in the unproductive sector that produce the same homogeneous final good as the high-productivity firms using a linear production function:

$$y_t^u = z_t^u k_{u,t}, \tag{64}$$

where $k_{u,t}$ represents capital installed in period $t-1$ that produces output in period t, and z_t^u is a productivity parameter. This sector is assumed to be financially unconstrained, and to pay out all profits as dividends:

$$d_t^u = y_t^u - q_t \left(k_{t+1}^u - k_t^u\right) \tag{65}$$

to their shareholders every period.

10.3 Aggregation

From (63) it follows that the aggregate stock of capital held by the high productivity firms is

$$K_{t+1} = \frac{A^e(1 + r_t) + Y^e}{q_t - \theta \frac{q_{t+1}}{1+r_{t+1}}}.$$

and aggregate output is

$$Y_t = Y_t^p + Y_t^u + Y^e = z_t K_t + z_t^u \left(\overline{K} - K_t\right).$$

Under the assumption that the high-productivity firms have the highest return on capital ($z_t > z_t^u$), but their resources are insufficient to absorb all the capital, $K_{t+1} < \overline{K}$, it follows that the low-productivity firms are willing to absorb all the capital not demanded by the high-productivity firms at a price equal to their marginal return on capital. the price of capital is:

$$q_t = z^u + \frac{1}{1 + r_{t+1} + \xi} q_{t+1}, \tag{66}$$

which, together with the assumption that $z_t > z_t^u$, proves the claim (62). $\xi \geq 0$ is a wedge that reduces the sensitivity of the price of capital to the interest rate, and summarizes the effect of factors included in the full model developed later, such as capital depreciation.

10.4 Steady State

We consider a steady state equilibrium and drop reference to the time subscript t. Total output is

$$Y = Y^p + Y^u + Y^e = zK + z^u\left(\overline{K} - K\right). \tag{67}$$

Aggregate capital holdings of the high-productivity firms in the steady state are:

$$K = \frac{A^e(1+r) + Y^e}{q\left(1 - \frac{\theta}{1+r}\right)}, \tag{68}$$

where

$$q = \frac{z^u}{r + \xi}. \tag{69}$$

Equations (1), (2), and (3) in the steady state equilibrium of Section 3 correspond to equations (67), (68) and (69) in this section.

11 Appendix B - Households

We derive below the solution of the households' optimization problem under the steady state, so that wages, dividends and interest rate are constant. Households have log utility. A representative old household still living at time t maximizes the following objective function:

$$V_t^o\left(b_t^o\right) = \max_{c_t^o, b_{t+1}^o} \sum_{j=0}^{\infty} (1-\varrho)^j \beta^j \log\left(c_{t+j}\right) \tag{70}$$

subject to

$$c_t^o = b_{t+1}^o + (1-\gamma)d_t - \frac{(1+r)}{(1-\varrho)} b_t^o.$$

Working backwards, we next consider the optimization problem of a young agent of age N in period t, who will become old in period $t+1$:[13]

$$V_{t,N}^y\left(b_{t,N}^y\right) = \max_{c_{t,N}^y, b_{t+1}^o} u\left(c_{t,N}^y\right) + \beta(1-\varrho)V_{t+1}^o\left(b_{t+1}^o\right) \tag{71}$$

subject to

$$c_{t,N}^y = \gamma d + w^{TOT} - (1+r)b_{t,N}^y + b_{t+1}^o. \tag{72}$$

where w^{TOT} is defined as:

$$w^{TOT} \equiv w + w^{uI} + w^{uT}$$

Then we consider the optimization problem for a young household of age $j < N$:

$$V_{t,j}^y\left(b_{t,j}^y\right) = \max_{c_{t,j}^y, b_{t+1,j+1}^y} u\left(c_{t,j}^y\right) + \beta V_{t+1,j+1}^y\left(b_{t+1,j+1}^y\right) \tag{73}$$

subject to

$$c_{t,j}^y = \gamma d + w^{TOT} - (1+r)b_{t,j}^y + b_{t+1,j+1}^y \tag{74}$$

11.1 Individual Problem of Old Households

The first order condition with respect to b_{t+1}^o is

$$c_{t+1}^o = \beta(1+r)c_t^o \tag{75}$$

We guess a consumption policy rule:

$$c_t^o = \Delta d + \Theta b_t^o,$$

and by using a guess and verify method on (75) we determine the value of the coefficients Δ and Θ and obtain:

$$c_t^o = (1-(1-\varrho)\beta)\left[\frac{(1-\gamma)(1+r)}{(\varrho+r)}d - \frac{(1+r)}{(1-\varrho)}b_t^o\right]. \tag{76}$$

Moreover, we can use the policy function to derive the evolution of the wealth of old households:

$$b_{t+1}^o = \frac{(1-\varrho)\left[1-\beta(1+r)\right]}{(\varrho+r)}(1-\gamma)d + (1+r)\beta b_t^o \tag{77}$$

[13] We assume that an agent can also die with probability ϱ in the transition between young and old.

which says that old households slowly consume savings if $\beta(1+r) < 1$, and do so at a faster rate the higher the dividends. In our simulations typically $\beta(1+r) < 1$.[14]

11.1.1 Derivation of Old Households' Consumption Rule

We guess a consumption policy rule:

$$c_t^o = \Delta d + \Theta b_t^o,$$

and plug it into the FOC

$$\begin{aligned}
\Delta d + \Theta\left[c_t^o - (1-\gamma)\,d + \frac{(1+r)}{(1-\varrho)} b_t^o\right] &= \beta(1+r)\left(\Delta d + \Theta b_t^o\right) \\
c_t^o &= \left[\frac{\beta(1+r)\Delta}{\Theta} + (1-\gamma) - \frac{\Delta}{\Theta}\right] d + (1+r)\left[\beta - \frac{1}{(1-\varrho)}\right] b_t^o,
\end{aligned}$$

and then solve for the unknown coefficients

$$\begin{aligned}
\Delta &= \frac{\beta(1+r)\Delta}{\Theta} + (1-\gamma) - \frac{\Delta}{\Theta} \\
\Theta &= (1+r)\left[\beta - \frac{1}{(1-\varrho)}\right]
\end{aligned}$$

$$\begin{aligned}
\Delta &= \frac{\beta\Delta}{\left[\beta - \frac{1}{(1-\varrho)}\right]} + (1-\gamma) - \frac{\Delta}{(1+r)\left[\beta - \frac{1}{(1-\varrho)}\right]} \\
\Delta &= \frac{(1-\gamma)\,(1+r)\,[1-\beta(1-\varrho)]}{\varrho + r}
\end{aligned}$$

The policy rule is:

$$\begin{aligned}
c_t^o &= \Delta d + \Theta b_t^o \\
&= [1-\beta(1-\varrho)]\left[\frac{(1-\gamma)\,(1+r)}{\varrho + r} d - \frac{(1+r)}{(1-\varrho)} b_t^o\right]
\end{aligned}$$

11.2 Individual Problem of Young Households

We first consider the optimization problem of an agent of age N in period t, who will become old in period $t+1$[15]:

$$V_{t,N}^y\left(b_{t,N}^y\right) = \max_{c_{t,N}^y,\, b_{t+1}^o} u\left(c_{t,N}^y\right) + \beta(1-\varrho)V_{t+1}^o\left(b_{t+1}^o\right) \tag{78}$$

[14] To see this more clearly, denote $a = -b$ as savings, and write

$$a_{t+1}^o = (1+r)\beta a_t^o - \frac{(1-\varrho)\,[1-\beta(1+r)]}{(\varrho + r)}(1-\gamma)d$$

[15] We assume that an agent can also die with probability ϱ in the transition between young and old.

such that:

$$c^y_{t,N} = \gamma d + w^{TOT} - (1+r)b^y_{t,N} + b^o_{t+1}. \tag{79}$$

The first order condition implies that

$$\frac{1}{c^y_{t,N}} + \beta\left(\frac{\partial V^o_{t+1}\left(b^o_{t+1}\right)}{\partial b^o_{t+1}} + \frac{\partial V^o_{t+1}\left(b^o_{t+1}\right)}{\partial b^o_{t+2}}\frac{\partial b^o_{t+2}}{\partial b^o_{t+1}}\right) = 0.$$

And applying the envelope theorem we obtain:

$$c^o_{t+1} = \beta\left(1+r\right)c^y_{t,N}$$

We substitute c^o_{t+1} using (76) and we obtain:

$$c^y_{t,N} = \left(\frac{1}{\beta} - (1-\varrho)\right)\left(\frac{(1-\gamma)d}{(\varrho+r)} - \frac{b^o_{t+1}}{(1-\varrho)}\right) \tag{80}$$

Then we consider the optimization problem for a young household of age $j<N$:

$$V^y_{t,j}\left(b^y_{t,j}\right) = \max_{c^y_{t,j},b^y_{t+1,j+1}} u\left(c^y_{t,j}\right) + \beta V^y_{t+1,j+1}\left(b^y_{t+1,j+1}\right) \tag{81}$$

such that:

$$c^y_{t,j} = \gamma d + w^{TOT} - (1+r)b^y_{t,j} + b^y_{t+1,j+1} \tag{82}$$

Which yields the standard Euler equation:

$$c^y_{t,j} = [\beta(1+r)]^{-(N-j)}\, c^y_{t+N-j,N} \tag{83}$$

Equations (80) and (83) fully characterize the lifecycle path of consumption of an household as a function of its assets when entering old age in period $t+1$, b^o_{t+1}.

11.3 Value of Savings of Oldest Young: b^o_{t+1}

We use the above equations, the budget constraint 74, and the assumption that newborn households have no endowment ($b^y_{t,1} = 0$) to determine the value of savings for retirement $b^o_{t+1} \equiv b^y_{t+1,N+1}$

We use the budget constraint for $j=1$ (a young of age=1), in which the debt brought over, $b^y_{t,1}$, is zero:

$$b^y_{t,1} = \frac{\left(\gamma d + w^{TOT}\right) - c^y_{t,1} + b^y_{t+1,2}}{(1+r)} = 0,$$

and we solve forward:

$$\begin{aligned} b^y_{t,1} &= \frac{\left(\gamma d + w^{TOT}\right) - c^y_{t,1}}{(1+r)} + \frac{\left(\gamma d + w^{TOT}\right) - c^y_{t+1,2} + b^y_{t+2,3}}{(1+r)^2} \\ &= \left(\gamma d + w^{TOT}\right)\sum_{j=1}^{N}\frac{1}{(1+r)^j} - \sum_{j=1}^{N}\frac{c^y_{t+j-1,j}}{(1+r)^j} + \frac{b^y_{t+N,N+1}}{(1+r)^N} \end{aligned}$$

Making use of the FOC:

$$c^y_{t,j} = [\beta(1+r)]^{-(N-j)}\, c^y_{t+N-j,N} \tag{84}$$

we get

$$\begin{aligned}
c^y_{t,j} &= [\beta(1+r)]^{-(N-j)}\, c^y_{t+N-j,N} \\
c^y_{t,1} &= [\beta(1+r)]^{-(N-1)}\, c^y_{t+N-1,N} \\
c^y_{t+1,2} &= [\beta(1+r)]^{-(N-2)}\, c^y_{t+1+N-2,N} = [\beta(1+r)]^{-(N-2)}\, c^y_{t+N-1,N} \\
c^y_{t+2,3} &= [\beta(1+r)]^{-(N-3)}\, c^y_{t+2+N-3,N} = [\beta(1+r)]^{-(N-3)}\, c^y_{t+N-1,N} \\
c^y_{t+N-1,N} &= [\beta(1+r)]^{-(N-N)}\, c^y_{t+N-1+N-N,N} = c^y_{t+N-1,N}
\end{aligned}$$

and plug in and simplify

$$\begin{aligned}
\sum_{j=1}^{N} \frac{c^y_{t+j-1,j}}{(1+r)^j} &= \frac{c^y_{t,1}}{(1+r)} + \frac{c^y_{t+1,2}}{(1+r)^2} + \frac{c^y_{t+2,3}}{(1+r)^3} + ... + \frac{c^y_{t+N-1,N}}{(1+r)^N} \\
&= \frac{c^y_{t+N-1,N}}{(1+r)^N} \left[\sum_{j=0}^{N-1} \beta^{-(N-1-j)}\right] = -\frac{c^y_{t+N-1,N}}{(1+r)^N} \frac{\beta^N - 1}{\beta^{N-1}(\beta-1)}
\end{aligned}$$

We substitute back in and keep simplifying

$$b^y_{t,1} = \left(\gamma d + w^{TOT}\right)\frac{1}{r}\left[1 - \frac{1}{(1+r)^N}\right] - \frac{c^y_{t+N-1,N}}{(1+r)^N}\frac{\beta^N - 1}{\beta^{N-1}(\beta-1)} + \frac{b^y_{t+N,N+1}}{(1+r)^N}$$

$$\begin{aligned}
0 = \left(\gamma d + w^{TOT}\right)\frac{1}{r}\left[(1+r)^N - 1\right] - \left(\frac{1}{\beta} - (1-\varrho)\right)\frac{\beta^N - 1}{\beta^{N-1}(\beta-1)}\frac{(1-\gamma)d}{(\varrho+r)} \\
+ \left[\left(\frac{1}{\beta} - (1-\varrho)\right)\frac{1}{(1-\varrho)}\frac{\beta^N - 1}{\beta^{N-1}(\beta-1)} + 1\right] b^y_{t+N,N+1} \\
\left(\frac{1}{\beta} - (1-\varrho)\right)\frac{\beta^N - 1}{\beta^{N-1}(\beta-1)}\frac{(1-\gamma)d}{(\varrho+r)} - \left(\gamma d + w^{TOT}\right)\frac{1}{r}\left[(1+r)^N - 1\right] \\
= \left[\left(\frac{1}{\beta} - (1-\varrho)\right)\frac{1}{(1-\varrho)}\frac{\beta^N - 1}{\beta^{N-1}(\beta-1)} + 1\right] b^y_{t+N,N+1}
\end{aligned}$$

Solve:

$$-b^y_{t+N,N+1} = \frac{\left(\gamma d + w^{TOT}\right)\frac{1}{r}\left[(1+r)^N - 1\right] - \Psi\frac{(1-\gamma)d}{(\varrho+r)}}{\frac{\Psi}{1-\varrho} + 1} \tag{85}$$

$$\Psi \equiv \left(\frac{1}{\beta} - (1-\varrho)\right)\frac{\beta^N - 1}{\beta^{N-1}(\beta-1)}$$

Equation (85) is very intuitive. Savings for retirement $-b^o_{t+1}$ increase in the difference between income before and after retirement. Moreover an increase in life expectancy (a drop in ϱ) reduces the value of the term Ψ and therefore increases $-b^o_{t+1}$.

11.4 Aggregate savings of the young

The previous section determines a sequence of optimal consumption at every age, $c^y_{t,1}, ..., c^y_{t,N}$, and applying the budget constraint (74) we can determine a sequence of assets for every age $b^y_{t,2}, ..., b^y_{t,N}$, which is constant for every period t. In equilibrium there is a measure 1 of households, a fraction $\frac{\phi}{\phi+\varrho}$ old, and a fraction $\frac{\varrho}{\phi+\varrho}$ young. Moreover there is a measure $\frac{\varrho}{\phi+\varrho}\frac{1}{N}$ of young households for each age. Therefore, after dropping the subscript t, we can define aggregate savings of the young households as:

$$B^y = \frac{\varrho}{\phi + \varrho} \frac{1}{N} \sum_{j=1}^{N} b^y_{j+1}. \tag{86}$$

Savings of a young household are:

$$b^y_{j+1} = c^y_j - \gamma d - w^{TOT} + (1+r) b^y_j \tag{87}$$

We solve for b^y_N (from now on for simplicity omit the superscript y) :

$$b_N = \frac{1}{1+r}\left(\gamma d + w^{TOT}\right) - \frac{1}{1+r} c_N + \frac{1}{1+r} b_{N+1}, \tag{88}$$

where both b_{N+1} and c_N are determined by (80) and (85) above. At age $N-1$ (we use $c_t = [\beta(1+r)]^{-1} c_{t+1}$):

$$b_{N-1} = \frac{1}{1+r}\left(\gamma d + w^{TOT}\right) - \frac{1}{1+r} c_{N-1} + \frac{1}{1+r}\left[\frac{1}{1+r}\left(\gamma d + w^{TOT}\right) - \frac{1}{1+r} c_N + \frac{1}{1+r} b_{N+1}\right] \tag{89}$$

$$\begin{aligned}
b_{N-1} &= \frac{1}{1+r}\left(\gamma d + w^{TOT}\right) - \frac{1}{1+r}[\beta(1+r)]^{-1} c_N + \frac{1}{1+r}\left[\frac{1}{1+r}\left(\gamma d + w^{TOT}\right) - \frac{1}{1+r} c_N + \frac{1}{1+r} b_{N+1}\right] \\
&= \frac{1}{1+r}\left(\gamma d + w^{TOT}\right) + \frac{1}{(1+r)^2}\left(\gamma d + w^{TOT}\right) - \frac{1}{1+r}[\beta(1+r)]^{-1} c_N - \frac{1}{(1+r)^2} c_N + \frac{1}{(1+r)^2} b_{N+1} \\
&= \left(\frac{1}{1+r} + \frac{1}{(1+r)^2}\right)\left(\gamma d + w^{TOT}\right) - \left(\frac{1}{\beta} + 1\right)\frac{c_N}{(1+r)^2} + \frac{1}{(1+r)^2} b_{N+1}
\end{aligned}$$

therefore at age $N-2$:

$$\begin{aligned}
b_{N-2} &= \frac{1}{1+r}\left(\gamma d + w^{TOT}\right) - \frac{1}{1+r} c_{N-2} + \frac{1}{1+r} b_{N-1} \\
&= \frac{1}{1+r}\left(\gamma d + w^{TOT}\right) - \frac{1}{1+r}[\beta(1+r)]^{-1}[\beta(1+r)]^{-1} c_N \\
&\quad + \frac{1}{1+r}\left(\left(\frac{1}{1+r} + \frac{1}{(1+r)^2}\right)\left(\gamma d + w^{TOT}\right) - \left(\frac{1}{\beta(1+r)^2} + \frac{1}{(1+r)^2}\right) c_N + \frac{1}{(1+r)^2} b_{N+1}\right) \\
&= \left(\frac{1}{1+r} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3}\right)\left(\gamma d + w^{TOT}\right) - \left(\frac{1}{\beta^2} + \frac{1}{\beta} + 1\right)\frac{c_N}{(1+r)^3} + \frac{1}{(1+r)^3} b_{N+1}
\end{aligned}$$

and at a generic age $N-t$:

$$b_{N-t} = \sum_{j=0}^{t} \frac{\gamma d + w^{TOT}}{(1+r)^{j+1}} - \frac{c_N}{(1+r)^{t+1}} \sum_{j=0}^{t} \frac{1}{\beta^j} + \frac{b_{N+1}}{(1+r)^{t+1}} \tag{90}$$

We use general formulas: $\sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$ and $\sum_{j=0}^{t} x^j = \left(1 - x^{t+1}\right)\frac{1}{1-x}$, or $\sum_{j=0}^{t} \frac{1}{(1+r)^j} =$

$\left(1-\frac{1}{(1+r)^{t+1}}\right)\frac{1+r}{r}$ and $\sum_{j=0}^{t}\frac{1}{(1+r)^{j+1}}=\left(1-\frac{1}{(1+r)^{t+1}}\right)\frac{1}{r}$, so that

$$
\begin{aligned}
\sum_{j=0}^{t}\frac{\gamma d+w^{TOT}}{(1+r)^{j+1}} &= \left(1-\frac{1}{(1+r)^{t+1}}\right)\frac{\gamma d+w^{TOT}}{r}\\
\frac{c_N}{(1+r)^{t+1}}\sum_{j=0}^{t}\frac{1}{\beta^j} &= \frac{c_N}{(1+r)^{t+1}}\left(1-\frac{1}{\beta^{t+1}}\right)\frac{\beta}{\beta-1}
\end{aligned}
$$

hence:

$$
b_{N-t}=\left(1-\frac{1}{(1+r)^{t+1}}\right)\frac{\gamma d+w^{TOT}}{r}-\frac{c_N}{(1+r)^{t+1}}\left(1-\frac{1}{\beta^{t+1}}\right)\frac{\beta}{\beta-1}+\frac{b_{N+1}}{(1+r)^{t+1}} \tag{91}
$$

Now we add up the savings/borrowing over all ages from b_2 to b_N

$$
\begin{aligned}
\sum_{t=0}^{N-2} b_{N-t} &= \frac{\gamma d+w^{TOT}}{r}\sum_{t=0}^{N-2}\left(1-\frac{1}{(1+r)^{t+1}}\right)-\frac{\beta}{\beta-1}c_N\sum_{t=0}^{N-2}\left(\frac{1}{(1+r)^{t+1}}\left(1-\frac{1}{\beta^{t+1}}\right)\right)\\
&\quad +b_{N+1}\sum_{t=0}^{N-2}\left(\frac{1}{(1+r)^{t+1}}\right)
\end{aligned}
$$

the value of each summation term is:

$$
\begin{aligned}
\sum_{t=0}^{N-2}\left(1-\frac{1}{(1+r)^{t+1}}\right) &= -\frac{1}{r}\left(r-(r+1)^{-N+1}-Nr+1\right)\\
&= -\frac{1}{r}\left(1+r(1-N)-\frac{1}{(1+r)^{N-1}}\right)\\
&= \frac{1}{r}\left(\frac{1}{(1+r)^{N-1}}+r(N-1)-1\right)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{t=0}^{N-2}\left(\frac{1}{(1+r)^{t+1}}\left(1-\frac{1}{\beta^{t+1}}\right)\right) &= \sum_{t=0}^{N-2}\frac{1}{(1+r)^{t+1}}-\sum_{t=0}^{N-2}\left(\frac{1}{[(1+r)\,\beta]^{t+1}}\right)\\
&= \frac{1-(1+r)^{-N+1}}{(1+r)-1}-\frac{1-[(1+r)\,\beta]^{-N+1}}{[(1+r)\,\beta]-1}
\end{aligned}
$$

$$
\sum_{t=0}^{N-2}\left(\frac{1}{(1+r)^{t+1}}\right)=\frac{1-(1+r)^{-N+1}}{(1+r)-1}
$$

Substituting them back:

$$
\begin{aligned}
\sum_{t=0}^{N-2} b_{N-t} &= \frac{\gamma d+w^{TOT}}{r}\frac{1}{r}\left(\frac{1}{(1+r)^{N-1}}+r(N-1)-1\right)-\\
&+\frac{\beta}{\beta-1}c_N\left[\frac{1-(1+r)^{-N+1}}{(1+r)-1}-\frac{1-[(1+r)\,\beta]^{-N+1}}{(1+r)\,\beta-1}\right]+b_{N+1}\frac{1-(1+r)^{-N+1}}{(1+r)-1}
\end{aligned}
$$

we rename terms:

$$\sum_{t=0}^{N-2} b_{N-t} = A_1 \frac{\gamma d + w^{TOT}}{r} - A_2 c_N + A_3 b_{N+1}$$

$$\begin{aligned}
A_1 &\equiv \frac{1}{r}\left(\frac{1}{(1+r)^{N-1}} + r(N-1) - 1\right) \\
A_2 &\equiv \frac{\beta}{\beta-1}\left[\frac{1-(1+r)^{-N+1}}{(1+r)-1} - \frac{1-[(1+r)\,\beta]^{-N+1}}{(1+r)\,\beta-1}\right] \\
A_3 &\equiv \frac{1-(1+r)^{-N+1}}{(1+r)-1}
\end{aligned}$$

11.5 Aggregate savings of the old

In equilibrium there are $\frac{\varrho}{\phi+\varrho}\frac{1}{N}$ households that become old every period, and $\frac{\varrho}{\phi+\varrho}\frac{1}{N}(1-\varrho)^j$ households that survived for j periods . Therefore aggregate savings of the old households are:

$$B^o = \frac{\varrho}{\phi+\varrho}\frac{1}{N}\sum_{j=1}^{\infty}(1-\varrho)^j b_j^0 \tag{92a}$$

also note that b_1^0 is the initial savings from young age as defined in (85)

Recall that (from (77)):

$$b_{j+1}^o = A + \beta(1+r)b_j^o \tag{93}$$

$$A \equiv \frac{(1-\varrho)\,[1-\beta(1+r)]}{(\varrho+r)}(1-\gamma)d$$

hence:

$$b_2^o = A + \beta(1+r)b_1^o \tag{94}$$

$$b_3^o = A + \beta(1+r)b_2^o = A + \beta(1+r)A + \beta^2(1+r)^2 b_1^o \tag{95}$$

$$b_t^o = A + \beta(1+r)A + ... + \beta^{t-2}(1+r)^{t-2}A + \beta^{t-1}(1+r)^{t-1}b_1^o \tag{96}$$

$$b_t^o = A\left[\sum_{j=0}^{t-2}\beta^j(1+r)^j\right] + \beta^{t-1}(1+r)^{t-1}b_1^o \tag{97}$$

$$b_t^o = \frac{(\beta\,(1+r))^{t-1}-1}{\beta\,(1+r)-1}A + [\beta(1+r)]^{t-1}\,b_1^o \tag{98}$$

$$
\begin{aligned}
\sum_{t=1}^{\infty}(1-\varrho)^t b_t^0 &= \sum_{t=1}^{\infty}(1-\varrho)^t \left[\frac{(\beta\,(r+1))^{t-1}-1}{\beta\,(1+r)-1}A + [\beta(1+r)]^{t-1}\,b_1^o\right] \\
&= \frac{A}{\beta\,(1+r)-1}\sum_{t=1}^{\infty}(1-\varrho)^t\left[(\beta\,(r+1))^{t-1}-1\right] + b_1^o\sum_{t=1}^{\infty}(1-\varrho)^t\,[\beta(1+r)]^{t-1} \\
&= \left[\frac{A}{\beta+r\beta-1}+b_1^o\right]\sum_{t=1}^{\infty}(1-\varrho)^t\,(\beta\,(r+1))^{t-1} - \frac{A}{\beta+r\beta-1}\sum_{t=1}^{\infty}(1-\varrho)^t \\
&= \left[\frac{A}{\beta+r\beta-1}+b_1^o\right]\frac{1}{\beta(1+r)}\sum_{t=1}^{\infty}[(1-\varrho)\beta(1+r)]^t - \frac{A}{\beta+r\beta-1}\sum_{t=1}^{\infty}(1-\varrho)^t
\end{aligned}
$$

$$
\sum_{t=1}^{\infty}[(1-\varrho)\beta(1+r)]^t = \frac{1}{1-(1-\varrho)\beta(1+r)} - 1 = \frac{(1-\varrho)\beta(1+r)}{1-(1-\varrho)\beta(1+r)}
$$

$$
\sum_{t=1}^{\infty}(1-\varrho)^t = \frac{1}{1-1+\varrho} - 1 = \frac{1-\varrho}{\varrho}
$$

hence:

$$
\sum_{t=1}^{\infty}(1-\varrho)^t b_t^0 = \left[\frac{A}{\beta+r\beta-1}+b_1^o\right]\frac{(1-\varrho)}{1-(1-\varrho)\beta(1+r)} - \frac{A}{\beta+r\beta-1}\frac{1-\varrho}{\varrho}
$$

11.6 Summing up aggregate household borrowing

aggregate household borrowing is:

$$
B = B^o + B^y \tag{99}
$$

Where savings of the old is:

$$
B^o = \frac{\varrho}{\phi+\varrho}\frac{1}{N}\left[\left(\frac{A}{\beta+r\beta-1}+b^{retirement}\right)\frac{(1-\varrho)}{1-(1-\varrho)\beta(1+r)} - \frac{A}{\beta+r\beta-1}\frac{1-\varrho}{\varrho}\right] \tag{100a}
$$

$$
A \equiv \left(\frac{(1-\varrho)\,(1-(1+r)\beta)}{(\varrho+r)}\right)(1-\gamma)d
$$

$$
b^{retirement} \equiv \frac{\Psi\frac{(1-\gamma)d}{(\varrho+r)} - \left(\gamma d + w^{TOT}\right)\frac{1}{r}\left[(1+r)^N-1\right]}{\frac{\Psi}{1-\varrho}+1} \tag{101}
$$

$$
\Psi \equiv \left(\frac{1}{\beta}-(1-\varrho)\right)\frac{\beta^N-1}{\beta^{N-1}\,(\beta-1)}
$$

And savings of the young is:

$$
B^y = \frac{\varrho}{\phi+\varrho}\frac{1}{N}\sum_{t=0}^{N-2} b_{N-t} = \frac{\varrho}{\phi+\varrho}\frac{1}{N}\left[A_1\frac{\gamma d + w^{TOT}}{r} - A_2 c_N + A_3 b^{retirement}\right]
$$

$$
\begin{aligned}
A_1 &\equiv \frac{1}{r}\left(\frac{1}{(1+r)^{N-1}}+r(N-1)-1\right)\\
A_2 &\equiv \frac{\beta}{\beta-1}\left[\frac{1-(1+r)^{-N+1}}{(1+r)-1}-\frac{1-[(1+r)\,\beta]^{-N+1}}{(1+r)\,\beta-1}\right]\\
A_3 &\equiv \frac{1-(1+r)^{-N+1}}{(1+r)-1}
\end{aligned}
$$

And

$$
c_N=\left(\frac{1}{\beta}-(1-\varrho)\right)\left(\frac{(1-\gamma)d}{(\varrho+r)}-\frac{b^{retirement}}{(1-\varrho)}\right)
$$