Likelihood Based Estimation and Inference for Unsolved DSGE Models

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Caveat emptor

- I congratulate the author for the nice paper for taking a novel approach to estimation (the approach finds me very sympathetic!)
- There are many technicalities, but I will try to highlight the main aspect of the paper trying to go easy with mathematical subtleties—not sure I will succeed!

The key idea

- Models are restrictions on probability distribution of random variables
- Such restrictions often involves certain moments, e.g.,

$$\int g(Y_{t+1};\theta_0)dF(Y_{t+1}|\mathscr{I}_t) = 0, \quad a.s.\mathscr{I}_t$$

  here $\mathscr{I}_t$ is a filtration.
- Given $\{Y_t\}_{t=1}^T$, the econometrician wants to
  - estimate θ_0— the "true" parameter vector
  - study of the "dynamics" of the model, e.g. impulse responses, counterfactual analysis, etc.

Estimation

- If all element of $\{Y_t\}_{t=1}^{T}$ are observables then estimation can be conducted by GMM, among other methods

$$\hat{\theta} = \arg\min_{\theta \in \Theta} \left(\sum_{t=1}^{T} \tilde{g}(y_{t+1}, \theta) \right)' W \left(\sum_{t=1}^{T} \tilde{g}(y_{t+1}, \theta) \right),$$

where

$$\tilde{g}(y_{t+1}, \theta) = A(\mathscr{I}_t) \otimes g(y_{t+1}, \theta)$$

and we know everything there is to know about $\hat{\theta}$.......

- Often, in very relevant cases, some of the elements of Y_{t+1} are not directly observables — they are latent
- To get the intuition, consider the case in which Y_{t+1} is fully observable

Probabilistic framework

- The probability distribution coherent with the model are given

$$\mathcal{P}(\theta) = \left\{ P : P \in \mathscr{P}, \int g(Y_{t+1}, \theta) dP = 0 \right\}$$

and

$$\underbrace{\mathcal{P} = \bigcup_{\theta \in \Theta} \mathcal{P}(\theta)}_{\text{conditional measures coherent with the model}}$$

- Correct specification

$$\underbrace{F \in \mathcal{P}}_{\text{"true" measure belongs to } \mathcal{P}}$$

- Identification

$$F \in \mathcal{P}(\theta_0) \text{ and } F \notin \mathcal{P}(\theta), \text{ for } \theta \neq \theta_0$$

Main idea

1. Postulate a working conditional measure parameterized by φ

$$Q(y_{t+1}|\mathscr{I}_t, \varphi), \text{ with density } q(y_{t+1}|\mathscr{I}_{t+1}, \varphi)$$

2. "Modify" Q in such a way that it belongs to $\mathcal{P}(\theta)$
3. This is accomplished by solving the following problem

$$\min_H \int \log\left(\frac{H}{Q}\right) dH, \quad \text{subject to } \int \mathrm{g}(y_{t+1},\theta)\mathrm{dH}=0$$

4. The solution to this problem is (in terms of densities)

$$h(y_{t+1}|\mathscr{I}_t, \theta, \varphi) = \exp(\lambda + \mu' g(y_{t+1}, \theta))q(y_{t+1}|\mathscr{I}_t, \phi)$$

Main Idea, ctd.

- Inference is based on

$$\prod_{t=0}^{T-1} h(y_{t+1}|\mathscr{I}_t, \theta, \varphi) = \prod_{t=0}^{T-1} \exp(\lambda + \mu' g(y_{t+1}, \theta)) q(y_{t+1}|\mathscr{I}_t, \varphi)$$

- For instance, by MLE

$$(\hat{\theta}, \hat{\varphi}) = \underset{\theta\in\Theta, \varphi\in\Phi}{\arg\max} \prod_{t=0}^{T-1} \exp(\lambda + \mu' g(y_{t+1}, \theta)) q(y_{t+1}|\mathscr{I}_t, \varphi)$$

- By "well known" regularity conditions

$$\hat{\theta} \xrightarrow{p} \theta^* \quad \hat{\varphi} \xrightarrow{p} \varphi^*$$

are pseudo-true values

Computational details

1. Draw from $y_{t+1}^{(s)} \sim q(y_{t+1}|\mathscr{I}_t, \varphi)$

2. Solve $\min_{(\lambda,\mu)\in\mathbb{R}^{M+1}} \frac{1}{S}\sum_{s=1}^{S} \exp(\lambda + \mu' g(y_{t+1}^{(s)}, \theta)) - \lambda$
3. Intuition, the First Order Conditions (FOC)

$$0 = \frac{1}{S}\sum_{s=1}^{S} \exp(\lambda + \mu' g(y_{t+1}^{(s)}, \theta)) g(y_{t+1}^{(s)}, \theta)$$

$$\xrightarrow[s\to\infty]{p} \int \exp(\lambda + \mu' g(y_{t+1}, \theta)) g(y_{t+1}, \theta) q(y_{t+1}|\mathscr{I}_t, \varphi) dy_{t+1}$$

$$1 = \frac{1}{S}\sum_{s=1}^{S} \exp(\lambda + \mu' g(y_{t+1}^{(s)}, \theta))$$

$$\xrightarrow[s\to\infty]{p} \int \exp(\lambda + \mu' g(y_{t+1}, \theta)) q(y_{t+1}|\mathscr{I}_t, \varphi) dy_{t+1}$$

2 Questions:

Correct specification

- When does $\hat{\theta} \xrightarrow{p} \theta_0$?
- When

$$\exists \varphi' \text{ such that } Q(y_{t+1}|\mathscr{I}_t, \varphi') = F(y_{t+1}|\mathscr{I}_t)$$

that is, the base measure is correctly specified for the truth.... (can be shown by simple KL arguments)

Identification

- Suppose that Q is correctly specified as defined above. Is θ_0 point identified — whenever the model is point identified? Yes

On Kullback-Leibler (or maximum entropy)

The method is based on being able to solve

$$\min_H \int \log\left(\frac{H}{Q}\right) dH, \quad \text{subject to } \int \mathrm{g}(y_{t+1},\theta)\mathrm{dH}{=}0$$

to obtain $h(y_{t+1}|\mathscr{I}_t, \theta, \varphi) = \exp(\lambda + \mu' g(y_{t+1}, \theta))q(y_{t+1}|\mathscr{I}_t, \phi)$.

1. When does a solution exists? Very challenging to establish necessary and sufficient conditions, only sufficient are usually available
2. Existing results are for the unconditional problem (which is much easier to deal with)
3. Whether a solution exists – crucially depends on the base density q and the form of g (e.g. Komunjer and Ragusa, 2014)

   $$\int \sup_{\lambda,\mu} \exp(\lambda + \mu' g(y_{t+1}, \theta))q(y_{t+1}|\mathscr{I}_t, \varphi)dy_{t+1} < \infty$$

   very strong requirement (all exponential moments of g w.r.t. q must exists)
4. Weaker conditions are possible (Komunjer and Ragusa, 2014)

Unsolved DSGE models

- The paper uses a simple model from Ireland (2004)
- The model has not dynamics and the moment condition depends only on observables

$$
\begin{aligned}
y_t &= A k_t^{\theta} (\eta^t h_t)^{1-\theta} \\
k_{t+1} &= (1-\delta) k_t + y_t - c_t \\
\gamma c_t h_t &= (1-\theta) y_t \\
1 &= \beta E_t \left\{ \frac{c_t}{c_{t+1}} \left(\theta \left(\frac{y_{t+1}}{k_{t+1}} \right) + 1 - \delta \right) \right\}
\end{aligned}
$$

- The idea is to postulate, for $x_t = (y_t, c_t, h_t, k_t)$

$$
\begin{aligned}
s_t &= B s_{t-1} + \varepsilon_t \\
x_t &= A s_t
\end{aligned}
$$

which gives

$$
x_t | s_{t-1} \sim N(A B s_{t-1}, A \Sigma A')
$$

Unsolved DSGE models

- Tilt $x_t | s_{t-1} \sim N(ABs_{t-1}, A\Sigma A')$ to satisfy the moment restriction, say

$$h(x_t | s_{t-1}, \theta, A, B) = \exp(\lambda + \mu' g(x_t, \theta)) N(ABs_{t-1}, A\Sigma A')$$

- Use particle filter to obtain

$$h(x_t | x_{1:t-1}, \theta, A, B), \quad t = 1, \dots, T$$

  using standard predict/update filter recursion

- Estimate unknown parameters by MLE (better do MCMC) on

$$h(x_{1:T}, \theta, A, B) = \prod_{t=1}^{T} h(x_t | x_{1:t-1}, \theta, A, B)$$

- Counterfactual analysis is possible (e.g., non-linear impulse responses)

Unsolved DSGE models

Remarks:

- Since the density upon which inference is based is the "closest" to the base one, the base one should be a good approximation
  - Why not choose the solution to the linear model as base model

$$
\begin{aligned}
s_t &= B(\theta)s_{t-1} + \Sigma^{1/2}(\theta)\varepsilon_t \\
x_t &= A(\theta)s_t
\end{aligned}
$$

    and then work with a density that only depends on θ?
  - We then are working with the distribution implied by the linear approximation to the model but which satisfies the moment condition

Why not solving can be useful

Something to think about

- The linearized DSGE model gives an approximate likelihood

$$\tilde{f}(y_t|y_{1:(t-1)}, \varphi(\theta))$$

- How this relate to the "true" yet unknown density?

$$f(y_t|y_{1:(t-1)}, \theta) = \tilde{f}(y_t|y_{1:(t-1)}, \varphi(\theta)) + error(\theta)$$

- The approximation error is not uniform — do MCMC converge?
  - Do forcing the approximate density to satisfy the moment condition ameliorate this?