GMM with latent variables

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- Two main applications:
  1. Moment condition models with time-varying parameters
  2. Estimating Dynamic Stochastic General Equilibrium models without solving the model

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- Problem: they assume that the model defines a likelihood. However:
- Moment condition models with time-varying parameters
    - Limited information so no likelihood without auxiliary assumptions
    - No estimation method currently exists

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  - Also need to deal with stochastic singularity and multiplicity of solutions
  - Desirable to have estimation methods that only use the information in equilibrium conditions

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- Apply a nonlinear filtering method to handle the latent variables
- Show that the use of an approximate density does not matter asymptotically

Assumption 1 - the model

- There exists a dynamic structural model but we have incomplete information in the form of m moment conditions

$$E\left[g\left(X_{t+1}, \Lambda_{t+1}, \theta_0\right) \middle| I_t\right] = 0$$

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- We observe a sample $X = (X_1, ..., X_T)$
- The remaining variables are latent, $\Lambda = (\Lambda_1, ..., \Lambda_T)$

Assumption 2 - the latent variables

- Assume we can draw from the transition density of the dynamic latent variables

$$\Lambda_{t+1} \sim P(\Lambda_{t+1}|\Lambda_t, \theta_0)$$

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- Different notion of latent variables than in microeconometrics

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- This is different from the usual distinction between state and control variables when solving a DSGE model

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  - Identification harder to discuss in nonlinear models + weak identification due to latent variables $\longrightarrow$ we might be worse off

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- Primitive assumption: sample moment condition asymptotically normal

$$Z_T = \left[\Sigma\left(X, \Lambda, \theta_0\right)\right]^{-1/2} g_T\left(X, \Lambda, \theta_0\right) \longrightarrow^d N(0, I)$$

$$g_T\left(X, \Lambda, \theta\right) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} g(X_t, \Lambda_t, \theta)$$

$\Sigma\left(X, \Lambda, \theta\right)$ asymptotic variance (maybe HAC)

Assumption 5 - Chernozhukov and Hong (2003)

- Consider the approximate density induced by GMM

$$p(X, \Lambda, \theta) =$$
$$(2\pi)^{-\frac{m}{2}} \exp\left\{-\frac{1}{2} g_T\left(X, \Lambda, \theta\right) \Sigma\left(X, \Lambda, \theta\right)^{-1} g_T\left(X, \Lambda, \theta\right)\right\}$$

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- If Λ were observable, the Chernozhukov and Hong (2003) result would hold:
  - equivalent to estimate θ by GMM or to draw from $p(X, \Lambda, \theta)$ using MCMC methods

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- Technical challenge: show that it doesn't matter asymptotically

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- Note that the method uses $p(X, \Lambda, \theta)$, even though in principle we need $p(\Lambda | X, \theta)$ for the particle filter and $p(\theta | \Lambda, X)$ for the Metropolis

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  - is assumed to equal 1 in particle filter
    - this is generally satisfied (primitive condition: $g(\cdot)$ unbounded wrt X) but, if not, one could compute it

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$$p\left(X,\Lambda,\theta\right)=\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}Z_T^2\right\}=$$

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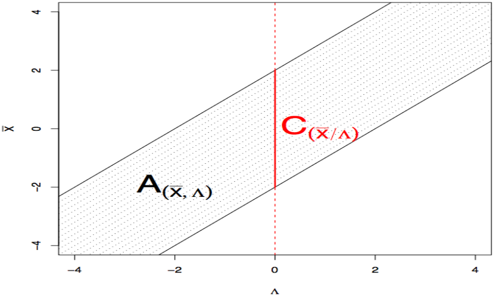
where $Z_T=\sqrt{T}\left(\overline{X}-\Lambda\right)\approx N(0,1)$
- Suppose Z_T exactly normal

Theoretical properties - key idea

- Exact density $f(X, \Lambda, \theta)$ assigns probability to rectangles (X, Λ), whereas $p(X, \Lambda, \theta)$ assigns probability to sets of the form

$$A = \left\{ \left(\overline{X}, \Lambda\right) : Z_T \in (a, b) \right\} = \left\{ \left(\overline{X}, \Lambda\right) : \frac{a}{\sqrt{T}} + \Lambda < \overline{X} < \frac{b}{\sqrt{T}} + \Lambda) \right\}$$



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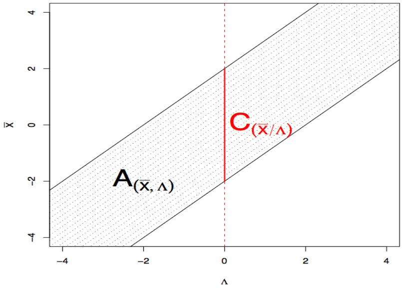

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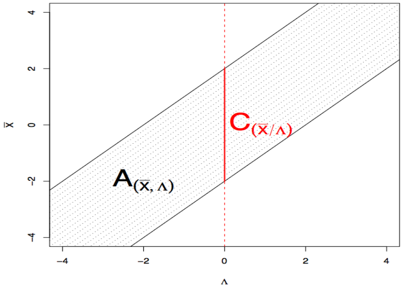

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- Similarly, $p(X|\Lambda, \theta)$ assigns probability to sets of the form C

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  - for sets of the form A, A^T, C the probability assigned by $p(\cdot)$ is the same as the probability assigned by the true density $f(\cdot)$
- Our main result uses this intuition + asymptotic normality of Z_T

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- STEP1: Use particle filter to obtain first "particle" $\Lambda_{1:T}^{(1)}$

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  - Set $t = T_0 + 1$

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  - For $i = 1, ..., N$, compute weights using the **GMM density** $w_t^{(i)} = p\left(X_{1:t}, \Lambda_{1:t}^{(i)}, \theta\right)$

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  - Increase t and repeat until reaching T
  - At T, output the particle $\Lambda_{1:T}^{(N)}$

Intuition

- For a fixed θ and X, the algorithm generates sequence of latent variables "most compatible" with the moment conditions

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  - Accept θ_{new} with probability that depends on the **GMM density**

$$\alpha = \min\left[1, \frac{p\left(X, \Lambda^{(i-1)}, \theta_{new}\right) T(\theta_{new}, \theta_{old})}{p\left(X, \Lambda^{(i-1)}, \theta_{old}\right) T(\theta_{old}, \theta_{new})}\right]$$

otherwise keep θ_{old}

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  otherwise keep θ_{old}
  - Iterate K times and set $\theta^{(i)}$ = last value of the chain

Why this works

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- The Metropolis works because, once conditioning on Λ, it's Chernozhukov and Hong (2003)

Why this works

- Theorem 1: The particle filter works because the method generates draws from the true conditional density (for large T)

$$f\left(\Lambda | X, \theta_0\right)$$

  - BUT, only for pairs of Λ, X that are "allowed" by the structural model
- The Metropolis works because, once conditioning on Λ, it's Chernozhukov and Hong (2003)
- That iterating the particle filter and Metropolis works follows from Andrieu, Douced and Holenstein (2010)

Does the method work in practice?

- Two examples

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- Two examples
  - Stochastic volatility

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  - A simplified DSGE

Does the method work in practice?

- Two examples
  - Stochastic volatility
  - A simplified DSGE
- In both cases we have a likelihood $\longrightarrow$ see what we lose by using GMM

Stochastic volatility example

- Data-generating process

$$
\begin{aligned}
X_t &= \rho X_{t-1} + \exp(\Lambda_t) u_t \\
\Lambda_t &= \phi \Lambda_{t-1} + \sigma e_t \\
e_t, u_t &\sim N(0,1) \text{ independent}
\end{aligned}
$$

Stochastic volatility example

- Data-generating process

$$\begin{aligned} X_t &= \rho X_{t-1} + \exp(\Lambda_t) u_t \\ \Lambda_t &= \phi \Lambda_{t-1} + \sigma e_t \\ e_t, u_t &\sim N(0,1) \text{ independent} \end{aligned}$$

- Best existing method is Flury-Shepherd particle filter

Choice of moment conditions

$$
\begin{aligned}
g_1 &= \left(X_t - \rho X_{t-1}\right) X_{t-1} \\
g_2 &= \left(\Lambda_t - \phi \Lambda_{t-1}\right) \Lambda_{t-1} \\
g_3 &= \left(\Lambda_t - \phi \Lambda_{t-1}\right)^2 - \sigma^2 \\
g_4 &= \left|X_t - \rho X_{t-1}\right| - \left(\frac{2}{\pi}\right)^2 \left[\exp\left(\Lambda_t\right)\right]^2 \\
g_5 &= \left|X_t - \rho X_{t-1}\right| \left|X_{t-1} - \rho X_{t-2}\right| \\
&\quad - \left(\frac{2}{\pi}\right)^2 \exp\left(\Lambda_t\right) \exp\left(\Lambda_{t-1}\right) \\
&\dots \\
g_{4+L} &= \left|X_t - \rho X_{t-1}\right| \left|X_{t-L} - \rho X_{t-L-1}\right| \\
&\quad - \left(\frac{2}{\pi}\right)^2 \exp\left(\Lambda_t\right) \exp\left(\Lambda_{t-L}\right)
\end{aligned}
$$

"Simulation" results

- Estimates of θ from our GMM method and the Flury-Shepherd ML

"Simulation" results

- Estimates of θ from our GMM method and the Flury-Shepherd ML
- Scatter plot of filtered latent variables against the true ones

Table 1. Parameter Estimates for the SV Model Moment Conditions (23) through (28) at both the Metropolis and Gibbs Steps.

Parameter	True Value	Mean	Mode	Standard Error
		With Jacobian Term		
ρ	0.25	0.30488	0.30961	0.074778
ϕ	0.8	0.09153	0.94851	0.660790
σ	0.1	0.09023	0.06702	0.050229
		Without Jacobian		
ρ	0.25	0.30271	0.30939	0.076758
ϕ	0.8	0.15348	0.85765	0.643400
σ	0.1	0.11400	0.08435	0.070081
		Flury and Shephard Estimator		
ρ	0.25	0.30278	0.28555	0.059320
ϕ	0.8	0.17599	0.89189	0.509780
σ	0.1	0.09737	0.07839	0.064661

Data of length $T = 250$ was generated by simulating the model of Subsection 6.1 at the parameter values shown in the column labeled "True Value". In the first two panels the model was estimated by using the Metropolis within Gibbs methods described in Section 2 with a one-lag HAC weighting matrix using $N = 1000$ particles for Gibbs and $K = 50$ draws for Metropolis. In the third panel the estimator is the Bayesian estimator proposed by Flury and Shepard (2010) with a flat prior. It is a standard maximum likelihood particle filter estimator except that the seed changes every time a new θ is proposed with N increased as necessary to control the rejection rate of the MCMC chain. The columns labeled mean, mode, and standard deviation are the mean, mode, and standard deviations of a Metropolis within Gibbs chain of length $R = 9637$ for the first two panels and the same from an MCMC chain of length $R = 500000$ with a stride of 5 for the third.

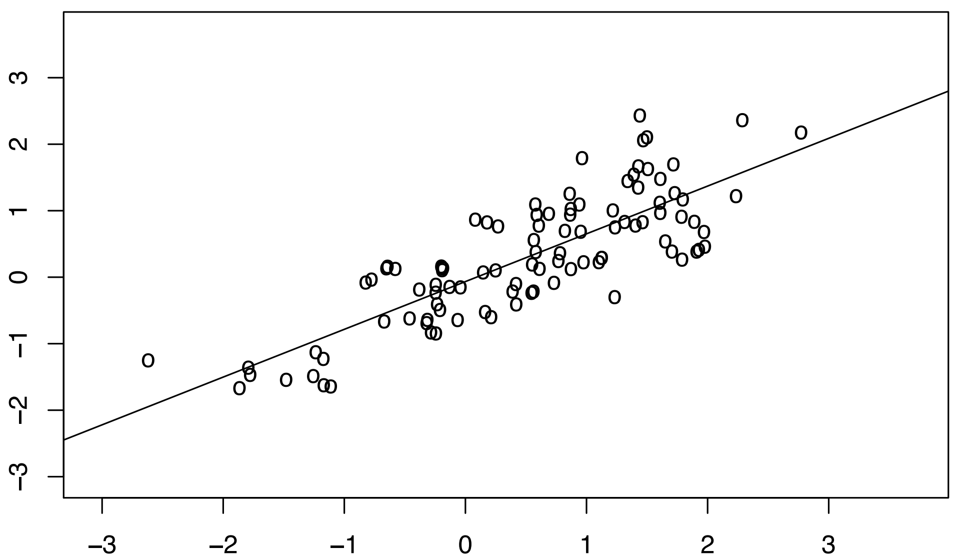

Figure 5. PF for Λ, without Jacobian, Scatter Plot, SV Model. As for Figure 4 except that plotted is the mean of the particles vs. the simulated Λ.

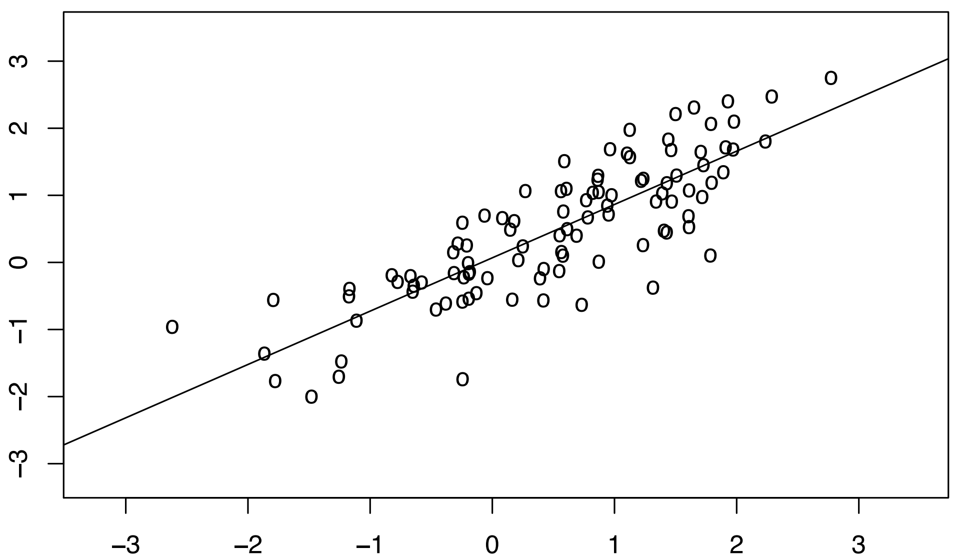

Figure 7. PF for Λ, Flurry-Shephard Method, Scatter Plot. As for Figure 6 except that plotted is the mean of the particles vs. the simulated Λ.

DSGE example

- Simplified version of Del Negro and Schorfheide (2008). First order conditions

$$
\begin{aligned}
E_t\left[y_{t+1}+\pi_{t+1}+z_{t+1}\right]-y_t-\frac{1}{\beta}\pi_t &= 0\\
\lambda_t+w_t &= 0\\
w_t-(1+v)y_t-\phi_t &= 0
\end{aligned}
$$

DSGE example

- Simplified version of Del Negro and Schorfheide (2008). First order conditions

$$
\begin{aligned}
E_t \left[y_{t+1} + \pi_{t+1} + z_{t+1} \right] - y_t - \frac{1}{\beta}\pi_t &= 0 \\
\lambda_t + w_t &= 0 \\
w_t - (1+v) y_t - \phi_t &= 0
\end{aligned}
$$

- Outputs: y_t output, w_t wages, π_t inflation

DSGE example

- Simplified version of Del Negro and Schorfheide (2008). First order conditions

$$
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E_t\left[y_{t+1}+\pi_{t+1}+z_{t+1}\right]-y_t-\frac{1}{\beta}\pi_t &= 0\\
\lambda_t+w_t &= 0\\
w_t-(1+v)y_t-\phi_t &= 0
\end{aligned}
$$

- Outputs: y_t output, w_t wages, π_t inflation
- Shocks

$$
\begin{aligned}
z_t &= \rho_z z_{t-1}+\sigma_z\varepsilon_{z,t} \text{ factor productivity}\\
\lambda_t &= \rho_\lambda z_{t-1}+\sigma_\lambda\varepsilon_{\lambda,t} \text{ consumption/leisure preference}\\
\phi_t &= \rho_\phi z_{t-1}+\sigma_\phi\varepsilon_{\phi,t} \text{ price elasticity}
\end{aligned}
$$

DSGE example

- λ_t pinned down by model so we have

$$\begin{aligned} \text{Observable variables } X_t &= (y_t, w_t, \pi_t) \\ \text{Latent variables } \Lambda_t &= (z_t, \phi_t) \end{aligned}$$

DSGE example

- λ_t pinned down by model so we have

$$\begin{aligned} \text{Observable variables } X_t &= (y_t, w_t, \pi_t) \\ \text{Latent variables } \Lambda_t &= (z_t, \phi_t) \end{aligned}$$

- Identification problems: likelihood reveals that only one of $\sigma_z, \sigma_\phi, v, \beta$ can be identified $\longrightarrow$ calibrate σ_z, σ_ϕ, v

DSGE example

- λ_t pinned down by model so we have

$$\begin{aligned}\text{Observable variables } X_t &= (y_t, w_t, \pi_t) \\ \text{Latent variables } \Lambda_t &= (z_t, \phi_t)\end{aligned}$$

- Identification problems: likelihood reveals that only one of $\sigma_z, \sigma_\phi, v, \beta$ can be identified $\longrightarrow$ calibrate σ_z, σ_ϕ, v
- In general, identification problems will cause the MCMC chain not to mix

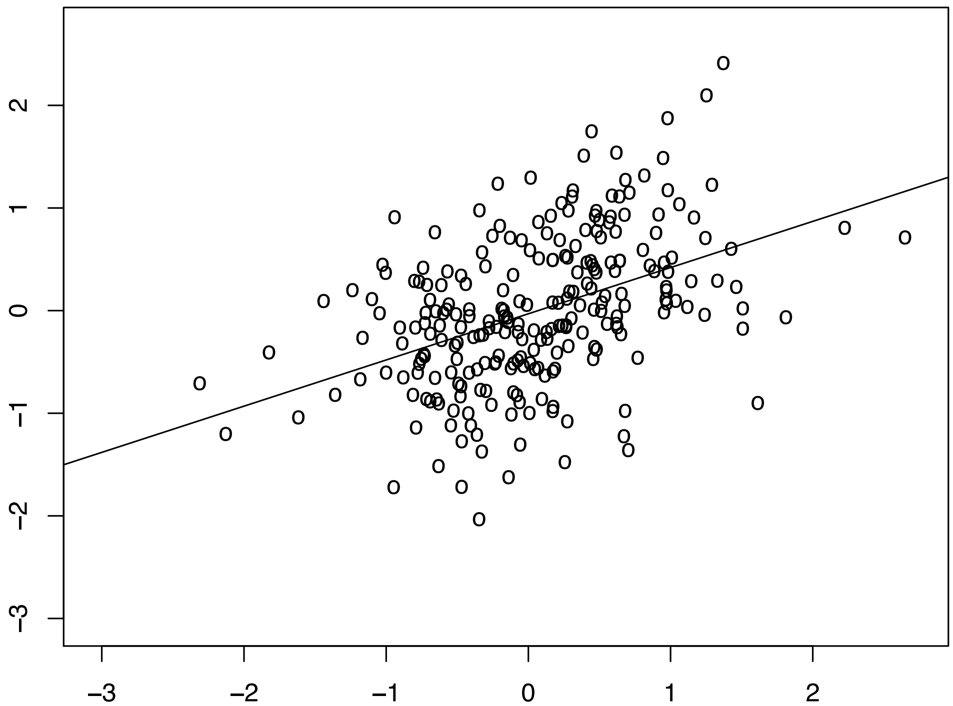
Choice of moment conditions

$$
\begin{aligned}
g_1 &= \left(w_t - \rho_\lambda w_{t-1}\right) w_{t-1} \\
g_2 &= \left(w_t - \rho_\lambda w_{t-1}\right)^2 - \sigma_\lambda^2 \\
g_3 &= \left[w_{t-1} - (1+v) y_{t-1}\right] \cdot \\
&\quad \left[w_t - (1+v) y_t - \rho_\phi \left(w_t - (1+v) y_{t-1}\right)\right] \\
g_4 &= \left[w_t - (1+v) y_{t-1}\right] \left(\phi_t - \rho_\phi \phi_{t-1}\right) \\
g_5 &= \left[w_t - (1+v) y_t\right]^2 - \sigma_\phi^2 \\
g_6 &= w_{t-1}\left(y_{t-1} + \frac{1}{\beta}\pi_{t-1} - y_t - \pi_t - \rho_z z_{t-1}\right) \\
g_7 &= y_{t-1}\left(y_{t-1} + \frac{1}{\beta}\pi_{t-1} - y_t - \pi_t - \rho_z z_{t-1}\right) \\
g_8 &= \pi_{t-1}\left(y_{t-1} + \frac{1}{\beta}\pi_{t-1} - y_t - \pi_t - \rho_z z_{t-1}\right)
\end{aligned}
$$

Table 2. Parameter Estimates for the DSGE Model Using Moment Conditions (32) through (40) at Both the Metropolis and Gibbs Steps.

Parameter	True Value	Mean	Mode	Standard Error
		With Jacobian		
ρ_z	0.15	0.21596	0.15006	0.08632
ρ_ϕ	0.68	0.60098	0.58945	0.04988
ρ_λ	0.56	0.50134	0.46443	0.28818
σ_λ	0.11	0.10827	0.08923	0.06494
β	0.996	0.98429	0.99603	0.01476
		Without Jacobian		
ρ_z	0.15	0.21887	0.23069	0.09179
ρ_ϕ	0.68	0.59967	0.60750	0.04988
ρ_λ	0.56	0.50884	0.31473	0.28981
σ_λ	0.11	0.10797	0.11613	0.06896
β	0.996	0.98201	0.99634	0.01834
		Maximum Likelihood		
ρ_z	0.15	0.15165	0.15087	0.00583
ρ_ϕ	0.68	0.59185	0.59419	0.05044
ρ_λ	0.56	0.56207	0.56549	0.05229
σ_λ	0.11	0.11225	0.11189	0.00508
β	0.996	0.99640	0.99643	0.00186

Data of length $T = 250$ was generated by simulating the model of Subsection 6.2 at the parameter values shown in the column labeled "True Value". In the first two panels the model was estimated by using the Metropolis within Gibbs method described in Section 2 with a two-lag HAC weighting matrix using $N = 1000$ particles for Gibbs and $K = 50$ draws for Metropolis. In the third panel the model was estimated by maximum likelihood. The columns labeled mean, mode, and standard deviation are the mean, mode, and standard deviations of a Metropolis within Gibbs chain of length $R = 9637$ for the first two panels and the same from an MCMC chain of length $R = 500000$ with a stride of 5 for the third.



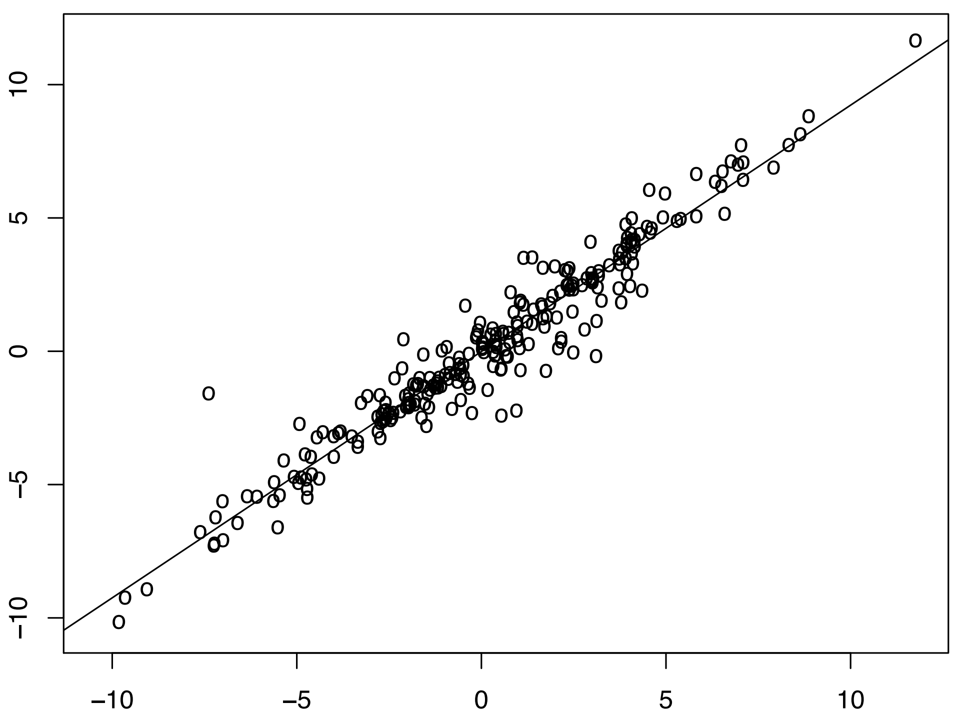


Figure 9. PF for Λ with Jacobian, Scatter Plot, DSGE Model. As for Figure 8 except that plotted is the mean of the particles vs. the simulated Λ for all 250 time points.

Conclusion

- Estimation method for moment-condition models with dynamic latent variables

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- Estimation method for moment-condition models with dynamic latent variables
- Implemented by a Metropolis within a particle filter algorithm
- Works in our limited experience
- Must better investigate effect of choice of moment conditions