Some unpleasant properties of log-linearized solutions when the nominal rate is zero.*†

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Abstract

Does fiscal policy have qualitatively different effects on the economy in a liquidity trap? We analyze a nonlinear stochastic New Keynesian model and compare the true and loglinearized equilibria. Using the loglinearized equilibrium conditions the answer to the above question is yes. However, for the true nonlinear model the answer is no. For a broad range of empirically relevant parameterizations labor falls in response to a tax cut in the loglinearized economy but rises in the true economy. While the government purchase multiplier is above two in the loglinearized economy it is about one in the true economy.

1 Introduction

The recent experiences of Japan, the United States, and Europe with zero/near-zero nominal interest rates have raised new questions about the conduct of monetary and fiscal policy in a liquidity trap. A large and growing body of new research has emerged that provides answers using New Keynesian (NK) frameworks that explicitly model the zero bound on the nominal interest rate. One conclusion that has emerged is that fiscal policy has different effects on the economy when the nominal interest rate is zero. Eggertsson (2011) finds that hours worked fall in response to a labor tax cut when the nominal interest rate is zero, a property that is referred to as the "paradox of toil," and Christiano, Eichenbaum, and Rebelo (2011), Woodford (2011) and Erceg and Lindé (2010) find that the size of the government purchase multiplier is substantially larger than one when the nominal interest rate is zero.

These and other results (see e.g. Del Negro, Eggertsson, Ferrero, and Kiyotaki (2010), Bodenstein, Erceg, and Guerrieri (2009), Eggertsson and Krugman (2010)) have been derived in setups that respect the nonlinearity in the Taylor rule but loglinearize the remaining equilibrium conditions about a steadystate with a stable price level. Loglinearized NK models require large shocks to generate a binding zero lower bound for the nominal interest rate and the shocks must be even larger if these models are to reproduce the measured declines in output and inflation that occurred during the Great Depression or the Great Recession of 2007-2009.[1] Loglinearizations are local solutions that only work within a given radius of the point where the approximation is taken. Outside of this radius these solutions break down (See e.g. Den Haan and Rendahl (2009)). The objective of this paper is to document that such a breakdown can occur when analyzing the zero bound.

We study the properties of a nonlinear stochastic NK model when the nominal interest rate is constrained at its zero lower bound. Our tractable framework allows us to provide a partial analytic characterization of equilibrium and to numerically compute all equilibria when the zero interest state is persistent. There are no approximations needed when computing equilibria and our numerical solutions are accurate up to the precision of the

[1] Eggertsson (2011) requires a 5.47% annualized shock to the preference discount factor in order to account for the large output and inflation declines that occurred in the Great Depression. Coenen, Orphanides, and Wieland (2004) estimate a NK model to U.S. data from 1980-1999 and find that only very large shocks produce a binding zero nominal interest rate.

computer. A comparison with the loglinearized equilibrium identifies a severe breakdown of the loglinearized approximate solution. This breakdown occurs when using parameterizations of the model that reproduce the U.S. Great Depression and the U.S. Great Recession. Conditions for existence and uniqueness of equilibrium based on the loglinearized equilibrium conditions are incorrect and offer little or no guidance for existence and uniqueness of equilibrium in the true economy. The characterization of equilibrium is also incorrect. These three *unpleasant properties* of the loglinearized solution have the implication that relying on it to make inferences about the properties of fiscal policy in a liquidity trap can be highly misleading. Empirically relevant parameterization/shock combinations that yield the paradox of toil in the log-linearized economy produce orthodox responses of hours worked in the true economy. The same parameterization/shock combinations that yield large government purchases multipliers in excess of two in the loglinearized economy, produce government purchase multipliers as low as 1.09 in the nonlinear economy. Indeed, we find that the most plausible parameterizations of the nonlinear model have the property that there is no paradox of toil and that the government purchase multiplier is close to one.

We make these points using a stochastic NK model that is similar to specifications considered in Eggertsson (2011) and Woodford (2011). The Taylor rule respects the zero lower bound of the nominal interest rate, and a preference discount factor shock that follows a two state Markov chain produces a state where the interest rate is zero. We assume Rotemberg (1996) price adjustment costs, instead of Calvo price setting. When loglinearized, this assumption is innocuous - the equilibrium conditions for our model are identical to those in Eggertsson (2011) and Woodford (2011), with a suitable choice of the price adjustment cost parameter. Moreover, the nonlinear economy doesn't have any endogenous state variables, and the equilibrium conditions for hours and inflation can be reduced to two nonlinear equations in these two variables when the zero bound is binding.[2] These two nonlinear equations are easy to solve and are the nonlinear analogues of what Eggertsson (2011) and Eggertsson and Krugman (2010) refer to as "aggregate demand" (AD) and "aggregate supply" (AS) schedules. This makes it possible for us to identify and relate the sources of the approximation errors associated with using loglinearizations to the shapes and slopes of these curves, and to also provide graphical intuition for the qualitative

[2]Under Calvo price setting, in the nonlinear economy a particular moment of the price distribution is an endogenous state variable and it is no longer possible to compute an exact solution to the equilibrium.

differences between the loglinear and nonlinear economies.

Our analysis proceeds in the following way. We first provide a complete characterization of the set of time invariant Markov zero bound equilibria in the loglinearized economy. Then we go on to characterize equilibrium of the nonlinear economy. Finally, we compare the two economies and document the nature and source of the breakdowns associated with using loglinearized equilibrium conditions. An important distinction between the nonlinear and loglinearized economy relates to the resource cost of price adjustment. This cost changes endogenously as inflation changes in the nonlinear model and modeling this cost has significant consequences for the model's properties in the zero bound state. In the nonlinear model a labor tax cut can increase hours worked and decrease inflation when the interest rate is zero. No equilibrium of the loglinearized model has this property. We show that this and other differences in the properties of the two models is precisely due to the fact that the resource cost of price adjustment is absent from the resource constraint of the loglinearized model.[3]

Our research is closest to research by Christiano and Eichenbaum (2012). They consider a similar setup to ours and reconfirm the evidence of multiple equilibria that we document here. They show that imposing a particular form of e-learnability rules out one of the two equilibria that occur in their model and find that the qualitative properties of the remaining equilibrium are close to the loglinearized solution under their baseline parameterization. The problems with the loglinear solution that we document here are not only related to issues of uniqueness. We produce a range of empirically relevant parameterizations of the nonlinear economy in which equilibrium is unique and yet the local dynamics are different from those that would arise in any zero bound equilibrium of the loglinear model. Moreover, we illustrate that these differences occur when the deviation of output from its steady state value is large and also when it is less small and less than 10%.

Our research is also related to a recent paper by Fernandez-Villaverde, Gordon, Guerron-Quintana, and Rubio-Ramirez (2012) that uses a projection method to solve for nonlinear policy functions and study the distribution of endogenous variables. They consider a model with Calvo price setting and a rich shock structure and find that the government purchase multiplier is above 1.5 but less than 2. An important distinction between their model and

[3]This distinction between the loglinearized and nonlinear resource constraint is not specific to our model of adjustment costs but also arises under Calvo price adjustment (see e.g. Braun and Waki (2010)).

ours is that we can compute an exact equilibrium and they cannot. There is no recourse but to use some form of approximation when solving their model. A second distinction is that we can compute all zero bound equilibria. This turns out to be important because it is not unusual to encounter multiple equilibria. A final distinction is that our tractable setup makes it possible to identify the specific source of the breakdown of the loglinearized solution and to provide intuition for why recognizing the resource costs of price adjustment matters.

Finally, our research is related to previous work by Braun and Waki (2010). They investigate the size of the government purchase multiplier at the zero bound in a model with capital accumulation. They assume perfect foresight and compute nonlinear equilibria under Rotemberg (1996) and Calvo (1983) price settings. Under both settings, the true multipliers are substantially lower than that in the loglinearized model. Although the true multipliers are often well above one, about one half of the response in excess of one results from a fall in the resource costs of price adjustment/dispersion and not from an increase in factor inputs.

The remainder of the paper proceeds as follows. Section 2 describes the model and the equilibrium concept. Section 3 provides a characterization of equilibrium in the loglinearized economy. Section 4 compares and contrasts the properties of the loglinearized economy with those of the true nonlinear economy. Section 5 contains our concluding remarks.

2 The model and equilibrium

We consider a NK model with a fixed capital stock. Price stickiness is introduced by Rotemberg (1996) quadratic price adjustment costs. We have two reasons for this choice. First, this choice is innocuous when the model is loglinearized around the zero inflation steady state. A suitable choice of parameters delivers the same loglinearized equilibrium conditions as the models with Calvo price setting considered by Eggertsson and Woodford (2003) and Eggertsson (2011). Second, the nonlinear equilibrium is easy to analyze. The equilibrium values of hours and inflation solve a system of two nonlinear equations when the shock follows a two state Markov chain. This facilitates characterization of equilibrium in the nonlinear model and comparisons with the loglinear equilibrium.

2.1 The model

Households: The representative household chooses consumption c_t, hours h_t and bond holdings b_t to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \Big(\prod_{j=0}^{t} d_j \Big) \left\{ \frac{c_t^{1-\sigma}}{1-\sigma} - \frac{h_t^{1+\nu}}{1+\nu} \right\} \tag{1}$$

subject to

$$b_t + c_t = \frac{b_{t-1}(1+R_{t-1})}{1+\pi_t} + w_t h_t (1-\tau_{w,t}) + T_t. \tag{2}$$

The variable d_t is a shock to the preference discount factor, and the one-step discount factor from period t to $t+1$ is βd_{t+1}. At the beginning of period t, the value of d_{t+1} is revealed. We assume $\beta < 1$, but βd_{t+1} is not necessarily less than one. The variable T_t includes transfers from the government and profit distributions from the intermediate producers.

The optimality conditions for the household's problem imply that consumption and labor supply choices satisfy

$$c_t^{\sigma} h_t^{\nu} = w_t (1-\tau_{w,t}), \tag{3}$$

and

$$1 = \beta d_{t+1} E_t \left\{ \frac{(1+R_t)}{1+\pi_{t+1}} \left(\frac{c_t}{c_{t+1}} \right)^{\sigma} \right\}. \tag{4}$$

Final good producers: Perfectly competitive final good firms use a continuum of intermediate goods $i \in [0,1]$ to produce a single final good that can be used for consumption and investment. The final good is produced using the following production technology

$$y_t = \left(\int_0^1 y_t(i)^{\frac{\theta-1}{\theta}} di \right)^{\frac{\theta}{\theta-1}}. \tag{5}$$

The profit maximizing input demands for final goods firms are

$$y_t(i) = \left(\frac{p_t(i)}{P_t} \right)^{-\theta} y_t, \tag{6}$$

where $p_t(i)$ denotes the price of the good produced by firm i. The price of the final good P_t satisfies

$$P_t = \left(\int_0^1 p_t(i)^{1-\theta} di\right)^{1/(1-\theta)}. \tag{7}$$

Intermediate goods producers: Intermediate goods producer i uses labor to produce output, using the technology: $y_t(i) = h_t(i)$. This production function implies that for all firms their real marginal cost is equal to the real wage w_t. Producer i sets prices to maximize

$$E_0 \sum_{t=0}^{\infty} \lambda_{c,t} \left[(1+\tau_s)p_t(i)y_t(i) - P_t w_t y_t(i) - \frac{\gamma}{2}\Big(\frac{p_t(i)}{p_{t-1}(i)} - 1\Big)^2 P_t y_t\right] / P_t \tag{8}$$

subject to the demand function (6). Here $\lambda_{c,t}$ is the stochastic discount factor and is equal to $\beta^t(\prod_{j=0}^t d_j)c_t^{-\sigma}$ in equilibrium. We assume that the intermediate good producers receive a sales subsidy τ_s. The last term in the brackets is a quadratic cost of price adjustment. This term is assumed to be proportional to the aggregate production y_t, so that the share of price adjustment costs in the aggregate production depends only on the inflation rate. The first order condition for this problem in a symmetric equilibrium is:

$$0 = \theta w_t + (1+\tau_s)(1-\theta) - \gamma\pi_t(1+\pi_t) + \beta d_{t+1} E_t \left\{ \left(\frac{c_t}{c_{t+1}}\right)^{\sigma} \frac{y_{t+1}}{y_t} \gamma\pi_{t+1}(1+\pi_{t+1}) \right\} \tag{9}$$

where $\pi_t = P_t/P_{t-1} - 1$.

Monetary policy: Monetary policy follows a Taylor rule that respects the zero lower bound on the nominal interest rate

$$R_t = \max(0, r_t^e + \phi_\pi \pi_t + \phi_y \hat{y}_t) \tag{10}$$

where $r_t^e \equiv 1/(\beta d_{t+1}) - 1$ and $\hat{y}_t$ is the log deviation of output from its steady state value.[4]

[4]The assumption that monetary policy responds directly to variations in d_t is made to facilitate comparison with other papers in the literature.

The aggregate resource constraint is given by

$$c_t = (1 - \kappa_t - \eta_t)y_t \tag{11}$$

where $\kappa_t \equiv \frac{\gamma}{2}(\pi_t)^2$ is the resource cost of price adjustment and government purchases $g_t = \eta_t y_t$.[5] It follows that gross domestic product in our economy, gdp_t, is given by:

$$gdp_t \equiv (1 - \kappa_t)y_t = c_t + g_t. \tag{12}$$

This definition of GDP assumes that the resource costs of price adjustment are intermediate inputs and are consequently subtracted from gross output when calculating GDP.

We will show below that loglinearizing equation (12) about a steadystate with a stable price level can introduce some large biases when analyzing the properties of this economy in a liquidity trap. To see why this is the case observe first that GDP falls when the term κ_t increases, even if the aggregate labor input h_t is unchanged. Both the sign and magnitude of movements GDP can differ from those of hours. When this equation is loglinearized about a steady-state with a stable price level, the term κ disappears, and GDP and hours are always equal. In the neighborhood of the steady state, κ is so small that it can safely be ignored. This is no longer the case when one analyzes this nonlinear economy in a liquidity trap. We wish to emphasize that the presence of the term κ is not specific to our model of price adjustment. An analogous term to κ also appears in the resource contraint under Calvo price setting. In that setting the term governs the resource costs of price dispersion (See e.g. the discussion in Yun (2005)).

2.2 Stochastic equilibrium with zero interest rates

We consider an equilibrium of the form proposed by Eggertsson and Woodford (2003). The preference shock d_{t+1} is assumed to follow a two-state, time-homogeneous Markov chain with state space $(d^L, 1)$, initial condition $d_1 = d^L$, and transition probabilities $P(d_{t+1} = d^L | d_t = d^L) = p < 1$ and $P(d_{t+1} = 1 | d_t = 1) = 1$. Fiscal policies (τ_w, η) are also assumed to be Markov with respect to d, i.e. $(\tau_{w,t}, \eta_t) = (\tau_w^L, \eta^L)$ when $d_{t+1} = d^L$, and $(\tau_{w,t}, \eta_t) = (\tau_w, \eta)$

[5] We introduce new notation for κ_t because it allows us to isolate the role of omitting the resource costs of price adjustment from the resource constraint.

when $d_{t+1} = 1$. We also assume $p\beta d^L < 1$ which is necessary for the existence of a Markov equilibrium.

Following Eggertsson and Woodford (2003) we focus exclusively on a Markov equilibrium with state d.[6] Such an equilibrium is characterized by two distinct values for prices and quantities: one value obtains when $d_{t+1} = d^L$ and the other obtains when $d_{t+1} = 1$. We will use the superscript L to denote the former value and no subscript to indicate the latter value. We assume further that the economy is in a steady state with zero inflation when $d_{t+1} = 1$, i.e.

$$h = \Big(\frac{(1+\tau_s)(\theta-1)(1-\tau_w)}{\theta(1-\eta)^\sigma}\Big)^{1/(\sigma+\nu)}, \quad \pi = 0.$$

We say that the economy is in the L state in period t if $d_{t+1} = d^L$. At this point we assume that the preference shock d^L is such that there exists an equilibrium in which the zero lower bound on the nominal interest rate binds when $d_{t+1} = d^L$. We subsequently refer to this type of equilibrium as a *zero bound equilibrium*.

2.3 Equilibrium hours and inflation

An attractive feature of our framework is that the equilibrium conditions for hours and inflation can be summarized by two equations in these two variables in the L state. These equations are nonlinear versions of what Eggertsson (2011) and Eggertsson and Krugman (2010) refer to as Aggregate Supply (AS) and Aggregate Demand (AD) schedules. In what follows we adopt the same shorthand when referring to these two equations.[7]

The AS schedule is an equilibrium condition that summarizes the intermediate good firm's price setting decision, the household's intratemporal first order condition, and the aggregate resource constraint. To obtain the AS schedule, we first rewrite the real wage w_t using the labor supply decision of the household, (3), and the resource constraint (11)

$$w_t = \frac{c_t^\sigma h_t^\nu}{(1-\tau_{w,t})} = \frac{(1-\kappa_t-\eta_t)^\sigma h_t^{\sigma+\nu}}{(1-\tau_{w,t})} \tag{13}$$

[6] We do not consider zero bound equilibria that depend on sunspot variables or time.

[7] One reason for the use of this terminology is that when interest rates are positive, under some weak regularity conditions described in Eggertsson (2011), the slope of AS equation is positive and the slope of the AD equation is negative.

Next we use expressions (11) and (13) to substitute the real wage and consumption out of the firm's optimal price setting restriction (9) and obtain the AS schedule:

$$
\begin{aligned}
0 =& \frac{\theta(1-\kappa_t-\eta_t)^\sigma}{(1-\tau_{w,t})} h_t^{\nu+\sigma} + (1+\tau_s)(1-\theta) - \gamma\pi_t(1+\pi_t) \\
&+ \beta d_{t+1} E_t \left\{ \left(\frac{1-\kappa_t-\eta_t}{1-\kappa_{t+1}-\eta_{t+1}} \right)^\sigma \left(\frac{h_t}{h_{t+1}} \right)^{\sigma-1} \gamma\pi_{t+1}(1+\pi_{t+1}) \right\}
\end{aligned} \tag{14}
$$

The AD schedule summarizes the household's Euler equation and the resource constraint. It is obtained by substituting consumption out of the household's intertemporal Euler equation (4) using the resource constraint (11). The resulting AD schedule is

$$
1 = \beta d_{t+1} E_t \left\{ \frac{1+R_t}{1+\pi_{t+1}} \left(\frac{1-\kappa_t-\eta_t}{1-\kappa_{t+1}-\eta_{t+1}} \right)^\sigma \left(\frac{h_t}{h_{t+1}} \right)^\sigma \right\}. \tag{15}
$$

If we impose the Markov property, then equilibrium hours and inflation in state L satisfy:

$$
0 = \theta \frac{(1-\kappa^L-\eta^L)^\sigma (h^L)^{\sigma+\nu}}{(1-\tau_w^L)} + (1+\tau_s)(1-\theta) - (1-p\beta d^L)\gamma\pi^L(1+\pi^L) \tag{16}
$$

$$
1 = p\left(\frac{\beta d^L}{1+\pi^L} \right) + (1-p)\beta d^L \left(\frac{(1-\kappa^L-\eta^L)^\sigma (h^L)^\sigma}{(1-\eta)^\sigma h^\sigma} \right) \tag{17}
$$

where $\kappa^L = \frac{\gamma}{2}(\pi^L)^2$.[8] Note that $R^L = 0$ has been imposed.

We find it convenient to express the aggregate demand and supply schedules in terms of labor input rather than GDP. This choice allows us to highlight how and why the response of labor differs according to the solution method.

[8] Other equilibrium objects are recovered as $y^L = h^L$, $gdp^L = (1-\kappa^L)y^L$, $c^L = (1-\kappa^L-\eta^L)y^L$, etc. Strictly speaking, not all pairs (h^L, π^L) that solve this system are equilibria: $(1-\kappa^L-\eta^L) \geq 0$ and $\hat{r}_L^e + \phi_\pi \pi^L + \phi_y \hat{y}^L \leq 0$ must also be satisfied.

3 A characterization of zero bound equilibria in the loglinearized economy

We start by providing a characterization of the types of time-invariant zero bound equilibria that arise under the loglinearized versions of the two previous equations. This analysis facilitates comparison with the existing literature and is also needed for comparison with the exact nonlinear economy.

Log-linearization of (16) and (17) about the zero inflation steady-state yields:

$$\pi^L = \frac{(1+\tau_s)(\theta-1)(\sigma+\nu)}{(1-p\beta)\gamma}\hat{h}^L - \frac{(1+\tau_s)(\theta-1)\sigma}{(1-p\beta)\gamma}\frac{\hat{\eta}^L}{1-\eta} + \frac{(1+\tau_s)(\theta-1)}{(1-p\beta)\gamma}\frac{\hat{\tau}_w^L}{1-\tau_w}, \quad (18)$$

$$\pi^L = \frac{1-p}{p}\sigma\hat{h}^L + \frac{1}{p}[-\hat{r}_L^e - (1-p)\sigma\frac{\hat{\eta}^L}{1-\eta}] \quad (19)$$

where $\hat{r}_L^e = 1/\beta - 1 - \hat{d}^L$, $\hat{\eta}^L = \eta^L - \eta$ and $\hat{\tau}_w^L = \tau_w^L - \tau_w$.[9] One consequence of loglinearizing around the zero inflation steady state is that the resource cost of price adjustment disappears from the resource constraint. This property of the loglinearization plays a central role in our analysis. It is also worth noting that the equations (18)-(19) can be derived from a variety of structural models including those with the Calvo price setting. It follows that our ensuing analysis of the loglinearized equilibrium applies to these models too.

Throughout the paper we put hours on the horizontal axis and inflation rate on the vertical axis whenever we draw the AD and AS schedule, and thus refer to the coefficients of $\hat{h}^L$ in (18) and (19) as $slope(AS)$ and $slope(AD)$, respectively. Since both the loglinearized AD and AS schedules are clearly upward sloping in state L, there are only two patterns for the way they intersect with each other. We now characterize these two types of zero bound equilibria. To make the exposition more transparent, suppose for now that $\hat{\eta}^L = \hat{\tau}_w^L = 0$. The first type of equilibrium occurs when the AD curve is steeper than the AS curve, and the shock is sufficiently large:

Proposition 1 Zero bound equilibrium with AD and AS upward sloping and AD steeper than AS (Type 1 equilibrium). *Suppose* $(\phi_\pi, \phi_y) \geq 0$ *and*

[9] Although we use the notation $\hat{r}_t^e$, it is not the deviation from the steady state but is the linearized version of $r_L^e = 1/(\beta d^L) - 1$.

1a) $\beta(1+\hat{d}^L) > 1$, *i.e.* $\hat{r}^e_L < 0$,

1b) $\frac{(1+\tau_s)(\theta-1)(\sigma+\nu)}{(1-p\beta)\gamma} < \sigma\frac{1-p}{p}$, *i.e. slope*$(AD) >$ *slope*(AS).

Then there is a unique zero bound equilibrium with $(\hat{h}^L, \pi^L) < 0$, *AD and AS upward sloping, and AD steeper than AS in the loglinearized economy.* [10]

We subsequently refer to equilibria in which the AD and AS schedules have the configuration described in Proposition 1 as "Type 1 equilibria" and we refer to configurations of parameters that satisfy 1a) and 1b) as "Type 1 configurations" of parameters. This distinction is very important because in the nonlinear model a Type 1 configuration of parameters doesn't necessarily deliver a Type 1 equilibrium.

Recent literature on the zero bound such as Eggertsson (2011), Woodford (2011), Eggertsson and Krugman (2010), and Christiano, Eichenbaum, and Rebelo (2011) has focused exclusively on this equilibrium. However, there is a second type of equilibrium that occurs when the shock is sufficiently small. This equilibrium, which we refer to as a "Type 2 equilibrium", has the property that the AS curve is steeper than the AD curve.

Proposition 2 Zero bound equilibrium with AD and AS upward sloping and AS steeper than AD (Type 2 equilibrium). *Suppose* $(\phi_\pi, \phi_y) \geq (p, 0)$ *and*

2a) $\beta(1+\hat{d}^L) < 1$, *i.e.* $\hat{r}^e_L > 0$,

2b) $\frac{(1+\tau_s)(\theta-1)(\sigma+\nu)}{(1-p\beta)\gamma} > \sigma\frac{1-p}{p}$, *i.e. slope*$(AD) <$ *slope*(AS).

Then there is a unique zero bound equilibrium with $(\hat{h}^L, \pi^L) < 0$, *AD and AS upward sloping, and AS steeper than AD in the loglinearized economy.*

We subsequently refer to configurations of parameters that satisfy 2a) and 2b) as "Type 2 configurations" of parameters.

The loglinearized AD and AS schedules also have intersections in which the parameters satisfy 1a) and 2b), or 2a) and 1b). For these configurations of parameters, the next proposition shows that the Taylor rule (10) implies that the nominal interest rate is positive, thereby contradicting our maintained assumption that it is zero.

[10] The proofs to this and all other propositions can be found in Appendix A.

Proposition 3 Non-existence of zero bound equilibrium *Suppose* 1a) *holds but* 1b) *is not satisfied. Then there is no equilibrium in the loglinearized economy with a binding zero bound if* $(\phi_\pi, \phi_y) \geq (p, 0)$. *Suppose instead 2a) holds but 2b) does not hold, then there is no equilibrium in the loglinearized economy with a binding zero bound if* $(\phi_\pi, \phi_y) \geq (0, 0)$.

We now ask whether a positive interest rate equilibrium exists under configurations in Proposition 1 and 2. The next proposition shows that there is none under a Type 1 configuration of parameters (hence the zero bound equilibrium is the unique time-invariant Markov equilibrium), whereas there is one under a Type 2 configuration (hence multiple time-invariant Markov equilibria exist).

Proposition 4 Existence of a positive interest rate time-invariant Markov equilibrium *Suppose conditions* 1a) *and* 1b) *of Proposition 1 are satisfied, and that* $(\phi_\pi, \phi_y) \geq (p, 0)$. *Then there is no time-invariant Markov equilibrium with a positive interest rate. Suppose instead that conditions* 2a) *and* 2b) *of Proposition 2 are satisfied and that* $(\phi_\pi, \phi_y) \geq (p, 0)$. *Then there is a time-invariant Markov equilibrium with a positive interest rate and* $(\hat{h}^L, \pi^L, R^L) = (0, 0, \hat{r}^e_L)$.

This result and the configuration of parameters under which multiple time-invariant Markov equilibria arise is closely related to the sunspot zero bound equilibria considered by Mertens and Ravn (2010). In our Type 2 equilibrium, the d^L shock plays two roles: it affects the preference discount factor and also is a coordination device for agents signaling that the economy is in the zero bound equilibrium and not the equilibrium with a positive nominal rate. Mertens and Ravn (2010) consider a nonlinear model and focus on the case of pure sunspot equilibria ($d^L = 1$). Proposition 4 implies that multiple equilibria and thus sunspot equilibria can also arise in the loglinearized economy. The next section shows that the Type 2 equilibrium in our loglinearized model produces the same policy responses as the nonlinear sunspot equilibrium considered by Mertens and Ravn (2010). In this sense, what's crucial for their result is not the nonlinearity or the type of the shock, but the configuration of parameters of the model.

3.1 Fiscal policy in the loglinearized model

Previous research by Christiano, Eichenbaum, and Rebelo (2011), Eggertsson (2011) and others has found that loglinearized NK models have the property that the government purchases multiplier exceeds one and hours fall if labour taxes are cut when the zero lower bound on the nominal rate is binding. The following proposition establishes that Type 1 equilibria exhibit these properties but that Type 2 equilibria do not.

Proposition 5 The effects of fiscal policy in the loglinear model

a) In a Type 1 equilibrium, a labor tax increase increases hours and inflation. The government purchase output multiplier is above 1.

b) In a Type 2 equilibrium a labor tax increase lowers hours and inflation.

c) Suppose parameters are such that an increase in η results in an increase in government purchases then in a Type 2 equilibrium the government purchase output multiplier is less than 1.[11]

Both types of equilibria have the property that inflation, output, hours and consumption fall in a zero bound equilibrium, yet they have exactly the opposite implications for fiscal policy. The unconventional responses to fiscal policy shocks in a Type 1 equilibrium have been documented elsewhere. It is still worthwhile to provide some intuition for these results. This helps to understand why the responses to the same shocks are so different in a Type 2 equilibrium.

Consider first how an increase in the labor tax translates into higher inflation for a given level of aggregate consumption and hours (h, or equivalently y). A higher labor income tax discourages households from working. At a given consumption level, an increase in the labor tax shifts the labor supply curve upward and to the left. This raises the pretax nominal wage and thus intermediate goods producers' marginal costs. In response to this, intermediate goods producers increase their relative prices. Because all firms are symmetric and aggregate output y is taken as given, their effort to increase their relative price simply ends up raising the nominal price level. Because the marginal price adjustment cost is

[11] In the proof we provide the exact condition in terms of parameters under which an increase in η results in an increase in government purchases.

increasing this adjustment in prices is only partial. Taking aggregate quantities as given, it follows that a labor income tax increase translates into higher inflation. Expected inflation also goes up, because the policy changes are persistent. Given that the nominal rate is constrained by the zero lower bound, higher expected inflation lowers the real interest rate. A lower real interest rate induces people to consumption more today, and reduces future consumption (savings). This pushes up the wage further for two reasons. First, the labor supply curve shifts left further due to increased consumption. Second, to satisfy increased demand for goods, intermediate firm's labor demand curve shifts rightward. This results into further inflation, a lower real rate, higher consumption, and so on, creating a virtuous cycle.

Proposition 5a) is the end result of this virtuous cycle. Positive feedback of inflation and consumption/hours occurs, resulting in higher inflation, consumption, and hours. What is crucial though is that this virtuous cycle damps. The adjustment of inflation in the second round is smaller than the first round and the response of consumption is also smaller in the second round.

Why doesn't this same dynamic occur in a Type 2 equilibrium? The key is that under 2b) of Proposition 2 the above virtuous cycle doesn't damp; the positive feedback mechanism is amplified when iterated, and diverges. Instead, in a Type 2 equilibrium consumption must *decline* sufficiently in response to an increase in the labor tax to create deflationary price pressure. This produces a dynamic with higher real rates, and lower consumption and lower inflation. Only a this type of vicious cycle damps. Finally observe that the vicious deflationary dynamic cannot occur in a Type 1 equilibrium, because it doesn't damp - this type of dynamic diverges when 1*b*) of Proposition 1 is satisfied.

To get a better handle on the nature of this damping effect and its relation to assumptions 1*b*) and 2*b*), it may be helpful to consider the above hypothetical adjustment process to (expected) inflation using the AD and AS schedules. An increase in τ_w shifts the AS schedule left and leaves the AD schedule unchanged (see the left panel of Figure 2(a)).[12] In a Type 1 equilibrium at the old equilibrium $\{\hat{c}_0, \pi_0^e\}$ the (expected) inflation rate is now too low. Suppose firms take $\hat{c}_0$ as given and adjust their prices as described above. Then inflation rises to the point $\{\hat{c}_0, \pi_1^e\}$ on the new AS schedule. At this new point the real

[12]This figure expresses the AD and AS schedules in terms of consumption instead of hours. This can be derived from (18)-(19) using the loglinearized aggregate resource to substitute consumption for hours.

interest rate is lower, and households find consumption $\hat{c}_0$ is too low. So they adjust their consumption to satisfy the Euler equation under a lower real interest rate, and we move back to the point on the AD schedule given by $\{\hat{c}_1, \pi_1^e\}$. The size of this adjustment in consumption is given by $1/slope(AD)$. Notice next that a unit increase in $\hat{c}$ in the second step implies a slope(AS) unit increase in expected inflation in the third step. This process converges if the increase in inflation at step three is lower than in step one, or in other words when $slope(AS)/slope(AD) < 1$, which is condition $1b$). Under this assumption in each successive step the responses of (expected) inflation and consumption are damped relative to the previous step and the dynamic process converges.

Now let's consider a Type 2 equilibrium. In this equilibrium a *higher* inflation in response to an increase in the labor tax produces explosive dynamics that diverge. This can readily be seen by positing an increase in expected inflation in the right panel of Figure 2(a). Instead, consumption and the (expected) inflation must fall. It then follows using a similar line of reasoning to above that the increase in the tax rate generates a deflationary cycle with lower consumption, lower marginal cost, a higher real interest rate, and lower inflation. This cycle damps when $slope(AS)/slope(AD) > 1$ which is equivalent to condition $2b$).

The response of hours/output in a Type 2 equilibrium depends on the type of fiscal shock. Hours unambiguously fall when the labor tax is increased. An increase in η, in contrast, may either produce an increase or reduction in hours depending on the parameterization of the model, while consumption drops regardless of the parameterization. When the AS shifts down more than the AD,[13] consumption drops but not so much as to offset the increase in government purchases; Hours increase and the government purchases output multiplier is positive but less than one since consumption has fallen. If instead the AD shifts down more than the AS, consumption declines by more than the increase in government purchases, and hours fall. The government purchase multiplier is negative if the decline in hours is moderate. However, it may actually be positive if the decline in hours is sufficiently large.[14]

[13] The exact condition for this to be the case is provided in the proof of Proposition 5.

[14] η is related to g by $\hat{g} = \hat{\eta}/\eta + \hat{h}$. If $\hat{h}$ falls by enough g also falls and the multiplier is positive.

We wish to emphasize that the convergent dynamics we have described are premised on a particular assumption about the initial guess of the new inflation rate. In a Type 1 equilibrium any guess that the new inflation rate is lower than the previous inflation rate diverges. In a Type 2 equilibrium this same guess produces convergent dynamics. We are assuming that agents know the entire structure of the economy. These results suggest though that the initial guess of the inflation rate can also be important in situations where agents have less information about the structure of the economy.

3.2 The effects of more price flexibility and longer expected duration on output and inflation

Previous research by Christiano, Eichenbaum, and Rebelo (2011) and Werning (2011) finds that increasing the degree of price flexibility has a counter-intuitive property when the nominal interest rate is zero: It increases the magnitude of the declines in output and inflation to a shock in d of a given size. We now show that Type 1 equilibria have this property but that Type 2 equilibria have the opposite property.

Proposition 6 Effects of more price flexibility on output and inflation responses *An increase in price flexibility (lower γ) magnifies the declines in output and inflation in a Type 1 equilibrium, but reduces the declines in output and inflation in a Type 2 equilibrium.*

From this Proposition we see that the result by Christiano, Eichenbaum, and Rebelo (2011) and Werning (2011) relies on the equilibrium being of Type 1. If instead the equilibrium is of Type 2, the response of hours and inflation fall as prices become more flexible.

Proposition 7 considers what happens as we increase p. A higher p implies a longer expected duration of the low interest rate state.

Proposition 7 Effects of a larger p on the magnitude of the output and inflation responses *An increase in p magnifies the declines in output and inflation in a Type 1 equilibrium, but reduces the declines in output and inflation in a Type 2 equilibrium.*

3.3 Discussion

To summarize, the loglinearized economy has two types of zero bound equilibria. These equilibria arise under distinct configurations of parameters and shocks. The local properties of the two equilibria are very different. In a Type 1 equilibrium an increase in price flexibility increases volatility as in De Long and Summers (1986). In a Type 2 equilibrium price volatility falls as prices become more flexible. Fiscal policy also has very different properties in the two equilibria. An increase in government purchases is highly stimulative in the Type 1 equilibrium. In a Type 2 equilibrium, however, the response of output to an increase in government purchases is muted or negative. Finally, a Type 1 equilibrium exhibits the paradox of toil (hours fall when the labor tax is increased), but there is no paradox of toil in a Type 2 equilibrium. Much of the recent literature has focused on Type 1 equilibria. Two notable exceptions are Mertens and Ravn (2010) and Bullard (2010).

4 Some unpleasant properties of loglinearized equilibria

We have seen that working with the loglinearized equilibrium conditions is a very convenient device for analyzing on the properties of the model. However, these results are only accurate in the neighborhood of the steady state with a stable price level. And there is a risk that the results based on the loglinearized equilibrium conditions do not obtain in empirically relevant situations. We now illustrate that such a breakdown does in fact occur when one considers parameterizations and shocks of a size that are needed to reproduce outcomes from the U.S. Great Depression and the U.S. Great Recession. We summarize the specific nature of this breakdown in the following statements.

I. *Existence of equilibrium* The conditions for existence of an equilibrium in the loglinearized economy do not apply to the true nonlinear economy. For instance, we will exhibit specifications of the nonlinear model which satisfy conditions 1a) and 2 b) and yet a zero bound equilibrium exists.

II. *Uniqueness of equilibrium* The conditions for uniqueness of equilibrium in the loglinearized economy are incorrect. When we use Type 1 configurations of the parameters in the nonlinear model we find multiple time-invariant Markov equilibria.

III. *Characterization of equilibrium* The characterization of equilibrium using the loglinearized equilibrium conditions is incorrect.

a) In the non-linear economy there are two classes of equilibrium that do not arise in the loglinearized economy.

b) Using the loglinearized economy conditions on parameters results in a miss-classification of the type of equilibrium. For instance, in cases were the loglinearized conditions indicate no equilibrium exists there are bonafide zero bound equilibria in the nonlinear model.

c) Using the loglinearized equilibrium conditions can severely exaggerate the response of endogenous variables to shocks of a given size.

4.1 Slopes of the AS and AD schedules in the nonlinear economy

The slopes of the AD and AS schedules are not constant in the nonlinear economy so we derive conditions for their *local* slopes at a given point (h^L, π^L). Before deriving these properties though we first need to dispense with two preliminary issues. The fact that equilibrium consumption and hours must be positive results in the following restrictions on the range of admissible π^L: $1-\kappa^L-\eta^L > 0$, $1-p\beta d^L/(1+\pi^L) > 0$, and $(1-p\beta d^L)\gamma\pi^L(1+\pi^L)+(\theta-1)(1+\tau_s) > 0$. The resource cost of price adjustment plays a central role in our results. To isolate this role it is helpful to consider the following *misspecified nonlinear economy.* In this economy κ is set to its steady-state value of zero and the AD and AS schedules are given by:

$$0 = \theta\frac{(1-\eta^L)^\sigma(h^L)^{\sigma+\nu}}{(1-\tau_w^L)} + (1+\tau_s)(1-\theta) - (1-p\beta d^L)\gamma\pi^L(1+\pi^L) \quad (20)$$

$$1 = p\left(\frac{\beta d^L}{1+\pi^L}\right) + (1-p)\beta d^L\left(\frac{(1-\eta^L)^\sigma(h^L)^\sigma}{(1-\eta)^\sigma h^\sigma}\right) \quad (21)$$

Let's start by considering the *local* slope properties of the AD schedule. In the loglinearized economy it is upward-sloping ($p > 0$) or vertical ($p = 0$). This is because in the loglinearized economy a fall in inflation in the L state reduces expected inflation, increases the real interest rate through the Fisher equation, depresses consumption through the Euler equation, and necessarily depresses hours through the resource constraint. In the nonlinear economy, however, the link between consumption and hours may be broken down: When the price adjustment costs decrease, hours may indeed increase. Proposition 8 establishes a condition under which the nonlinear AD schedule is downward sloping.

Proposition 8 *The AD schedule is downward sloping at (h^L, π^L) if and only if*

$$\frac{1}{\sigma}\frac{p\beta d^L/(1+\pi^L)^2}{1-p\beta d^L/(1+\pi^L)} - \frac{-(\kappa^L)'}{1-\kappa^L-\eta} < 0, \tag{22}$$

where $(\kappa^L)' = \gamma\pi^L$. It is upward-sloping (vertical) if and only if the left hand side is positive (zero).

Intuitively, the first term in (22) is the inflation elasticity of consumption (implied by the Euler equation) and the second term is the inflation elasticity of the consumption-hours ratio (implied by the resource constraint). The first term is always non-negative, and the second term is positive when $\pi^L < 0$. More importantly the second term drops out in the misspecified model and the loglinearized model, hence the presence of the price adjustment costs in the resource constraint is responsible for the possibility of a downward sloping AD schedule.

Corollary 1 *In the misspecified nonlinear economy the AD schedule is vertical if $p = 0$ and upward sloping if $p > 0$.*

The case where $p = 0$ and $\pi^L < 0$ provides an example of a situation where the characterization of equilibrium provided by the loglinearized model is incorrect. It is incorrect in two respects. The possibility of a downward sloping AD schedule does not arise in equilibrium in the loglinearized economy (*unpleasant property IIIa*). It follows immediately that the loglinearized classification of the type equilibrium is incorrect (*unpleasant property IIIb*).

Now consider the AS schedule. In the loglinearized economy it is upward sloping. The reasoning for this is as follows. Price setting producers charge a markup over their marginal cost of production which is given by the real wage. It follows that a higher inflation rate is associated with a higher real wage. Since the after-tax real wage is in turn equal to the marginal rate of substitution between consumption and hours it follows that the marginal rate of substitution (after the consumption is substituted out using the resource constraint) is unambiguously increasing in hours in the loglinearized economy. Thus higher inflation is associated with higher hours. In the true nonlinear economy, in contrast, the marginal rate of substitution increases when the inflation rate increases from a negative value, even if hours stay constant. Thus higher inflation is not necessarily associated with higher hours. The following proposition formalizes this point.

Proposition 9 *The AS schedule is downward sloping at (h^L, π^L) if and only if*

$$\frac{\gamma(1+2\pi^L)(1-p\beta d^L)}{\gamma\pi^L(1+\pi^L)(1-p\beta d^L)+(1+\tau_s)(\theta-1)} - \frac{-\sigma(\kappa^L)'}{1-\kappa^L-\eta^L} < 0. \tag{23}$$

It is upward-sloping (vertical) if and only if the left hand side is positive (zero).

The first term is the inflation elasticity of real wage (implied by the NK Phillips curve[15]) and the second term is the inflation elasticity of the marginal rate of substitution (the right hand side of (13)). The gap between these two terms has to be absorbed by changes in hours, through (13). The former is positive when $\pi^L > -\frac{1}{2}$, and when $\pi^L < 0$ the latter is positive. In the misspecified model and loglinearized model, the second term drops out and the AS schedule is unambiguously upward sloping. The fact that the AS schedule can be downward sloping in a zero bound equilibrium in the nonlinear economy but is always upward sloping in zero bound equilibria in the loglinearized economy creates a distinct way for *unpleasant properties IIa* and *IIIb* to occur.

Corollary 2 *The misspecified AS nonlinear schedule is upward sloping for $\pi^L > -\frac{1}{2}$.*

These propositions suggests that when the term $-(\kappa^L)'/(1-\kappa^L-\eta^L)$ is a large, positive number, then both AD and AS schedules are likely to be downward sloping. Since $(\kappa^L)' = \gamma\pi^L$, this happens for example when the price adjustment cost parameter γ is large and an equilibrium inflation rate π^L is a big negative number. In Section 4.2 we show that this happens when one attempts to reproduces the Great Depression with this model as in Eggertsson (2011).

[15]We are referring to the version: $\gamma\pi^L(1+\pi^L) = \theta w^L + (1+\tau_s)(1-\theta) + p\beta d^L\gamma\pi^L(1+\pi^L)$.

Taken together these results highlight the central role played by the resource costs of price adjustment. When one abstracts from these costs the AD and AS schedules are both upward sloping under the weak restrictions that $p > 0$ and that the inflation rate in the low state exceeds -50%. However, once the resource cost or price adjustment is recognized the slope of each schedule depends on the size of the shock as well as a variety of model parameters.

A final point that we wish to mention pertains to the shape of the AD and AS schedules. In the loglinearized economy the schedules were linear, and thus the intersection is unique if it exists, except for the non-generic case where the AS and the AD coincide. In the true model the AD and AS schedules are well-defined functions when viewed as mappings from π^L to h^L, but they may be correspondences when viewed as mappings from h^L to π^L. We report results below that indicate that it is not unusual to encounter situations where there are indeed multiple zero bound equilibria.

4.2 Numerical results

The results in the previous subsection point out some potentially important differences between the properties of the true economy and the loglinearized economy. We now demonstrate that these same differences arise in more empirically relevant settings with $p >> 0$.

4.2.1 Great Depression parameterization of the loglinearized model

We start by considering a parameterization of the model that facilitates comparison with other recent research on the zero bound. Using the parameterization reported in Table 1, our model has the identical loglinearized equilibrium conditions as the specifications in Eggertsson (2011) and Woodford (2011).[16] This parameterization is designed so that the loglinearized model reproduces two facts from the Great Depression: a 30% decline in output and a 10% decline in annualized inflation rate. We refer to these as the "Great Depression" facts.

[16] The parameter γ that governs the adjustment cost of prices is set according to the formula $\gamma = \frac{(1+\tau_s)(\theta-1)(1+\nu\theta)\alpha}{(1-\alpha)(1-\beta\alpha)}$, where α is the Calvo parameter. This formula is derived by comparing the coefficient on output in the AS schedule of Eggertsson (2011) with the same coefficient in the loglinear version of our model. For other parameters we use values in Eggertsson (2011).

Table 2 reports equilibrium GDP, hours and inflation in the L state, as well as the slopes of the AD and AS schedules, for the loglinearized, nonlinear misspecified, and the nonlinear true economies. In all three economies there is a unique, time-invariant Markov zero bound equilibrium. Column 1 reports results for the loglinearized economy. The parameterization of the model is a Type 1 configuration, hence the equilibrium is a Type 1 equilibrium. It reproduces the Great Depression facts, and the GDP and hours responses are identical. Column 2 reports the slopes of the AD and AS schedules in the nonlinear misspecified economy. As shown in Corollaries 1 and 2, the AD and AS schedules are upward sloping. The AD schedule is steeper than the AS schedule, giving rise to a Type 1 equilibrium.

The properties of the true nonlinear economy reported in column 3 of Table 2 illustrate that *unpleasant property IIIb* also occurs when p is much larger than zero. The assumptions of Proposition 1 are satisfied, yet the AD and the AS schedules are both downward sloping in the nonlinear economy! This pattern of slopes cannot be supported as an equilibrium in the loglinearized economy. The true model also produces much smaller responses of output and inflation that are not at all close to the magnitudes needed to account for the Great Depression facts (*unpleasant property IIIc*). It is quite clear that the loglinearized model provides a very poor approximation to the true model when using this parameterization. The fact that both of the loglinear and misspecified nonlinear model produce responses of output and inflation that are quantitatively similar and that both are very different from the true nonlinear model highlights the important role played by the resource costs of price adjustment.

Next we vary d^L holding the other parameters fixed. We have two objectives in doing this. First, as the size of the shock to d^L gets smaller the quality of the approximation provided by the loglinearized system should improve. Second, we are interested in understanding whether the nonlinear model can reproduce the Great Depression facts with larger shocks. Results from simulations that vary d^L are reported in Table 3. The column with $d^L = 1.0134$ corresponds to column 3 in Table 2.

When the loglinear approximation is working well the true model should also exhibit the properties described by Proposition 1. This does indeed occur for very small values of d^L that range from 1.0031 to 1.004. However, values of d^L in this range are too small to account for the large declines in output and inflation that occurred in the Great Depression.

They are also too small to account for the declines in output in inflation that occurred in the recent U.S. Great Recession. Christiano, Eichenbaum, and Rebelo (2011) present evidence that GDP fell by 7% and inflation by 1% and Denes, Eggertsson, and Gilbukh (2012) find that output fell by 10% and inflation by 2%. The Great Recession outcomes are more consistent with shocks to d^L that range between 1.007 and 1.0096. In this interval the AD schedule is downward sloping and the AS schedule is upward sloping.

Observe next that when d^L reaches 1.0172, a second zero bound equilibrium occurs. We refer to the equilibrium with a lower inflation rate as the "first equilibrium", and the other with a higher inflation rate as the "second equilibrium." As can be seen in the table, both equilibria appear to change continuously with d^L. In the first equilibrium, GDP and inflation monotonically decrease as we increase d^L and the AD and AS schedules are both downward sloping, with the AS steeper than the AD. In the second equilibrium, however, GDP monotonically decreases while inflation monotonically increases. Moreover, inflation is positive. Both the AD and AS schedules are upward sloping, but the AS is steeper across all simulations reported in this table. It is worth emphasizing that all specifications reported in this table are Type 1 configurations and yet *none* of these configurations have the property that AD and AS are both upward sloping with AD steeper than AS. It is also clear that *unpleasant property II* applies here. Conditions for uniqueness of equilibrium in the loglinearized equilibrium economy do not obtain if $d^L \geq 1.0172$ in the nonlinear economy.

When d^L is 3% per quarter the first equilibrium reproduces the 30% drop in GDP associated with the Great Depression. It is interesting that multiple equilibria are occurring in the precise region of the shock space needed to reproduce the GDP decline from the Great Depression. Moreover, neither of these equilibria are Type 1 equilibria. The inflation rate in the first equilibrium is around -5.2% or about half the size needed to account for the 10% decline that occurred in the Great Depression. Finally, hours exceed their steady state level by about 9%.

4.2.2 Great Depression calibrations of the nonlinear model

We now report a range of parameterizations of the nonlinear model that are *empirically relevant* in the sense that they reproduce both of the Great Depression facts. Our main objective is to show that the unpleasant properties of the loglinearized solution also arise in empirically relevant parameterizations of the nonlinear model. A second objective is to see whether Type 1 equilibria occur in empirically relevant parameterizations of the nonlinear model and if so when. We start from the specification reported in the final column of Table 2, which sets steady state profits to zero, and adjust γ, p, d^L, θ and τ_s in a way that allows the model to reproduce the Great Depression facts, i.e. $(1+\pi^L)^4 - 1 = -0.1$ and $\log(gdp^L/gdp) = -0.3$. The other structural parameters (β, σ, ν) are held fixed at their baseline values reported in Table 1. Our rationale for this strategy is as follows. On the one hand, the values of (β, σ, ν) are broadly consistent with values encountered in other NK analysis so they are kept fixed. On the other hand, the value of γ reported in Table 1 is very large. For instance, the implied resource cost share of output κ exceeds one when evaluated at $\pi = -10\%$ per annum in the loglinearized model. It is thus quite unlikely that the loglinearization solution would approximate the true solution in this type of situation. We thus would like to entertain smaller values of γ. Varying (p, d^L, θ) at the same time allows us to do this and also to match the targeted Great Depression facts exactly.[17] Altering θ, however, also alters steady state hours. We control for this effect by adjusting τ_s to keep steady state profits constant at zero. This adjustment ensures that steady state hours are the same for each parameterization of the model.

The upper panel of Table 4 reports results using the baseline value of $p = 0.903$. We experiment with a wide range of $\gamma' s$ that range from 160 to 1209.[18] The first two columns report γ and the corresponding value of the Calvo parameter α using the formula in footnote 16 and the third and fourth columns display the associated values of (d^L, θ) that reproduce the Great Depression facts. The next three columns show the resource costs of price adjustment κ^L as a share of GDP, the value of τ_s that is required to set profits to zero (this is also the size of the gross markup), the slopes of the nonlinear AD and AS schedules and the number of equilibria in the nonlinear model.

Results reported in the upper panel of Table 4 have several noteworthy properties. Perhaps the most important result pertains to the slope of the AD schedule. It is upward sloping when γ is low, but turns negative when γ exceeds 330. The AS schedule is also upward sloping for γ's less than 400. Three distinct configurations of the AD and AS

[17] Appendix B explains how this is accomplished.

[18] Ireland (2003) estimates γ to be 161.8 using US Post-1979 data.

schedules occur in the upper panel. One of them is the Type 1 equilibrium that also occurs in the loglinearized model. However, the other two configurations (AD schedule downward sloping with AS schedule upward sloping and AD schedule downward sloping with AS schedule downward sloping and steeper) are not equilibria in the loglinearized model. Another striking feature about the results is that all of the parameterizations except one ($\gamma = 160$) violate the conditions for existence of equilibrium in the loglinearized model. This is because these parameterizations satisfy conditions 1a) and 2b) in Propositions 1 and 2. From this we see that *unpleasant property I* also obtains in empirically relevant parameterizations of the model.

A somewhat surprising result is that a lower value of γ is associated with a higher α. The reason for this can be seen by noting that $\gamma = \theta(1+\nu\theta)\alpha/((1-\alpha)(1-\beta\alpha))$ and that a lower value of γ is also associated with a smaller value of θ. It turns out that the decline in θ needed to reproduce the Great Depression facts is of a sufficient magnitude that $\gamma/[(1+\tau_s)(\theta-1)(1+\nu\theta)]$ increases and thus α increases as well. Finally, observe that the resource cost of price adjustment is large in most cases. It exceeds 10% of GDP for all but the two lowest settings of γ.

Let's consider the various parameterizations in more detail starting with the bottom three rows of the upper panal of Table 4. These three rows illustrate that one way to reproduce the Great Depression Facts is with large values of θ that are close to its baseline value of 12.7721. Both of these specifications exhibit a configuration of the AD and AS schedule that does not arise in the loglinearized economy. The resource costs of price adjustment are very large ($\kappa^L > 0.3$). They are so large, in fact, that hours and GDP move in opposite directions just as we observed above using the baseline parameterization of the model. For values of γ in the range from 400 to 800, though, both hours and GDP fall.

Consider next the first three rows with $\gamma = \{160, 250, 300\}$. Interestingly, the non-linear model produces a Type 1 equilibrium (upward sloping AD and AS, with the AD steeper than the AS). From this we see that it is indeed possible to reconcile a Type 1 configuration of the AD and AS schedules with the Great Depression facts. The resource costs of price adjustment in these calibrations are also much lower. A troubling feature of these specifications though is that the value of θ is close to or even less than one. These parameterizations require values of the markup that are in excess of 200% to account for the Great Depression Facts.

Finally, note that for $330 \leq \gamma < 800$, the AD schedule is downward sloping and the AS schedule is upward sloping. These last two observations are similar to what we documented above using the baseline parameterization of the model. Type 1 equilibria are associated with smaller values of d^L and in the same neighborhood there is also a range of values of d^L for which the AD schedule is downward sloping and the AS schedule is upward sloping. The particular range of values of d^L that produces this configuration of AD and AS is somewhat larger here as compared to in Table 3.

The lower panel of Table 4 reports results that set $p = 0.8$. With a lower value of p all specifications in the lower panel of the Table are a Type 1 configuration. A lower value of p results in higher values of d^L and θ. However, all of the specifications now have a downward sloping AD schedule. Therefore it is not possible to obtain Type 1 equilibria that reproduce the Great Depression facts with this value of p. Finally, observe that there are situations where two equilibria occur. We do not report results for the second equilibrium because they have the property that the inflation rate is positive. However, we do wish to point out that when two equilibria occur, the second equilibrium is a Type 2 equilibrium.

4.2.3 Great Recession Calibrations of the Nonlinear Model

Now we recalibrate the nonlinear model to reproduce the Great Recession in the U.S. Following Christiano, Eichenbaum, and Rebelo (2011) we target a 7% decline in GDP and a 1% annualized decline in the inflation rate.

First we report results for $p = 0.903$. which appear in the upper panel of Table 5. Observe that the inflation decline is so small that the resource costs of price adjustment κ are negligible for a large range of γ's. The size of the θ's is much lower as compared to the Great Depression scenario. For instance, the baseline value of $\theta = 12.77$ arises when $\gamma = 8142$.

The AD schedule is upward sloping for a broad range of parameterizations of the model and only has a negative slope for specifications with $\gamma \geq 3000$. Moreover, the AS schedule is upward sloping for all specifications. This results in a Type 1 equilibrium in the true model for a large range of values of γ.[19] All of these Type 1 equilibrium parameterizations

[19] All of the parameterizations in this table satisfy the conditions of Proposition 1 and are thus a Type

require large markups that exceed 50%.

When we decrease p to 0.8 as before, the implied values of d^L and θ both increase. Results are reported in the lower panel of Table 5. The AD now becomes downward sloping when γ reaches 1200. However, the AS schedule is once again upward sloping throughout. Interestingly, the model now produces a conventional configuration of AD and AS for values of γ that range from 1200 to 3000. For values of γ that range from 800 to 1000, the equilibrium is a Type 1 equilibrium. Once again though, this type of equilibrium is associated with large values of the markup that range from 50% to over 80%.

4.2.4 Discussion

Our explorations of the parameter space produced four distinct configurations of the AD and AS schedules. Two of them also arise in the loglinearized economy (Type 1 and Type 2). However, two new configurations occur that do not occur in the loglinearized model. One is the conventional configuration with an upward sloping AD schedule and a downward sloping AS schedule. The other is a configuration where both schedules are downward sloping with the slope of the AS schedule steeper than the AD schedule. We now turn to analyze the properties of fiscal policy for each of these configurations of the AS and AD schedules.

4.3 Fiscal policy in the nonlinear model

We saw in the loglinearized economy that the relative slopes of the AD and AS schedules had important implications for the effects of fiscal policy on the economy in the zero bound state. This continues to be the case in the nonlinear economy. Now, however, there is a richer set of configurations of the AD and AS schedules that occur in equilibrium. We now turn to describe how the (local) slopes of these schedules affects the responses of the economy to small changes in the labor tax and government purchases.

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1 equilibrium for the loglinearized economy.

4.3.1 A labor tax cut

We first study a labor tax cut. We begin by deriving qualitative results that relate the effects of a labor tax cut to the slopes of the AD and AS schedules. We have encountered four distinct patterns of the AD and AS schedules in the nonlinear model. The following proposition summarizes how hours and inflation respond to a change in the labor tax, for each of these patterns.

Proposition 10 Response of hours and inflation to a change in the labor tax

a) *If both the AS and the AD schedule are upward sloping and the AD schedule is steeper, then a labor tax cut lowers hours and inflation. Consumption also falls if* $\pi^L < 0$.

b) *If both the AS and the AD schedule are upward sloping and the AS schedule is steeper, then a labor tax cut increases hours and inflation. Consumption also rises if* $\pi^L < 0$.

c) *If both the AS and AD schedule is downward sloping and the AS schedule is steeper or if the AS schedule is upward sloping and the AD schedule is downward sloping, then a labor tax cut increases hours and lowers inflation.*

Cases a) and b) are generalizations of Proposition 5 a) and b). They are Type 1 and Type 2 equilibria, and their responses are qualitatively the same as those in the loglinearized model. Case c) is unique to the nonlinear model. Moreover, it implies a conventional response of hours to labor income tax: Hours increase in response to a labor tax cut.

From Proposition 10 we see that in three out of the four distinct type of equilibria, there is no paradox of toil in the true model. Labor responds in the conventional way to a labor tax cut. Proposition 10 can also be applied to understand under what situations the paradox of toil is likely to arise in the true nonlinear economy. For instance, in Table 2 we see that there is no paradox of toil in the true model when we use the baseline parameterization of the model. There is also no paradox of toil in any of the First equilibrium specifications reported in Table 3, except for $d^L < 1.0040$.[20] In Table 4 the only specifications where the paradox of toil occurs is when $160 \leq \gamma \leq 300$ (upper panel). For the Great Recession

[20] As we noted above we do not discuss the second equilibrium because the inflation rate is positive.

calibrations reported in Table 5, in contrast, there is a larger range of parameterizations of the nonlinear model that produces the paradox of toil.

How large is the paradox of toil in the true economy when it occurs? How close are the loglinear and nonlinear labor tax multipliers more generally? The analysis in Eggertsson (2011) implies that a one percent increase in the labor tax increases output by about 1.0% during the Great Depression. Table 6 reports labor tax multipliers associated with a 1% increase in the labor tax for the true nonlinear and loglinear economies for the Great Depression calibrations.[21] The upper panel reports the results for the case of $p = 0.903$ and the lower panel for $p = 0.8$. In the upper panel all of the parameterizations, but one ($\gamma = 160$), fall in the region where there is no equilibrium for the loglinearized model. As a consequence there are no results reported for the loglinear model for these parameterizations. Consistent with Proposition 10 an increase in the labor tax lowers hours in all cases but three. In the three cases where hours increases the magnitude of the increase is less then 0.5%. In the one case where an equilibrium exists for the loglinear model, the labor tax multiplier exceeds 15% in the loglinearized model whereas it is less than 0.5% in the true economy.

In the lower panel where results are available for both the true and loglinear economies it is immediately clear that the loglinearized solution is giving highly misleading results. The sign of the responses of hours is flipped and the magnitude and pattern of hours responses are very different.

Notice next that consumption generally increases in the nonlinear economy in response to an increase in the labor tax even though hours falls. This occurs even when Proposition 10c) obtains, and arises in both the upper and lower panels of Table 6. The resource costs of price adjustment are responsible for the seemingly puzzling response of consumption. In this setting a labor tax increase results in higher inflation (see Table 6). Since we are starting from a situation with a negative inflation rate, a higher labor tax acts to increase the price level and reduce price adjustment costs. This frees up proportionately more resources for consumption relative to what is forgone by working fewer hours and the result is that consumption increases. This mechanism is absent in the loglinearized model.

[21] In the proof of Proposition 10 we derive the expression for the tax multiplier in terms of parameters and the slopes of the AD and AS schedule. We use this expression to calculate the tax multiplier (27)

Table 7 reports the tax multipliers under the Great Recession calibrations. When p is set to 0.903, the tax multipliers on hours is indeed positive for a wide range of parameters. However, the magnitude of the response of hours is very small. Its maximum response in the nonlinear model is 0.22%. More importantly, the size of the response of labor is highest in situations where the markup is very large (see Table 5). The lowest value of the markup that produces a positive hours response is 57% ($\theta = 2.76$) in the upper panel and 56% ($\theta = 2.77$) in the lower panel. As the size of the markup is reduced below these levels the sign of the hours response turns negative in the nonlinear model and there is an increasing gap beween the properties of the nonlinear model and the loglinearized model. For purposes of comparison Denes, Eggertsson, and Gilbukh (2012) find that the labor tax output multiplier is about 0.17% when they fit a loglinearized NK model to data from the Great Recession using an (estimated) value of $\theta = 13.22$.

To summarize, it is very difficult to make the case that the paradox of toil is an important and robust prediction of the true economy when the model is calibrated to the Great Depression. The breakdown of the loglinear solution is particularly dramatic in this setting. For the Great Recession calibration, there are a range of parameterizations where the loglinearized model works well in the sense that both the nonlinear and nonlinear models produce a paradox of toil and the magnitude of the hours response is about the same. However, these parameterizations have very large markups. In the parameterizations where the markup is less than 50%, there is no paradox of toil in the nonlinear model.

A more robust conclusion that emerges from our analysis is that consumption generally increases in response to a tax increase in both the loglinear and the nonlinear model.

4.3.2 An increase in government purchases

Christiano, Eichenbaum, and Rebelo (2011) find that the government purchases output multiplier is much larger than one when the nominal interest rate is zero. They derive this result in a log-linearized economy and limit attention to configurations of shocks and parameters that satisfy Proposition 1. We now turn to analyze the properties of increases in government purchases in the nonlinear economy.

We have explored the possibility of deriving analytical responses in the nonlinear economy. It is not difficult to derive analytic conditions that characterize the qualitative responses of hours, consumption and GDP to a change in government purchases. However, the key question of interest is not the sign of the government purchase output multiplier but its magnitude. The magnitude of the response of GDP and other variables to a change in government purchases is difficult to pin down analytically in the true economy. We consequently resort to numerical methods. Tables 8 and 9 report the responses of hours, inflation, consumption and GDP to a change in the level of government purchases in the nonlinear economy and the loglinearized economy. Table 8 contains results for parameterizations that reproduce the Great Depression Facts and Table 9 contains results for parameterizations that reproduce the Great Recession Facts.

Table 8 reveals some rather dramatic differences between the government purchase multipliers in the nonlinear and loglinear models. Since hours and GDP are identical in the loglinear economy only responses of hours are reported for that specification. Direct comparisons of the two solutions are only available in the case where $\gamma = 160$ when $p = 0.903$ (upper panel). In this simulation GDP (hours) multiplier is 4.2 in the loglinear model or over twice the magnitude of the GDP multiplier in the nonlinear model (1.8). Another noteworthy feature of the results reported in the upper panel of Table 8 is the large difference between the response of hours and GDP in the nonlinear model. This difference is most pronounced when $\theta \geq 2$. In these simulations the hours multiplier is less than one and turns negative when $\theta \geq 7.5$. These simulations are associated with larger values for γ. In these scenarios the reason that the GDP multiplier exceeds one is the large savings in the resource costs of price adjustment associated with a higher inflation rate (less deflation).

The lower panel of Table 8 provides a direct side by side comparison of the nonlinear and loglinear models. All of the parameterizations reported here satisfy the conditions for a Type 1 equilibrium in the loglinear model and it follows from Proposition 5 that the government purchase output multiplier is greater than one. Table 8 indicates that the government purchase output (and hours) multipliers in the loglinear economy are not all that large for most parameterizations. It is only for the largest values $\gamma \geq 800$ that the output multiplier exceeds 1.5. Interestingly, the pattern of GDP multipliers in the nonlinear model has exactly the opposite pattern. It is less than 1.2 in all cases and falls as γ increases. Overall, the government purchase GDP multiplier is always less than two in the nonlinear model when the model is calibrated to reproduce the Great Depression Facts. If one restricts attention to parameterizations with a markup of less than 50% its magnitude is less than 1.5.

Table 9 reports the same information for the case where the two models are calibrated to reproduce the Great Recession facts. All parameterizations in both panels are a Type 1 equilibrium for the loglinear model. There are two noteworthy facts. First, the gap between the loglinear and nonlinear model once again increases with the size of d^L. The second and more interesting fact is that the overall size of the government purchase multiplier is small in both models. Consider first the GDP multipliers for the loglinear model. In the upper panel they are only greater than 1.2 when $\gamma = 8142$ and in the lower panel they are all less than 1.08. The government purchase GDP multipliers are also small in the nonlinear model. They are less than 1.17 if we limit attention to parameterizations where the markup is less than 100% in the upper panel and less than 1.062 when we impose the same restriction in the lower panel.

Looking across these various parameterizations of the model we see that the government purchase GDP multiplier only exceeds 1.5 in the nonlinear model when the shock to d^L is very large and the markup is very large (see the upper panel of Table 8). Moreover, in most cases the configuration of parameters/shocks lie in a region where no equilibrium exists in the loglinear economy.

4.4 The paradox of flexibility in the nonlinear model

We have also investigated how the properties of the nonlinear model change as we reduce the costs of price adjustment holding fixed the other parameters at their baseline values. The properties of the nonlinear model in response to more price flexibility are qualitatively consistent with the loglinear model in the following sense. More price flexibility is stabilizing only if condition $2b$) of Proposition 2 holds and both schedules are upward sloping with the AS being more steeper.[22]

Figure 2 provides an example of this situation.[23] Observe that there are two zero bound

[22]The results in this section can be established using both numerical and analytical methods. The analytical results are established using the same techniques we used to derive results on fiscal policy.

[23]These equilibria were found setting $d^L = 1.002$ using the parameterization of the model with $\gamma = 1209.2$ and $p = 0.9030$ reported in Table 8.

equilibria. One of them has the property that more price flexibility is stabilizing (AD and AS are upward sloping with the AS steeper) and the other has the property that more price flexibility is destabilizing (AS and AD downward sloping with AS steeper). Both equilibria exhibit moderate declines in hours of less than 1%. Starting from this situation if the value of γ is reduced the equilibrium in which price flexibility is destabilizing eventually disappears and there is a unique zero bound equilibrium. It is a Type 2 equilibrium, has the property that increased price flexibility is stabilizing and it continues to exist in the limiting case as γ tends to zero.

4.5 Robustness

Before we conclude we briefly discuss the robustness of our conclusions to some of our modeling assumptions and relate our findings to other research. The size of the government purchase multipliers reported here for the Great Recession are smaller than results reported in Christiano and Eichenbaum (2012) who report a government purchase multiplier of about 2 for the equilibrium that is stable under learning in their baseline parameterization. We have seen here the size of the shock plays an important role in determining the slope of the AD and AS schedules. That is also true in their model. If we calibrate their model to the Great Recession facts we find that the size of the government purchases GDP multiplier falls to 1.22 using their setting of $\theta = 3$. For purposes of comparison if $\theta = 6$ instead the size of the GDP multiplier is 1.18. These results are qualitatively similar to results we have reported here.[24]

A second issue pertains to their strategy for ruling out one of the two equilibria they encounter. The e-stability criteria may be a useful device for restoring uniqueness in a setting where equilibrium is not unique. However, we have found that nonuniqueness is only one of the ways in which the loglinear solution breaks down. We have documented many empirically relevant cases where equilibrium is unique in the nonlinear model and there is no equilibrium in the loglinear model. We have also provided a range of examples where the local properties of the nonlinear model are unique and different from any zero

[24] Although the size of the government purchase multiplier is similar there are some differences between our specification and the specification considered by Christiano and Eichenbaum (2012). They hold fixed the level of government purchases across the low and high states and normalize the price adjustment costs by GDP instead of gross production.

bound equilibrium in the nonlinear economy. As we have documented above these problems arise both for large shocks that produce 30% declines in GDP and also for smaller shocks that produce declines in GDP of 7% or less.

We have assumed a stochastic probability model of the zero bound. Carlstrom, Fuerst, and Paustian (2012) find that the size of the government purchase multiplier can be much larger in this type of setup as compared to a deterministic simulation and that the size of the multiplier grows with the expected duration of the zero bound state. In other experiments not reported here we have also considered deterministic simulations. We get a similar result when we hold the rest of the parameterization including the size of the shocks fixed. However, when we recalibrate the deterministic model to a given to reproduce a particular set of observations such as the Great Depression facts then the difference to the two specifications is much smaller. The reason for this is that the deterministic model requires much larger shocks to reproduce the same set of facts.

Another question that arises is whether our results are robust to the form of price rigidity. It is not possible to provide exact solutions to a stochastic model like ours using Calvo pricing. The duration of the episode of zero interest rates is exogenous in our model. Under Calvo price setting relative price dispersion is an endogenous state variable and it follows that the duration of zero interest rates is endogenous. To get an idea about what might happen we have performed perfect foresight simulations of a similar model with Calvo pricing. The problems we have documented here also arise under Calvo price adjustment. In particular, we have also found reversals of the paradox of toil, substantial reductions in the size of the government purchase GDP multiplier and multiple equilibria.

5 Concluding remarks

In this paper we have documented that it can be very misleading to rely on the loglinearized economy to make inferences about existence of an equilibrium, uniqueness of equilibrium or to characterize the local dynamics of equilibrium. We have illustrated that these problems arise in empirically relevant parameterizations of the model that have been chosen to match observations from the Great Depression and Great Recession.

We have also documented the response of the economy to fiscal shocks in calibrated versions of our nonlinear model. We found that the paradox of toil is not a robust property of the nonlinear model and that it is quantitatively small even when it occurs. Similarly, the evidence presented here suggests that the government purchase GDP multiplier is not much above one in our nonlinear economy.

Although we encountered situations where the loglinearized solution worked reasonably well and the model exhibited the paradox of toil and a government purchase multiplier above one, the magnitude of these effects was quantitatively small. This result was also very tenuous. There is no simple characterization of when the loglinearization works well. Breakdowns can occur in regions of the parameter space that are very close to ones where the loglinear solution works. In fact, it is hard to draw any conclusions about when one can safely rely on loglinearized solutions in this setting without also solving the nonlinear model. For these reasons we believe that the safest way to proceed is to entirely avoid the common practice of loglinearizing the model around a stable price level when analyzing liquidity traps.

This raises a question. How should one proceed with solution and estimation of medium or large scale NK models with multiple shocks and endogenous state variables when considering episodes with zero nominal interest rates? One way forward is proposed in work by Adjemian and Juillard (2010) and Braun and Körber (2011). These papers solve NK models using extended path algorithms.

We conclude by briefly discussing some extensions of our analysis. In this paper we assumed that the discount factor shock followed a time-homogeneous two state Markov chain with no shock being the absorbing state. In our current work we relax this final assumption and consider general Markov switching stochastic equilibria in which there are repeated swings between episodes with a positive interest rate and zero interest rates. We are also interested in understanding the properties of optimal monetary policy in the nonlinear model. Eggertsson and Woodford (2003), Jung, Teranishi, and Watanabe (2005), Adam and Billi (2006), Nakov (2008), and Werning (2011) consider optimal monetary policy problems subject to a non-negativity constraint on the nominal interest rate, using implementability conditions derived from loglinearized equilibrium conditions. The results documented here suggest that the properties of an optimal monetary policy could be different if one uses the nonlinear implementability conditions instead.

6 Apendix A: Proofs

(For Online Publication)

Proof of Proposition 1 To derive the first restriction observe that the AS schedule passes through $(\hat{h}^L, \pi^L) = (0, 0)$ and that the intercept of the AD schedule determines the relative position of two curves at $\hat{h}^L = 0$. At $\hat{h}^L = 0$, the AD schedule is above AS if $1 - \frac{1}{\beta} + \hat{d}^L > 0$. This final restriction is equivalent to condition 1a) in the proposition.

Condition 1b) states that the AD is steeper than the AS. It follows from condition 1a) and the linearity of the two schedules that $\hat{h}^L < 0$ and $\pi^L < 0$ at their intersection. Given that $(\hat{h}^L, \pi^L) < 0$, the linear part of the Taylor rule in equation (10) prescribes $\hat{r}^e + \phi_\pi \pi^L + \phi_y \hat{y}^L \leq \hat{r}^e_L := 1/\beta - 1 - \hat{d}^L < 0$, which implies that the nominal interest rate is zero. □

Proof of Proposition 2 Proof is almost the same as the proof for Proposition 1 above. The only difference is that ϕ_π needs be sufficiently large to have $\hat{r}^e_L + \phi_\pi \pi^L + \phi_y \hat{y}^L < 0$, since $\hat{r}^e_L$ is positive under 2a). Since the AD is upward sloping, π^L is smaller than the intercept of the AD, $-\hat{r}^e_L/p$. Thus

$$\hat{r}^e_L + \phi_\pi \pi^L + \phi_y \hat{y}^L \leq \hat{r}^e - \phi_\pi \hat{r}^e_L/p = (1 - \phi_\pi/p)\hat{r}^e_L.$$

If $\phi_\pi \geq p$, then the rightmost term is non-positive under 2a). □

Proof of Proposition 3 Suppose 1a) holds and $(\phi_\pi, \phi_y) \geq (p, 0)$, but 1b) is not satisfied. Then the AD is no steeper than the AS, and the intercept at $\hat{h}^L = 0$ is strictly higher for the AD. When the AD and the AS are parallel, then there is no intersection and thus no equilibrium with $R = 0$. When the AS is strictly steeper than the AD, then their intersection satisfies $(\hat{h}^L, \pi^L) > 0$. Since the AD is upward sloping, $(\hat{h}^L, \pi^L) > (0, \frac{-\hat{r}^e_L}{p})$. Thus,

$$\hat{r}^e_L + \phi_\pi \pi^L + \phi_y \hat{y}^L > \hat{r}^e_L + \frac{\phi_\pi}{p}(-r^e) = \frac{\phi_\pi - p}{p}(-\hat{r}^e_L) \geq 0,$$

where the last inequality follows from 1a) and $(\phi_\pi, \phi_y) \geq (p, 0)$. Thus the zero bound is not binding at this intersection; There is no equilibrium with a binding zero bound in this case.

The same argument goes through for the case where 2a) holds but 2b) doesn't. One difference is that because $\hat{r}_L^e$ is positive in this case, the zero bound doesn't bind at any intersection on the positive orthant as long as $(\phi_\pi, \phi_y) \geq (0, 0)$. $\square$

Proof of Proposition 4 Consider the loglinearized AD without imposing $R = 0$:

$$\begin{aligned} \frac{1}{\beta} - 1 &= p(\hat{d}^L + R^L - \pi^L) + (1-p)(\hat{d}^L + R^L + \sigma \hat{h}^L) \\ &= (\hat{d}^L + R^L) - p\pi^L + (1-p)(\sigma \hat{h}^L) \end{aligned}$$

(Note that we are assuming $\hat{\eta}^L = 0$.) In a Markov equilibrium with a positive interest rate (if it exists), we also have $R^L = \hat{r}_L^e + \phi_\pi \pi^L + \phi_y \hat{h}^L$. Then the AD reduces to $0 = (\phi_\pi - p)\pi^L + (\phi_y + (1-p)\sigma)\hat{h}^L$, which is downward sloping when $\phi_\pi > p$ and vertical when $\phi_\pi = p$. Thus the AD and the AS has a unique intersection at $(\hat{h}^L, \pi^L) = (0, 0)$ and the implied nominal rate is $R^L = \hat{r}_L^e$. Therefore if there is a Markov equilibrium with a positive nominal rate, it is unique and $(\hat{h}^L, \pi^L, R^L) = (0, 0, \hat{r}_L^e)$.

Consider first the Type 1 configuration. Then condition 1a) implies $\hat{r}_L^e < 0$ and setting $R^L = \hat{r}_L^e$ violates the zero bound. Therefore a time invariant Markov equilibrium is unique, and the zero bound binds in that equilibrium.

Consider now the Type 2 configuration. If 2a) is satisfied then $\hat{r}_L^e > 0$ and it is straightforward to show that $(\hat{h}^L, \pi^L, R^L) = (0, 0, \hat{r}_L^e)$ satisfies the AD, the AS and the Taylor rule. Thus there is a second time-invariant Markov equilibrium in which the nominal interest rate is positive. The allocations are the same as in the steady-state allocations and the nominal rate is below its steady-state level but still positive.

Proof of Proposition 5 We first solve for $(\hat{h}^L, \pi^L)$, allowing non-zero $\hat{\eta}^L$ and τ_w^L. Let

$$\begin{aligned} icept(AS) &= -slope(AS)\frac{\sigma}{\sigma+\nu}\frac{1}{1-\eta}\hat{\eta}^L + slope(AS)\frac{1}{\sigma+\nu}\frac{1}{1-\tau_w}\hat{\tau}_w^L \\ icept(AD) &= \frac{-\hat{r}_L^e}{p} - slope(AD)\frac{1}{1-\eta}\hat{\eta}^L. \end{aligned}$$

Then

$$\begin{aligned}\hat{h}^L &= \frac{icept(AD) - icept(AS)}{slope(AS) - slope(AD)} \\ \pi^L &= icept(AS) + slope(AS)\hat{h}^L \quad (= icept(AD) + slope(AD)\hat{h}^L)\end{aligned}$$

As far as the policy changes are concerned, both $slope(AD)$ and $slope(AS)$ are unaffected.

Consider a paradox of toil: By differentiating $(\hat{h}^L, \pi^L)$ by $\hat{\tau}_w^L$, we obtain

$$\frac{d\hat{h}^L}{d\hat{\tau}_w^L} = \frac{-slope(AS)\frac{1}{\sigma+\nu}\frac{1}{1-\tau_w}}{slope(AS) - slope(AD)}, \quad \frac{d\pi^L}{d\hat{\tau}_w^L} = slope(AD)\frac{d\hat{h}^L}{d\hat{\tau}_w^L}.$$

Under the Type 1 configuration, both hours and inflation increase in response to an increase in labor tax ($\frac{d\hat{h}^L}{d\hat{\tau}_w^L}$ and $\frac{d\pi^L}{d\hat{\tau}_w^L}$ are positive), thereby establishing the paradox of toil. Under the Type 2 configuration, however, both $\frac{d\hat{h}^L}{d\hat{\tau}_w^L}$ and $\frac{d\pi^L}{d\hat{\tau}_w^L}$ are negative, and both hours and inflation decline in response to an increase in labor tax.

Next consider a change in government purchase share. Differentiating by $\hat{\eta}^L$, we obtain

$$\frac{d\hat{h}^L}{d\hat{\eta}^L} = \frac{1}{1-\eta}\frac{slope(AS)\frac{\sigma}{\sigma+\nu} - slope(AD)}{slope(AS) - slope(AD)}, \quad \frac{d\pi^L}{d\hat{\eta}^L} = slope(AD)[-\frac{1}{1-\eta} + \frac{d\hat{h}^L}{d\hat{\eta}^L}].$$

Under the Type 1 configuration, $slope(AD) > slope(AS) > slope(AS)\frac{\sigma}{\sigma+\nu} > 0$, and thus $\frac{d\hat{h}^L}{d\hat{\eta}^L}$ is positive and greater than $1/(1-\eta)$. This further implies $\frac{d\pi^L}{d\hat{\eta}^L}$ is positive. An increase in the government purchase share η thus increases both hours and inflation.

Under the Type 2 configuration, there are two cases: (1) $slope(AD) - slope(AS)\frac{\sigma}{\sigma+\nu} > 0$ and (2) $slope(AD) - slope(AS)\frac{\sigma}{\sigma+\nu} < 0$. In the first case, both $\frac{d\hat{h}^L}{d\hat{\eta}^L}$ and $\frac{d\pi^L}{d\hat{\eta}^L}$ are negative. In the second case, we have

$$0 < \frac{d\hat{h}^L}{d\hat{\eta}^L} < \frac{1}{1-\eta}, \quad \frac{d\pi^L}{d\hat{\eta}^L} < 0.$$

We are more interested in the response of GDP to a change in government purchases than a change in its share. Suppose for a moment that the government purchase g increases in response to an increase in η. In this case, the government purchase multiplier on GDP is greater than one if and only if consumption increases in response to η. From the resource constraint $c = (1-\eta-\kappa)h$, we have

$$\frac{d\hat{c}^L}{d\eta^L} = -\frac{1}{1-\eta} + \frac{d\hat{h}^L}{d\eta^L}.$$

From the above discussion we know that under the Type 1 (2) configuration the right hand side is positive (negative). Thus the government purchase multiplier is greater than one under the Type 1 configuration, and less than one under the Type 2 configuration, as long as the government purchases increase in response to an increase in η.

Finally, the government purchases decreases in response to an increase in η only when hours respond negatively. This requires $slope(AS) > slope(AD) > slope(AS)\frac{\sigma}{\sigma+\nu}$. From $g = \eta h$, we have $d\hat{g}^L/d\eta^L = 1/\eta + d\hat{h}^L/d\eta^L$. This is less than zero if and only if

$$(1-\eta+\eta\frac{\sigma}{\sigma+\nu})slope(AS) < slope(AD).$$

Proof of Proposition 6 Assuming that $\hat{\eta}^L = \hat{\tau}_w^L = 0$ and that $slope(AS) \neq slope(AD)$,

$$\frac{d\hat{h}^L}{d\gamma} = \frac{-dslope(AS)/d\gamma}{(slope(AS)-slope(AD))^2}\frac{-\hat{r}_L^e}{p}$$

Since $dslope(AS)/d\gamma < 0$, the right hand side is positive (negative) if $-\hat{r}_L^e > 0 \quad (< 0)$, which is 1$a$) in Proposition 1 (2$a$) in Proposition 2). The result for inflation and for part b) of the proof can be derived in a similar way.

Proof of Proposition 7 We use the analytical expressions for $(\hat{h}^L, \pi^L)$ in the proof of Proposition 6, and differentiate them with respect to p. Because

$$\frac{dslope(AS)}{dp} = slope(AS)\frac{\beta}{1-p\beta} \text{ and } \frac{dslope(AD)}{dp} = slope(AD)\frac{-1}{p(1-p)},$$

we have

$$\begin{aligned}\frac{d\hat{h}^L/dp}{-\hat{r}_L^e} &= \frac{[\frac{-1}{p^2}\{slope(AS)-slope(AD)\} - \frac{1}{p}\{\frac{dslope(AS)}{dp} - \frac{dslope(AD)}{dp}\}]}{[slope(AS)-slope(AD)]^2} \\ &= \frac{\frac{1}{p^2}[\frac{-1}{1-p\beta}slope(AS) - \frac{p}{1-p}slope(AD)\}]}{[slope(AS)-slope(AD)]^2} < 0\end{aligned}$$

In a Type-1 equilibrium, $-\hat{r}_L^e$ is positive and $\frac{d\hat{h}^L}{dp} < 0$, while in a Type-2 equilibrium $\frac{d\hat{h}^L}{dp} > 0$.

Since

$$\frac{d\pi^L}{dp} = slope(AS)\frac{d\hat{h}^L}{dp} + \frac{dslope(AS)}{dp}\hat{h}^L,$$

the right hand side is clearly negative in a Type-1 equilibrium. The RHS is also equal to

$$slope(AS)\frac{\hat{h}^L}{slope(AS) - slope(AD)}[-slope(AS) - \frac{1+\beta-2p\beta}{(1-p\beta)(1-p)}slope(AD)],$$

which is positive in a Type-2 equilibrium, where $\hat{h}^L < 0$ and $slope(AS) > slope(AD)$.

Proof of Proposition 8: We loglinearize the AD relationship (17) around a reference point (h^L, π^L) which satisfies (17). Let $\hat{h}^L$ be the log deviation of hours and $\hat{\pi}^L$ be the level deviation of inflation rate from this reference point. This yields

$$\hat{h}^L = \Big\{\frac{1}{\sigma}\frac{p\beta d^L/(1+\pi^L)^2}{1-p\beta d^L/(1+\pi^L)} + \frac{(\kappa^L)'}{1-\kappa^L-\eta^L}\Big\}\hat{\pi}^L. \tag{24}$$

This implies Proposition 8.

Proof of Proposition 9 We loglinearize the AS relationship around a reference point (h^L, π^L) which satisfies (16). Let $(\hat{h}^L, \hat{\pi}^L)$ be the same as in the proof of Proposition 8. Then it yields

$$(\sigma+\nu)\hat{h}^L = \Big\{\frac{(1-p\beta d^L)\gamma(1+2\pi^L)}{\gamma(1-p\beta d^L)\pi^L(1+\pi^L)+(1+\tau_s)(\theta-1)} + \frac{\sigma(\kappa^L)'}{1-\kappa^L-\eta^L}\Big\}\hat{\pi}^L. \tag{25}$$

Since $\sigma + \nu > 0$, Proposition 9 is implied.

Proof of Proposition 10 To analyze the effect of labor tax changes, we loglinearize the AS (16) with respect to the tax change too. We obtain

$$(\sigma+\nu)\hat{h}^L = \Big\{\frac{(1-p\beta d^L)\gamma(1+2\pi^L)}{\gamma(1-p\beta d^L)\pi^L(1+\pi^L)+(1+\tau_s)(\theta-1)} + \frac{\sigma(\kappa^L)'}{1-\kappa^L-\eta^L}\Big\}\hat{\pi}^L - \frac{1}{1-\tau_w^L}\hat{\tau}_{w,L}, \tag{26}$$

where $\hat{\tau}_{w,L}$ is the level deviation from τ_w^L. Thus we have

$$\hat{\pi}^L = slope(AD)\hat{h}^L = slope(AS)\hat{h}^L + slope(AS)\frac{1/(1-\tau_w^L)}{\sigma+\nu}\hat{\tau}_w^L$$

We emphasize here that when the slope of the AS schedule is negative, labor tax cut shifts the AS curve *upward*. Solving for $(\hat{\pi}^L, \hat{h}^L)$, the policy effect is summarized as

$$\hat{h}^L = \frac{slope(AS)}{slope(AD) - slope(AS)}\frac{1/(1-\tau_w^L)}{\sigma+\nu}\hat{\tau}_w^L, \quad \hat{\pi}^L = \frac{slope(AS)slope(AD)}{slope(AD) - slope(AS)}\frac{1/(1-\tau_w^L)}{\sigma+\nu}\hat{\tau}_w^L. \tag{27}$$

When the AD is downward sloping, inflation rate and hours move in the opposite direction.

Consumption response is

$$\hat{c}^L = \frac{-(\kappa^L)'}{1-\kappa^L-\eta^L}\hat{\pi}^L + \hat{h}^L.$$

When inflation and hours move in the same direction, so does the consumption if $\pi^L < 0$.

7 Appendix B: Calibration

(For Online Publication)

We fix (β, σ, ν) and calibrate (γ, p, d^L, θ) to hit the target (π^L, gdp^L). We adjust the sales subsidy τ_s so that the steady state hours and GDP are unaffected. Thus we know the value $h^L = gdp^L/(1-\kappa^L)$.

For a given pair (γ, p), solve the AD for d^L:

$$d^L = [\frac{p\beta}{1+\pi^L} + (1-p)\beta\Big(\frac{(1-\kappa^L-\eta^L)h^L}{(1-\eta)h}\Big)^\sigma]^{-1}.$$

Solving the AS for θ, we have

$$\theta = (1-p\beta d^L)\gamma\pi^L(1+\pi^L)\Big[\frac{(1-\kappa^L-\eta^L)^\sigma(h^L)^{\sigma+\nu}}{1-\tau_w^L} + \frac{(1-\theta)(1+\tau_s)}{\theta}\Big]^{-1}.$$

Because we are assuming τ_s adjusts with θ to make the ratio $\frac{(1-\theta)(1+\tau_s)}{\theta}$ constant, the RHS is independent of θ and is computable for a given triple (γ, p, d^L).

References

Adam, K., and R. Billi (2006): "Optimal Monetary Policy under Commitment with a Zero Lower Bound on Nominal Interest Rates," *Journal of Money, Credit and Banking*, 38(7), 1877–1905. 5

Adjemian, S., and M. Juillard (2010): "Dealing with ZB in DSGE Models an Application to the Japanese Economy," ESRI Discussion Paper No. 258. 5

Bodenstein, M., C. J. Erceg, and L. Guerrieri (2009): "The effects of foreign shocks when interest rates are at zero," mimeo. 1

Braun, R. A., and L. M. Körber (2011): "New Keynesian Dynamics in a Low Interest Rate Environment," *Journal of Economic Dynamics and Control*, 35(12), 2213–2227, Federal Reserve Bank of Atlanta Working Paper. 5

Braun, R. A., and Y. Waki (2010): "On the Size of the Fiscal Multiplier When the Zero Bound on Nominal Interest Rates is Binding," mimeo. 3, 1

Bullard, J. (2010): "Seven Faces of "The Peril"," *Federal Reserve Bank of St. Louis Review*, 92(5), 339–352. 3.3

Calvo, G. A. (1983): "Staggered prices in a utility-maximizing framework," *Journal of Monetary Economics*, 12(3), 383–398. 1

Carlstrom, C., T. Fuerst, and M. Paustian (2012): "Fiscal Multipliers under an Interest Rate Peg of Deterministic and Stochastic Duration," mimeo. 4.5

Christiano, L., and M. Eichenbaum (2012): "Notes on Linear Approximations, Equilibrium Multiplicity and E-learnability in the Analysis of the Zero Bound," mimeo. 1, 4.5, 24

Christiano, L., M. Eichenbaum, and S. Rebelo (2011): "When is the Government Spending Multiplier Large?," *Journal of Political Economy*, 119(1), 78–121. 1, 3, 3.1, 3.2, 3.2, 4.2.1, 4.2.3, 4.3.2

Coenen, G., A. Orphanides, and V. Wieland (2004): "Price Stability and Monetary Policy Effectiveness when Nominal Interest Rates are Bounded at Zero," *Advances in Macroeconomics*, 4(1), 1187–1187. 1

De Long, J. B., and L. H. Summers (1986): "Is Increased Price Flexibility Stabilizing?," *American Economic Review*, 76(5), 1031–44. 3.3

Del Negro, M., G. Eggertsson, A. Ferrero, and N. Kiyotaki (2010): "The Great Escape? A Quantitative Evaluation of the Fed's Non-Standard Policies," mimeo. 1

Den Haan, W., and P. Rendahl (2009): "Solving the Incomplete Markets Model with Aggregate Uncerctainty using Explicit Aggregation," mimeo. 1

Denes, M., G. Eggertsson, and S. Gilbukh (2012): "Deficits, Public Debt Dynamics and Tax and Spending Multipliers," mimeo. 4.2.1, 4.3.1

Eggertsson, G. (2011): "What Fiscal Policy is Effective at Zero Interest Rates?," in *NBER Macroconomics Annual 2010, Volume 25*, NBER Chapters, pp. 59–112. National Bureau of Economic Research, Inc. 1, 1, 2, 2.3, 7, 3, 3.1, 4.1, 4.2.1, 16, 4.3.1

Eggertsson, G., and P. Krugman (2010): "Debt, Deleveraging and the Liquidity Trap," mimeo. 1, 2.3, 3

Eggertsson, G., and M. Woodford (2003): "The Zero Interest Rate Bound and Optimal Monetary Policy," *Brookings Panel on Economic Activity*. 2, 2.2, 5

Erceg, C., and J. Lindé (2010): "Is There a Fiscal Free Lunch in a Liquidity Trap?," Board of Governors Board of Governors International Finance Discussion Papers No. 1003. 1

Fernandez-Villaverde, J., G. Gordon, P. A. Guerron-Quintana, and J. Rubio-Ramirez (2012): "Nonlinear Adventures At The Zero Lower Bound," *NBER Working Paper 18058*. 1

Ireland, P. N. (2003): "Endogenous money or sticky prices?," *Journal of Monetary Economics*, 50(8), 1623–1648. 18

Jung, T., Y. Teranishi, and T. Watanabe (2005): "Optimal Monetary Policy at the Zero-Interest-Rate Bound," *Journal of Money, Credit, and Banking*, 37(5), 813–835. 5

Mertens, K., and M. O. Ravn (2010): "Fiscal Policy in an Expectations Driven Liquidity Trap," CEPR Discussion Papers 7931, C.E.P.R. Discussion Papers. 3, 3.3

Nakov, A. (2008): "Optimal and Simple Monetary Policy Rules with Zero Floor on the Nominal Interest Rates," *International Journal of Central Banking*, 4(2), 73–127. 5

Rotemberg, J. (1996): "Prices, Output and Hours: An Empirical Analysis based on a Sticky Price Model,," *Journal of Monetary Economics*, 37, 503–533. 1, 2

Werning, I. (2011): "Managing a Liquidity Trap: Monetary and Fiscal Policy," mimeo. 3.2, 3.2, 5

Woodford, M. (2011): "Simple Analytics of the Government Expenditure Multiplier," *American Economic Journal: Macroeconomics*, 3(1), 1–35. 1, 3, 4.2.1

Yun, T. (2005): "Optimal Monetary Policy with Relative Price Distortions," *American Economic Review*, 95(1), 89–109. 2.1

Table 1: Parameterization

Symbol	Description	Values
β	Discount factor	0.997
σ	Consumption curvature	1.1599
ν	Leisure curvature	1.5692
p	Probability of a low state in the next period	0.9030
θ	Elasticity of substitution of intermediate goods	12.7721
α	Calvo Parameter	0.7747
γ	Implied price adjustment cost parameter	3741.92
τ_w	Labour tax rate	0.2
d^L	Annualized shock to discount rate in state L	5.47%
η	Government purchase share of output	0.2
ϕ_π	Inflation coefficient on the Taylor rule	1.5
ϕ_y	Output coefficient on the Taylor rule	0.125

All values except for γ are taken from Eggertsson (2011). The price adjustment cost parameter γ is calibrated to reproduce the same slope of loglinearized AS schedule in Eggertsson (2011). We have converted the shock to the Wicksellian real interest rate $\hat{r}^e = -0.0104$ to the discount factor shock d^L using $d^L = 1/\beta - \hat{r}^e$.

Table 2: Zero Bound Equilibrium Across Different Models

	No Sales Subsidy ($\tau_s = 0$)			Zero Steady State Profits ($\tau_s = 0.0849$)
	Log-linearized	Misspecified	Nonlinear	Nonlinear
% change in GDP	-29.92%	-31.73%	-13.72%	-13.43%
% change in hours	-29.92%	-31.73%	-2.51%	-2.11%
Inflation (annualized)	-9.92%	-8.23%	-2.98%	-2.87%
Slope of AD schedule	0.12	0.083	-0.033	-0.031
Slope of AS schedule	0.086	0.043	-0.220	-0.182

For all models the time-invariant zero bound equilibrium is unique. Slopes of AD and AS in the nonlinear models are computed by loglinearizing the model around the low state equilibrium, not around the zero inflation steady-state. See equations (25) and (26).

Table 3: Varying Discount Factor Shock d^L In The Nonlinear Model

Discount factor shock d^L	1.03	1.0172	1.0134	1.0096	1.0069	1.0047	1.0040	1.0031
First Equilibrium								
% deviation in GDP	-30.4	-17.8	-13.7	-9.45	-6.19	-3.2	-2.08	-0.259
% deviation in hours	9.16	-0.988	-2.51	-3.21	-3.03	-2.15	-1.59	-0.25
Inflation (annualized)	-5.18	-3.59	-2.98	-2.26	-1.62	-0.938	-0.645	-0.09
loglinear slope AD	-0.011	-0.0238	-0.0326	-0.0511	-0.0902	-0.363	1.53	0.143
loglinear slope AS	-0.0307	-0.103	-0.220	3.28	0.26	0.139	0.118	0.0916
Second Equilibrium								
% deviation in GDP	-5.04	0.0438	NA	NA	NA	NA	NA	NA
% deviation in hours	26.9	18.2	NA	NA	NA	NA	NA	NA
Inflation (annualized)	4.92	3.82	NA	NA	NA	NA	NA	NA
loglinear slope AD	0.0105	0.0156	NA	NA	NA	NA	NA	NA
loglinear slope AS	0.0231	0.0317	NA	NA	NA	NA	NA	NA

We vary d^L only, keeping other parameters unchanged. The column with $d^L = 1.0134$ corresponds to the baseline parameterization reported in Table 1.

Table 4: Great Depression Calibration with Nonlinear Model

Great Depression Calibration with $p = 0.903$

γ	α	d^L	θ	κ^L	$1+\tau_s$	slope(AD)	slope(AS)	# of eqm.
160	0.9093	1.0086	0.7073	0.0541	-2.4165	0.1514	0.0775	1
250	0.8946	1.0093	1.1403	0.0845	8.128	0.3099	0.0976	1
300	0.8867	1.0097	1.3945	0.1014	3.5349	0.8182	0.1146	1
330	0.8819	1.0100	1.5524	0.1115	2.8104	-35.9093	0.1282	1
400	0.8707	1.0106	1.9384	0.1352	2.0656	-0.3233	0.1794	1
800	0.7941	1.0149	4.9387	0.2703	1.2539	-0.0387	-0.1091	1
1000	0.7343	1.0177	7.5468	0.3379	1.1527	-0.0237	-0.0539	1
1200	0.6339	1.0211	12.434	0.4055	1.0875	-0.0156	-0.0331	1
1209.2	0.6275	1.0213	12.773	0.4086	1.0849	-0.0153	-0.0324	1

Great Depression Calibration with $p = 0.8$

γ	α	d^L	θ	κ^L	$1+\tau_s$	slope(AD)	slope(AS)	# of eqm.
160	0.8560	1.0445	1.2833	0.0541	4.5292	-2.9290	0.0775	1
250	0.8305	1.0461	2.0678	0.0845	1.9365	-0.2636	0.0976	1
300	0.8164	1.0470	2.5279	0.1014	1.6545	-0.1712	0.1146	1
330	0.8081	1.0476	2.8135	0.1115	1.5514	-0.1405	0.1282	1
400	0.7889	1.0490	3.5114	0.1352	1.3982	-0.0976	0.1794	1
800	0.6673	1.0586	8.9093	0.2703	1.1264	-0.0298	-0.1091	2
1000	0.5814	1.0649	13.571	0.3379	1.0795	-0.0199	-0.0539	2
1200	0.4519	1.0727	22.26	0.4055	1.0470	-0.0138	-0.0331	2
1209.2	0.4442	1.0731	22.86	0.4086	1.0457	-0.0135	-0.0324	2

We calculate the Calvo parameter α so that the loglinearization around the zero inflation steady state yields the same slope for the AS schedule, i.e. $\gamma/\{(1+\tau_s)(1-\theta)(1+\nu\theta)\} = \alpha/\{(1-\alpha)(1-\beta\alpha)\}$.

Table 5: Great Recession Calibration with Nonlinear Model

Great Recession Calibration with $p = 0.903$

γ	α	d^L	θ	κ^L	$1+\tau_s$	slope(AD)	slope(AS)	# of eqm.
160	0.9601	1.0084	0.2130	0.0005	-0.2706	0.1213	0.0330	1
500	0.9503	1.0084	0.6697	0.0016	-2.0275	0.1393	0.0337	1
800	0.9429	1.0084	1.0775	0.0025	13.901	0.1603	0.0345	1
1000	0.9385	1.0084	1.3520	0.0031	3.8413	0.1783	0.0350	1
1200	0.9342	1.0084	1.6285	0.0038	2.5911	0.2010	0.0355	1
1500	0.9283	1.0085	2.0472	0.0047	1.9549	0.2483	0.0363	1
2000	0.9192	1.0085	2.7560	0.0063	1.5695	0.4096	0.0377	1
3000	0.9030	1.0086	4.2158	0.0094	1.3110	-1.3335	0.0408	1
8142	0.8352	1.0090	12.774	0.0256	1.0849	-0.0569	0.0730	1

Great Recession Calibration with $p = 0.8$

γ	α	d^L	θ	κ^L	$1+\tau_s$	slope(AD)	slope(AS)	# of eqm.
160	0.9358	1.0168	0.4365	0.0005	-0.7747	0.3070	0.0330	1
500	0.9125	1.0169	1.3728	0.0016	3.6825	0.4568	0.0337	1
800	0.8962	1.0170	2.2088	0.0025	1.8272	0.8035	0.0345	1
1000	0.8867	1.0170	2.7715	0.0031	1.5645	1.6304	0.0350	1
1200	0.8779	1.0170	3.3338	0.0038	1.4276	-53.2709	0.0355	1
1500	0.8657	1.0171	4.1969	0.0047	1.3128	-1.0322	0.0363	1
2000	0.8476	1.0172	5.6501	0.0063	1.2150	-0.3910	0.0377	1
3000	0.8162	1.0173	8.6436	0.0094	1.1308	-0.1738	0.0408	1
8142	0.6943	1.0183	26.201	.0256	1.0397	-0.0443	0.0730	2

Table 6: Policy Effects: Labor Tax Multipliers Under Great Depression Calibration

Great Depression Calibration: $p = 0.903$

		Nonlinear Model			Loglinear Model		
γ	θ	hours	inflation	consumption	hours	inflation	consumption
160	0.7073	0.4807	0.07276	0.8864	15.394	1.918	15.394
250	1.1403	0.2105	0.06522	0.8028	NA	NA	NA
300	1.3945	0.07456	0.06101	0.7556	NA	NA	NA
330	1.5524	-0.001629	0.05851	0.7274	NA	NA	NA
400	1.9384	-0.1634	0.05284	0.6630	NA	NA	NA
800	4.9387	-0.7098	0.02746	0.368	NA	NA	NA
1000	7.5468	-0.8168	0.01935	0.2716	NA	NA	NA
1200	12.4340	-0.8660	0.0135	0.2008	NA	NA	NA
1209.2	12.773	-0.8673	0.01326	0.1980	NA	NA	NA

Great Depression Calibration: $p = 0.8$

		Nonlinear Model			Loglinear Model		
γ	θ	hours	inflation	consumption	hours	inflation	consumption
160	1.2833	-0.01181	0.03459	0.1810	0.2724	0.0790	0.2724
250	2.0678	-0.1237	0.03262	0.1725	0.2863	0.0830	0.2863
300	2.5279	-0.1836	0.03143	0.1673	0.2951	0.0856	0.2951
330	2.8135	-0.2186	0.03070	0.1640	0.3009	0.0872	0.3009
400	3.5114	-0.2966	0.02894	0.1561	0.3159	0.0916	0.3159
800	8.9093	-0.6307	0.01883	0.1085	0.4919	0.1427	0.4919
1000	13.571	-0.7262	0.01446	0.08720	0.7834	0.2272	0.7834
1200	22.26	-0.7844	0.01079	0.06901	2.8750	0.8337	2.8750
1209.2	22.86	-0.7863	0.01064	0.06825	3.3298	0.9656	3.3298

Slopes reported are those in the nonlinear model. Multipliers are calculated as $\Delta\hat{X}/\Delta\hat{\tau}_w^L$, i.e. when the multiplier for variable X is equal to x, it means when the labor tax rate increases by 1% the variable X increases by x%. The loglinear model multipliers are the multipliers that arise when the model is loglinearized around the zero inflation steady state.

Table 7: Policy Effects: Labor Tax Multipliers Under Great Recession Calibration

Great Recession Calibration: $p = 0.903$

		Nonlinear Model			Loglinear Model		
γ	θ	hours	inflation	consumption	hours	inflation	consumption
160	0.2130	0.1711	0.02074	0.1815	0.1893	0.0236	0.1893
500	0.6697	0.1465	0.02040	0.1785	0.1910	0.0238	0.1910
800	1.0775	0.1254	0.02010	0.1760	0.1925	0.0240	0.1925
1000	1.3520	0.1117	0.01991	0.1744	0.1935	0.0241	0.1935
1200	1.6285	0.09814	0.01972	0.1727	0.1945	0.0242	0.1945
1500	2.0472	0.07831	0.01944	0.1703	0.1961	0.0244	0.1961
2000	2.7560	0.04637	0.01800	0.1665	0.1988	0.0248	0.1988
3000	4.2158	-0.01361	0.01815	0.1592	0.2045	0.0255	0.2045
8142	12.774	-0.2573	0.01465	0.1291	0.2409	0.0300	0.2409

Great Recession Calibration: $p = 0.8$

		Nonlinear Model			Loglinear Model		
γ	θ	hours	inflation	consumption	hours	inflation	consumption
160	0.4365	0.05511	0.01692	0.06360	0.0666	0.0193	0.0666
500	1.3728	0.03653	0.01669	0.06276	0.0670	0.0194	0.0670
800	2.2088	0.02052	0.01649	0.06203	0.0675	0.0196	0.0675
1000	2.7715	0.01003	0.01636	0.06155	0.0678	0.0196	0.0678
1200	3.3338	-0.00030	0.01623	0.06108	0.0681	0.0197	0.0681
1500	4.1969	-0.01554	0.01604	0.06038	0.0685	0.0199	0.0685
2000	5.6501	-0.04024	0.01573	0.05925	0.0693	0.0201	0.0693
3000	8.6436	-0.08714	0.01515	0.05709	0.0709	0.0205	0.0709
8142	26.201	-0.2851	0.01262	0.04779	0.0806	0.0234	0.0806

Slopes reported are those in the nonlinear model. Multipliers are calculated as $\Delta\hat{X}/\Delta\hat{\tau}_w^L$, i.e. when the multiplier for variable X is equal to x, it means when the labor tax rate increases by 1% the variable X increases by x%. The loglinear model multipliers are the multipliers that arise when the model is loglinearized around the zero inflation steady state.

Table 8: Policy Effects: Government Purchase Multipliers Under Great Depression Calibration

Great Depression Calibration: $p = 0.903$

		Nonlinear Model				Loglinear Model		
γ	θ	hours	inflation	consumption	GDP	hours	inflation	consumption
160	0.7073	1.5033	0.1079	0.7781	1.7782	4.1778	2.7239	3.1778
250	1.1403	1.3055	0.1031	0.7447	1.7447	NA	NA	NA
300	1.3945	1.1921	0.09996	0.7224	1.7224	NA	NA	NA
330	1.5524	1.1237	0.09791	0.7080	1.708	NA	NA	NA
400	1.9384	0.9649	0.09278	0.6717	1.6717	NA	NA	NA
800	4.9387	0.1976	0.06076	0.4440	1.444	NA	NA	NA
1000	7.5468	-0.0556	0.04688	0.3447	1.3447	NA	NA	NA
1200	12.434	-0.2309	0.03557	0.2637	1.2637	NA	NA	NA
1209.2	12.773	-0.2374	0.03511	0.2604	1.2604	NA	NA	NA

Great Depression Calibration: $p = 0.8$

		Nonlinear Model				Loglinear Model		
γ	θ	hours	inflation	consumption	GDP	hours	inflation	consumption
160	1.2833	1.0448	0.05803	0.1798	1.1798	1.2561	0.1219	0.2561
250	2.0678	0.9550	0.05647	0.1752	1.1752	1.2682	0.1298	0.2682
300	2.5279	0.9026	0.05542	0.1721	1.1721	1.2759	0.1349	0.2759
330	2.8135	0.8705	0.05473	0.1700	1.1700	1.2809	0.1382	0.2809
400	3.5114	0.7941	0.05297	0.1648	1.1648	1.2940	0.1470	0.2940
800	8.9093	0.3646	0.04021	0.1263	1.1262	1.4397	0.2585	0.4397
1000	13.571	0.1843	0.03337	0.1054	1.1054	1.6575	0.4846	0.6575
1200	22.26	0.03857	0.02699	0.08597	1.0860	2.6769	5.3370	1.6769
1209.2	22.86	0.03272	0.02672	0.08512	1.0851	2.8214	7.9240	1.8214

Slopes reported are those in the nonlinear model. Multipliers are in levels, $\Delta X/\Delta g$, where $\Delta X = X^L \times \hat{X}^L$ and $\Delta g = g^L \times \hat{g}^L$. For the loglinear model only hours is reported because hours and GDP are identical.

Table 9: Policy Effects: Government Purchase Multipliers Under Great Recession Calibration

Great Recession Calibration: $p = 0.903$

		Nonlinear Model				Loglinear Model		
γ	θ	hours	inflation	consumption	GDP	hours	inflation	consumption
160	0.2130	1.1653	0.02643	0.1747	1.1747	1.1814	0.0298	0.1814
500	0.6697	1.1436	0.02614	0.1729	1.1729	1.1830	0.0301	0.1830
800	1.0775	1.1248	0.02590	0.1712	1.1712	1.1844	0.0303	0.1844
1000	1.3520	1.1125	0.02574	0.1702	1.1702	1.1853	0.0305	0.1853
1200	1.6285	1.1002	0.02558	0.1691	1.1691	1.1863	0.0307	0.1863
1500	2.0472	1.0821	0.02534	0.1676	1.1676	1.1877	0.0309	0.1877
2000	2.7560	1.0526	0.02495	0.1650	1.1650	1.1902	0.0314	0.1902
3000	4.2158	0.9957	0.02420	0.1601	1.1601	1.1954	0.0323	0.1954
8142	12.774	0.7441	0.02084	0.1380	1.1380	1.2281	0.0381	0.2281

Great Recession Calibration: $p = 0.8$

		Nonlinear Model				Loglinear Model		
γ	θ	hours	inflation	consumption	GDP	hours	inflation	consumption
160	0.4365	1.0551	0.02217	0.06299	1.0630	1.0657	0.0252	0.0657
500	1.3728	1.038	0.02197	0.06243	1.0624	1.0662	0.0254	0.0662
800	2.2088	1.0231	0.02180	0.06194	1.0619	1.0666	0.0255	0.0666
1000	2.7715	1.0132	0.02168	0.06161	1.0616	1.0669	0.0257	0.0669
1200	3.3338	1.0035	0.02157	0.06129	1.0613	1.0672	0.0258	0.0672
1500	4.1969	0.9890	0.02140	0.06081	1.0608	1.0676	0.0259	0.0676
2000	5.6501	0.9653	0.02112	0.06002	1.0600	1.0684	0.0262	0.0684
3000	8.6436	0.9194	0.02058	0.05849	1.0585	1.0699	0.0269	0.0699
8142	26.201	0.7112	0.01810	0.05146	1.0515	1.0793	0.0307	0.0793

Slopes reported are those in the nonlinear model. Multipliers are in levels, $\Delta X/\Delta g$, where $\Delta X = X^L \times \hat{X}^L$ and $\Delta g = g^L \times \hat{g}^L$. For the loglinear model only hours is reported because hours and GDP are identical.

Figure 1: Dynamic adjustments of consumption and expected inflation to a higher labor tax in a Type 1 equilibrium (left panel) and in a Type 2 equilibrium (right panel).

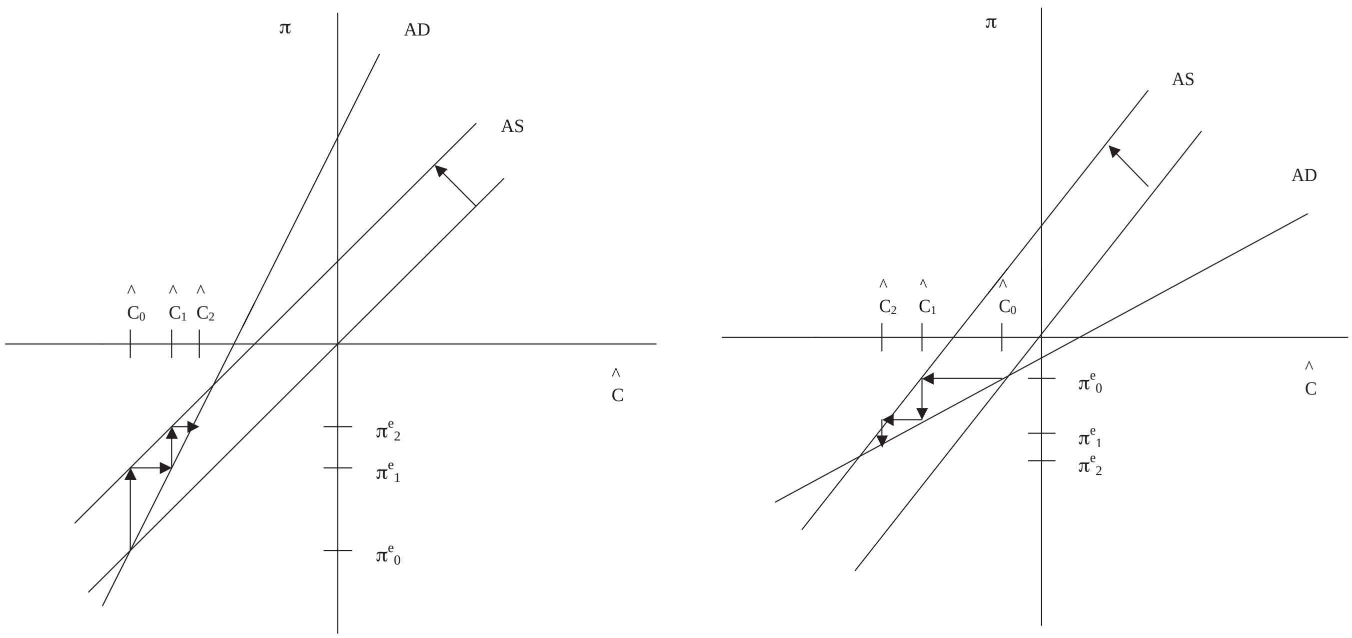

Figure 2: AD and AS schedules from a parameterized version of the model with $p = 0.903$, $\gamma = 1209.2$, and a value of $d^L = 1.002$.