Stochastic Cycles in VAR Processes

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Second International Conference in Memory of Carlo Giannini
Banca d'Italia, 19-20 January 2010.

Main contributions of this paper

1. This paper presents an additive decomposition of the MA representation of VAR processes into cyclical components, associated with the characteristic roots of the VAR polynomial.

2. It is a Beveridge-Nelson type of decomposition in which the contribution of each root to the dynamics of the process is explicit. All the coefficients of the MAD representation are characterized in terms of the VAR coefficients.

3. Relations with structural time series models, see e.g. Harvey (1990), and with common features literature, see e.g. Engle and Kozicki (JBES, 1993), are discussed.

VAR representation and characteristic roots

Consider a

$$\text{VAR: } X_t + \Pi_1 X_{t-1} + \cdots + \Pi_{d_\Pi} X_{t-d_\Pi} = \epsilon_t;$$

let

$$\Pi(z) := \sum_{n=0}^{d_\Pi} \Pi_n z^n, \qquad z \in \mathbb{C}, \qquad \Pi_n \in \mathbb{R}^{p \times p}, \qquad \Pi_0 = I,$$

and

$$\det \Pi(z) = \prod_{u=1}^{q} (1 - w_u z)^{a_u}, \qquad a_u > 0,$$

where $z_u := 1/w_u$ is a characteristic root and q is the total number of roots; then

$$\operatorname{adj} \Pi(z) =: G(z) \prod_{u=1}^{q} (1 - w_u z)^{b_u}, \qquad 0 \leq b_u < a_u, \qquad G(z_u) \neq 0.$$

Inverse function and poles

Hence

$$C(z) := \operatorname{inv}\Pi(z) = \frac{\operatorname{adj}\Pi(z)}{\det\Pi(z)} = \frac{G(z)}{\prod_{u=1}^{q}(1 - w_u z)^{m_u}}, \qquad G(z_u) \neq 0,$$

where $m_u := a_u - b_u > 0$ is the order of the pole of $\operatorname{inv}\Pi(z)$ at z_u.

MA and BN

The complex roots come in conjugate pairs; let

$$w_u =: \rho_u e^{\mathrm{i}\lambda_u}, \qquad 0 \leq \lambda_u < 2\pi,$$

and index

a complex pair by $u \,:\, 0 < \lambda_u < \pi$

and

a real root by $u \,:\, \lambda_u \in \{0, \pi\}$.

Moving Average Decomposition representation

Theorem

The MAD representation of X_t is

$$X_t = \sum_{u:0<\lambda_u<\pi} A_u(L)c_u(L)\epsilon_t + \sum_{u:\lambda_u\in\{0,\pi\}} B_u(L)d_u(L)\epsilon_t + R(L)\epsilon_t,$$

where

$$M(L) := \sum_{n=0}^{d_M} M_n L^n, \qquad M_n \in \mathbb{R}^{p\times p}, \qquad M = A_u, B_u, R,$$

is a matrix polynomial of finite degree d_M and

$$s(L) := \sum_{n=0}^{d_s} s_n L^n, \qquad s_n \in \mathbb{R}, \qquad s = c_u, d_u,$$

is a scalar polynomial of degree d_s.

Moving Average Decomposition representation ctd

Theorem ctd

Moreover,

$$\det B_u(0) = 0$$

and $A_u(L)$, $B_u(L)$, $R(L)$ have finite degree

$$d_{A_u} = 2m_u - 1, \quad d_{B_u} = m_u - 1, \quad d_R = d_G - d_g,$$

where m_u is the order of the pole of $\operatorname{inv}\Pi(z)$ at z_u and

$$d_s = \infty \qquad \Longleftrightarrow \qquad |z_u| > 1, \qquad s = c_u, d_u.$$

Related results

1. $I(1)$ and cointegration $z_1 = 1, m_1 = 1$, see Engle and Granger (ECTA, 1987), Stock and Watson (JASA, 1988).

2. $I(2)$ and cointegration $z_1 = 1, m_1 = 2$: Johansen (ET, 1992).

3. Non stationary seasonal roots $z_u = \pm 1, z_u = \pm i, m_u = 1$: Hylleberg, Engle, Granger, and Yoo (JoE, 1990), Cubadda (JAE, 1999), Johansen and Schaumburg (JoE, 1999).

4. Co-dependence $z_u = \infty, m_u \geq d_\Pi$: Gourieroux and Paucelle (WP, 1988), Vahid and Engle (JAE, 1993), Vahid and Engle (JoE, 1997), Franchi and Paruolo (WP, 2009).

Example from Benati and Surico (AER, 2009)

Let $X_t = (r_t, \pi_t, y_t)'$ and consider

$$\text{VAR: } X_t = A_1 X_{t-1} + A_2 X_{t-2} + \epsilon_t$$

where A_1, A_2 Go; because

$$g(z) = c_z(z-1.24)(z-1.57)(z-2.18)(z-2.38)(z-2.95)(z-20.95),$$

each z_u is real and stable, $m_u = 1$ and $d_{B_u} = 0$. Moreover, because

$$d_R = d_G - d_g = 2 - 6 < 0,$$

the finite MA part $R(L)\epsilon_t$ is absent from the MAD.

Example from Benati and Surico (AER, 2009) ctd

Hence one has

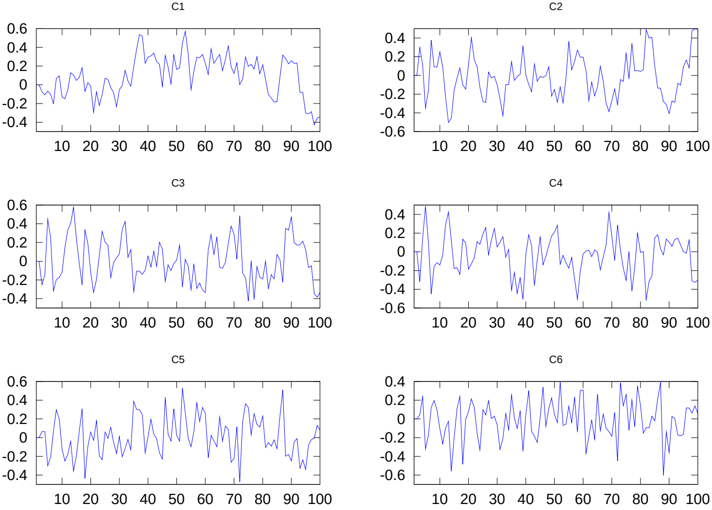
$$\text{MAD: } X_t = \sum_{u=1}^{6} B_u d_u(L)\epsilon_t, \quad B_u = \gamma_u \delta_u', \quad d_u(z) = \sum_{n=0}^{\infty} \left(\frac{1}{z_u}\right)^n z^n,$$

for γ_u, δ_u of dimension 3×1 Go; that is,

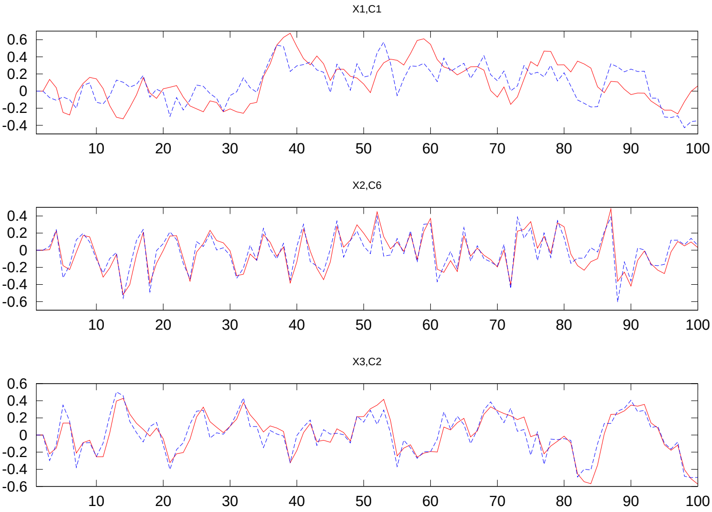
$$X_t = \sum_{u=1}^{6} \underset{(3\times 1)}{\gamma_u} \underset{(1\times 1)}{c_{u,t}},$$

and we call $c_{u,t} := d_u(L)\delta_u' \epsilon_t$ the u^{th} stochastic cycle in X_t.

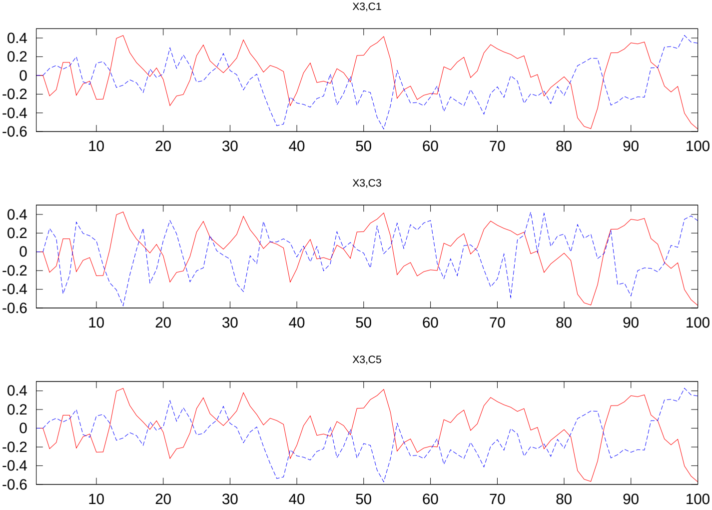
Stochastic cycles in X_t



Concordance between $X_{j,t}$ (red line) and $c_{u,t}$



Concordance between $X_{3,t}$ (red line) and $c_{u,t}$ ctd



Consider another example

$$\text{VAR: } X_t = \begin{pmatrix} -1 & -4/3 \\ 2 & 5/3 \end{pmatrix} X_{t-1} + \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} X_{t-2} + \epsilon_t$$

with

$$g(z) = -\frac{1}{3}(2z - 3);$$

hence $z_1 = 3/2$, $m = 1$ and $d_B = 0$. Moreover, because

$$d_R = d_G - d_g = 2 - 1 = 1,$$

the finite MA part $R_0 \epsilon_t + R_1 \epsilon_{t-1}$ is present in the MAD.

Hence one has

$$\text{MAD: } X_t = B\, d(L)\epsilon_t + R_0\epsilon_t + R_1\epsilon_{t-1}, \quad d(L) = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n L^n,$$

$$B = \frac{1}{8}\begin{pmatrix} -7 \\ 11 \end{pmatrix}(3:1) =: \gamma\delta';$$

that is, one has the factor structure

$$X_t = \underset{(2\times 1)}{\gamma} \quad \underset{(1\times 1)}{c_t} + R_0\epsilon_t + R_1\epsilon_{t-1},$$

where $c_t := d(L)\delta'\epsilon_t$ is the only stochastic cycle in X_t.

A natural choice of

$$A: \quad u_t = A\epsilon_t, \quad Var(u_t) = I,$$

is

$$A = \begin{pmatrix} (\delta'\Omega\delta)^{-1/2}\delta' \\ (\delta_\perp'\Omega^{-1}\delta_\perp)^{-1/2}\delta_\perp'\Omega^{-1} \end{pmatrix}, \quad Var(\epsilon_t) = \Omega.$$

This implies

$$c_t = d(L)\delta'\epsilon_t = d(L)\delta' A^{-1} u_t = d(L)(\delta'\Omega\delta)^{1/2} u_{1,t}$$

so that $u_{1,t}$ is the business cycle shock and $u_{2,t}$ is the idiosyncratic shock.

Conclusions and work in progress

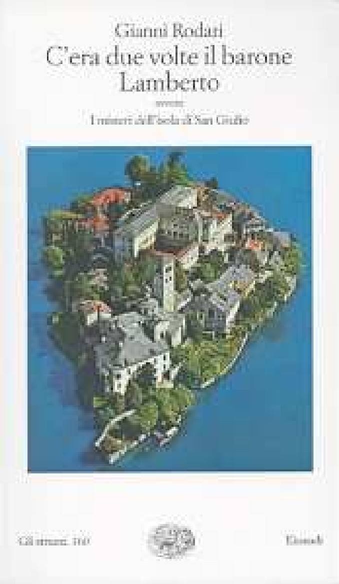
1. MAD includes many different representations as particular cases;

2. Its coefficients are explicit, non recursive functions of the VAR coefficients;

3. Inference in a likelihood based framework;

4. VARMA processes.

Dedicatory and thanks



MA and BN representations

If $|z_u| > 1$, then the expansion of $C(z)$ around 0 defines the

$$\text{MA: } X_t = \sum_{n=0}^{\infty} C_n \epsilon_{t-n}, \qquad C_n \in \mathbb{R}^{p \times p}, \qquad C_0 = I;$$

if $z_u = 1$ or $|z_u| > 1$, then the expansion of $C(z)$ around 1 defines the

$$\text{BN: } X_t = C(1) \sum_{n=0}^{t} \epsilon_{t-n} + (1 - L) \sum_{n=0}^{\infty} C_n^* \epsilon_{t-n} + \text{in. values.}$$

Back

A_1, A_2 matrices

$$A_1 = \begin{pmatrix} 1.21 & 0.01 & 0.14 \\ -0.03 & 0.47 & 0.07 \\ -0.11 & -0.05 & 1.02 \end{pmatrix}, A_2 = \begin{pmatrix} -0.32 & -0.01 & -0.05 \\ 0.02 & -0.02 & -0.02 \\ 0.08 & 0.00 & -0.23 \end{pmatrix}.$$

Back

B_u matrices

$$B_1 = \begin{pmatrix} 1 \\ -0.03 \\ -0.13 \end{pmatrix} (1.78 : -0.27 : 1.87)$$

$$B_2 = \begin{pmatrix} 1 \\ -0.22 \\ -1.18 \end{pmatrix} (0.61 : 0.69 : -2.76)$$

$$B_3 = \begin{pmatrix} 1 \\ -1.44 \\ -2.28 \end{pmatrix} (-0.51 : -0.71 : 0.58)$$

Back

B_u matrices ctd

$$B_4 = \begin{pmatrix} 1 \\ 0.7 \\ -0.87 \end{pmatrix} (-0.97 : 0.25 : 0.28)$$

$$B_5 = \begin{pmatrix} 1 \\ 1.8 \\ -14.4 \end{pmatrix} (0.07 : 0.04 : 0.03)$$

$$B_6 = \begin{pmatrix} 1 \\ -31.5 \\ 0.82 \end{pmatrix} \tfrac{1}{1000}(0.16 : 4 : -0.4)$$

Back