STOCHASTIC CYCLES IN VAR PROCESSES

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ABSTRACT. This paper presents an additive decomposition of the moving average representation of VAR processes into cyclical components. We give a definition of cyclical component that encompasses seasonal components as well as (stationary) long-run and short-run fluctuations. Each cyclical component is univariate and is characterized by a spectrum with peak at a given spectral frequency. The representation is unique and it provides explicit formulae for the the (dynamic) loadings of the variables onto the different cyclical components. We discuss relations with existing definitions of stochastic cycles. Finally we discuss how to obtain cancelations of the cyclical components by linear combinations and/or by filtering of the observed series.

1. INTRODUCTION

This paper presents an additive decomposition of the moving average representation of VAR processes into cyclical components. We give a definition of cyclical component that encompasses seasonal components as well as (stationary) long-run and short-run fluctuations. Each cyclical component is univariate and is characterized by a spectrum with peak at a given spectral frequency. The representation is unique and it provides explicit formulae for the the (dynamic) loadings of the variables onto the different cyclical components. The paper discusses how to obtain cancelations of the cyclical components by linear combinations and/or by filtering of the observed series. We discuss relations with existing definitions of stochastic cycles.

The rest of the paper is organized as follows: Section 2 introduces notation and definitions of structures of interest, Section 3 present the additive cycle decomposition, Section 4 discusses its relation with the n-order stochastic cycle of Harvey and Trimbur (2003). Section 5 describes the spectral properties of the stochastic cycles, Section 6 discusses how to obtain cancelations of the cyclical components by linear combinations and/or by filtering of the observed series. Section 7 presents examples, while Section 8 reports conclusions. In the Appendices we present some mathematical results that are needed for the derivation of the representation results.

A final word on notation. In the following, $a := b$ and $b =: a$ indicate that a is defined by b; any sum $\sum_{n=a}^{b} \cdot$ where $b < a$ is defined equal to 0. For any matrix polynomial $\pi(z) := \sum_{n=0}^{d_\pi} \pi_n z^n$, $z \in \mathbb{C}$, $\pi_n \in \mathbb{C}^{p\times r}$ where $0 < r \leq p$, we indicate its degree by d_π, i.e.

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$d_\pi := \deg \pi(z)$ and $0 < d_\pi < \infty$; when $\pi_n \in \mathbb{R}^{p\times r}$ we say that $\pi(z)$ has real coefficients. For $z_u \in \mathbb{C}$, $|z_u|$ indicates its modulus.

For any full column rank matrix $\gamma \in \mathbb{C}^{p\times r}$, γ^* indicates its complex conjugate and γ' its conjugate transpose; in case γ is real, γ' reduces to the transpose. We indicate by $\mathrm{col}(\gamma)$ the linear span of the columns of γ with coefficients in the field $\mathbb{C}$ or $\mathbb{R}$ if γ is complex or real, respectively. $\gamma_\perp$ indicates a basis of $\mathrm{col}^\perp(\gamma)$, the orthogonal complement of $\mathrm{col}(\gamma)$. $\bar{\gamma} := \gamma(\gamma'\gamma)^{-1}$ so that $P_\gamma := \bar{\gamma}\gamma' = \gamma\bar{\gamma}'$ denotes the orthogonal projector matrix onto $\mathrm{col}(\gamma)$ and $M_\gamma := I - P_\gamma$ the orthogonal projector matrix onto $\mathrm{col}^\perp(\gamma)$. For a matrix A we often employ a rank factorization of the type $A = -\alpha\beta'$ where α and β are bases of $\mathrm{col}(A)$ and $\mathrm{col}(A')$, and the negative sign is chosen for convenience in the calculations. Finally, $1_{j,k}$ is the indicator function equal to 1 if $j = k$ and 0 otherwise.

2. Setup and definitions

In this section we introduce notation and state the autoregressive (AR) and moving average (MA) representation of a VAR system. We consider the vector autoregressive process (VAR) of finite order d_Π

$$\sum_{n=0}^{d_\Pi} \Pi_n X_{t-n} = \epsilon_t \tag{2.1}$$

where $\Pi_n \in \mathbb{R}^{p\times p}$, $\Pi_0 = I$ and ϵ_t is a p-dimensional martingale difference sequence (with respect to the natural filtration generated by X_t) with positive definite conditional covariance matrix Ω. A leading example of this is when ϵ_t are Gaussian i.i.d. random vectors. Deterministic components D_t are omitted from (2.1) for ease of exposition; they could be included by replacing X_t with $X_t - D_t$ or by replacing ϵ_t with $\epsilon_t + D_t$.

Indicate the AR polynomial in (2.1) by $\Pi(z) := \sum_{n=0}^{d_\Pi} \Pi_n z^n$, $z \in \mathbb{C}$, and let $k(z) := \det \Pi(z)$, $K(z) := \mathrm{adj}\, \Pi(z)$ be respectively its characteristic and adjoint polynomials, where $\mathrm{inv}\, \Pi(z) = K(z)/k(z)$. Remark that, because $\Pi(z)$ has real coefficients, so do $k(z)$ and $K(z)$. It is useful to factorize the characteristic polynomials in terms of its roots; because $\Pi(0) = I$, one can write $k(z) = \prod_{u=1}^{q} (1 - w_u z)^{a_u}$, where $w_u := z_u^{-1}$ and z_u is a root of $k(z)$ with multiplicity $a_u > 0$. We also define $\rho := \min_u |z_u|$ and observe that $\rho > 0$ because $\Pi(0) = I$.

The power series representation of $\mathrm{inv}\, \Pi(z)$ has real coefficients and it is written as

$$C(z) := \mathrm{inv}\, \Pi(z) = \sum_{n=0}^{d_C} C_n z^n, \quad |z| < \rho,$$

where $d_C < \infty$ if and only if $d_g = 0$, i.e. $\Pi(z)$ is unimodular. Here $C(0) = C_0 = \mathrm{inv}\, \Pi_0 = \mathrm{inv}\, I = I$. It is well known (see e.g. Brockwell and Davis, 1987, page 408) that if $\Pi(z)$ has stable roots $\rho > 1$, so that $C(z)$ is holomorphic on a disk larger than the unit disk, then the following moving average (MA) form corresponds to a linear process with second moments,

$$X_t = C(L)\epsilon_t. \tag{2.2}$$

3. Additive cycles decomposition representation

Some of the factors in $k(z)$ could be common to $K(z)$; we state the cancelation of their common factors as the following lemma, for ease of later reference. The same lemma gives also the order of the pole of $\mathrm{inv}\, \Pi(z)$ at $z = z_u$, labeled m_u.

Lemma 3.1 (inv $\Pi(z)$ has pole of order m_u at $z = z_u$). *One has*

$$K(z) =: G(z) \prod_{u=1}^{q} (1 - w_u z)^{b_u}, \quad 0 \leq b_u < a_u, \quad G(z_u) \neq 0,$$

where $G(z)$ has real coefficients,

$$\operatorname{inv} \Pi(z) = \frac{G(z)}{g(z)}, \quad z \in \mathbb{C} \setminus \{z_1, \dots, z_q\},$$

where

$$g(z) := \prod_{u=1}^{q} (1 - w_u z)^{m_u}, \quad m_u := a_u - b_u > 0,$$

has real coefficients, and inv $\Pi(z)$ *has a pole of order m_u at $z = z_u$.*

In Theorem 3.2 below, we introduce a novel representation, which we call the additive cycles (ACD) representation of X_t. This is derived from the Laurent series representation of $C(z)$ and gives an additive decomposition of the MA form $X_t = \sum_{n=0}^{\infty} C_n \epsilon_{t-n}$, where the contribution of each root to the dynamics of the process is made explicit, see Franchi and Paruolo (2009) for details and proofs. In (3.1) below, X_t is written as the sum of matrix polynomials $A_u(L)$, $B_u(L)$ which respectively load the MA(∞) processes $c_u(L)\epsilon_t$, $d_u(L)\epsilon_t$ plus an additional finite MA part $R(L)\epsilon_t$, which is present only when $d_G \geq d_g$. All the coefficients of such a representation are real and uniquely determined by $\Pi(z)$.

We represent the reciprocal of both real and complex roots in polar form, i.e. we define (λ_u, ρ_u) from $w_u =: \rho_u e^{\mathrm{i}\lambda_u}$ with $0 \leq \lambda_u < 2\pi$, and order them using a lexicographic order[1] on the pairs (λ_u, ρ_u). In Theorem 3.2 below we present the additive cycles decomposition representation of X_t.

Theorem 3.2 (Additive cycles decomposition (ACD) representation). *Let $w_u =: \rho_u e^{\mathrm{i}\lambda_u}$ with $0 \leq \lambda_u < 2\pi$; then one has*

$$X_t = \sum_{u\,:\,0<\lambda_u<\pi} A_u(L) c_u(L) \epsilon_t + \sum_{u\,:\,\lambda_u \in \{0,\pi\}} B_u(L) d_u(L) \epsilon_t + R(L)\epsilon_t, \tag{3.1}$$

with $A_u(z)$, $B_u(z)$, $R(z)$ matrix polynomials with real coefficients and degree $d_{A_u} = 2m_u - 1$, $d_{B_u} = m_u - 1$, $d_R = d_G - d_g$ and

$$c_u(z) := \left(\sum_{n=0}^{\infty} \frac{\sin(n+1)\lambda_u}{\sin \lambda_u} \rho_u^n z^n \right)^{m_u}, \qquad d_u(z) := \left(\sum_{n=0}^{\infty} w_u^n z^n \right)^{m_u}, \tag{3.2}$$

where these power series representations converge for $|z| < \rho$.

Proof. See Franchi and Paruolo (2009). □

The following remarks are in order:

i) Eq. (3.1) gives an additive decomposition of the MA form $X_t = \sum_{n=0}^{\infty} C_n \epsilon_{t-n}$, where the contribution of each root to the dynamics of the process is made explicit. All the coefficients in (3.1) are real and uniquely determined by $\Pi(z)$.

[1]This means $t < s$ if and only if $\lambda_t < \lambda_s$ or ($\lambda_t = \lambda_s$ and $\rho_t < \rho_s$).

ii) In the first term one finds the contribution of the complex roots. In particular, the component of the dynamics of X_t which is due to the complex pair z_u, z_u^* is given by $A_u(L)c_u(L)\epsilon_t$, where $y_{u,t} := c_u(L)\zeta_t$, with ζ_t a univariate white noise, is the MA representation of the univariate AR($2m_u$) process

$$(1 - 2\rho_u \cos\lambda_u L + \rho_u^2 L^2)^{m_u} y_{u,t} = \zeta_t$$

and

$$1 - 2\rho_u \cos\lambda_u z + \rho_u^2 z^2 = (1 - w_u z)(1 - w_u^* z),$$

using $w_u = \rho_u(\cos\lambda_u + \mathrm{i}\sin\lambda_u)$.

iii) When $m_u = 1$, $c_u(L)\zeta_t$ describes a cycle with period $2\pi/\lambda_u$ and amplitude ρ_u^2. The coefficients of $c_u(z)$ are $\varphi_n := \frac{\sin(n+1)\lambda_u}{\sin\lambda_u}\rho_u^n$, composed of a cyclical function $\frac{\sin(n+1)\lambda_u}{\sin\lambda_u}$ times a damping factor ρ_u^n. If $m_u = 2$, $c_u(z)$ has a power series representation with coefficients given by the convolution of the coefficients $\{\varphi_n\}$ with themselves, i.e. the n-th coefficient in the power series $c_u(z)$ is $c_{u,n} = \sum_{j=0}^n \varphi_j \varphi_{n-j}$. Similarly for the cases $m_u = 3, \ldots$ one obtains the m_u-th order convolution.

iv) In the second term one finds the contribution of the positive ($\lambda_u = 0$) and negative ($\lambda_u = \pi$) real roots. The contribution of z_u is given by $B_u(L)d_u(L)\epsilon_t$, where $y_{u,t} := d_u(L)\zeta_t$, with ζ_t a univariate white noise, is the MA representation of the univariate AR(m_u) process

$$(1 - w_u L)^{m_u} y_{u,t} = \zeta_t.$$

When $m_u = 1$, $d_u(L)\zeta_t$ gives either a dampened oscillation for $w_u < 0$ or a geometric decay if $w_u > 0$. We observe that the remark in *iii)* applies here substituting $c_u(z)$, φ_n with $d_u(z)$, w_n. This gives the generic coefficient of $d_u(z)$ as the m_u-th order convolution of $\{w_n\}$.

v) When $d_G \geq d_g$, one finds the additional term $R(L)\epsilon_t$ of finite degree $d_G - d_g$. Hence only the first two terms are responsible for the presence of infinite memory in X_t, i.e. for $cov(X_t, X_{t-n}) \neq 0$ for all n.

4. Relationship with the stochastic cycles of Harvey and Trimbur

In this section we discuss the relationship between univariate AR$(2)^n$ processes and the stochastic cycles of Harvey (1990), Harvey and Trimbur (2003), Trimbur (2006), see also Luati and Proietti (2009) for extensions. Both processes have the same AR polynomial, and they differ because of the presence of a MA component present in the stochastic cycles of Harvey and Trimbur (2003), which is absent in the AR$(2)^n$ processes.

The MAD representation involves AR$(2)^n$ processes $y_{n,t}$ as stochastic cycles or order n, where $y_{n,t}$ is defined by

$$\left(1 - 2\rho\cos\lambda L + \rho^2 L^2\right)^n y_{n,t} = \zeta_t \tag{4.1}$$

with ζ_t an uncorrelated univariate white noise with mean 0 and covariance matrix σ_ζ^2. The AR polynomial of (4.1) is $a(L)^n$ where $a(L)$ is the polynomial $a(z) := 1 - (2\rho\cos\lambda)\, z + \rho^2 z^2 = \left(1 - \rho e^{i\lambda} z\right)\left(1 - \rho e^{-i\lambda} z\right)$, with two complex conjugate roots at $\rho^{-1} e^{\pm i\lambda}$.

Harvey and Trimbur (2003), building on Harvey (1990), consider the following bivariate processes $\psi_t^{[j]} := \left(\psi_{1,t}^{[j]} : \psi_{2,t}^{[j]}\right)'$

$$\psi_t^{[j]} = G\psi_{t-1}^{[j]} + S\psi_t^{[j-1]}, \qquad G := \rho \begin{pmatrix} \cos\lambda & \sin\lambda \\ -\sin\lambda & \cos\lambda \end{pmatrix}, \qquad S := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.2}$$

for $j = 1, 2, \ldots, n$, where $0 \leq \lambda \leq \pi$ is a given frequency, $0 < \rho \leq 1$, and $\psi_t^{[0]} := (\kappa_{1t} : \kappa_{2t})'$ is an uncorrelated white noise with mean 0 and covariance matrix $\Sigma := \operatorname{diag}(\sigma_1^2, \sigma_2^2)$. Because of the selection matrix S, there is no loss of generality in setting $\sigma_2^2 = 0$. They identify $\psi_{1,t}^{[n]}$ as the n-th order stochastic cycle. Trimbur (2006) studied the properties of such processes when $S = I_2$ and $\sigma_1^2 = \sigma_2^2$. We consider here the original setup (4.2) of Harvey and Trimbur (2003), with $\sigma_2^2 = 0$, which involves a single input disturbance κ_{1t}, as in the AR$(2)^n$ case.

The univariate representation of $\psi_{1,t}^{[n]}$ is an ARMA($2n$,n) of the type

$$\left(1 - 2\rho\cos\lambda L + \rho^2 L^2\right)^n \psi_{1,t}^{[n]} = (1 - \rho\cos\lambda L)^n \kappa_{1,t}, \tag{4.3}$$

as can be obtained by computing the final equations form of (4.2), see Harvey and Trimbur (2003)[2]. Comparing (4.1) and (4.3) one sees that $\psi_{1,t}^{[n]}$ and $y_{n,t}$ share the same AR polynomial $a(L)^n$; however, while $y_{n,t}$ has no MA polynomial, $\psi_{1,t}^{[n]}$ involves the MA(n) polynomial $(1 - \rho\cos\lambda L)^n$.

Another way to discuss the relationship between $\psi_{1,t}^{[n]}$ and $y_{n,t}$ is by comparing the companion form of $y_{n,t}$ directly with the definition (4.2) of $\psi_{1,t}^{[n]}$. To this end, define $Y_t^{[j]} := (y_{j,t} : y_{j,t-1})'$ as state vector for (4.1); one finds

$$Y_t^{[j]} = FY_{t-1}^{[j]} + SY_t^{[j-1]}, \qquad F := \begin{pmatrix} 2\rho\cos\lambda & -\rho^2 \\ 1 & 0 \end{pmatrix} \tag{4.4}$$

for $j = 1, 2, \ldots, n$ with $Y_t^{[0]} := (\zeta_t : 0)'$, and covariance $\mathbb{E}(Y_t^{[0]} Y_t^{[0]\prime}) = \operatorname{diag}(\sigma_\zeta^2, 0)$. The following theorem shows that the matrices G in (4.2) and F in (4.4) are similar.

Theorem 4.1. *The matrices G in* (4.2) *and A in* (4.4) *are similar, i.e. $G = HFH^{-1}$ or $F = H^{-1}GH$ with*

$$H := \left(\rho^2 + 1\right)^{\frac{1}{2}} \begin{pmatrix} -(\rho\sin\lambda)^{-1} & \cot\lambda \\ 0 & 1 \end{pmatrix}, \qquad H^{-1} = \left(\rho^2 + 1\right)^{-\frac{1}{2}} \begin{pmatrix} -\rho\sin\lambda & \rho\cos\lambda \\ 0 & 1 \end{pmatrix},$$

where both F and G have as matrix of eigenvalues $\Lambda := \operatorname{diag}(\rho e^{i\lambda}, \rho e^{-i\lambda})$.

We observe that F, G, H are all real matrices, unlike the matrix of eigenvalues $\Lambda := \operatorname{diag}(\rho e^{i\lambda}, \rho e^{-i\lambda})$, which is complex. The following corollary shows that this implies that the generating mechanisms in (4.2) and (4.4) are related by the nonsingular transformation H above.

Corollary 4.2. *Consider the AR*(2) *process $y_{1,t}$ defined in* (4.1) *with companion representation* (4.4) *and the matrix H as defined in Theorem* 4.1*; then $\psi_t^{[1]} = HY_t^{[j]}$ satisfies eq.* (4.2) *with $n = 1$ and*

$$\sigma_1^2 = \sigma_\zeta^2 \frac{\rho^2 + 1}{\rho^2 \sin^2\lambda}, \qquad \sigma_2^2 = 0. \tag{4.5}$$

Vice-versa one can generate $\psi_t^{[1]}$ as in (4.2) *with* (4.5) *and set $Y_t^{[1]} = H^{-1}\psi_t^{[1]}$ which satisfies* (4.4) *for $n = 1$.*

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[2]See pag. 247 there, the lines preceding eq. (11).

Let $H =: (H_1 : H_2)'$ and $H^{-1} =: (H^1 : H^2)'$, where H_j' and $H^{j\prime}$ are the j-th rows of H and H^{-1} respectively. The Corollary above implies that the bivariate generating mechanisms of $Y_t^{[1]}$ and $\psi_t^{[1]}$ correspond 1 to 1, and $\psi_{1,t}^{[1]}$ and $y_{1,t}$ can be obtained by different sampling schemes from this bivariate process. In fact, one could generate the stochastic cycle $\psi_t^{[1]}$ and obtain $\psi_{1,t}^{[1]}$ and $y_{1,t}$ as $\psi_{1,t}^{[1]} = (1 : 0)\psi_t^{[1]}$ and $y_{1,t} = H^{1\prime}\psi_t^{[1]}$, where $(1 : 0)$ and $H^{1\prime}$ are the sampling vectors. Symmetrically, one could generate the stochastic cycle $Y_t^{[1]}$ and obtain $\psi_{1,t}^{[1]}$ and $y_{1,t}$ as $\psi_{1,t}^{[1]} = H_1'Y_t^{[1]}$ and $y_{1,t} = (1 : 0)Y_t^{[1]}$, where H_1' and $(1 : 0)$ are the sampling vectors from the same bivariate process $Y_t^{[1]}$.

The comparison of the companion form with (4.2) in the general case $n > 1$ is less straightforward. In fact, pre-multiplying (4.4) by H one finds

$$\left(HY_t^{[n]}\right) = G\left(HY_{t-1}^{[n]}\right) + HSY_t^{[n-1]} \tag{4.6}$$

where, however, $HS \neq SH$, i.e. the two matrices do not commute. If they did, then (4.6) would be equal to (4.2) when setting $\psi_t^{[n]} = HY_t^{[n]}$.

5. Spectra of stochastic cycles

As it is well know, see e.g. Fuller (1996) Section 4.3, the spectral density $f_y(\theta)$ of an AR$(1)^n$

$$(1 - w_u L)^n y_{n,t} = \zeta_t \tag{5.1}$$

is given by

$$f_y(\theta) = \frac{\sigma_\zeta^2}{2\pi}\left(\frac{1}{1 - 2w_u\cos\theta + w_u^2}\right)^n, \qquad -\pi \leq \theta \leq \pi.$$

Similarly, the spectral $f_y(\theta)$ of an AR$(2)^n$ process (4.1) is given by

$$f_y(\theta) = \frac{\sigma_\zeta^2}{2\pi}\left(\frac{1}{1 - 2\rho_u\cos(\lambda_u - \theta) + \rho_u^2}\right)\left(\frac{1}{1 - 2\rho_u\cos(\lambda_u + \theta) + \rho_u^2}\right), \qquad -\pi \leq \theta \leq \pi$$

[Include figures and description]

6. Filtering

First we present a procedure called 'polynomial rank factorization' of $\Pi(z)$ at $z = z_u$; it consists in performing a sequence of m_u rank factorizations on the matrices in (6.2).

Definition 6.1 (Polynomial rank factorizationof $\Pi(z)$ at $z = z_u$)**.** *Let z_u, $u = 1, \ldots, q$, be a characteristic root of $\Pi(z) = \sum_{n=0}^{d_\Pi}\Pi_n^{(u)}(z - z_u)^n$ and define $\alpha_{u,0}$ and $\beta_{u,0}$ of dimension $p \times r_{u,0}$, where $0 < r_{u,0} < p$, from the matrix rank factorization*

$$\Pi_0^{(u)} = -\alpha_{u,0}\beta_{u,0}'. \tag{6.1}$$

For $j = 1, \ldots, m_u$, let $a_{u,j} := (\alpha_{u,0} : \cdots : \alpha_{u,j-1})$, $b_{u,j} := (\beta_{u,0} : \cdots : \beta_{u,j-1})$ and $r_{u,j}^{\max} := p - \sum_{n=0}^{j-1} r_{u,n}$ and define $\alpha_{u,j}$ and $\beta_{u,j}$ of dimension $p \times r_{u,j}$, where $0 \leq r_{u,j} < r_{u,j}^{\max}$ for $j \neq m_u$ and $0 < r_{u,m_u} = r_{u,m_u}^{\max}$, from the matrix rank factorization

$$M_{a_{u,j}}\Pi_{j,1}^{(u)}M_{b_{u,j}} = -\alpha_{u,j}\beta_{u,j}', \tag{6.2}$$

where $\Pi_{j,k}^{(u)}$ is defined for $j, k \geq 1$ from the recursions

$$\Pi_{j,k}^{(u)} := \Pi_{j-1,k+1}^{(u)} + \Pi_{j-1,1}^{(u)}\sum_{n=0}^{j-2}\bar{\beta}_{u,n}\bar{\alpha}_{u,n}'\Pi_{n+1,k}^{(u)} \tag{6.3}$$

with initial values $\Pi_{0,k}^{(u)} := \Pi_{k-1}^{(u)}$.

The polynomial rank factorization in Definition 6.1 gives a characterization of the set of reduced rank restrictions that are satisfied by the coefficients of a matrix polynomial whose inverse function has a pole of given order at a specific point. That is, if $\Pi(z)$ and its derivatives at $z = z_u$ satisfy those conditions then $\operatorname{inv}\Pi(z)$ has a pole of order m_u at the same point; the converse is also true, i.e. if $\operatorname{inv}\Pi(z)$ has a pole of order m_u at $z = w_u$ then $\Pi(z)$ and its derivatives at $z = z_u$ satisfy the rank restrictions of the polynomial rank factorization at that point. Hence the polynomial rank factorization is a one to one and onto map from the structure of the matrix polynomial to the nature of the singularity of its inverse function. This result is based on the recursive algorithm developed in Franchi (2009) and further analyzed in Franchi and Paruolo (2009).

The following additional remarks are in order:

Remark 6.2. *Eq.* (6.1), (6.2) *define* $\alpha_{u,j}$, $\beta_{u,j}$ *up to a conformable change of bases of the row and column spaces; this does not affect the results.*

Remark 6.3. *The* $p \times p$ *matrices* $(\alpha_{u,0} : \cdots : \alpha_{u,m_u})$ *and* $(\beta_{u,0} : \cdots : \beta_{u,m_u})$ *are non-singular with orthogonal blocks, i.e.* $\alpha_{u,j}'\alpha_{u,k} = \beta_{u,j}'\beta_{u,k} = 0$ *for* $j \neq k$.

Remark 6.4. *The conditions* (6.2) *are reduced-rank conditions for* $j = 1, \cdots, m_u - 1$, *while the terminal condition for* $j = m_u$ *is a full-rank condition.*

Remark 6.5. *For* $z_u = 1$, *and* $m_u = 1$, $m_u = 2$ *these conditions were derived by Johansen (1992) and are called the* $I(1)$ *and* $I(2)$ *conditions.*

Here we present implications of the polynomial rank factorization; in particular, using the coefficients defined above, we construct polynomials $\gamma_{u,j}(z)$ such that, for any $\varphi \neq 0$, $\varphi'\gamma_{u,j}'(z) \operatorname{inv}\Pi(z)$ has a pole of order $j = 0, \ldots, m_u$ at $z = z_u$, see Theorem 6.6 below.

Theorem 6.6 (Filtering)**.** *For* $u = 1, \ldots, q$, *let*

$$\gamma_u'(z) := \beta_{u,0}' - \bar{\alpha}_{u,0}' \sum_{k=1}^{m_u-1} (-z_u)^k \Pi_k^{(u)} (1 - w_u z)^k, \tag{6.4}$$

where $\alpha_{u,0}$, $\beta_{u,0}$ *and* $\Pi_{0,k}^{(u)}$ *are defined by the polynomial rank-decomposition of* $\Pi(z)$ *at* $z = z_u$*; then*

$$\gamma_u'(L)X_t$$

does not contain the contribution of z_u.

That is, the linear combinations in 6.4 eliminate the component of the dynamics which is due to a specific characteristic root. In the nex corollary, we report the leading case where $m_u = 1$ for all u.

Corollary 6.7 (Example with $m_u = 1$)**.** *Let* $m_u = 1$, $u = 1, \ldots, q$*; then one has*

$$X_t = \sum_{u=1}^{q} B_u d_u(L)\epsilon_t + R(L)\epsilon_t,$$

with $R(z)$ *a matrix polynomial with real coefficients and degree* $d_R = d_G - d_g$, $d_u(z) := \sum_{n=0}^{\infty} w_u^n z^n$, $w_u \in \mathbb{C}$, *converges for* $|z| < \rho$ *and*

$$B_u = -\bar{\beta}_{u,1}\bar{\alpha}_{u,1}'.$$

This shows that $\beta_{u,0}'X_t$ does not contain the contribution of z_u.

7. Examples

For $\ell = \omega, \psi$, let $1_{\ell,\psi} = 0$ if $\ell = \omega$ and $1_{\ell,\psi} = 1$ if $\ell = \psi$ and consider $\Pi^\ell(L) X_t^\ell = \epsilon_t$, where

$$\Pi^\ell(z) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 3 & 4 \\ -6 & -5 + 1_{\ell,\psi} \end{pmatrix} z - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} (1 : 1)\, z^2,$$

with MA representations $X_t^\ell = \sum_{n=0}^{\infty} C_n^\ell \epsilon_{t-n}$, where

$$C_0^\omega = C_0^\psi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C_1^\omega = C_1^\psi = \frac{1}{3} \begin{pmatrix} -3 & -4 \\ 6 & 5 - 1_{\ell,\psi} \end{pmatrix},$$

and for $n = 2, 3, \ldots$, C_n^ψ is non-singular and

$$C_n^\omega = \frac{c_n}{18} \begin{pmatrix} -7 \\ 11 \end{pmatrix} (3 : 1) \text{ where } c_n := \left(\frac{2}{3}\right)^{n-2}.$$

Note that the two processes are indistinguishable from the perspective of the column spaces of the VAR coefficients, while they are very different from the MA perspective: the first has got a factor structure that is absent from the second one. The reason becomes evident by looking at their ACD representation; first we compute

$$k^\omega(z) := \det \Pi^\omega(z) = -\frac{1}{3}(2z - 3), \qquad k^\psi(z) := \det \Pi^\psi(z) = -\frac{1}{6}(z^3 - 2z^2 + 2z - 6)$$

and

$$K^\ell(z) := \operatorname{adj} \Pi^\ell(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} -5 + 1_{\ell,\psi} & -4 \\ 6 & 3 \end{pmatrix} z - \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} (1 : -1)\, z^2,$$

Because $k^\omega(z)$ has only one root with multiplicity one, i.e. $z_1 = \frac{3}{2}$ and $m_1^\omega = 1$, and $d_{K^\omega} - d_{k^\omega} = 1$, one has

$$X_t^\omega = B \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \epsilon_{t-n} + R_0 \epsilon_t + R_1 \epsilon_{t-1},$$

where $B = \frac{1}{8} \begin{pmatrix} -7 \\ 11 \end{pmatrix} (3 : 1)$, $R_0 = \frac{1}{8} \begin{pmatrix} 29 & 7 \\ -33 & -3 \end{pmatrix}$ and $R_1 = \frac{3}{4} \begin{pmatrix} 1 \\ -1 \end{pmatrix} (1 : -1)$. Because $k^\psi(z)$ has a pair of complex roots with $\rho \simeq 0.62$, $\lambda \simeq 0.53\pi$ and a real root $z_3 \simeq 2.28$ all with multiplicity one and $d_{k^\psi} > d_{k^\psi}$, one has

$$X_t^\psi \simeq (A_0 + A_1 L) \sum_{n=0}^{\infty} \frac{\sin(n+1)0.53\pi}{\sin 0.53\pi} 0.62^n \epsilon_{t-n} + B \sum_{n=0}^{\infty} 0.44^n \epsilon_{t-n}$$

where $A_0 \simeq \begin{pmatrix} 2.44 & 0.14 \\ -2.23 & 0.78 \end{pmatrix}$, $A_1 \simeq \begin{pmatrix} -0.11 & -1.26 \\ 0.78 & 1.32 \end{pmatrix}$ and $B \simeq \begin{pmatrix} -1 \\ 1.55 \end{pmatrix} (1.44 : 0.13)$. Because A_0, A_1 are non singular, this explains the absence of a factor structure.

8. Conclusions

The ACD representation provides a decomposition of a VAR process into cyclical components, which are closely connected with existing definitions of stochastic cycles. Cancelations of the cyclical components by linear combinations and/or by filtering of the observed series is discussed.

Appendix A. Proofs

Proof of Lemma 3.1. Because $\det \Pi(z_u) = 0$ one has $0 \leq \operatorname{rank} \Pi(z_u) \leq p-1$; when $0 \leq \operatorname{rank} \Pi(z_u) < p-1$ one has $\operatorname{adj} \Pi(z_u) = 0$ and thus each entry of $\operatorname{adj} \Pi(z)$ contains the factor $(1-z/z_u)^{b_u}$ for some $0 < b_u < a_u$; if $\operatorname{rank} \Pi(z_u) = p-1$ then $\operatorname{adj} \Pi(z_u) \neq 0$ and thus $b_u = 0$. If $\operatorname{Im} z_u \neq 0$ then the same applies to z_u^*. Let $g(z) := \prod_{u=1}^{q}(1-z/z_u)^{m_u} =: (1-z/z_u)^{m_u} g_u(z)$; because $\operatorname{inv} \Pi(z) := \frac{\operatorname{adj} \Pi(z)}{\det \Pi(z)}$ and $G(z_u), g_u(z_u) \neq 0$ one has the last statement. This completes the proof. $\square$

Proof of Theorem 4.1. The eigenvalue decomposition of the companion matrix F is $FV = V\Lambda$ with $\Lambda = \operatorname{diag}(\rho e^{i\lambda}, \rho e^{-i\lambda})$ and

$$V := \left(\rho^2+1\right)^{-\frac{1}{2}} \begin{pmatrix} \rho e^{i\lambda} & \rho e^{-i\lambda} \\ 1 & 1 \end{pmatrix}, \qquad V^{-1} = \frac{(\rho^2+1)^{\frac{1}{2}}}{2i \sin\lambda} \begin{pmatrix} \rho^{-1} & -e^{-i\lambda} \\ -\rho^{-1} & e^{i\lambda} \end{pmatrix},$$

and the one of the matrix G is $GU = U\Lambda$ with

$$U = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}, \qquad U^{-1} := \frac{1}{2} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}.$$

Hence $V^{-1}FV = \Lambda = U^{-1}GU$, from which $G = HFH^{-1}$ or $F = H^{-1}GH$ for $H := UV^{-1}$ with

$$H := UV^{-1} = \frac{(\rho^2+1)^{\frac{1}{2}}}{2i \sin\lambda} \begin{pmatrix} -2i\rho^{-1} & 2i\cos\lambda \\ 0 & 2i\sin\lambda \end{pmatrix} = \left(\rho^2+1\right)^{\frac{1}{2}} \begin{pmatrix} -\left(\rho \sin\lambda\right)^{-1} & \cot\lambda \\ 0 & 1 \end{pmatrix}.$$

$\square$

Proof. of Corollary 4.2 From Theorem 4.1 we see that $HY_t^{[1]} = (HFH^{-1})HY_{t-1}^{[1]} + HSY_t^{[0]}$, where $HSY_t^{[0]} = HY_t^{[0]}$ has covariance

$$\mathbb{E}(HY_t^{[0]}Y_t^{[0]\prime}H') = \sigma_1^2 H \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} H' = \sigma_1^2 \frac{\rho^2+1}{\rho^2 \sin^2\lambda} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

$\square$

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