In Search of a Theory of Debt Management*

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Abstract

We report stylised facts on recent OECD debt management practice, evaluate the ability of a recently proposed framework for debt management to account for these findings and draw conclusions about the required ingredients for a successful theory of debt management that can offer guidance to policymakers.

We show that across the OECD, governments issue positive amounts of debt at all maturities; the maturity profile is U shaped with largest issuance being of money market instruments and long term debt; portfolio shares are always substantially less than 100% and display limited variability but substantial persistence; and that gross and net issuance of debt per period is small compared to the stock of debt. It has been suggested that one can build a theory of debt management by studying how the government can effectively complete the markets by issuing the correct amount of debt at various maturities and exploiting movements in the yield curve. We show the implications of this approach to debt management are dramatically at variance to the observed behaviour of OECD debt management practice and non-robust to slight variations in assumptions. We show across a broad range of models (with or without capital accumulation, habits, matching variability of long rates, no government buybacks) how the complete market approach implies governments taking huge positive or negative positions on debt of different maturities. These positions sometimes show enormous volatility across small changes in maturity and across time. The presumption that complete market models imply a government issuing long term debt and holding short term securities is not robust. We conclude by suggesting that an empirically succesful model of debt management needs to focus on incomplete market specifications - such as transaction costs, refinancing risk and short sale constraints.

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1 Introduction

A key influence on the behaviour of optimal fiscal policy is the government's ability to offset unexpected fluctuations in government expenditure or revenue. If the government has access to a complete set of Arrow contingent securities then tax volatility can be minimised by variation in the market value of debt, the so called "complete market" outcome.[1] Angeletos (2002) and Buera and Nicolini (2004) demonstrate how the complete market outcome can be achieved by holding the right amount of non-contingent bonds at different maturities. By exploiting movements in the term structure of interest rates, a portfolio of bonds at different maturities can generate changes in the market value of government debt to fully offset unexpected shocks to the government's budget constraint to implement the desired tax policy. Angeletos (2002) therefore proposes a policy of debt management where the government can effectively complete the markets by issuing long term debt and investing in short term assets. The idea that governments should issue long debt is recently been found in other contributions, such as Nosbusch (2006) and Barro (1999) and (2003).

The validity of models that complete the markets has been recently challenged. Aiyagari et al (2002) show how, in the absence of complete markets, taxes contain a unit root and are far more volatile than under complete markets. Scott (2006) examines a number of implications for optimal labour taxes under complete and incomplete markets, compares them with OECD data and finds substantial support for the incomplete market model. Marcet and Scott (2005) show how complete markets has implications for the behaviour of debt (debt should fall in response to primary deficit shocks and should be less persistent than other variables) which run counter to the behaviour of US fiscal policy. However, these papers focus on the failure of the complete market model to account for observed behaviour of taxes and debt. By contrast, the purpose of this paper is to evaluate the ability of the complete market model to account for observed debt management practice, to consider the robustness of its implications and its ability to provide insights that are useful for the actual practice of debt management.

Evaluating the complete market model requires first documenting patterns in debt management, which we do in Section 2. We find that OECD governments tend to issue positive amounts of debt at all maturities; that debt positions are a smooth U-shaped function of maturity with the largest positions being in longer term debt and money market instruments and the smallest in short term debt; debt positions vary in particular instruments from 0% to a maximum of 19% GDP and that

[1] Some authors have shown that markets can be effectively completed by the right amount of ex-post variation in some policy variable, such as capital taxes, or exchange rates, or inflation. We assume throughout the paper that such an instrument is not available and that the government can only complete the markets by debt policy.

debt compositions are highly persistent and show limited volatility. Finally we show that gross and net new issuance is a small proportion of existing outstanding debt.

Section 3 outlines our basic model of the complete markets approach to debt management in a world without capital, in essence the same model as used by Angeletos (2002) and Buera and Nicolini (2004). Buera and Nicolini (2004) show how this model, because of the limited volatility in the term structure of interest rates it produces, predicts that governments should hold long and short positions on government debt that are extremely large multiples of GDP; the figures for the positions of some maturities often are of the order of five times GNP. Angeletos (2002) desribes this result as "disturbing" but conjectures that it is a result of the endowment economy and that allowing for capital accumulation should result in more plausible bond holdings[2]. In this paper we investigate how sensitive is this result to a number of variations in the endowment economy. We introduce capital accumulation in section 4, we find that the performance of the model worsens in several areas: not only debt positions are often larger than without capital, now they are very volatile across time and models. A major problem in assessing the performance of the complete markets model is the fact that many canonical business cycle models generate limited interest rate volatility. Given the key ingredient that effectively completes markets in the by debt management are the fluctuations in the yield curve, a model with little variability in the interest rates exaggerates extreme positions in debt holdings because of this data mismatch. To overcome this potential bias in our evaluation of effectively complete market models we incorporate in Section 5 habits explicitly in order to match the volatility of yield curves between simulations and data. We show how even in this case the complete market model of debt management is inconsistent with many of the observed features of OECD debt management. A common, clearly counterfactual, assumption often made in models of optimal debt management is that the government every period can buyback without cost the entire stock of government debt and restructure through reissuance. To see how sensitive the results of the model are to this assumption, in Section 6 we investigate the claim that this buyback assumption is without loss of generality and show not only that the model is even more at odds with the data and predicts even larger fluctuations. A final section concludes by summarising our evaluation of the complete market approach to debt management and suggests potentially fruitful areas for future research.

[2] *"However, this disturbing result [of debt holdings exploding to plus and minus infinity] is mostly an artefact of an economy without capital"* Angeletos (2002)

2 OECD Debt Management Practice

Figure 1 shows the average debt composition across 14 OECD countries[3] for the period 1994-2003 using data taken from the OECD's *Central Government Debt Statistical Yearbook 1994-2003* (see Missale (1999) for a survey of OECD debt issuance from 1960 and related economic theories). Governments issue positive amounts of debt at all maturities and debt composition is a smooth U shaped function of maturity, initially falling with maturity but then rising and, for most countries, peaking for long term debt (more than 5 years maturity). Focusing on these averages the largest debt component is for long term debt which amounts to 38% of total debt, equivalent to 19% of GDP given that average debt across these countries for this sample period was 50% GDP.

HERE FIGURE 2 TO 5

Figures 2-5 show the time variation for each country, recording the average and the minimum and maximum proportion of total debt accounted for by each maturity. Although these charts reveal some variation in debt composition over time and across countries our main conclusions hold - debt is issued in positive amounts at all maturities, debt positions are not extreme relative to GDP, governments issue more money market instruments than short term debt but issue larger amounts of medium and long term debt than short debt. The limited variation in debt positions is shown in Table 1 which reports the coefficient of variation of portfolio shares, with the vast majority of these being well below 0.5. 3. Table 2 shows that the majority of portfolio shares show substantial persistence.

HERE TABLE 1 AND 2

3 Model without capital

We start our analysis of the complete market approach to debt management by using the Lucas and Stokey (1983) model of a barter economy under full commitment. This is the model used by Angeletos (2002) and Buera and Nicolini (2004) and in this section we both replicate their results for our calibration based on US data and create the foundations from which we consider a number of extensions and robustness issues. The economy produces a single good that cannot be stored

[3] Australia, Belgium, Canada, Denmark, Finland, France, Germany, Italy, Japan, Mexico, Netherlands, New Zealand, Spain, US

and every period the agent is endowed with one unit of time that it allocates between leisure and labour, the latter being the only input of production. Technology for every period t is given by:

$$c_t + g_t \leq \theta_t \left(1 - x_t\right), \tag{1}$$

where x_t, c_t and g_t represent respectively leisure, private consumption and government expenditure and θ_t represents a productivity shock.

We assume $h_t \equiv (g_t, \theta_t)$ are stochastic and exogenous and represent the only sources of uncertainty in the model. In every period there is a finite number, N, of possible realizations of these shocks $\overline{h}^n \equiv (\overline{g}^n, \overline{\theta}^n)$, $n = 1, ..., N$. As usual, $h^t = (h_0, h_1, ..., h_t)$ represent the history of shocks up to and including period t.

Preferences are given by

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[U\left(c_t\left(h^t\right)\right) + V\left(x_t\left(h^t\right)\right)\right], \tag{2}$$

where $0 < \beta < 1$. For simplicity we assume utility is separable in consumption and leisure and that U and V are strictly increasing and strictly concave in their respective arguments.

The government has two instruments to finance government expenditure. They can impose a flat tax on labour or they can issue debt/lend to the consumer.

To reproduce the results of Angeletos, and Buera and Nicolini, we shall first of all consider the complete market case where governments can issue a full set of contingent claims and then show how this approach can be used to pin down the optimal debt structure for a government in the case where it does not have access to these contingent securities. The case of complete markets requires that the government issue N distinct contingent bonds, each contingent on $\overline{h}^n$ for $n = 1, ...N$. The quantity $b_t\left(h^t, \overline{h}^n\right)$ denotes the amount of government bonds $b_t\left(h^t, \overline{h}^n\right)$ issued in period t that will pay one unit of consumption in period $t+1$ in the event that $h_{t+1} = \overline{h}^n$ is realized. Just like all choices at t, it depends on the given the history of the shocks h^t.

The budget constraint of the consumer is given by:

$$\begin{aligned} c_t\left(h^t\right) + & \sum_{n=1}^{N} q_t\left(h^t, \overline{h}^n\right) \ b_t\left(h^t, \overline{h}^n\right) \\ \leq \quad & \left(1 - \tau_t^x\left(h^t\right)\right) w_t\left(h^t\right)\left(1 - x_t\left(h^t\right)\right) + b_{t-1}\left(h^{t-1}, h_t\right), \end{aligned} \tag{3}$$

where $q_t\left(h^t, \overline{h}^n\right)$ is the price in terms of consumption of one bond $b_t\left(h^t, \overline{h}^n\right)$, $\tau_t^x\left(h^t\right)$ is the tax on labour and $w_t\left(h^t\right)$ is the wage earned by the consumer.

Finally, the government faces the constraint:

$$g_t\left(h^t\right)+b_{t-1}\left(h^{t-1},h_t\right)\leq\tau_t^x\left(h^t\right)w_t\left(h^t\right)\left(1-x_t\left(h^t\right)\right)+\sum_{n=1}^{N}q_t\left(h^t,\overline{h}^n\right)b_t\left(h^t,\overline{h}^n\right). \tag{4}$$

Let c denote the sequence of all consumptions $\{c_0,c_1,...\}$, and similarly for all other variables. A competitive equilibrium is defined as a feasible allocation (c,x,g), a price system (w,q) and a government policy (g,τ^x,b) that, given the price system and government policy, solves the firm's and consumer's first order conditions and also satisfies the sequence of government budget constraints (4).

The optimal Ramsey problem is to choose policy by selecting the competitive equilibrium that maximizes (2). As shown, for example, in Chari and Kehoe (1999), this is equivalent to maximizing utility with (1) and the constraint

$$E_0\sum_{t=0}^{\infty}\beta^t\left[c_t\left(h^t\right)U_{c,t}\left(h^t\right)-(1-x_t\left(h^t\right))V_{x,t}\left(h^t\right)\right]=b_{-1}U_{c,0}\left(h_0\right), \tag{5}$$

where b_{-1} is the amount of liabilities/assets inherited by the government in period 0, U_c is the marginal utility of consumption and V_x is the marginal utility of leisure.

Equation (5) turns out to summarize all equilibrium constraints. Under the assumption of complete markets, given c,x that satisfy (5) it is always possible to back out the contingent bond holdings that complete the markets. This is done in the following way: given c,x, define a sequence of random variables z as

$$z_t\left(h^{t-1},h_t\right)\equiv E_t\sum_{s=0}^{\infty}\beta^s\left[c_{t+s}\left(h^{t+s}\right)\frac{U_{c,t+s}\left(h^{t+s}\right)}{U_{c,t}\left(h^t\right)}-(1-x_{t+s}\left(h^{t+s}\right))\frac{V_{x,t+s}\left(h^{t+s}\right)}{U_{c,t}\left(h^t\right)}\right], \tag{6}$$

If the government issues in period $t-1$ an amount of debt/credit such that, for every n,

$$b_{t-1}\left(h^{t-1},\overline{h}^n\right)=z_t\left(h^{t-1},\overline{h}^n\right) \tag{7}$$

this amount of bonds precisely satisfies the budget constraints period by period for the sequences c,x considered, equilibrium prices q and tax rates.

Angeletos, Buera and Nicolini use these z to derive the optimal structure of government debt when there is no contingent debt but there are bonds at different maturities. The trick is that despite the fact that the government cannot issue explicitly state contingent debt variations in equilibrium bond prices enable the government to create through their debt management a "synthetic" portfolio that effectively achieves the complete market outcome. These are the details.

Assume now that the government can issue only a sequence of non contingent bonds of different maturities. We assume throughout the paper that the number of maturities equals N, for now assume that the maturities are given by $j = 1, ..., N$, for each j the bond issued in period t that promises to pay one unit of consumption in period $t+j$ is denoted b_t^j, and let p_t^j denote the market price of this bond in terms of consumption in period t, both variables a function of h^t. Moreover, assume that in every period the government buys back the entire stock of outstanding debt, so that the budget constraint of the government is

$$g_t\left(h^t\right) + \sum_{j=0}^{N-1} p_t^j\left(h^t\right)\ b_{t-1}^j\left(h^{t-1}\right) \leq \tau_t^x\left(h^t\right) w_t\left(h^t\right)\left(1 - x_t\left(h^t\right)\right) + \sum_{j=1}^{N} p_t^j\left(h^t\right)\ b_t^j\left(h^t\right) \tag{8}$$

and symmetrically for the consumer, where $p_t^0 \equiv 1$. Equilibrium prices satisfy $p_t^j(h^t) = \beta^j \frac{E_t\left(U_{c,t+j}\left(h^{t+j}\right)\right)}{U_{c,t}(h^t)}$.

Buera and Nicolini (2004) and Angeletos (2002) prove that if there are at least as many maturities as there are states of the world and if bond prices are sufficiently variable, then one can choose each period a portfolio of maturities $(b_t^1, ..., b_t^N)$ such that

$$\sum_{j=0}^{N-1} p_t^j\left(h^t\right)\ b_{t-1}^j\left(h^{t-1}\right) = z_t\left(h^t\right) \tag{9}$$

almost surely, for all t. This can be done because even though the stock of bonds issued in $t-1$ is not a function of the realization of h_t, today's value of last period's debt $\sum_{j=0}^{N-1} p_t^j\left(h^t\right)\ b_{t-1}^j\left(h^{t-1}\right)$ is state contingent by the fact that bond prices vary according to the state of nature today. In other words, despite the non-availability of contingent bonds, the government can effectively complete the markets and reproduce the Ramsey allocation of the complete market problem.

To see how markets can be effectively completed with this approach take as a special case $\theta_t = \bar{\theta}$ and that government expenditure can take only two values: $\overline{g}^H > \overline{g}^L > 0$ and it is a two-state Markov process with probabilities π_{HH} and π_{LL} of remaining in the same state.

If we further assume $b_{-1}^j = 0$ for $j = 1, 2$, it is well known that in this case the variables dated t in the Ramsey allocation depend only on the shock g_t and not on past history, so that we have $z_t\left(h^{t-1}, \overline{g}^i\right) = \overline{z}^i$, and also that $p_t^1\left(h^{t-1}, \overline{g}^i\right) \equiv \overline{p}^i$ for $i = H, L$ and for all t. Assuming in addition that $g_0 = \overline{g}^H$ it turns out $\overline{z}^H = 0 < \overline{z}^L$.

The bondholdings that insure (9) with the time-invariance just mentioned satisfy for all t

$$b_{t-1}^1\left(h^{t-1}\right) + \overline{p}^i\ b_t^2\left(h^{t-1}\right) = \overline{z}^i \quad \text{for } i = H, L$$

This gives two equations to solve for bond-holdings that deliver

$$\begin{pmatrix} b^1_{t-1}(h^{t-1}) \\ b^2_{t-1}(h^{t-1}) \end{pmatrix} = \begin{pmatrix} 1 & \overline{p}^H \\ 1 & \overline{p}^L \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \overline{z}^L \end{pmatrix} = \begin{pmatrix} \frac{\overline{p}^H \overline{z}^L}{\overline{p}^H - \overline{p}^L} \\ \frac{-\overline{z}^L}{\overline{p}^H - \overline{p}^L} \end{pmatrix} \equiv \begin{pmatrix} B^1 \\ B^2 \end{pmatrix} \tag{10}$$

for all t. The necessary and sufficient condition for this problem to have a solution is that $\overline{p}^L \neq \overline{p}^H$. Notice that this implies that for this particular model the amount of debt issued in each maturity is the same in all periods. Moreover, in the standard case where the utility function implies that the Ramsey solution satisfies $c_H < c_L$, if U' is not constant $\overline{p}^H < \overline{p}^L$ and the above equation gives $B^2 > 0$ and $B^1 < 0$.

Therefore, the optimal policy is to hold short term assets and issue long term liabilities -as stressed by Angeletos (2002) and also Barro (1999) and (2003) and Nosbusch (2005). The reason is that the long bonds are the ones that have a contingent payoff, since their payoff is next period's bond price that depends on g_{t+1}. The above formula requires that $B^2 \left(\overline{p}^H - \overline{p}^L\right) = \overline{z}^H - \overline{z}^L$, so that the long bond is solely responsible for matching the variability of the z's. The level of short bonds B^1 is chosen to insure that the total value of debt is in line with the level of discounted surpluses z. It is because $\overline{p}^L \neq \overline{p}^H$ that a constant level of bonds is able to insure government against shocks, but since the one-period ahead variability of long rates $\left(\overline{p}^H - \overline{p}^L\right)$ is not very large (this happens both in the model and in the real world), a very large value for B^2 may be needed for this equation to hold.

In order to quantify this solution and to explore a broader range of assumptions we utilise numerical methods. We assume the utility function:

$$\frac{c_t^{1-\gamma_1}}{1-\gamma_1} + \eta \frac{x_t^{1-\gamma_2}}{1-\gamma_2}$$

and set $\beta = 0.98$, $\gamma_1 = 1$[4] and $\gamma_1 = 2$ and we set η such that the government's deficit equals zero in the non stochastic steady state. We also use the steady state condition to fix the fraction of leisure as 30% of the total time endowment. We assume that $b_{-1} = 0$; there is no debt inherited from the past. We calibrate the process of government spending using annual US data of government consumption expenditures from 1950 to 2005 (NIPA source). We approximate a two state symmetric Markov process of the logarithm, of government consumption. Under this formulation the level of government spending in the two states, $i = H, L$, is $\overline{g}^i = \exp(\log(g^*) + \xi_i)$, the transition probabilities are $\pi^g_{HH} = \pi^g_{LL} = 0.975$ and $\xi_H = -\xi_L = 0.157$, with g^* equal to 25% of GDP

[4] We choose utility to be logaritmic in consumption in order to simplify in the next section. Under log utility, capital will be taxed only in periods 0 and 1 and they will be zero for all future periods.

in the non stochastic steady state. We estimate a similar process for technology shocks, using the same period and the US Solow residual[5]. The level of productivity in the two states is $\theta_i = \exp\left(\log\left(\theta^*\right) + \phi_i\right)$ where $\phi_H = -\phi_L = 0.0351$, $\theta^* = 1$ and we assume $\pi^{\theta}_{HH} = \pi^{\theta}_{LL} = 0.9875$.[6] We report results for an economy facing either productivity or expenditure shocks alone and we also consider the case where both shocks can occur simultaneously. In the latter case we assume that the two processes are independent.

To check for robustness we also explore also different levels of persistence of the shocks. As the volatility of shocks is a critical component for the volatility of the yield curve, which is in turn a key ingredient in determining the optimal debt structure. We assume that the variance of the intermediate cases is equal to the unconditional variance of the calibrated case. In particular we show results for $\rho\Delta + (1-\rho)\, I$ where $\Delta = \begin{pmatrix} \pi_{HH} & \pi_{HL} \\ \pi_{LH} & \pi_{LL} \end{pmatrix}$ and $I = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$.

Table 3 reports the results from simulating these various different versions of the basic model of Angeletos, Buera and Nicolini. We show the unconditional average of the ratio of the value of debt positions with total output i.e 7.50 means a position of 750% of GDP.

We report the cases only with the g shock, only θ shock, and both shocks, and for each case we consider the calibration for a serially correlated shock as in the data and the iid or close to iid shock for comparison. Because the maturity structure of debt depends critically on the term structure of the interest rates these are also reported and are as expected. The table reports the rates in the high (H) and the low (L) case of the short interest rate, R_1, and the long rate, 30 year rate, R_{30}.

HERE TABLE 3

In the case where the economy is subject to either government expenditure shocks or productivity shocks the debt management agency need only issue two different maturities. When we allow for both expenditure and productivity shocks there are four possible realizations and four different

[5] We have estimated the Solow residual using US real GDP (NIPA source), real non residential fixd assets (NIPA source), total hours worked (OECD dataset).

[6] We estimate the following process for government spending:

$$\log\left(g_t\right) = \rho_0 + \rho_1 \log\left(g_{t-1}\right) + \varepsilon_t$$

with $\varepsilon_t \sim N\left(0, \sigma_g^2\right)$ .From our estimate $\rho_1 = 0.95$ and $\sigma_g = 0.045$.We calibrate a simmetric two state Markov process matching the variance and covariance of the process.

Similarly for the technology shock:

$$\log\left(\theta_t\right) = \varphi_0 + \varphi_1 \log\left(\theta_{t-1}\right) + \zeta_t$$

with $\zeta_t \sim N\left(0, \sigma_\theta^2\right)$ .From our estimate $\sigma_\theta = 0.0078$. We calibrate a simmetric two state Markov process assuming $\varphi_1 = 0.975$ matching the variance and covariance of the process.

maturities need to be issued in order to obtain the complete market outcome. As in Buera and Nicolini (2002) the maturities issued are selected by choosing the combination that yields the *lowest* absolute value for the debt positions; e.g., in the case of serially correlated g, if the maturity of the long bond is different from 30 it leads to even larger debt holdings.

Reassuringly the simulations confirm the results of both Angeletos and Buera and Nicolini. In the case of serially correlated g, the positions are between fourteen and fifteen times total production. Confirming the result in (10) the government holds short term assets and long term liabilities. The size of the positions are enormous, the reason is the limited fluctuations in the yield curve achieved in the case of persistent shocks. These small variations in yields/bond prices need to be magnified by extreme holdings in order to help the government reach the complete market outcome in the face of fiscal shocks.

With i.i.d g, bond holdings are around 80% of GDP, so the positions are less extreme. This is because of two reasons: first, with less persistence the effect on the permanent wealth of the government is less (z is less volatile), second, the long interest rate is more volatile. In the case of only productivity shock a similar pattern arises.

When we allow for both shocks to occur simultaneously the results are even more disturbing: the optimal positions are even larger and the government does not get in debt on long bonds. The government now issues short term debt and holds lond term assets, but also issues medium-long term debt (20 period bonds) and holds short-medium term assets (6 periods). Positions at each maturity are extremely large relative to GDP (issuing debt 1271 times GDP into 20 year bonds!).[7]

The case of simultaneous expenditure and productivity shocks highlights another possible difficulty with the approach of effectively completing markets: the maturity structure that minimises the sum of the absolute positions is very sensitive to the exact nature of the shocks. The table shows that the less persistent are shocks the more the government likes to use shorter maturities. If government in the $\rho = 0.333$ case were constrained to using the same maturities as in the case of correlated shocks (1-, 6-, 20- and 30-period bonds) then the matrix of returns becomes near singular and the optimal positions in the close to i.i.d case are almost plus and minus infinity.

[7] The pattern of asset and liablities in this case is different than the one reported by Buera Nocolini (2002). In their simulation with four bonds, the one with shorter maturity is an asset, the second a liability, the third an asset and finally the last a liability. The case reported hee is the opposite. This is due to the slope of the yield curve in the HH state and LL state. In our case the first is downward sloping and the latter upward sloping. In their case it is the opposite. This is due to the serial correlation of the technology shock with respect to the the serial correlation of the government spending shock.

4 Allowing for Capital Accumulation

In this section we use the complete market optimal tax model of Chari et al (1994) to consider Angeletos' (2002) claim that capital mitigates the extreme positions needed to effectively complete markets by using debt management. We once more first outline the complete market Ramsey equilibria where a full set of contingent claims exist and use the solution to this problem to determine the optimal debt structure when governments have access to non-contingent bonds of different maturities.

Assume there are two factors of production : labour $(1-x)$ and capital k, with output being produced through a Cobb Douglas function such that the resource constraint of the economy is given by:

$$c_t + g_t + k_t - (1-\delta)k_{t-1} \leq \theta_t k_{t-1}^{\alpha}(1-x_t)^{1-\alpha} = \theta_t F\left(k_{t-1}, x_t\right), \tag{11}$$

where δ is the depreciation rate of capital. The exogenous shocks h, g and θ are as before. The government now has three policy instruments to finance g: taxes on labour τ^x, taxes on capital τ^k and debt/credit.

For this problem to be of interest we need to restrict capital taxes in two ways. First we need to bound the first period capital tax, otherwise the planner can achieve the first best with no tax distortions by taxing initial capital, so we add the constraint $\tau_0^k \leq \overline{\tau}^k$ for a fixed constant $\overline{\tau}^k$. Also, we need to assume that capital taxes are decided one period in advance, otherwise debt and taxes in equilibrium would be undetermined and the role of debt management would be marred by the possibility of effectively completing markets with the capital tax.[8]

The budget constraint of the consumer is now:

$$\begin{aligned} c_t\left(h^t\right) + k_t\left(h^t\right) + \sum_{n=1}^{N} q_t\left(h^t, \overline{h}^n\right)\ b_t\left(h^t, \overline{h}^n\right) &\leq \left(1-\tau_t^k\left(h^{t-1}\right)\right) r_t\left(h^t\right) k_{t-1}\left(h^{t-1}\right) \\ &\quad + \left(1-\tau_t^x\left(h^t\right)\right) w_t\left(h^t\right)\left(1-x_t\left(h^t\right)\right) + b_{t-1}\left(h^{t-1}, h_t\right) \end{aligned} \tag{12}$$

and for the government:

$$\begin{aligned} g_t\left(h^t\right) + b_{t-1}\left(h^{t-1}, h_t\right) \quad \leq \quad & \tau_t^k\left(h^{t-1}\right)\ r_t\left(h^t\right)\ k_{t-1}\left(h^{t-1}\right) \\ & + \tau_t^x\left(h^t\right)\ w_t\left(h^t\right)\ \left(1-x_t\left(h^t\right)\right) \\ & + \sum_{n=1}^{N} q_t\left(h^t, \overline{h}^n\right)\ b_t\left(h^t, \overline{h}^n\right) \end{aligned} \tag{13}$$

[8] See, for example, Chari and Kehoe (1999) for a detailed discussion of this issue. Note, also, that we denote τ_t^k the tax that is applied to capital income in period t even though this tax is set with information on h^t.

where $r_t\left(h^t\right)$ denotes the return on capital.

The Ramsey problem is as before but augmented with the consumer's Euler equation with respect to capital, viz.,

$$U_{c,t}\left(h^t\right)=\beta E_t\left\{U_{c,t+1}\left(h^{t+1}\right)\left[\left(1-\tau_{t+1}^k\left(h^{t+1}\right)\right)r_{t+1}\left(h^{t+1}\right)+1-\delta\right]\right\}. \tag{14}$$

As before, we first study the case that there is a full array of contingent claims. Again, the results of Chari and Kehoe (1999) guarantee that the implementability constraint

$$\begin{aligned}E_0\sum_{t=0}^{\infty}\beta^t\left[c_t\left(h^t\right)U_{c,t}\left(h^t\right)-(1-x_t\left(h^t\right))V_{x,t}\left(h^t\right)\right] &= U_{c,0}\left(h^0\right)\left[(\left(1-\tau_0^k\left(h_0\right)\right)F_{k,0}\left(h_0\right)\right. \\ &\quad \left.+1-\delta)k_{-1}+b_{-1}\right],\end{aligned} \tag{15}$$

plus feasibility are necessary and sufficient conditions for an optimum.

For given sequences c,k,x that satisfy the above constraint, we build the expected discounted sum of future surpluses of the government z:

$$\begin{aligned}z_t^k\left(h^{t-1},h_t\right) &\equiv E_t\sum_{s=0}^{\infty}\beta^s\left[c_{t+s}\left(h^{t+s}\right)\frac{U_{c,t+s}\left(h^{t+s}\right)}{U_{c,t}\left(h^t\right)}-(1-x_{t+s}\left(h^{t+s}\right))\frac{V_{x,t+s}\left(h^{t+s}\right)}{U_{c,t}\left(h^t\right)}\right] \\ &\quad -\left[\left(1-\tau_t^k\left(h^t\right)\right)F_{k,t}\left(h^t\right)+1-\delta\right]\ k_{t-1}\left(h^{t-1}\right).\end{aligned}$$

As before, (7) gives the portfolio that insure the period by period budget constraint given c,k,x.

The addition of capital means it is no longer possible to solve our simple two state Markov example of the previous section to reach qualitative conclusions regarding optimal debt structure. This is because $z_t^{k,H}$ and $z_{t-1}^{k,L}$ now are time varying. More precisely, it is well known that the Ramsey solution satisfies the following recursive structure:

$$\begin{bmatrix}k_t\\ c_t\\ x_t\\ \tau_t^x\\ \tau_t^k\end{bmatrix}=G(h_t,k_{t-1})$$

for all $t\geq 1$ for some time-invariant function G, using proposition 1 A) in Marcet and Scott (2005), we conclude that there is a time-invariant function $D:R^3\rightarrow R$ such that

$$D(k_{t-1},\overline{h}^n)=z_t^k(h^{t-1},\overline{h}^n)$$

for all $t \geq 1$, all h^t and all n. In other words, even though $z_t(h^{t-1}, \overline{h}^n)$ depends on all past shocks, the recursive structure of the Ramsey solution, all past shocks are summarized in the last capital stock. Furthermore, (7) implies that $D(k_{t-1}, \overline{h}^n) = b_{t-1}(h^{t-1}, \overline{h}^n)$.

Let us now consider the case of N maturities. It is easy to see that the government using N maturities is able to effectively complete markets if it can find bond holdings such that

$$\sum_{j=0}^{N-1} p_t^j(h^{t-1}, \overline{h}^n)\ b_{t-1}^j\left(h^{t-1}\right) = D(k_{t-1}, \overline{h}^n)$$

for all t, all h^t and all n. Since the recursive structure of the Ramsey solution implies $P^j\left(k_{t-1}, \overline{h}^n\right) = p_t^j(h^t)$ for N time-invariant functions P^j, for all $t \geq 1$, all h^{t-1} and all n, this gives N equations to solve for the unknowns $(b_{t-1}^1\left(h^{t-1}\right), ..., b_{t-1}^N\left(h^{t-1}\right))$ in each period. More precisely, letting $\Pi : R_+ \rightarrow R^{N\times N}$ be defined as

$$\Pi(k) \equiv \begin{bmatrix} 1 & P^1\left(k, \overline{h}^1\right) & P^{N-1}\left(k, \overline{h}^1\right) \\ \vdots & & \vdots \\ 1 & P^1\left(k, \overline{h}^N\right) & P^{N-1}\left(k, \overline{h}^N\right) \end{bmatrix}$$

and assuming $\Pi(k_t)$ is invertible with probability one,[9] then the time-invariant function $B : R_+ \rightarrow R^N$ given by

$$\begin{bmatrix} b_{t-1}^1\left(h^{t-1}\right) \\ \vdots \\ b_{t-1}^N\left(h^{t-1}\right) \end{bmatrix} = \Pi(k_{t-1})^{-1} \begin{bmatrix} D(k_{t-1}, \overline{h}^1) \\ \vdots \\ D(k_{t-1}, \overline{h}^N) \end{bmatrix} \equiv B(k_{t-1}) \tag{16}$$

gives the portfolio that effectively completes the markets for all $t \geq 1$, all h^t.[10]

This says that under capital the amount issued of maturity j at time t is not constant as in the ABN case, but now it is a time-invariant function only of this period's capital.

Table 4 summarizes the results for simulations of the model with capital accumulation. We set $\alpha = 0.4$, the depreciation rate $\delta = 0.05$, assume that the initial value of government debt is always zero, the initial capital stock is set equal to its deterministic steady state value.[11] and no capital taxes in the first period. Unlike in the ABN case, now bond holdings of each maturity are not

[9] The "probability" statement is with respect to the distribution induced by the Ramsey solution.

[10] Notice that even though the model is only fully recursive for $t \geq 1$, because variables such as consumption or capital are only time-invariant functions after period 1, the portfolio that completes the markets turns out to be time-invariant for $t \geq 0$.

[11] We define the deterministic steady steate as the steady state where shocks are constant (g^* and θ^*), there are no capital taxes and labour taxes are constant.

constant, they vary with capital. We therefore report both the average structure of the value of debt and also the average of 5% minimum and maximum values of the values of position taken. We consider the same possibilities for the structure of the exogenous shocks as in Table 3.

HERE TABLE 4

Focusing first on the cases with either only g or only θ shocks, the most noticeable feature of Table 4 is that the addition of capital accumulation has made the optimal debt positions even more extreme. With only expenditure shocks the optimal portfolio is now (-24.89, 21.52) compared with (-14.62, 15.12) for the endowment economy, and (-9.16, 7.09) compared with (-0.79, 0.82) for iid government spending shocks. The range of variation in the debt positions is also substantial - especially for the case of serially correlated shocks. For instance, the desired short position on one-period bonds now varies between -30.25 times GDP to -17.90 times GDP. The intuition for why capital accumulation leads to even more extreme positions is simple enough - capital accumulation offers another channel through which to smooth consumption and so leads to less variation in the long bond prices, and larger debt positions. This is highlighted by the flatter term structure (R_1 and R_{30}) in the high government spending (H) and low government spending (L) when capital is introduced.

Another interesting feature of these simulations is that the introduction of capital can reverse the portfolio recommendation of complete markets, outlined in the previous section, that governments should issue long term debt and invest in short term liabilities.

HERE FIGURE 7

In the case of correlated productivity shocks it is now optimal to invest in long term assets and issue short term debt. Figure 7 shows the average of the positions at different maturities. The value of debt with the different maturity combination is reported (one period bond-two period bond, one period bond-three period bond...) in the first panel, as well as the absolute value of each combination in the second panel and the term structure of interest rates in the H and L case. With a high productivity shock expected consumption growth is above average and so the return on short term bonds exceeds that on long term bonds. By contrast, with a negative productivity shock the government benefits from lower interest rates on short term debt as the yield curve slopes upwards. Buera and Nicolini (2004) offer an example in the case of an endowment economy where this reversal of the standard prediction (short short end and issue long term) is achieved but

Table 4 shows us this is a very natural consequence of allowing for capital accumulation. Moreover the minimum of the positions is achieved with a combination of a one and a sixteen period bond differently from the other cases where the minimum is reached with the combination of a one period bond and the thirty period bond.

The addition of capital accumulation does lead to a small reduction in the absolute magnitude of the desired debt positions in the case where both shocks can occur simultaneously. With productivity shocks reversing their impact on the term structure under capital accumulation they now exert an offsetting influence on the impact of government expenditure shocks and as a consequence positions for optimal debt are less extreme in this case compared to the endowment economy. However this remains a Phyrric victory for the complete market approach. Not only are the positions are extremely large but they continue to contradict OECD data as the debt positions here zig-zag between positive and negative values with only small changes in maturity. Secondly, the range of variation in the optimal debt positions is now huge. For example the sixteen period bond in the case of technology and government spending shocks moves between -1113 times GDP and -628 times GDP.

Therefore the addition of capital accumulation only serves to worsen the disparity between the implications of the complete markets approach to debt management and observed OECD practice. It also gives positions that are very sensitive to the exact nature of the model. The model predicts positions that are too extreme, too volatile and that vary too much across maturities and across periods of time to be a useful guide for debt management.

5 Habits and Term Structure Volatility

Our previous results could be criticised because what matters for the size of the positions is the variability of long rates and the above model generates low observed levels of volatility in the term structure. By underestimating the volatility in the yield curve we overestimate the required positions necessary to achieve the complete market outcome.

It is clear that what matters for the size of positions is the 'conditional' variability of interest rates. In the case $N = 2$ equation (16) gives

$$B^2(k_t) = \frac{D(k_t, \overline{h}^1) - D(k_t, \overline{h}^2)}{P^1\left(k_t, \overline{h}^1\right) - P^1\left(k_t, \overline{h}^2\right)} \tag{17}$$

and in the denominator there is the difference in interest rates at $t + 1$ for a given history at time t, this quantity is related to the *conditional* variance of interest rates.

To assess the potential scale of this bias we how well the model explains the observed one step ahead forecast error of the term spread (spr_t).

In our model without capital the ratio of the expected variance of the term spread between a one and ten period bond over the average of the short term interest rate $\left(\frac{\sqrt{E(var_{t-1}(spr_t))}}{Er_t}\right)$ is small, 0.00105 for the model with government spending shocks, 0.00015 for the model with technology shocks and only 0.048 even if we allow for both shocks.

To find this ratio in the data we specify an equation to predict the spread between the ten- and one- year yield for US bonds between 1949 and 2004. We use the following regression:

$$spr_t = \alpha_1 + \alpha_2 spr_{t-1} + \alpha_3 \frac{def_{t-2}}{gdp_{t-2}} + \alpha_4 r_{t-2} + \varepsilon_t$$

where $\frac{def_t}{gdp_t}$ is the primary deficit/GDP ratio and r_t is the one year real interest rate[12]. We interpret the residual ε as the one-step-ahead forecast error of the spread, and the variance of ε is our measure of $var_{t-1}(s_t)$. This leads to an estimate of $\frac{\sqrt{E(var_{t-1}(\varepsilon_t))}}{Er_t}$ equal to 0.341 confirming the inadequacy of our theoretical models to match this volatility.

To overcome this problem and in order to raise the volatility of the yield curve we introduce habits into our utility function. This approach has been widely used as a means of matching asset market puzzles in the literature e.g Constantinides (1990), Campbell and Cochrane (1999). In essence it makes the term structure of interest rates a function of the rate of change of consumption and in this way provides additional volatility.

With habits the utility function of the consumer becomes:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[U\left(c_t\left(h^t\right), c_{t-1}\left(h^{t-1}\right)\right) + V\left(x_t\left(h^t\right)\right)\right], \tag{18}$$

The resource and budget constraints are the same as in the endowment model developed in section 3..

With habits the marginal utility of consumption of the consumer becomes:

$$U_{c_t}\left(h^t\right) \equiv \frac{\partial\, U\left(c_t\left(h^t\right), c_{t-1}\left(h^{t-1}\right)\right)}{\partial c_t\left(h^t\right)} + \beta E_t \left[\frac{\partial\, U\left(c_{t+1}\left(h^{t+1}\right), c_t\left(h^t\right)\right)}{\partial c_t\left(h^t\right)}\right] \tag{19}$$

which depends on both past and expected future levels of consumption explicitly.

[12] This is the final specification reached through standard model selection procedures starting with a model with four lags of the spread, the deficit/GDP ratio, nominal interest rates and inflation.The criteria of selection of the model has been to maximize the R^2 of the regression.

In the case with habits and no capital accumulation the optimal solution satisfies

$$\begin{bmatrix} c_t \\ x_t \end{bmatrix} = G(h_t, c_{t-1})$$

for $t > 0$, for some time-invariant function G (different from the function in the previous section) and we have similarly as before that the bond positions are a function of current consumption only. We set up the Ramsey problem as in section 3 with the above definition for U_{c_t}, we find the value of the outstanding debt under complete markets and use this to determine optimal debt structure. As was the case with capital accumulation, the introduction of habits means that both the level of debt and its composition vary, in this case it varies with consumption and we have an equation like (16) determining the portfolio of bonds that effectively completes the markets, but now c_t plays the role of k_t.

We incorporate habits into the utility function by assuming:

$$\frac{(c_t - \chi c_{t-1})^{1-\gamma_1}}{1-\gamma_1} + \eta \frac{x_t^{1-\gamma_2}}{1-\gamma_2},$$

and we calibrate the degree of habit persistence, χ, in order to match the forecast error of the term spread in the US data. Matching this moment should remove any bias against complete market models of debt management by underprediction of the conditional variability of interest rates.

HERE TABLE 5

Table 5 summarizes the results from simulations of our various cases. We report results only for just the government expenditure shock or expenditure and productivity shocks. We do not report results for the case of just technology shocks as we were unable to find a value for χ capable of matching the one step ahead forecast error to the one in the data. The positions are for the case in which the government issues only a one and a ten period bond and the rest is calculated as in Table 3.

As suggested the introduction of habits and the greater volatility of the term structure we thereby achieve does reduce the extreme nature of the optimal debt positions required. However they remain extremely large - for instance, in the case of correlated expenditure shocks the desired positions decline from around 31 times output to “only” 800% of output. The introduction of habits, as was the case for capital accumulation, leads to substantial volatility in desired debt structure and once again optimal debt holdings should vary between positive and negative values as maturity increases, according to the results of Table 5. We also once more find that it is no longer optimal to invest heavily in short term assets and issue long term debt. Now the average outcome is rather to short the one period bond as well as the 10 period bond and the 26 period bond, the longest maturity.

Finally, the variation across time in debt positions is the highest of all the models we have seen so far, it is so extreme that governments in some periods issue long term debt worth 76 times output and sometimes go short by 275% in long term debt.

HERE FIGURE 8

Figure 8 reports the policy functions of the value of the debt positions as percentage of total output in the four states. There exist a level of consumption such that the matrix of the returns is not invertible. This point is where the value of the 10 period, 20 period and 27 period bond change sign. The introduction of habits only seems to strengthen the inconsistencies between the implications we derive for the complete market model of debt management, observed OECD debt management practice, and what seems like a reasonable recommendation for debt management.

6 Ruling out Debt Buyback

So far we have assumed that the whole stock of bonds issued in the past was bought back each period (and that all bond holdings were sold). As we have seen in the empirical part of the paper however, period by period government transactions in its own debt are small as a proportion of the total stock of debt and buybacks occurs only rarely. An obvious motivation for this behaviour would be the existence of transaction costs in the secondary market.

For someone familiar with the complete market literature it might seem at first glance that whether the government buys back the whole debt or not is innocuous for the results. It is well known that under no transaction costs and no arbitrage an agent can achieve the same allocations whether the debt is bought back every period or all debt is held until maturity. But the quantity of bonds that needs to be held is different in each case and the evolution across time can be quite different. In this section we analyse the behaviour of debt under the assumption that the government holds the debt until maturity.

For simplicity, let us go back to the endowment case, (no habits, no capital), only g varies and takes only two values $\overline{g}^L$ and $\overline{g}^H$, and initial $g_0 = \overline{g}^H$. The government issues two bonds: a one-period and an M-period bond (denoted $\mathbf{b}^1$ and $\mathbf{b}^M$ respectively).

The period t budget constraint of the consumer is now:

$$c_t - (1-\tau_t^x) w_t (1-x_t) + p_t^1 \mathbf{b}_t^1 + p_t^M \mathbf{b}_t^M = \mathbf{b}_{t-1}^1 + \mathbf{b}_{t-M}^M. \tag{20}$$

where as before $\mathbf{b}_t^1$ and $\mathbf{b}_t^M$ are functions of h^t.

In this case the government issues two kinds of bonds, but it really holds M kinds of bonds every period: in addition to the bonds that mature and produce income at t ($\mathbf{b}_{t-1}^1, \mathbf{b}_{t-M}^M$), the government also holds long bonds that have not yet matured: namely, $\mathbf{b}_{t-M+1}^M$,...,$\mathbf{b}_{t-1}^M$. Even though these non-maturing bonds do not show up in the government's and consumer's budget constraint at t they may nonetheless affect the actions of the government since they influence the income that will be available in the future. It is shown in the appendix that, without buybacks, the government can still effectively complete markets by an equation analogous to (9): in this case it is the value of all currently held bonds that has to be set equal to the discounted sums z:

$$\begin{aligned} \mathbf{b}_t^1 + \sum_{i=0}^{M-2} \overline{p}^{M-i-1,H}\ \mathbf{b}_{t-i}^M + \mathbf{b}_{t+1-M}^M &= 0 \\ \mathbf{b}_t^1 + \sum_{i=0}^{M-2} \overline{p}^{M-i-1,L}\ \mathbf{b}_{t-i}^M + \mathbf{b}_{t+1-M}^M &= \overline{z}^L. \end{aligned} \tag{21}$$

for all $t \geq 0$ given initial conditions $\mathbf{b}_{-1}^1, \mathbf{b}_{-i}^M$ for $i = 1, ..., M-1$. Next we characterize the steady state of bonds and later the evolution of debt.

6.1 Bonds at Steady State

Adapting the analysis in section 3 to any long maturity M we see that

$$\begin{pmatrix} b_t^1 \\ b_t^M \end{pmatrix} = \begin{pmatrix} B^1 \\ B^M \end{pmatrix} \equiv \begin{pmatrix} \frac{\overline{p}^{M-1,H}\ \overline{z}^L}{\overline{p}^{M-1,H} - \overline{p}^{M-1,L}} \\ \frac{-\overline{z}^L}{\overline{p}^{M-1,H} - \overline{p}^{M-1,L}} \end{pmatrix} \tag{22}$$

for all $t \geq 0$. Therefore, B^1, B^M give the steady state of bonds with buybacks.

To find the steady state without buybacks, we set $\mathbf{b}_t^M = \mathbf{B}_{ss}^M$ and $\mathbf{b}_t^1 = \mathbf{B}_{ss}^1$ for all t in (21) and we have

$$\begin{pmatrix} 1\ , & \sum_{i=1}^{M-1} \overline{p}^{i,H} + 1 \\ 1\ , & \sum_{i=1}^{M-1} \overline{p}^{i,L} + 1 \end{pmatrix} \begin{pmatrix} \mathbf{B}_{ss}^1 \\ \mathbf{B}_{ss}^M \end{pmatrix} = \begin{pmatrix} 0 \\ \overline{z}^L \end{pmatrix}. \tag{23}$$

yielding $\mathbf{B}_{ss}^M = \frac{-z_L}{\sum_{i=1}^{M-1} \left(\overline{p}^{M-i,H} - \overline{p}^{M-i,L}\right)}$. For standard utility functions $\overline{p}^{i,H} < \overline{p}^{i,L}$ for all $i = 1, ..., M$ so that, as before, $\mathbf{B}_{ss}^M > 0$ showing that the government issues long bonds as in section 3 and the long bond is responsible for insuring that the variability of the portfolio payoff equals the variability of z.[13]

In the case of $M = 2$ it is clear that $\mathbf{B}_{ss}^M = B^M$ so that having no buyback makes no difference in the steady state. However, in the case of $M > 2$ we can expect $\sum_{i=1}^{M-1} \left(\overline{p}^{i,H} - \overline{p}^{i,L}\right) > \overline{p}^{M-1,H} - \overline{p}^{M-1,L}$ implying $B^M > \mathbf{B}_{ss}^M$, that is, the government now has to issue a lower amount of long bonds every period. How much lower the position with no buyback is depends on M and the equilibrium prices. This means that the transaction costs (if there were any) of issuing long bonds would be lower now.

But the quantity $\mathbf{B}_{ss}^M$ per se is not an indicator of the total debt of the government, since the government has all the unmatured long debt to redeem. It is relevant, therefore, to study the ratio of the *value* of total long debt with and without buyback when bonds are at steady state

$$RVLD^j \equiv \frac{\overline{p}^{M-1,j}\ B^M}{\left(\sum_{i=1}^{M-1} \overline{p}^{i,j} + 1\right) \mathbf{B}_{ss}^M}$$

for $j = H, L$. Even if bonds are at steady state, this ratio is not constant due to the fact that prices change with the realization. To gain some insight on likely values of this ratio, we first use the (rough) approximation $E\left(p_t^i\right) \approx \beta^i$ to claim

$$E\left(RVLD^j\right) \approx \frac{\beta^{M-1}}{\sum_{i=0}^{M-1} \beta^i} \frac{B^M}{\mathbf{B}_{ss}^M}$$

Assuming, in addition, that g is iid, we can use the exact formula (33) for $B^M/\mathbf{B}_{ss}^M$ as derived in the appendix, and taking the limit for the (relevant) case that β is close to one we have

$$E(RVLD) \approx 1 - \frac{1}{M} \quad . \tag{24}$$

Therefore, this approximation suggests that the ratio of debt in long bonds is increasing in M and that it is one half for the lower maturities $M = 2$, and that it is one for very long maturities $M = \infty$.

This formula also gives an insight on the steady state position for short bonds. The ratio of short debt with and without buybacks

$$\frac{B^1}{\mathbf{B}_{ss}^1} = RVLD^H$$

[13] In real business cycle models one has to shut down all shocks in order to obtain constant values for the state variables in the steady state. But in our case one does not have to shut down uncertainty to obtain a constant amount of bonds, because bonds are constant in the steady state distribution, even if the shocks to g occur every period.

where the equality uses the first equations in (23) and in (22), and the definition of $RVLD^H$. Therefore, except for the discrepancy between $RVLD^H$ and $E(RVLD)$, which is likely to be small, we can claim that for iid case and β close to 1

$$\frac{B^1}{\mathbf{B}^1_{ss}} \approx 1 - \frac{1}{M}$$

Again, this ratio goes from .5 to 1 as M goes from 2 to infinity

To summarize, the various approximations above suggest that, in steady state, compared to the financial arrangement of section 3, in the case without buybacks:

- positions on long and short bonds are of the same sign
- positions on long bonds are smaller
- value of total outstanding long debt is larger, the ratio going from 2 to 1 as the length of the maturity goes from the lowest possible to infinity
- position on short bonds is (idem as last bullet point)

6.2 Stability of the steady state

We now analyse the possibility of a transition of the government's debt position towards the steady state starting from arbitrary initial conditions.

Under the buyback case of section 3 convergence is very fast. Notice that the steady state of each type of bond is not influenced by the composition of initial debt b^1_{-1}, b^M_{-1} (see equation (22)) Therefore, the portfolio jumps to steady state in the very first period, and debt positions do not move from then on.[14]

However, with the assumption of no buy back as in the present section convergence to the steady state is less obvious. Simple algebra gives, from (21) that

$$\mathbf{b}^M_t = \frac{z_L}{\overline{p}^{L,M-1} - \overline{p}^{H,M-1}} + \sum_{i=1}^{M-2} \frac{\overline{p}^{H,M-i-1} - \overline{p}^{L,M-i-1}}{\overline{p}^{L,M-1} - \overline{p}^{H,M-1}} \mathbf{b}^M_{t-i}. \tag{25}$$

for all $t \geq 0$, given initial $\mathbf{b}^M_{-i}$, $i = 1, ..., M-2$. This shows that $\mathbf{b}^M_t$ is given by a linear difference equation of order $M-2$. Notice that this difference equation is deterministic, so convergence to steady state can be studied in the standard way. The value of the short bond can then be backed out from (21).

[14] Only the total initial value of debt $b^1_{-1} + \overline{p}^{M-1,H} b^M_{-1}$ matter to determine the discounted sum.

It is easy to see that when $M = 2$ steady state is achieved from any initial condition in two periods[15]. But for the case $M = 3$ (25) gives

$$\mathbf{b}_t^3 = \frac{\overline{z}^L}{\overline{p}^{L,2} - \overline{p}^{H,2}} + \frac{\overline{p}^{H,1} - \overline{p}^{L,1}}{\overline{p}^{L,2} - \overline{p}^{H,2}} \mathbf{b}_{t-1}^3.$$

Using the prices for iid shocks derived in the appendix, the coefficient that multiplies $\mathbf{b}_{t-1}^3$ is:

$$\frac{\overline{p}^{H,1} - \overline{p}^{L,1}}{\overline{p}^{L,2} - \overline{p}^{H,2}} = -\frac{1}{\beta}$$

and as a result the optimal debt structure does not converge, but instead shows increasing oscillations, jumping from positive to negative values of increasing absolute value.

6.3 Simulations

For comparison purposes we calibrate our no buyback model using the same parameters as our endowment economy with buyback of Section 3. Table 6 reports the results when the model reaches the steady state. We assume the only source of uncertainty is government expenditure which follows a two state Markov process.

HERE TABLE 6

We already know from our previous analysis that the government would like to issue fewer bonds every period under the case of no buyback but the value of bond positions that we suggested were based on certain approximations and for the iid case. Table 6 shows that for a more reasonable calibration the value of positions held in short and long bonds are twice as large as the case with buyback. With buyback the government issues long term bonds worth 1512% of output but without buyback the total value of long term outstanding debt is 2778%. On the other hand the government has t issue in every period "only" 0.66 times of GDP of debt. Removing the possibility of buyback therefore only worsens the match between the implications of the complete markets approach and observed debt management practice. A similar result holds in the case of i.i.d shocks although the increase in the position is less extreme.

[15] This happens because in this case $\mathbf{b}_t^2$ does not depend on past $\mathbf{b}^2$'s, therefore $\mathbf{b}_0^2$ is set to its steady state value and it does not change after that. For the short bond, (**??**) says that $\mathbf{b}_0^1$ depends on $\mathbf{b}_{-1}^2$, but $\mathbf{b}_t^1$ is a function only of $\mathbf{b}_{t-2}^2$ which is at steady state for $t \geq 1$. Summarizing, if $M = 2$

$$\begin{aligned} \mathbf{b}_t^1 &= \mathbf{B}_{ss}^1 \quad \text{for } t \geq 1 \\ \mathbf{b}_t^2 &= \mathbf{B}_{ss}^2 \quad \text{for } t \geq 0. \end{aligned}$$

HERE FIGURE 9 AND 10

Figure 9 shows the results for the optimal number of bonds and the market value of debt with and without buyback in steady state when the shocks are correlated. We report the results for all combinations of one period bonds and an M period bond where $M = 1, .., 30$. The upper panel of the figure reports the relative value and the number of new issues every period between the buyback and no buyback cases. The solid line refers to the short bond, the dotted line to the total value of the long bond (axis on the left) and the dashed line to the new issues of the long bond (axis on the right). The ratio of the market values is always close to 0.5 and increases with the maturity of the long term bond . As predicted by our analysis in the last subsection the number of long terms bonds issued in the no buy back case is lower than under buy back. However, the government responds by issuing a greater number of short term bonds when buyback is prohibited as shown by the lower panel. This is because the short bond has to take the opposite sign from the value of the long term bond in order to rebalance the total value of debt. Figure 9 shows the results for the case where the expenditure shock is i.i.d ($\rho = 0$). The difference in the value of debt between buyback and no buyback disappears as the maturity of the long bond rises although even though the value converges the lower panel shows that the number of bonds issued per period remains different.

We can also use our simulations to consider the issues relating to the transition towards the steady state. In our previous analysis we showed that the convergence of the long bond position depends on a $M - 2$ difference equation. For the calibraton behind Table 6 the model converges all the bonds with maturity up to $M = 30$. When the shock is iid the model always diverges.

7 Conclusion

Recent developments in optimal taxation have offered an appealing theory of debt management. Governments should exploit variations in the term structure by issuing bonds of different maturities so that fluctuations in the market value of debt offset fluctuations to government finances without the need to vary tax rates. In this paper we have considered the ability of this (effectively) complete markets model to account for observed debt management practice in the OECD, the sensitivity of the model to changes in its assumptions and usefulness of the policy insights it gives.

Using a wide range of simulations we find that debt positions that effectively complete markets have to be extreme multiples of GDP and always require investing heavily in debt of some maturity. In this way, we generalize the result of Buera and Nicolini to many different environments.

Whereas in practice debt in each maturity is a smooth U shaped function of maturity the complete market model requires asset holdings that change sharply over the maturity profile. Adding capital accumulation or habits only exacerbates the discrepancy between the predictions of the complete market approach and observed debt management practice. As well as leading to even more extreme debt positions, they also recommend huge volatility across time in these positions. Under capital and habits the recommendation of issuing long term debt and investing in short term assets is no longer derived from optimal policy. In some cases the policy advice is the exact opposite and in other models whether the government should go short or long at different maturities varies sharply from period to period. We also show that the simplifying assumption that governments can buyback debt every period and then restructure is not without loss of generality. Ruling out buyback makes the government want to take even more extreme positions and creates significant stability problems in debt management, leading to wildly oscillatory and non-persistent debt issuance practice.

The sharp discrepancy between the implications of the complete market approach to debt management and the observed practice could of course simply reflect the sub-optimality of existing practice. However, we tend to believe that the discrepancy with observed behaviour is so large, and the fragility of the results so great, that this framework is not useful for policy analysis. If governments were to try and implement the policy recommendations that come out of the models under complete markets they would have to place enormous amounts of debt in the market every period and they would have to purchase large amounts of other kind of debt every period. This would entail all kinds of transaction costs, refinancing risks, and it would force some private agents in the economy to take the opposite of the huge positions the government decided to take, possibly facing credit constraints. The exact amounts would not be robust to small changes in the model.

All of this suggests that the approach of analysing debt management by effectively completing the markets through a portfolio of different maturities needs to be modified. Some elements that have been left out of the model should play a prominent role if the optimal policy paradigm is to give useful policy recommendations for debt management. Introducing transaction costs and short sales constraints, refinancing risk, and other frictions seems to be crucial in order to better match the data and to come up with policy recommendations that do not give recommendations that imply enormous costs of this kind. This would lead to a model of truly incomplete markets, where governments have to use debt as a buffer stock, as in the incomplete markets literature.

It may be a good idea for actual governments to get in debt on long bonds only, but this recommendation can not be based on the elements that are captured in the complete markets model. Perhaps issues having to do with transaction costs, refinancing risks etc. will also give this recommendation, but aside from the sign of the long bond position, the kind of policy behaviour that will arise from those models is likely to be very different from the policy that effectively completes the markets with different maturities.

APPENDIX A

Here we give some algebraic details of the formulae used in section 6 under the assumptions that the only uncertainty comes from g, that this can take two values and that $g_0 = g^H$.

Debt positions

First of all, we show that (21) gives the debt portfolio that effectively completes markets with different maturities under no buyback. For this purpose we first show that a sequence c, x satisfies all period t constraint (20) and equilibrium conditions

$$w_t(1-\tau_t^x) = \frac{V_{x,t}}{U_{c,t}} \tag{26}$$

for some portfolio of maturities $\left\{\mathbf{b}_t^1, \mathbf{b}_t^M\right\}_{t=0}^{\infty}$ that satisfies a standard no-ponzi scheme condition if and only if c, x satisfies the constraint in period zero

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[c_t U_{c,t} - (1-x_t)V_{x,t}\right] = \left(\mathbf{b}_{-1}^1 + \sum_{i=0}^{M-2} \overline{p}^{M-i-1,H} \ \mathbf{b}_{-i}^M + \mathbf{b}_{-(M-1)}^M\right) U_{c,0} \tag{27}$$

To prove that this equation is necessary, consider a c, x that satisfies (20) and (26). Add and subtract to both sides of (20) the value of old unmatured bonds held by the government (AE: note that there was an algebraic mistake in the previous formulation, sorry about that. The sum in the right of the equation should start at $i = 1$ instead of at $i = 0$ as it used to, I have changed it throughout the paper, I also think it did not affect the formulas used to compute equilibria, but it would be safe to double check that the computations were done with the right formula):

$$c_t - (1-x_t)\frac{V_{x,t}}{U_{c,t}} + p_t^1 \mathbf{b}_t^1 + \sum_{i=0}^{M-1} p_t^{M-i} \mathbf{b}_{t-i}^M = \mathbf{b}_{t-1}^1 + \sum_{i=1}^{M-1} p_t^{M-i} \mathbf{b}_{t-i}^M + \mathbf{b}_{t-M}^M. \tag{28}$$

where p_t^i is *defined* as its equilibrium value if a secondary market did exist[16]

$$p_t^i \equiv \beta^i E_t \left(\frac{U_{c,t+i}}{U_{c,t}}\right) \quad i = 1, ...M-1 \tag{29}$$

for the particular sequence c being considered. Now we have

[16] We insist that p is "defined" by this equation because in fact these unmatured bonds are not sold so there is no market for them.

$$
\begin{aligned}
p_t^1 \mathbf{b}_t^1 + \sum_{i=0}^{M-1} p_t^{M-i} \mathbf{b}_{t-i}^M &= p_t^1 \mathbf{b}_t^1 + \sum_{i=0}^{M-1} \beta^{M-i} E_t \left(\frac{U_{c,t+M-i}}{U_{c,t}} \right) \mathbf{b}_{t-i}^M \qquad (30) \\
&= \beta E_t \left[\frac{U_{c,t+1}}{U_{c,t}} \left(\mathbf{b}_t^1 + \sum_{i=0}^{M-1} \beta^{M-i-1} \frac{U_{c,t+M-i}}{U_{c,t+1}} \mathbf{b}_{t-i}^M \right) \right] \\
&= \beta E_t \left[\frac{U_{c,t+1}}{U_{c,t}} \left(\mathbf{b}_t^1 + \sum_{i=0}^{M-1} \beta^{M-i-1} E_{t+1} \left(\frac{U_{c,t+M-i}}{U_{c,t+1}} \right) \mathbf{b}_{t-i}^M \right) \right] \\
&= \beta E_t \left[\frac{U_{c,t+1}}{U_{c,t}} \left(\mathbf{b}_t^1 + \sum_{i=1}^{M-1} p_{t+1}^{M-i} \mathbf{b}_{t+1-i}^M + \mathbf{b}_{t+1-M}^M \right) \right],
\end{aligned}
$$

where the first equality follows from the equilibrium formula for p_t^1 and simple algebra, the second equality follows from applying the law of iterated expectations, and the third from simple algebra and the pricing equation for p_t^{M-i}.

Defining total wealth as $W_t = \mathbf{b}_{t-1}^1 + \sum_{i=1}^{M-1} p_t^{M-i} \mathbf{b}_{t-i}^M + \mathbf{b}_{t-M}^M$ we then have

$$
c_t - (1-x_t) \frac{V_{x,t}}{U_{c,t}} + \beta E_t \left(\frac{U_{c,t+1}}{U_{c,t}} W_{t+1} \right) = W_t
$$

and iterating forward and assuming no Ponzi games yields

$$
E_t \sum_{j=0}^{\infty} \beta^j \frac{U_{c,t+j}}{U_{c,t}} \left(c_{t+j} - (1-x_{t+j}) \frac{V_{x,t+j}}{U_{c,t+j}} \right) = \mathbf{b}_{t-1}^1 + \sum_{i=1}^{M-1} p_t^{M-i} \mathbf{b}_{t-i}^M + \mathbf{b}_{t-M}^M \qquad (31)
$$

for all $t = 0, 1, ...$ This proves that (27) holds and is, therefore, a necessary condition.

Now we check that this is also a sufficient condition. For this purpose, take any c, x that satisfies (27); consider the discounted values z defined by (6) and the hypothetical prices p defined by (29) associated with these sequences, assume the matrix

$$
\begin{bmatrix} 1, & p_t^{M-1}(h^{t-1}, g^H) \\ 1, & p_t^{M-1}(h^{t-1}, g^L) \end{bmatrix}
$$

is invertible a.s. for all t all h^t. For a given $t > 0$ and h^t define the portfolios $\widetilde{\mathbf{b}}$ as

$$
\begin{aligned}
\begin{bmatrix} \widetilde{\mathbf{b}}_{t-1}^1(h^{t-1}) \\ \widetilde{\mathbf{b}}_{t-1}^M(h^{t-1}) \end{bmatrix} &= \begin{bmatrix} 1 & p_t^{M-1}(h^{t-1}, g^H) \\ 1 & p_t^{M-1}(h^{t-1}, g^L) \end{bmatrix}^{-1} \\
& \begin{bmatrix} z_t(h^{t-1}, g^H) - \sum_{i=1}^{M-2} p_t^{M-i-1}(h^{t-1}, g^H)\, \widetilde{\mathbf{b}}_{t-1-i}^M(h^{t-1-i}) - \widetilde{\mathbf{b}}_{t-M}^M(h^{t-M}) \\ z_t(h^{t-1}, g^L) - \sum_{i=1}^{M-2} p_t^{M-i-1}(h^{t-1}, g^L)\, \widetilde{\mathbf{b}}_{t-1-i}^M(h^{t-1-i}) - \widetilde{\mathbf{b}}_{t-M}^M(h^{t-M}) \end{bmatrix}
\end{aligned}
$$

given initial conditions $\widetilde{\mathbf{b}}_{-i}^M = \mathbf{b}_{-i}^M$ $i = 1, ..., M$. Notice that, given c, x (and the corresponding z, p) this equation fully defines the evolution of $\widetilde{\mathbf{b}}$ for given initial conditions, for all realizations and all periods. We now show that precisely these portfolios satisfy the period t budget constraints. Clearly, the above equation implies

$$\widetilde{\mathbf{b}}_{t-1}^1(h^{t-1}) + \sum_{i=1}^{M-1} p_t^{M-i}(h^{t-1}, g^j)\widetilde{\mathbf{b}}_{t-i}^M(h^{t-i}) + \widetilde{\mathbf{b}}_{t-M}^M(h^{t-M}) = z_t(h^{t-1}, g^j)$$

for $j = H, L$ all $t > 0$ all h^t so that

$$\widetilde{\mathbf{b}}_{t-1}^1 + \sum_{i=1}^{M-1} p_t^{M-i}\widetilde{\mathbf{b}}_{t-i}^M + \widetilde{\mathbf{b}}_{t-M}^M = E_t \sum_{j=0}^{\infty} \beta^j \frac{U_{c,t+j}}{U_{c,t}} \left(c_{t+j} - (1 - x_{t+j}) \frac{V_{x,t+j}}{U_{c,t+j}} \right) \tag{32}$$

for all $t > 0$ all h^t. This implies

$$\begin{aligned}
\widetilde{\mathbf{b}}_{t-1}^1 + & \sum_{i=1}^{M-1} p_t^{M-i}\widetilde{\mathbf{b}}_{t-i}^M + \widetilde{\mathbf{b}}_{t-M}^M = c_t - (1 - x_t)\frac{V_{x,t}}{U_{c,t}} + E_t \sum_{j=1}^{\infty} \beta^j \frac{U_{c,t+j}}{U_{c,t}} \left(c_{t+j} - (1 - x_{t+j}) \frac{V_{x,t+j}}{U_{c,t+j}} \right) \\
= \; & c_t - (1 - x_t)\frac{V_{x,t}}{U_{c,t}} + \beta E_t \left[\frac{U_{c,t+1}}{U_{c,t}} \sum_{j=0}^{\infty} \beta^j \frac{U_{c,t+1+j}}{U_{c,t+1}} \left(c_{t+1+j} - (1 - x_{t+1+j}) \frac{V_{x,t+1+j}}{U_{c,t+1+j}} \right) \right] \\
= \; & c_t - (1 - x_t)\frac{V_{x,t}}{U_{c,t}} + \beta E_t \left[\frac{U_{c,t+1}}{U_{c,t}} E_{t+1} \left(\sum_{j=0}^{\infty} \beta^j \frac{U_{c,t+1+j}}{U_{c,t+1}} \left(c_{t+1+j} - (1 - x_{t+1+j}) \frac{V_{x,t+1+j}}{U_{c,t+1+j}} \right) \right) \right] \\
= \; & c_t - (1 - x_t)\frac{V_{x,t}}{U_{c,t}} + \beta E_t \left[\frac{U_{c,t+1}}{U_{c,t}} \left(\widetilde{\mathbf{b}}_t^1 + \sum_{i=1}^{M-1} p_{t+1}^{M-i}\widetilde{\mathbf{b}}_{t+1-i}^M + \widetilde{\mathbf{b}}_{t+1-M}^M \right) \right] \\
= \; & c_t - (1 - x_t)\frac{V_{x,t}}{U_{c,t}} + p_t^1\widetilde{\mathbf{b}}_t^1 + \sum_{i=0}^{M-1} p_t^{M-i}\widetilde{\mathbf{b}}_{t-i}^M
\end{aligned}$$

where the first and second equalities follow from (32) and algebra, the third equality is the law of iterated expectations, the fourth equality applies (32) again, and the last equality is because (30) holds for $\widetilde{\mathbf{b}}$ also. Subtracting $\sum_{i=1}^{M-1} p_t^{M-i}\widetilde{\mathbf{b}}_{t-i}^M$ from both the first and last expression of this string of equalities we get that the budget constraint in period t is satisfied.

Therefore, the set of c, x that satisfy the period by period equilibrium constraints is the same as the set of c, x that satisfy the (27), so that the Ramsey optimizer with no buybacks is found, as in the complete markets case, by considering (27) as the only constraint that summarizes all equilibrium conditions. This means that the quantities in this case are the same as with complete markets. Also, from our previous derivation it is clear that the bond portfolios that implement the optimum are given by (32) with the Ramsey allocations and, therefore, that (21) gives the equilibrium bond portfolios.

Formulae for ratio of bonds with and without buybacks

Note that in the case that g is iid bond prices are $\overline{p}^{M-i,j} = \beta^{M-i}\frac{K}{\overline{U}_c^j}$ for $j = H, L$, where $\overline{U}_c^j$ is the marginal utility today if the shock $j = H, L$ is realized and $K \equiv EU_{c,t}$ is expected marginal utility, which is a constant in the iid case, independent of M, g and t. In the iid case, therefore

$$\frac{B^M}{\mathbf{B}_{ss}^M} = \frac{\sum_{i=1}^{M-1}\beta^i\left(\frac{1}{\overline{U}_c^H} - \frac{1}{\overline{U}_c^L}\right)}{\beta^{M-1}\left(\frac{1}{\overline{U}_c^H} - \frac{1}{\overline{U}_c^L}\right)} = \frac{\beta - \beta^M}{\beta^{M-1}(1-\beta)} \tag{33}$$

Using the approximation

$$E\left(p_t^i\right) = \beta^i E\left(E_t \frac{U_{c,t+i}}{U_{c,t}}\right) \approx \beta^i \frac{E\left(E_t U_{c,t+i}\right)}{E\left(U_{c,t}\right)} = \beta^i$$

and (33) we have

$$E(RVLD) \approx \frac{\beta^{M-1}}{\left(\sum_{i=0}^{M-1}\beta^i\right)}\frac{\beta - \beta^M}{\beta^{M-1}(1-\beta)} = \frac{\beta - \beta^M}{1 - \beta^M}$$

Simple algebra shows that this ratio is increasing in M, going from $\frac{\beta-\beta^2}{1-\beta^2}$ for the shortest maturity $M = 2$ to β as $M \to \infty$. Therefore,

$$\frac{\beta - \beta^2}{1 - \beta^2} < \frac{\beta - \beta^M}{1 - \beta^M} < \beta$$

Furthermore, using l 'Hôpital rule,

$$\lim_{\beta \to 1}\frac{\beta - \beta^M}{1 - \beta^M} = 1 - \frac{1}{M}$$

APPENDIX B

Numerical solution of the endowment economy

The Lagrangian of the Ramsey problem of the endowment economy is:

$$\begin{aligned} L &= \sum_{t=0}^{\infty} \beta^t \{U(c_t) + V(x_t) + \lambda [U_{c,t} c_t - V_{x,t}(1-x_t)] \\ &\quad +\nu_t [\theta_t (1-x_t) - c_t - g_t] - \lambda U_{c,0} b_{-1}\} \end{aligned}$$

The first order conditions of the problem are:

for $t > 0$

$$\begin{aligned} U_{c,t} + \lambda (U_{cc,t} c_t + U_{c,t}) - \nu_t &= 0 \\ V_{x,t} - \lambda (V_{xx,t}(1-x_t) - V_{x,t}) + \nu_t &= 0 \\ \theta_t (1-x_t) - c_t - g_t &= 0 \end{aligned} \tag{34}$$

for $t = 0$

$$\begin{aligned} U_{c,0} + \lambda (U_{cc,0} c_0 + U_{c,0}) - \nu_0 + \lambda U_{cc,0} b_{-1} &= 0 \\ V_{x,0} - \lambda (V_{xx,0}(1-x_0) - V_{x,0}) + \nu_0 &= 0 \\ \theta_0 (1-x_0) - c_0 - g_0 &= 0 \end{aligned}$$

We assume $b_{-1} = 0$. This assumption guarantees that there is no difference between period zero and the other periods. Moreover assume that the shocks follow a Markov process of N^2 states, $\{g_i, \theta_j\}$ with $i = 1..N$, $j = 1..N$.

The numerical procedure that we follow is

1) we guess a value for λ;

2) we solve system (34) for every state. We get N^2 values of c, x and the surplus;

3) given the values form point two and the transition probabilities of the states, we compute the infinite sum of the expected value of the surpluses;

4) we check the value of the implementability constraint. We change λ accordingly until

$$E_0 \sum_{s=0}^{\infty} \beta^s \left[c_{t+s} \frac{U_{c,t+s}}{U_{c,0}} - (1 - x_{t+s}) \frac{V_{x,t+s}}{U_{c,0}} \right] = 0$$

5) we repeat 2)-4) until convergence on λ.

6) Given λ and values of c, x and of the surpluses, we compute the prices in all the states of the bonds with different maturities, computing the expectation on marginal utilities;

7) For every maturity we can calculate the value of the matrix of returns and compute:

$$b = Pz$$

where $\underset{(N^2 \times 1)}{b}$ is the vector of bonds, $\underset{(N^2 \times N^2)}{P}$ is the matrix of the returns and $\underset{(N^2 \times 1)}{z}$ is the vector of surpluses.

Numerical solution of the economy with capital

The model with capital is complicated by a difference in the solution between period 0 and the following periods.

Assuming $\tau_0^k = 0$, the Lagrangian of the Ramsey problem is:

$$\begin{aligned} L &= \sum_{t=0}^{\infty} \beta^t \{ U(c_t) + V(x_t) + \lambda [U_{c,t} c_t - V_{x,t}(1 - x_t)] \\ &\quad + \nu_t [F(k_{t-1}, 1 - x_t, \theta_t) + (1-\delta) k_{t-1} - c_t - g_t - k_t] \\ &\quad - \lambda [b_{-1} + (F_{k,0} + 1 - \delta) k_{-1}] U_{c,0} \} \end{aligned}$$

and the first order conditions are:

for $t > 0$:

$$\begin{aligned} U_{c,t} + \lambda (U_{cc,t} c_t + U_{c,t}) - \nu_t &= 0 \\ V_{x,t} - \lambda (V_{xx,t}(1 - x_t) - V_{x,t}) + \nu_t F_{x,t} &= 0 \\ \nu_t - \beta E_t [\nu_{t+1} (F_{k,t+1} + 1 - \delta)] &= 0 \\ F(k_{t-1}, 1 - x_t, \theta_t) + (1-\delta) k_{t-1} - c_t - g_t - k_t &= 0 \end{aligned} \tag{35}$$

for $t = 0$:

$$\begin{aligned} U_{c,0} + \lambda (U_{cc,0} c_0 + U_{c,0}) - \nu_0 - \lambda [b_{-1} + (F_{k,0} + 1 - \delta) k_{-1}] U_{cc,0} &= 0 \\ V_{x,0} - \lambda (V_{xx,0}(1 - x_0) - V_{x,0}) + \nu_0 F_{x,0} - \lambda F_{kx,0} k_{-1} &= 0 \\ \nu_0 - \beta E_t [\nu_1 (F_{k,1} + 1 - \delta)] &= 0 \\ U_{c,0} - \beta E_t \left[U_{c,1} \left(\tau_1^k F_{k,1} + 1 - \delta \right) \right] &= 0 \\ F(k_{-1}, 1 - x_0, \theta_0) + (1-\delta) k_{-1} - c_0 - g_0 - k_0 &= 0 \end{aligned} \tag{36}$$

We assume log utility and $b_{-1} = 0$.

The numerical procedure that we follow is

1) we guess a value for λ;

2) given proposition 1 A) in Marcet and Scott (2005), the structure of system (35) suggests that it is natural to parameterize the function

$$E_t \left[\nu_{t+1} \left(F_{k,t+1} + 1 - \delta\right)\right] = \Phi \left(k_{t-1}, g_t, \theta_t\right),$$

as a function of the states of the economy (k_{t-1}, g_t, θ_t). Given the assumption of log utility: $U_{c,t} = \nu_t = \Phi \left(k_{t-1}, g_t, \theta_t\right)$.

We draw a long realization (10000 periods) of the shocks and we use system (35) in order to converge on the parameters of $\Phi \left(k_{t-1}, g_t, \theta_t\right)$;

3) period 0 is different from the other periods. Now $U_{c,0} \neq \nu_0$.

We guess a value for k_0. For every value of g, θ we solve period 1 using system (35) and we compute $E_0 \left[\nu_1 \left(F_{k,1} + 1 - \delta\right)\right]$, $E_0 \left(U_{c,1} F_{k,1}\right)$, and $E_0 \left(U_{c,1}\right)$.

We solve then system (36) setting $\tau_1^k = \left(1 - \frac{U_{c,0}}{\beta E_0\left(U_{c,1} F_{k,1}\right)} + (1-\delta) \frac{E_0(U_{c,1})}{E_0\left(U_{c,1} F_{k,1}\right)}\right)$, the level of capital tax that satisfies the first order conditions of the consumer. From the solution of the system we get a new value of k_0. We repeat point 3) converging on k_0;

4) long simulation: we perform a long simulation (100000 periods) of the model given k_0 and $\Phi \left(k_{t-1}, g_t, \theta_t\right)$;

5) given the realization for (c_t, x_t) from point 4), we approximate the infinite sum of the surpluses as a function of the states:

$$E_t \sum_{j=t+1}^{\infty} \beta^{j-t} \left\{U_{c,j} c_j + V_{x,j} \left(1 - x_j\right)\right\} = \Omega \left(k_{t-1}, g_t, \theta_t\right);$$

6) short simulation: we draw 10000 realizations of the shocks for the first 50 periods. We solve (35) given k_0 and we compute the infinite sum of the expected surplus in period 0 as an average of the infinite sums using the short simulations for the first 50 periods and $\Omega \left(k_{t-1}, g_t, \theta_t\right)$ for $t = 51$;.

7) we check the value of the implementability constraint and we change λ accordingly;

8) we repeat the procedure from 2) to 7) and converge on λ.

9) Given λ, $\Omega \left(k_{t-1}, g_t, \theta_t\right)$ and the realizations of (c_t, x_t, k_t) of the long simulation of point 4), in order to get the bond prices at different maturities (from 1 to 30 years) we approximate the expectations of future marginal utilities as a function of the current states of the economy.

10) We select 10000 consecutive periods of the long simulation and we calculate the value of (c_t, x_t) and the surplus for every possible realization of the shocks in every period given k_{t-1}.

11) We can now calculate for every period the value of the debt positions selecting any composition of the maturity. In the present paper we always have chosen to have a one period bond and different maturities of the longer bonds.

For every t we solve the system:

$$b_{t-1} = P_t z_t$$

Numerical solution of the economy with consumption habits

As for the endowment economy, the Lagrangian of the Ramsey problem is:

$$\begin{aligned} L &= \sum_{t=0}^{\infty} \beta^t \{U(c_t) + V(x_t) + \lambda [U_{c,t} c_t - V_{x,t}(1-x_t)] \\ &\quad + \nu_t [\theta_t (1-x_t) - c_t - g_t] - \lambda U_{c,0} b_{-1}\} \end{aligned}$$

The first order conditions of the problem are:

for $t > 0$

$$\begin{aligned} U_{c,t} + \lambda (U_{cc,t} c_t + U_{c,t}) - \nu_t &= 0 \\ V_{x,t} - \lambda (V_{xx,t}(1-x_t) - V_{x,t}) + \nu_t &= 0 \\ \theta_t (1-x_t) - c_t - g_t &= 0 \end{aligned} \tag{37}$$

where $U_{c,t} \equiv \frac{\partial\ U(c_t, c_{t-1})}{\partial c_t} + \beta E_t \left[\frac{\partial\ U(c_{t+1}, c_t)}{\partial c_t}\right]$ and $U_{cc,t} = \frac{\partial\ U(c_t, c_{t-1})}{\partial^2 c_t} + \beta E_t \left[\frac{\partial\ U(c_{t+1}, c_t)}{\partial^2 c_t}\right] + \beta \frac{E_t \left[\frac{\partial\ U(c_{t+1}, c_t)}{\partial c_t}\right]}{\partial c_t}$

If we assume that $b_{-1} = 0$, there is no difference between the first and the following periods.

1) we guess a value for λ;

2) given proposition 1 A) in Marcet and Scott (2005), the structure of system (37) suggests that it is natural to parameterize the functions

$$\begin{aligned} E_t \left[\frac{\partial\ U(c_{t+1}, c_t)}{\partial c_t}\right] &= \Phi_1 (c_{t-1}, g_t, \theta_t) \\ E_t \left[\frac{\partial\ U(c_{t+1}, c_t)}{\partial^2 c_t}\right] + \frac{E_t \left[\frac{\partial\ U(c_{t+1}, c_t)}{\partial c_t}\right]}{\partial c_t} &= \Phi_2 (c_{t-1}, g_t, \theta_t) \end{aligned}$$

We draw a long realization (10000 periods) of the shocks and we use system (37) in order to converge on the parameters of Φ_1 and Φ_2;

3) long simulation: we perform a long simulation (100000 periods) of the model, given Φ_1 and Φ_2;

4) given the realization for (c_t, x_t) form point 4), we approximate the infinite sum of the surpluses as a function of the states:

$$E_t \sum_{j=t+1}^{\infty} \beta^{j-t} \left\{ U_{c,j} c_j + V_{x,j} \left(1 - x_j\right) \right\} = \Omega_1 \left(c_{t-1}, g_t, \theta_t\right);$$

5) we check the value of the implementability constraint and we change λ accordingly;

6) we repeat the procedure from 2) to 5) and converge on λ.

7) Given λ, $\Omega_1 \left(c_{t-1}, g_t, \theta_t\right)$ and the realizations of (c_t, x_t) of the long simulation of point 3), in order to get the bond prices at different maturities (form 1 to 30 years), we approximate the expectations of future marginal utilities as a function of the current states of the economy.

8) We select 10000 consecutive periods of the long simulation and we calculate the value of (c_t, x_t) and the surplus for every possible realization of the shocks in every period given c_{t-1}.

9) We can now calculate for every period the value of the debt positions selecting any composition of the maturity. In the present paper we always have chosen to have a one period bond and different maturities of the longer bonds.

For every t we solve the system:

$$b_{t-1} = P_t z_t$$

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Table 1 - Coefficient of Variation

	Money Market	Short	Medium	Long	Indexed	Variable	Other	Non-Market
Australia	0.421	0.203	0.123	0.068	0.412	0.842	0.621	0.818
Belgium	0.233	0.406	0.110	0.072		0.396	1.054	0.518
Canada	0.253	0.180	0.166	0.058		0.059	0.322	0.106
Denmark	0.198	0.116	0.102	0.118		0.550	0.230	
Finland	0.418	0.510	0.134	0.127		0.458		0.210
France	0.227		0.067	0.036		0.260		0.210
Germany	0.656		0.112	0.071		0.416		0.232
Italy	0.379		0.065	0.206		0.183		0.129
Japan	0.118	0.224	0.185	0.123		1.741		0.132
Mexico	0.276		0.743	0.107	0.713	0.676	0.559	0.321
Netherlands	0.597	0.813	0.188	0.164				0.746
New Zealand	0.178	0.276	0.147	0.164	0.825	0.839		0.519
Spain	0.401		0.047	0.309	0.239		0.574	0.409
US	0.118	0.115	0.133	0.055	0.042			0.084

Table 2 - Autocorrelation Coefficients

	Money Market	Short	Medium	Long
Australia	0.863	0.265	1.055	0.358
Belgium	0.723	0.433	0.787	0.510
Canada	0.667	0.845	0.325	0.666
Denmark	0.874	-0.193	0.361	0.553
Finland	0.757	-0.088	-0.32	-0.008
France	0.929		0.788	0.844
Germany	0.791		0.492	0.900
Italy	0.791		0.652	0.964
Japan	0.393	0.766	0.905	0.881
Mexico	0.743		0.940	0.709
Netherlands	1.040	0.822	0.745	0.609
New Zealand	0.471	0.342	-0.504	0.083
Spain	0.884		0.396	0.976
US	0.820	0.330	0.850	0.670

Table 3 - Endowment economy

							H	**L**		
g	ρ =1	**b_1**	**b_{30}**			**R_1**	2.25	1.81		
		-14.62	15.12			**R_{30}**	2.15	1.92		
	ρ =0	**b_1**	**b_{30}**			**R_1**	6.35	-2.27		
		-0.79	0.82			**R_{30}**	2.25	1.82		
							H	**L**		
θ	ρ =1	**b_1**	**b_{30}**			**R_1**	1.92	2.15		
		-7.34	7.50			**R_{30}**	2.12	1.96		
	ρ =0	**b_1**	**b_{30}**			**R_1**	-2.49	6.58		
		-0.17	0.17			**R_{30}**	1.87	2.20		
							HH	**HL**	**LH**	**LL**
g and θ	ρ =1	**b_1**	**b_6**	**b_{20}**	**b_{30}**	**R_1**	2.11	2.37	1.71	1.91
		447.22	-1055.32	1271.71	-578.42	**R_{30}**	2.06	2.24	1.84	2.00
	ρ =0.333	**b_1**	**b_2**	**b_3**	**b_{30}**	**R_1**	1.57	8.24	-3.81	1.83
		15.88	-210.91	498.94	-301.64	**R_{30}**	2.02	2.35	1.73	2.04

Table 4 - Model with capital

g			**b_1**	**b_{30}**				**H**	**L**		
	ρ =1		-24.89	21.52			**R_1**	2.09	1.92		
		min	-30.25	15.1			**R_{30}**	2.10	1.95		
		max	-17.90	28.27							
			b_1	**b_{30}**							
	ρ =0		-9.16	7.09			**R_1**	2.08	1.93		
		min	-9.56	6.68			**R_{30}**	2.04	2.01		
		max	-8.76	7.49							
θ			**b_1**	**b_{16}**				**H**	**L**		
	ρ =1		12.34	-14.77			**R_1**	2.26	1.84		
		min	9.35	-17.63			**R_{16}**	2.14	1.94		
		max	15.37	-11.87							
			b_1	**b_{30}**							
	ρ =0		-3.50	1.48			**R_1**	2.01	2.07		
		min	-3.87	1.24			**R_{30}**	2.03	2.06		
		max	-3.17	1.77							
g and θ			**b_1**	**b_5**	**b_{16}**	**b_{30}**		**HH**	**HL**	**LH**	**LL**
	ρ =1		-327.93	811.11	-900.02	414.17	**R_1**	2.83	1.70	2.58	1.45
		min	-389.64	619.32	-1113.65	272.41	**R_{30}**	2.21	2.08	2.03	1.89
		max	-244.70	976.68	-628.20	523.27					
			b_1	**b_7**	**b_{14}**	**b_{29}**					
	ρ =0.333		-16.44	20.36	-18.82	12.78	**R_1**	2.12	2.02	2.02	1.92
		min	-23.10	11.49	-22.12	11.42	**R_{29}**	2.04	2.07	2.00	2.03
		max	-11.51	29.36	-13.54	13.69					

The positions and the interst rates are obtained as average of 10000 period simulation. The minima and maxima are 5% of the lowest and highest values of the bonds in the simulation.

Table 5: Model with consumption habits

		χ										
				b_1	b_{10}				H	L		
	No habits	0		-31.14	31.64			R_1	2.25	1.81		
			min	-	-			R_{10}	2,21	1.86		
			max	-	-							
g												
				b_1	b_{10}							
	Habits	0.507		-8.11	7.68			R_1	-1.11	6.22		
			min	-8.50	7.27			R_{10}	1.48	2.70		
			max	-7.76	8.09							
				b_1	b_{10}	b_{20}	b_{30}		HH	HL	LH	LL
	No habits	0		283.95	-1083.87	1504.97	-760.92	R_1	2.11	2.37	1.71	1.91
			min	-	-	-	-	R_{30}	2.06	2.24	1.84	2.00
			max	-	-	-	-					
g and θ												
				b_1	b_{10}	b_{19}	b_{27}					
	Habits	0.365		-0.18	-63.62	117.96	-54.81	R_1	2.00	8.37	-1.40	1.77
			min	-0.19	-117.34	-66.45	-275.50	R_{27}	2.08	2.47	1.67	1.97
			max	-0.16	-4.62	508.94	76.04					

The positions and the interst rates are obtained as average of 10000 period simulation. The minima and maxima are 5% of the lowest and highest values of the bonds in the simulation

Table 6: Model without buyback - steady state values

			$\mathbf{b_1}$	$\mathbf{b_{30}}$	
				tot long term	*new issues*
g	*corr*	*no buy back*	-26.28	27.78	0.66
		buy back	-14.62	15.12	15.12
	iid	*no buy back*	-0.82	0.86	0.02
		buy back	-0.79	0.82	0.82

Figure 1: **Money Market Instruments (% of Debt) 1993-2003**

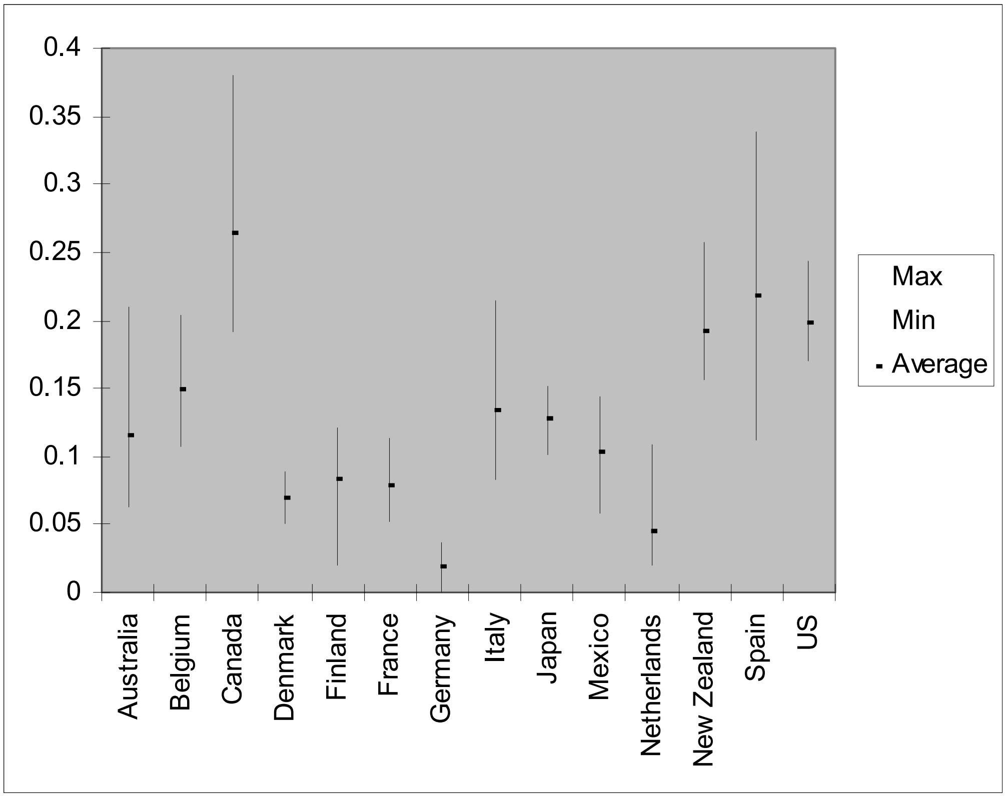


Figure 2: **Short Term Debt (% of Debt) 1993-2003**

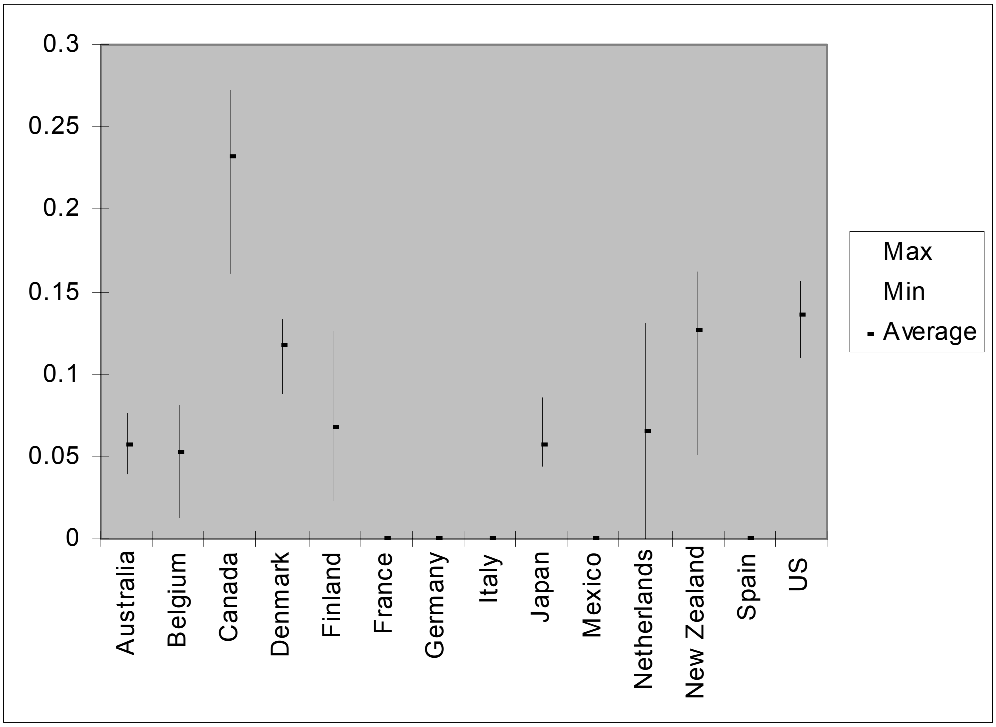


Figure 3: Medium Term Debt (% of Debt) 1993-2003

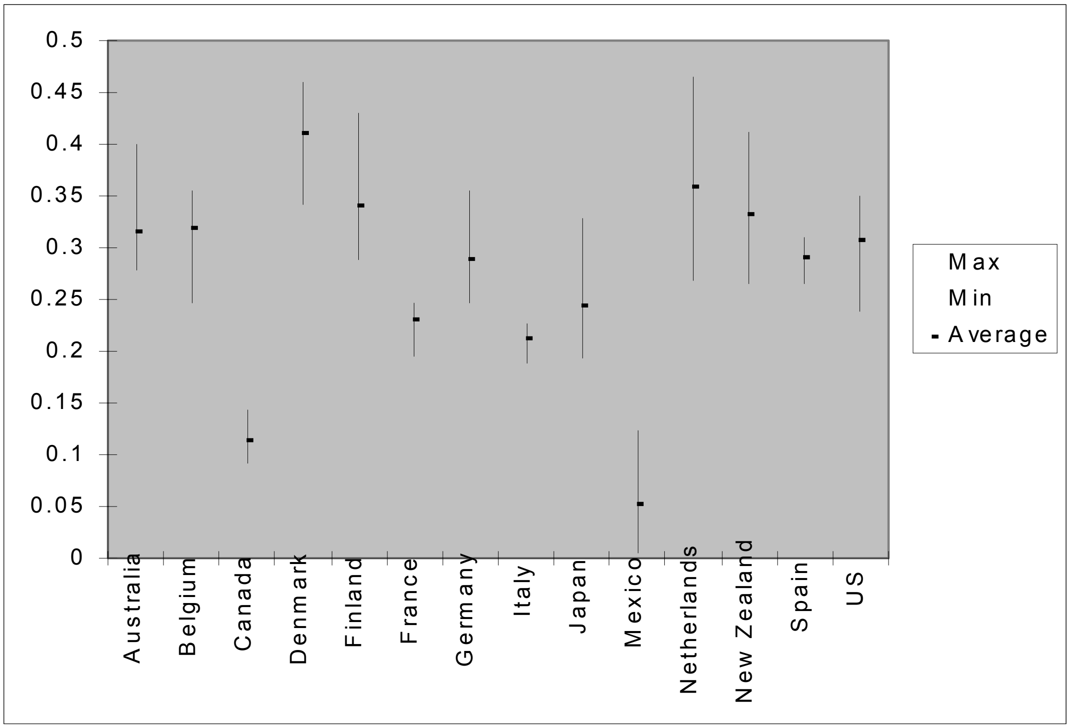


Figure 4: Long Term Debt (% of Debt) 1993-2003

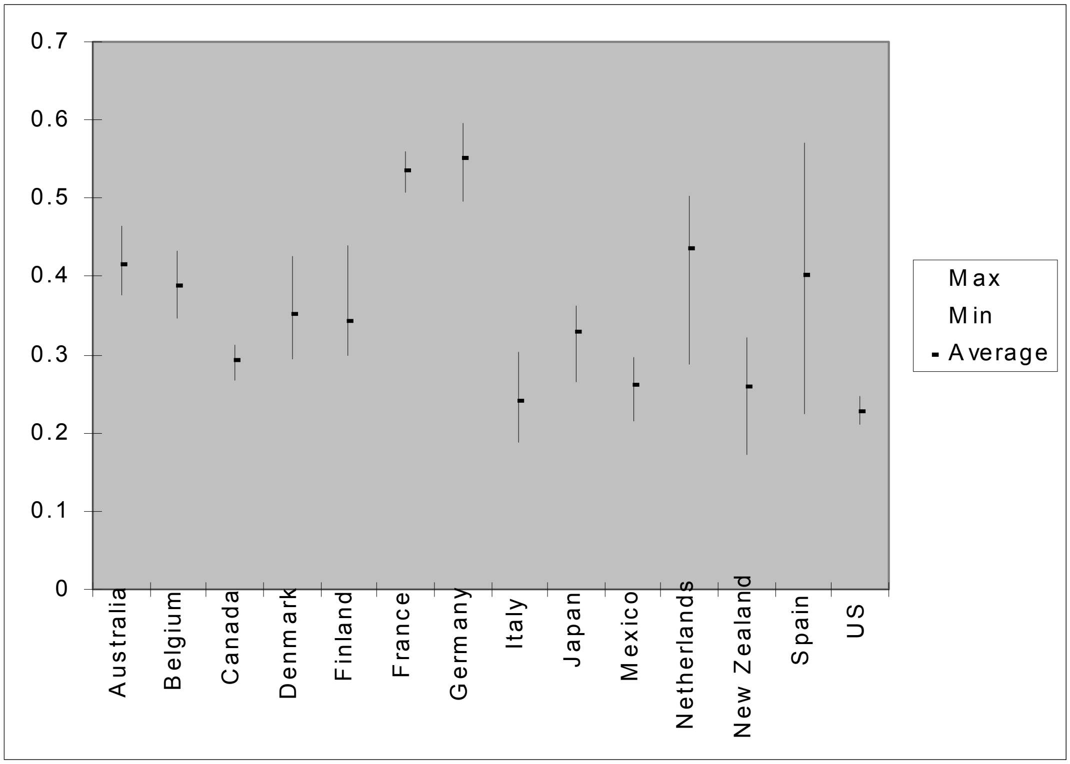


Figure 5: **Average OECD: Debt Composition 1993-2003**

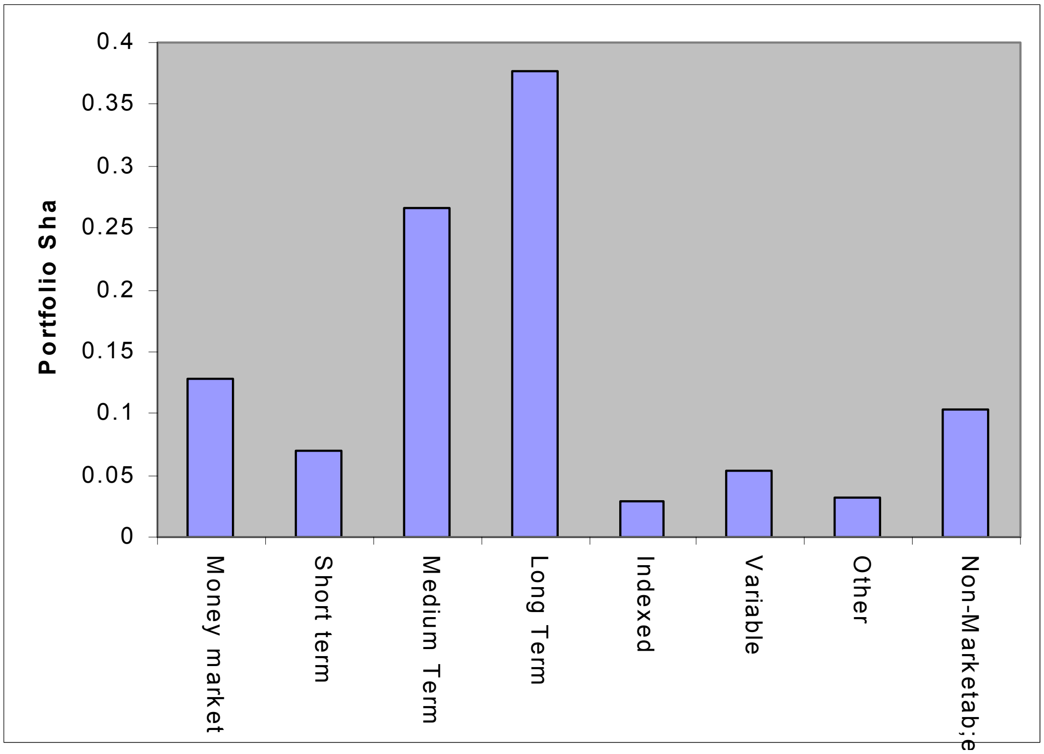


Figure 6: **Evolution of US Marketable Debt Composition**

Figure 7: **Capital - Tech-shocks: corr. shocks**

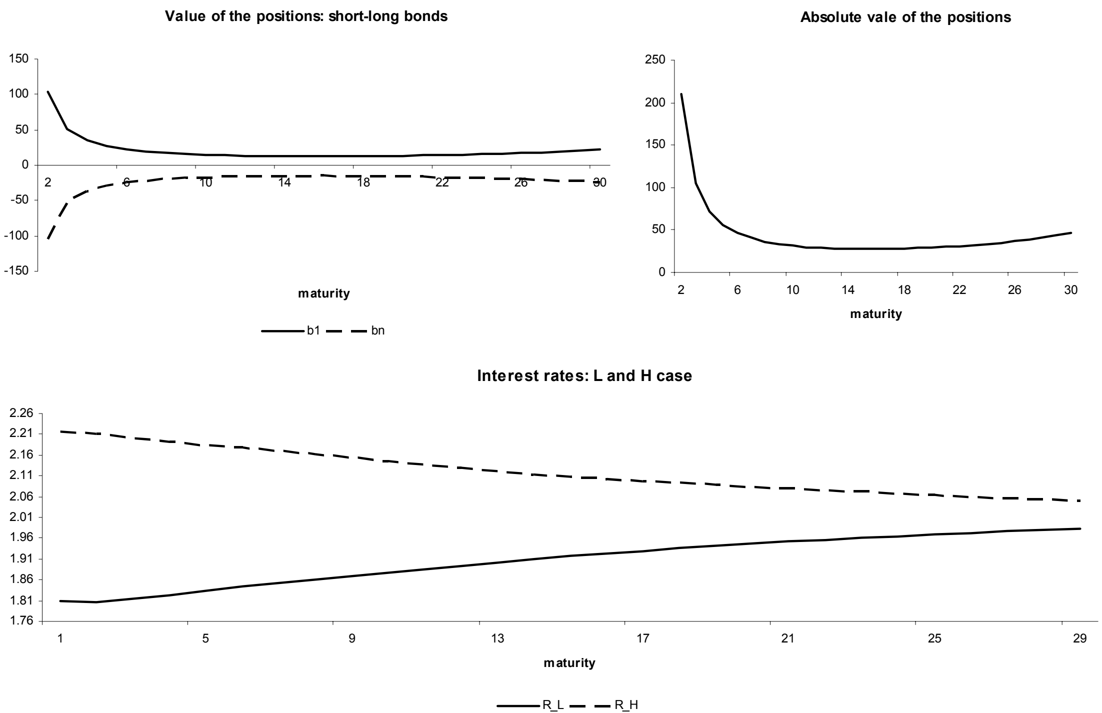


Figure 8: Policy functions consumption habits 4 shocks

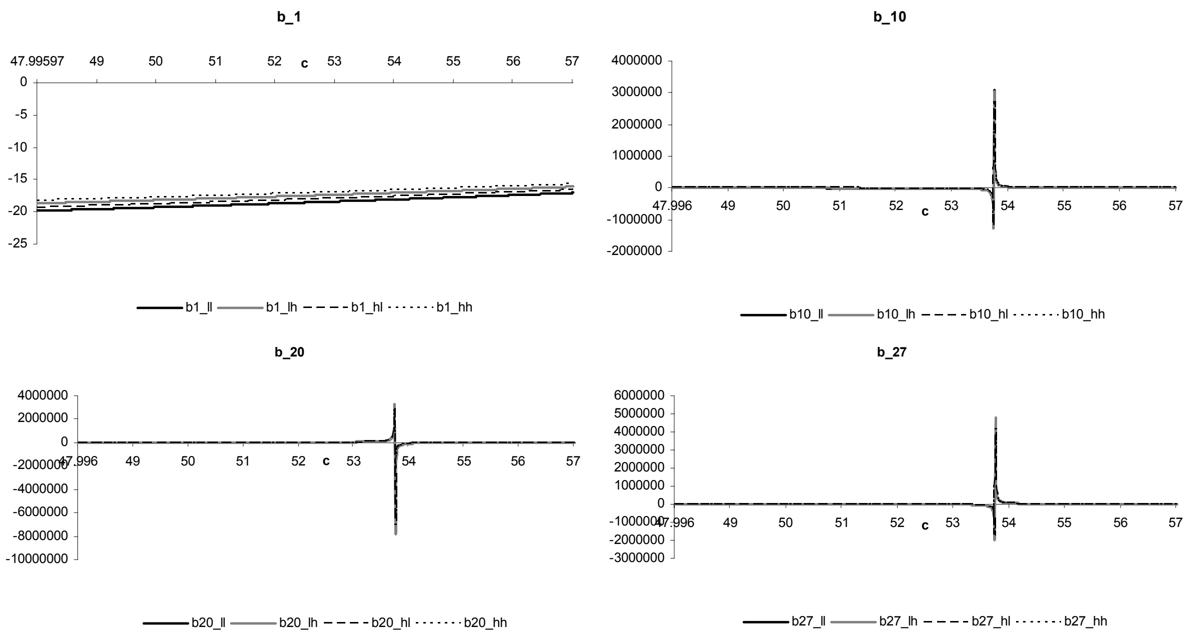
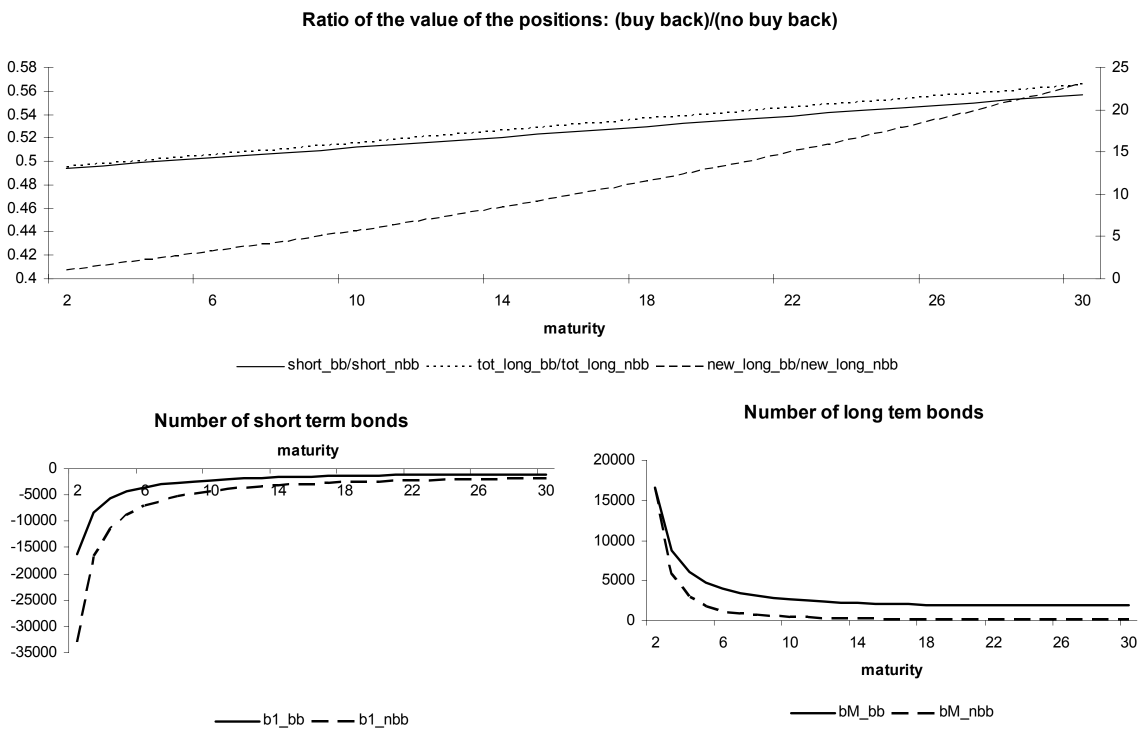
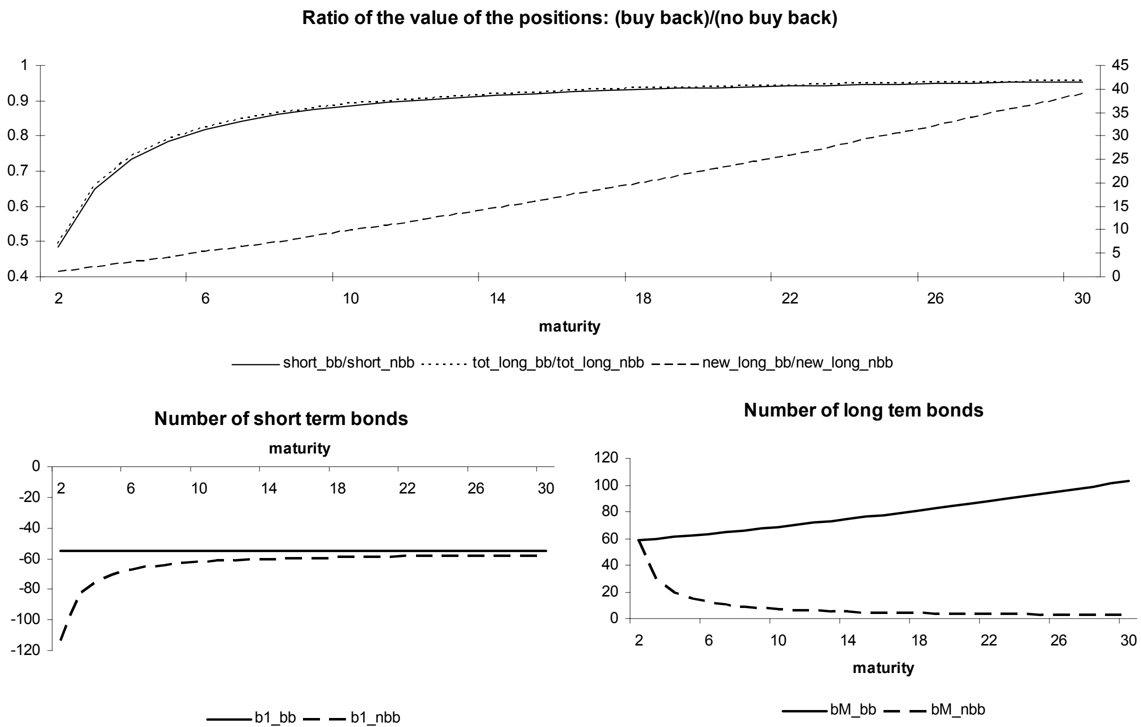


Figure 9: **No buy back - g-shocks: correlated shocks**



bb - buy back, nbb - no buy back

Figure 10: **No buy back - G-shocks: iid shocks**



bb - buy back, nbb - no buy back